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► **To cite this version:**

| Francois Cuvelier. Geometrical optics formulaes for Helmholtz equation. 2013. hal-00785774

**HAL Id: hal-00785774**

**<https://inria.hal.science/hal-00785774>**

Preprint submitted on 7 Feb 2013

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# GEOMETRICAL OPTICS FORMULAE FOR HELMHOLTZ EQUATION

FRANÇOIS CUVELIER

ABSTRACT. The present work deals with high frequency Helmholtz equation resolution using geometrical optics. We give formulae in dimension 2 and 3 for mixed Dirichlet, Neumann and Robin boundaries conditions.

## 1. INTRODUCTION

Let  $(K_j)_{j=1,\dots,N}$  be a set of regular, disjoint and strictly convex compacts in  $\mathbb{R}^d$   $d = 2$  or  $3$ . We suppose that for  $j \neq l$ ,  $K_j \cap K_l = \emptyset$ . Let  $\Omega_j = K_j^c$  and  $\Omega = \bigcap_{j=1}^N \Omega_j$ . We note  $\Gamma_j$  the boundary of  $K_j$  and

$$\Gamma = \bigcup_{j=1}^N \Gamma_j.$$

Let  $u^{inc}$  be the incident wave given by

$$u^{inc}(x) = e^{-ik\varphi^{inc}(x)} a^{inc}(x) \quad (1.1)$$

which satisfy Helmholtz equation in  $\mathbb{R}^d$ . We want to solve, at *high frequency*, Helmholtz equation

$$\Delta u + k^2 u = 0, \text{ in } \Omega \quad (1.2)$$

with boundaries conditions

$$u(\gamma) = -u^{inc}(\gamma), \quad \forall \gamma \in \Gamma_D, \quad (1.3)$$

$$\frac{\partial u}{\partial n}(\gamma) = -\frac{\partial u^{inc}}{\partial n}(\gamma), \quad \forall \gamma \in \Gamma_N, \quad (1.4)$$

$$\alpha(k) \frac{\partial u}{\partial n}(\gamma) + \beta(k) u(\gamma) = -\left( \alpha(k) \frac{\partial u^{inc}}{\partial n}(\gamma) + \beta(k) u^{inc}(\gamma) \right), \quad \forall \gamma \in \Gamma_R, \quad (1.5)$$

and outgoing Sommerfeld radiation condition

$$r^2 \left( \frac{\partial u}{\partial r} - iku \right) \text{ bound for } r = |x| \rightarrow \infty. \quad (1.6)$$

We assume that

$$\alpha(k) = \sum_{j=0}^{\infty} \frac{\alpha_j}{k^j}, \quad \alpha_j \in \mathbb{C}, \quad (1.7)$$

$$\beta(k) = \sum_{j=-1}^{\infty} \frac{\beta_j}{k^j}, \quad \beta_j \in \mathbb{C}, \quad (1.8)$$

and

$$\beta_{-1} + \iota \alpha_0 \mu \neq 0 \quad \forall \mu \in [0; 1] \quad (1.9)$$

Here we take  $\Gamma_D, \Gamma_N$  and  $\Gamma_R$  such that  $\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_R$ ,  $\overset{\circ}{\Gamma}_D \cap \overset{\circ}{\Gamma}_N = \emptyset$ ,  $\overset{\circ}{\Gamma}_D \cap \overset{\circ}{\Gamma}_R = \emptyset$  and  $\overset{\circ}{\Gamma}_N \cap \overset{\circ}{\Gamma}_R = \emptyset$ .

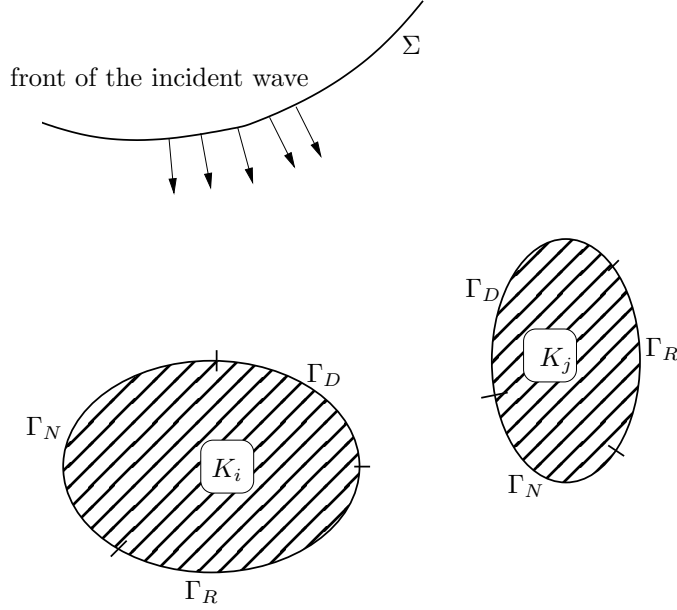


FIGURE 1. Exterior Helmholtz problem

We assume that,  $\Sigma$ , the front of the incident wave is given by

$$\Sigma = \{x \in \mathbb{R}^d \mid \varphi^{inc}(x) = \varphi_\Sigma\}$$

is a regular and orientable surface ( $d = 3$ ) or curve ( $d = 2$ ). We chose the orientation  $N(\sigma) = \frac{\nabla \varphi^{inc}(\sigma)}{\|\nabla \varphi^{inc}(\sigma)\|}$ .

Let  $\Sigma_+ = \{x \in \mathbb{R}^d \mid \exists \sigma \in \Sigma, \exists t > 0, x = \sigma + t \nabla \varphi^{inc}(\sigma)\}$ , we also made following assumption :

- (i)  $\bigcup_{j=1}^N K_j \subset \Sigma_+$
- (ii) curvature radius of  $\Sigma$  are negatives

At *high frequency* (means that wave length is small with respect to the obstacle boundary curvature radius), geometrical optics approximation for this problem is based on  $k^{-1}$  asymptotic expansion of  $v(x) = u(x) + u^{inc}(x)$ , where  $u$  is solution of problem (1.2,1.3,1.4,1.5,1.6). The first term of the  $k^{-1}$  expansion is the geometrical optics ansatz.

## 2. NOTATIONS AND RESULTS

**Definition 1** (Geometrical optic ray). Let  $\rho = (\sigma, \gamma_1, \dots, \gamma_l) = (\gamma_0, \gamma_1, \dots, \gamma_l) \in \Sigma \times \Gamma^l$ , we say that  $\rho$  is a geometrical optic ray coming from  $\Sigma$ , going through  $x = \gamma_{l+1}$  and reflected  $l$  times if

- (1)  $\gamma_j \neq \gamma_{j-1}, \forall j \in \{1, \dots, l+1\}$ ,
- (2)  $\left(\bigcup_{j=1}^{l+1} ]\gamma_{j-1}; \gamma_j[ \right) \cap \Gamma = \emptyset$ ,
- (3)  $\frac{\gamma_1 - \gamma_0}{|\gamma_1 - \gamma_0|} = \nabla \varphi^{inc}(\gamma_0)$ ,
- (4) **reflections conditions** :  $\forall j \in \{1, \dots, l\}$

$$\frac{\gamma_{j+1} - \gamma_j}{|\gamma_{j+1} - \gamma_j|} = \frac{\gamma_j - \gamma_{j-1}}{|\gamma_j - \gamma_{j-1}|} - 2 \left\langle \frac{\gamma_j - \gamma_{j-1}}{|\gamma_j - \gamma_{j-1}|}, \mathbf{n}(\gamma_j) \right\rangle \mathbf{n}(\gamma_j),$$

- (5) **non grazing conditions** :  $\forall j \in \{1, \dots, l\}$

$$\left\langle \frac{\gamma_j - \gamma_{j-1}}{|\gamma_j - \gamma_{j-1}|}, \mathbf{n}(\gamma_j) \right\rangle \neq 0,$$

where  $\mathbf{n}(\gamma)$  note the exterior normal at  $\Gamma$  in  $\gamma$ .

We note  $\mathcal{R}_l(x)$  a such set of  $\rho$  and  $\mathcal{R}(x) = \bigcup_{l \in \mathbb{N}^*} \mathcal{R}_l(x)$ .

**Definition 2** (Grazing ray). Let  $\rho = (\sigma, \gamma_1, \dots, \gamma_l) = (\gamma_0, \gamma_1, \dots, \gamma_l) \in \Sigma \times \Gamma^l$ , we say that  $\rho$  is a grazing ray coming from  $\Sigma$ , going through  $x = \gamma_{l+1}$  and reflected  $l$  times if conditions (1) to (4) of definition 1 are satisfied and not condition (5). We note  $\mathcal{G}_l(x)$  a such set of  $\rho$  and  $\mathcal{G}(x) = \bigcup_{l \in \mathbb{N}^*} \mathcal{G}_l(x)$ .

**Definition 3** (Propagation operator). Let  $t > 0$ , we note  $S_t^d \subset \mathbb{R}^{d \times d}$  the set of matrix  $\mathbb{A}$  such that  $\mathbb{I} + t\mathbb{A}$  is regular. We note  $\mathcal{S}_t$  the following application :

$$\mathcal{S}_t : \begin{array}{l} S_t^d \longrightarrow \mathbb{R}^{d \times d} \\ \mathbb{A} \longmapsto \mathbb{A}(\mathbb{I} + t\mathbb{A})^{-1} \end{array}$$

**Definition 4** (Reflection operator). Let  $\mathbb{B} \in \mathbb{R}^{d \times d}$  a symmetric matrix. Let  $\eta \in \mathbb{R}^d$  and  $\zeta \in \mathbb{R}^d$  such that  $\langle \zeta, \eta \rangle \neq 0$ . We note  $\mathcal{T}_{\mathbb{B}, \eta, \zeta}^d : \mathbb{R}^{d \times d} \longrightarrow \mathbb{R}^{d \times d}$  given by :

$$\begin{aligned} (\mathcal{T}_{\mathbb{B}, \eta, \zeta}^d(\mathbb{A}))x &= (\mathbb{A} - 2\langle \zeta, \eta \rangle \mathbb{B})x - 2\langle \eta, x \rangle (\mathbb{A}\eta + \mathbb{B}\zeta) \\ &\quad - 2\langle \mathbb{A}\eta + \mathbb{B}\zeta, x \rangle \eta + 2 \left[ 2\langle \mathbb{A}\eta, \eta \rangle - \frac{\langle \mathbb{B}\zeta, \zeta \rangle}{\langle \zeta, \eta \rangle} \right] \langle \eta, x \rangle \eta \end{aligned}$$

**Definition 5** (Reflection coefficient). Let  $\eta \in \mathbb{R}^d$ , we note  $b_\eta$  the function defined on  $\Gamma$  by

$$b_\eta(\gamma) = \begin{cases} -1, & \text{if } \gamma \in \Gamma_D, \\ 1, & \text{if } \gamma \in \Gamma_N, \\ \frac{i\alpha_0 \langle \eta, \mathbf{n}(\gamma) \rangle + \beta_{-1}}{i\alpha_0 \langle \eta, \mathbf{n}(\gamma) \rangle - \beta_{-1}}, & \text{if } \gamma \in \Gamma_R. \end{cases}$$

**Theorem 1.** Let  $x \in \Sigma_+$ , such that  $\mathcal{G}(x) = \emptyset$ , and  $\rho = (\gamma_0, \gamma_1, \dots, \gamma_l) \in \mathcal{R}_l(x)$ ,  $l \in \mathbb{N}^*$ . We note  $\mathbb{B}(\gamma)$  the curvature matrix of  $\Gamma$  at  $\gamma$  and  $\mathbf{n}(\gamma)$  the exterior normal to  $\Gamma$  at  $\gamma$ . Let  $u_\rho^{G.O.}(x)$  be the ray contribution :

$$u_\rho^{G.O.}(x) = a_\rho(x) e^{-ik\varphi(x; \rho)}$$

with

$$\varphi(x; \rho) = \varphi_\rho^{(l)}(\gamma_l) + \|x - \gamma_l\|$$

$$a_\rho(x) = \frac{a_\rho^{(l)}(\gamma_l)}{\sqrt{\det(\mathbb{I} + \|x - \gamma_l\| \mathbb{A}_\rho^{(l)}(\gamma_l))}}$$

and for  $j = l$  to 1,

$$\begin{cases} \varphi_\rho^{(j)}(\gamma_j) &= \varphi_\rho^{(j-1)}(\gamma_{j-1}) + \|\gamma_j - \gamma_{j-1}\| \\ a_\rho^{(j)}(\gamma_j) &= b_{\frac{\gamma_j - \gamma_{j-1}}{\|\gamma_j - \gamma_{j-1}\|}}(\gamma_j) \frac{a_\rho^{(j-1)}(\gamma_{j-1})}{\sqrt{\det(\mathbb{I} + \|\gamma_j - \gamma_{j-1}\| \mathbb{A}_\rho^{(j-1)}(\gamma_{j-1}))}} \\ \mathbb{A}_\rho^{(j)}(\gamma_j) &= \mathcal{T}_{\mathbb{B}(\gamma_j), \mathbf{n}(\gamma_j), \frac{\gamma_j - \gamma_{j-1}}{\|\gamma_j - \gamma_{j-1}\|}}^d \circ \mathcal{S}_{\|\gamma_j - \gamma_{j-1}\|} \left( \mathbb{A}_\rho^{(j-1)}(\gamma_{j-1}) \right) \end{cases}$$

where  $\varphi_\rho^{(0)}(\gamma_0) = \varphi^{inc}(\gamma_0)$ ,  $\mathbb{A}_\rho^{(0)}(\gamma_0) = \text{Hess } \varphi^{inc}(\gamma_0)$  and  $a_\rho^{(0)}(\gamma_0) = a_0^{inc}(\gamma_0)$ .

Then, we have

$$u^{inc}(x) = \sum_{\rho \in \mathcal{R}_0(x)} u_\rho^{G.O.}(x) + O\left(\frac{1}{k}\right)$$

and

$$u(x) = \sum_{\rho \in \mathcal{R}(x)} u_\rho^{G.O.}(x) + O\left(\frac{1}{k}\right)$$

### 3. PROOF

To obtain theorem 1, we first give geometrical optics formulae without obstacle, then with a stricly convov compact and finally with an union of stricly convov compacts.

3.1. **Without obstacles.** We want to solve, at high frequency, Helmholtz equation

$$\Delta u + k^2 u = 0 \text{ in } \mathbb{R}^d \quad (3.1)$$

using an asymptotic expansion in  $k^{-1}$  of the solution

$$u(x; k) = \sum_{j \in \mathbb{N}} \frac{a_j(x)}{k^j} e^{-ik\varphi(x)} \quad (3.2)$$

where we suppose that  $u$  is known on  $\Sigma$ .

By substituting expansion (3.2) in Helmholtz equation, we obtain

$$\begin{aligned} & k^2(1 - \|\nabla \varphi(x)\|^2) a_0(x) \\ & + k \left[ -i \{2\langle \nabla \varphi(x), \nabla a_0(x) \rangle + a_0(x) \Delta \varphi(x)\} + (1 - \|\nabla \varphi(x)\|^2) a_1(x) \right] \\ & + \sum_{j \in \mathbb{N}} k^{-j} \left( -i \{2\langle \nabla \varphi(x), \nabla a_{k+1}(x) \rangle + a_{k+1}(x) \Delta \varphi(x)\} + \Delta a_k(x) \right) = 0 \end{aligned}$$

Equating the coefficients of different powers of  $k$ , we have :

- **Eikonal equation**

$$\|\nabla \varphi(x)\|^2 = 1 \quad (3.3)$$

- **Transport equation for  $a_0$**

$$\langle \nabla \varphi(x), \nabla a_0(x) \rangle + \frac{1}{2} a_0(x) \Delta \varphi(x) = 0 \quad (3.4)$$

- **Transport equation for  $a_j$  in function of  $a_{j-1}$**

$$\langle \nabla \varphi(x), \nabla a_j(x) \rangle + \frac{1}{2} a_j(x) \Delta \varphi(x) = \frac{i}{2} \Delta a_{j-1}(x) \quad (3.5)$$

3.1.1. *Solution of Eikonal equation (3.3).* Derivating eikonal equation (3.3) give

$$\text{Hess } \varphi(x) \nabla \varphi(x) = 0 \quad (3.6)$$

This equation can be solved by the characteristics method :

Let  $X'(t) = \nabla \varphi(X(t))$  with  $X(0) = \sigma \in \Sigma$  then we have :

$$\begin{aligned} \frac{d}{dt} (\nabla \varphi(X(t))) &= \text{Hess } \varphi(X(t)) X'(t) \\ &= \text{Hess } \varphi(X(t)) \nabla \varphi(X(t)) \\ &= 0 \end{aligned}$$

That is to say

$$\nabla \varphi(X(t)) = \nabla \varphi(X(0))$$

hence

$$X(t) = X(0) + t \nabla \varphi(X(0)).$$

and  $\nabla \varphi(X(t))$  is constant along the line  $X(t) = X(0) + t \nabla \varphi(X(0))$ .

Owing to the construction of  $\Sigma$ , we have

$$X(t) = \sigma + t \mathbf{N}(\sigma)$$

with  $\sigma \in \Sigma$  and  $\mathbf{N}(\sigma) = \nabla \varphi(\sigma)$  unitary normal vector to  $\Sigma$  at  $\sigma$ .

We also have :

$$\begin{aligned} \frac{d}{dt} (\varphi(X(t))) &= \langle \nabla \varphi(X(t)), X'(t) \rangle \\ &= \langle \nabla \varphi(X(0)), \nabla \varphi(X(0)) \rangle \\ &= 1 \end{aligned}$$

Thus

$$\varphi(X(t)) = \varphi(X(0)) + t.$$

We have proved following lemma :

**Lemma 1.** (1) *Characteristic curves of Eikonal equation are geometrical optic rays,*  
(2) *Phase is linear along geometrical optic rays :*

$$\forall \sigma \in \Sigma, \forall t \geq 0 \quad \varphi(\sigma + t \mathbf{N}(\sigma)) = \varphi(\sigma) + t. \quad (3.7)$$

(3) *We have  $\forall \sigma \in \Sigma$*

$$\nabla \varphi(\sigma) = \mathbf{N}(\sigma) \quad (3.8)$$

and

$$\text{Hess } \varphi(\sigma) \mathbf{N}(\sigma) = 0. \quad (3.9)$$

3.1.2. *Solution of transport equation (3.4)* . We want to solve

$$\begin{cases} \langle \nabla \varphi(x), \nabla a_0(x) \rangle + \frac{1}{2} a_0(x) \Delta \varphi(x) = 0 \\ a_0 \text{ given on } \Sigma \end{cases} \quad (3.10)$$

In fact, we only have to solve this equation along the ray  $\sigma + t\mathbf{N}(\sigma)$  where  $\sigma \in \Sigma$  and  $\mathbf{N}(\sigma) = \nabla \varphi(\sigma)$ .

Thus, transport equation for  $a_0$  becomes

$$\begin{cases} \frac{d}{dt}(a_0(\sigma + t\mathbf{N}(\sigma))) + \frac{1}{2} \Delta \varphi(\sigma + t\mathbf{N}(\sigma)) a_0(\sigma + t\mathbf{N}(\sigma)) = 0, \\ a_0(\sigma) \text{ given on } \Sigma. \end{cases} \quad (3.11)$$

We now have to compute  $\Delta \varphi(\sigma + t\mathbf{N}(\sigma)) = \text{Tr Hess } \varphi(\sigma(t))$ .

Let  $\sigma \in \Sigma$ , we note for  $d = 3$  (resp.  $d = 2$ )  $\mathcal{B}_\Sigma(\sigma) = \{\mathbf{u}, \mathbf{v}, \mathbf{N}\}(\sigma)$  (resp.  $\mathcal{B}_\Sigma(\sigma) = \{\mathbf{u}, \mathbf{N}\}(\sigma)$ ) the **direct orthonormal curvature basis** of  $\Sigma$  at  $\sigma$ . Here  $\mathbf{u}$  and  $\mathbf{v}$  are the direction of maximum and minimal principal curvature  $k_1^{(0)}(\sigma)$  and  $k_2^{(0)}(\sigma)$  (resp.  $\mathbf{u}$  is the tangent vector and  $k^{(0)}(\sigma)$  the curvature). By hypothesis on  $\Sigma$ , we have

$$k_2^{(0)}(\sigma) \leq k_1^{(0)}(\sigma) \leq 0 \quad (\text{resp. } k^{(0)}(\sigma) \leq 0).$$

With these notations, we have in basis  $\mathcal{B}_\Sigma(\sigma)$

$$\text{Hess } \varphi(\sigma) = \begin{pmatrix} -k_1^{(0)}(\sigma) & 0 & 0 \\ 0 & -k_2^{(0)}(\sigma) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{resp. } \text{Hess } \varphi(\sigma) = \begin{pmatrix} -k^{(0)}(\sigma) & 0 \\ 0 & 0 \end{pmatrix})$$

**Lemma 2.** *Let  $\sigma \in \Sigma$ . We note  $\sigma(t) = \sigma + t\mathbf{N}(\sigma)$  then*

$$\forall t \geq 0, (\text{Hess } \varphi)(\sigma(t)) = \mathcal{S}_t(\text{Hess } \varphi(\sigma)) \quad (3.12)$$

*In basis  $\mathcal{B}_\Sigma(\sigma)$  for  $d = 3$  (resp.  $d = 2$ )*

$$\text{Hess } \varphi(\sigma(t)) = \begin{pmatrix} -k_1^{(0)}(\sigma(t)) & 0 & 0 \\ 0 & -k_2^{(0)}(\sigma(t)) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{resp. } \text{Hess } \varphi(\sigma(t)) = \begin{pmatrix} -k^{(0)}(\sigma(t)) & 0 \\ 0 & 0 \end{pmatrix})$$

*with*

$$0 \geq k_j^{(0)}(\sigma(t)) = \frac{k_j^{(0)}(\sigma)}{1 - tk_j^{(0)}(\sigma)} \geq k_j^{(0)}(\sigma) \quad j = 1, 2$$

$$(\text{resp. } 0 \geq k^{(0)}(\sigma(t)) = \frac{k^{(0)}(\sigma)}{1 - tk^{(0)}(\sigma)} \geq k^{(0)}(\sigma))$$

*Proof.* To simplify notations, we note  $\forall t \geq 0$

$$\mathbb{A}_t(\sigma) = (\text{Hess } \varphi)(\sigma + t\mathbf{N}(\sigma)).$$

Let  $\mathcal{V}(\sigma) \subset \mathbb{R}^d$  be a neighborhood of  $\sigma \in \Sigma$ . We have,  $\forall x \in \mathcal{V}(\sigma)$

$$\nabla \varphi(x) = \nabla \varphi(\sigma) + \mathbb{A}_0(\sigma)(x - \sigma) + O(\|x - \sigma\|) \quad (3.13)$$

Substituting  $\nabla \varphi(x)$  by (3.13) in Eikonal equation (3.3) give,  $\forall x \in \mathcal{V}(\sigma)$  :

$$\begin{aligned} 1 &= \left\| \nabla \varphi(\sigma) + \mathbb{A}_0(\sigma)(x - \sigma) + O(\|x - \sigma\|^2) \right\|^2 \\ &= \|\nabla \varphi(\sigma)\|^2 + 2 \langle \mathbb{A}_0(\sigma)(x - \sigma), \nabla \varphi(\sigma) \rangle + O(\|x - \sigma\|^2) \\ &= 1 + 2 \langle \mathbb{A}_0(\sigma)(x - \sigma), \mathbf{N}(\sigma) \rangle + O(\|x - \sigma\|^2). \end{aligned}$$

Using  $\mathbb{A}_0(\sigma)$  symmetry, we obtain

$$\forall x \in \mathcal{V}(\sigma), \langle \mathbb{A}_0(\sigma)\mathbf{N}(\sigma), x - \sigma \rangle = 0$$

Thus

$$\mathbb{A}_0(\sigma)\mathbf{N}(\sigma) = 0. \quad (3.14)$$

We now want to obtain a similar formula for  $\mathbb{A}_t(\sigma)$ .

We know that  $\mathbb{A}_0(\sigma)$  is the curvature matrix of  $\Sigma$  at  $\sigma$  and in basis  $\mathcal{B}_\Sigma(\sigma)$  we have for  $d = 3$  (resp.  $d = 2$ )

$$\mathbb{A}_0(\sigma) = \begin{pmatrix} -k_1^\Sigma(\sigma) & 0 & 0 \\ 0 & -k_2^\Sigma(\sigma) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\text{resp. } \mathbb{A}_0(\sigma) = \begin{pmatrix} -k^\Sigma(\sigma) & 0 \\ 0 & 0 \end{pmatrix}).$$

By hypothesis, curvature radius of  $\Sigma$  are negative and

$$k_2^\Sigma(\sigma) \leq k_1^\Sigma(\sigma) \leq 0 \quad (\text{resp. } k^\Sigma(\sigma) \leq 0)$$

Thus,  $\forall t \geq 0$ ,  $\mathbb{I} + t\mathbb{A}_0(\sigma)$  is regular. Let  $x = \sigma + t\mathbf{N}(\sigma)$  with  $t > 0$ , then, there exists a neighborhood  $\mathcal{V}(x) \subset \mathbb{R}^d$  of  $x$  such that

$$\forall x' \in \mathcal{V}(x), \exists \sigma' \in \mathcal{V}(\sigma) \text{ and } t' \text{ in a neighborhood of } t \text{ verifying } x' = \sigma' + t'\mathbf{N}(\sigma')$$

So we obtain by Taylor's developpement

$$\nabla \varphi(x') = \nabla \varphi(x) + \mathbb{A}_t(\sigma)(x' - x) + O(\|x' - x\|) \quad (3.15)$$

Substituing  $\nabla \varphi(x')$  for (3.15) in eikonal equation (3.3) give

$$\mathbb{A}_t(\sigma)\mathbf{N}(\sigma) = 0 \quad (3.16)$$

To prove formula (3.12) we write

$$\begin{aligned} x' - x &= \sigma' + t'\mathbf{N}(\sigma') - \sigma - t\mathbf{N}(\sigma) \\ &= \sigma' - \sigma + (t' - t)\mathbf{N}(\sigma) + t'(\mathbf{N}(\sigma') - \mathbf{N}(\sigma)). \end{aligned}$$

But, due to (3.13)

$$\mathbf{N}(\sigma') - \mathbf{N}(\sigma) = \mathbb{A}_0(\sigma)(\sigma' - \sigma) + O(\|\sigma' - \sigma\|) \quad (3.17)$$

and, thus

$$x' - x = \sigma' - \sigma + (t' - t)\mathbf{N}(\sigma) + t'\mathbb{A}_0(\sigma)(\sigma' - \sigma) + O(\|\sigma' - \sigma\|). \quad (3.18)$$

Substituing  $x' - x$  for (3.18) in (3.15) and using (3.16) give

$$\mathbf{N}(\sigma') = \mathbf{N}(\sigma) + \mathbb{A}_t(\sigma)(\sigma' - \sigma) + t'\mathbb{A}_t(\sigma)\mathbb{A}_0(\sigma)(\sigma' - \sigma) + O(\|\sigma' - \sigma\| + \|t' - t\|).$$

Now, with formula (3.17) we found

$$\begin{aligned} &\mathbb{A}_0(\sigma)(\sigma' - \sigma) + O(\|\sigma' - \sigma\|) \\ &= \\ &\mathbb{A}_t(\sigma)(\sigma' - \sigma) + t'\mathbb{A}_t(\sigma)\mathbb{A}_0(\sigma)(\sigma' - \sigma) + O(\|\sigma' - \sigma\| + \|t' - t\|) \end{aligned}$$

That is to say

$$\begin{aligned} \mathbb{A}_0(\sigma) &= \mathbb{A}_t(\sigma) + t\mathbb{A}_t(\sigma)\mathbb{A}_0(\sigma) \\ &= \mathbb{A}_t(\sigma)(\mathbb{I} + t\mathbb{A}_0(\sigma)). \end{aligned}$$

But,  $\forall t \geq 0$ , the matrix  $(\mathbb{I} + t\mathbb{A}_0(\sigma))$  is regular and formula (3.12) is proved.  $\square$

**Lemma 3.** *Let  $\sigma \in \Sigma$ . Along the ray  $\sigma(t) = \sigma + t\mathbf{N}(\sigma)$ ,  $\forall t \geq 0$ , we have*

$$a_0(\sigma(t)) = \frac{a_0(\sigma)}{\sqrt{\det(\mathbb{I} + t\text{Hess } \varphi(\sigma))}} \quad (3.19)$$

and  $\forall t > 0$ ,  $\forall t' \geq 0$  such that  $t > t'$

$$|a_0(\sigma(t))| \leq |a_0(\sigma(t'))|. \quad (3.20)$$

The previous inequality becomes strict if, at least, one of the curvature radius of  $\Sigma$  is strictly negative.

*Proof.* By definition of  $\mathbb{A}_t(\sigma)$  (see proof of lemma 2)

$$\Delta \varphi(\sigma(t)) = \text{Tr}(\mathbb{A}_t(\sigma))$$

Then we solve equation (3.11):

$$\begin{aligned} a_0(\sigma(t)) &= a_0(\sigma)e^{-\frac{1}{2} \int_0^t \text{Tr } \mathbb{A}_\tau(\sigma) d\tau} \\ &= a_0(\sigma)e^{-\frac{1}{2} \int_0^t \text{Tr}[\mathbb{A}_0(\sigma)(\mathbb{I} + \tau\mathbb{A}_0(\sigma))^{-1}] d\tau} \\ &= a_0(\sigma)e^{-\frac{1}{2} [\log \det(\mathbb{I} + \tau\mathbb{A}_0(\sigma))]_0^t} \\ &= \frac{a_0(\sigma)}{\sqrt{\det(\mathbb{I} + t\mathbb{A}_0(\sigma))}}. \end{aligned}$$

We have seen, in proof of lemma 2, that eigenvalues of  $\mathbb{A}_0(\sigma)$  are positive, thus

$$\det(\mathbb{I} + t'\mathbb{A}_0(\sigma)) \geq \det(\mathbb{I} + t\mathbb{A}_0(\sigma)) \geq 1.$$

and inequality (3.20) is immediatly proved.

We can remark that if, at least, one of the curvature of  $\Sigma$  at  $\sigma$  is strictly negative then, for  $d = 3$  (resp.  $d = 2$ )

$$k_2^\Sigma(\sigma) < k_1^\Sigma(\sigma) \leq 0 \quad (\text{resp. } k^\Sigma(\sigma) < 0)$$

and thus inequality (3.20) become strict.  $\square$

3.1.3. *Conclusion.* The knowledge of  $a_0$ ,  $\varphi$ ,  $\nabla \varphi$  and  $\text{Hess } \varphi$  on  $\Sigma$  is sufficient to compute them for all  $x$  in  $\Sigma_+$  using formulae along geometrical optic rays comming through  $x$  and we have

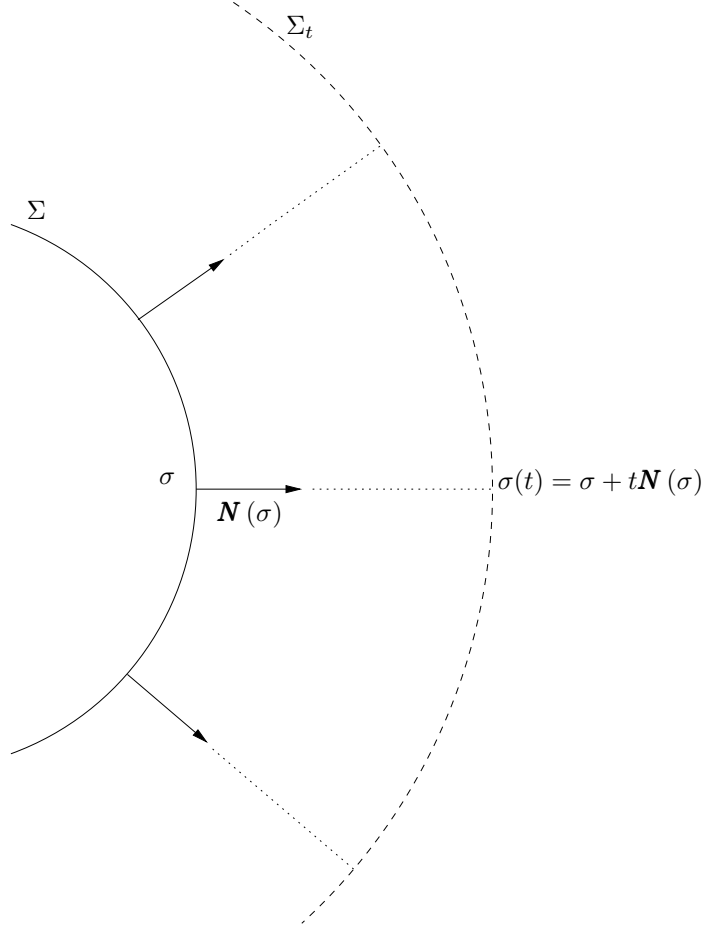


FIGURE 2. Wave Propagation

**Theorem 2.** *Let  $x \in \Sigma_+$ , then there exist an unique  $\sigma \in \Sigma$  such that*

$$x = \sigma + \|x - \sigma\| \mathbf{N}(\sigma) \quad (3.21)$$

*and solution of (3.1-3.2) is given by*

$$u(x; k) = \frac{a_0(\sigma)}{\sqrt{\det(\mathbb{I} + \|x - \sigma\| \text{Hess } \varphi(\sigma))}} e^{-ik(\varphi(\sigma) + \|x - \sigma\|)} + O\left(\frac{1}{k}\right) \quad (3.22)$$

*Proof.* By definition of  $\Sigma_+$ , we have existence of  $\sigma \in \Sigma$  satisfying (3.21).

For unicity, we suppose there exists  $\sigma_1 \in \Sigma$  and  $\sigma_2 \in \Sigma$  satisfying (3.21). By convexity hypothesis on  $\Sigma$ , we have

$$\langle \sigma_2 - \sigma_1, \mathbf{N}(\sigma_1) \rangle \leq 0$$

and then

$$\langle \sigma_2 - \sigma_1, x - \sigma_1 \rangle \leq 0.$$

But

$$\langle \sigma_2 - \sigma_1, x - \sigma_1 \rangle = \langle \sigma_2 - \sigma_1, x - \sigma_2 \rangle + \|\sigma_2 - \sigma_1\|^2$$

so we obtain

$$\langle \sigma_2 - \sigma_1, x - \sigma_2 \rangle \leq 0. \quad (3.23)$$

By convexity hypothesis on  $\Sigma$ , we have

$$\langle \sigma_1 - \sigma_2, \mathbf{N}(\sigma_2) \rangle \leq 0$$



and then

$$\langle \sigma_1 - \sigma_2, x - \sigma_2 \rangle \leq 0. \quad (3.24)$$

Thus, inequalities (3.23) and (3.24) give us  $\sigma_1 \equiv \sigma_2$ .

Formula (3.22) is just an application of propagation Lemmas (lemma 1 and lemma 3) along the ray  $\sigma \in \mathcal{R}_0(x)$ .  $\square$

**3.2. Outside a strictly convex compact.** Let  $K \subset \mathbb{R}^d$  be a regular and strictly convex compact,  $\Omega = K^c$  and  $\Gamma = \partial K$  be its boundary.

**Remark 1.** Owing to the strict convexity of  $K$  and the hypotheses on  $\Sigma$ , we have

$$\forall x \in \Sigma_+ \cap \Omega, \forall l > 1, \mathcal{R}_l(x) = \emptyset.$$

There is only one reflexion on  $K$ .

We denote by

$$\Gamma^s = \{\gamma \in \Gamma \mid \mathcal{R}_0(\gamma) = \emptyset\} \text{ and } \Gamma^e = (\Gamma^s)^c$$

Let  $\gamma \in \Gamma^e$  and  $\sigma \in \mathcal{R}_0(\gamma)$ . Using previous formulas give us  $a_0^{(0)}$ ,  $\varphi^{(0)}$ ,  $\nabla \varphi^{(0)}$  and  $\text{Hess } \varphi^{(0)}$  on  $\gamma$ .

If we can compute reflected wave on  $\gamma$  (i.e.  $a_0^{(1)}$ ,  $\varphi^{(1)}$ ,  $\nabla \varphi^{(1)}$  and  $\text{Hess } \varphi^{(1)}$  on  $\gamma$ ) then we can use propagation formulae along the reflected ray given by  $\gamma + t \nabla \varphi^{(1)}(\gamma)$ ,  $t > 0$

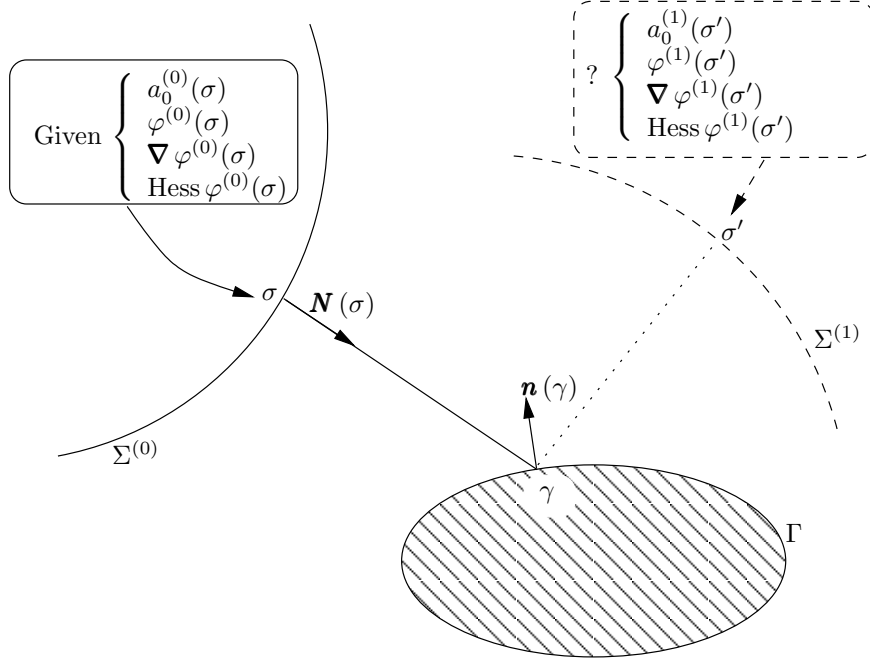


FIGURE 3. Wave reflection

So, the local problem is only to find how to compute the reflected wave in  $\gamma$ . That is to say :

$$\begin{cases} a_0^{(0)}(\gamma) \\ \varphi^{(0)}(\gamma) \\ \nabla \varphi^{(0)}(\gamma) \\ \text{Hess } \varphi^{(0)}(\gamma) \end{cases} \text{ given, how to compute } \begin{cases} a_0^{(1)}(\gamma) \\ \varphi^{(1)}(\gamma) \\ \nabla \varphi^{(1)}(\gamma) \\ \text{Hess } \varphi^{(1)}(\gamma) \end{cases} ?$$

3.2.1. Computation of  $\varphi^{(1)}$  and  $\nabla \varphi^{(1)}$  in  $\gamma$ .

**Lemma 4.** Let  $\gamma \in \Gamma$ , then

$$\varphi^{(1)}(\gamma) = \varphi^{(0)}(\gamma) \quad (3.25)$$

*Proof.* Due to boundary condition :

- If  $\gamma \in \Gamma_D$ , we have

$$a^{(1)}(\gamma)e^{-ik\varphi^{(1)}(\gamma)} = -a^{(0)}(\gamma)e^{-ik\varphi^{(0)}(\gamma)} \quad (3.26)$$

- If  $\gamma \in \Gamma_N$ , we have

$$\begin{aligned} & \left( \frac{\partial a^{(1)}}{\partial n}(\gamma) - ika^{(1)}(\gamma) \frac{\partial \varphi^{(1)}}{\partial n}(\gamma) \right) e^{-ik\varphi^{(1)}(\gamma)} \\ & \quad = \\ & - \left( \frac{\partial a^{(0)}}{\partial n}(\gamma) - ika^{(0)}(\gamma) \frac{\partial \varphi^{(0)}}{\partial n}(\gamma) \right) e^{-ik\varphi^{(0)}(\gamma)} \end{aligned} \quad (3.27)$$

- If  $\gamma \in \Gamma_R$ , we have

$$\begin{aligned} & \left[ \alpha(k) \left( \frac{\partial a^{(1)}}{\partial n}(\gamma) - ika^{(1)}(\gamma) \frac{\partial \varphi^{(1)}}{\partial n}(\gamma) \right) + \beta(k)a^{(1)}(\gamma) \right] e^{-ik\varphi^{(1)}(\gamma)} \\ & \quad = \\ & - \left[ \alpha(k) \left( \frac{\partial a^{(0)}}{\partial n}(\gamma) - ika^{(0)}(\gamma) \frac{\partial \varphi^{(0)}}{\partial n}(\gamma) \right) + \beta(k)a^{(0)}(\gamma) \right] e^{-ik\varphi^{(0)}(\gamma)} \end{aligned} \quad (3.28)$$

By identification, we immediately obtain (3.25).  $\square$

**Lemma 5.** Let  $\gamma \in \Gamma$ , and  $\mathbf{n}(\gamma)$  the exterior normal to  $\Gamma$  at  $\gamma$ . We have

$$\nabla \varphi^{(1)}(\gamma) = \nabla \varphi^{(0)}(\gamma) - 2 \left\langle \nabla \varphi^{(0)}(\gamma), \mathbf{n}(\gamma) \right\rangle \mathbf{n}(\gamma) \quad (3.29)$$

*Proof.* Let  $\mathcal{B}_\Gamma(\gamma) = \{\mathbf{u}(\gamma), \mathbf{v}(\gamma), \mathbf{n}(\gamma)\}$  be the **direct orthonormal curvature basis** of  $\Gamma$  at  $\gamma$ . Then we have the local parametrization of  $\Gamma$  at  $\gamma$  :

$$\gamma(u, v) = \gamma + (u, v, g(u, v))$$

with  $g(u, v) = \frac{1}{2}(k_1^\Gamma(\gamma)u^2 + k_2^\Gamma(\gamma)v^2) + o(u^2 + v^2)$

Taylor's expansion at order 1 of  $\varphi^{(0)}$  and  $\varphi^{(1)}$  on  $\Gamma$  at point  $\gamma$  are

$$\begin{aligned} \varphi^{(0)}(\gamma(u, v)) &= \varphi^{(0)}(\gamma) + \left\langle \nabla \varphi^{(0)}(\gamma), u\mathbf{u}(\gamma) + v\mathbf{v}(\gamma) \right\rangle + O(u^2 + v^2) \\ \varphi^{(1)}(\gamma(u, v)) &= \varphi^{(1)}(\gamma) + \left\langle \nabla \varphi^{(1)}(\gamma), u\mathbf{u}(\gamma) + v\mathbf{v}(\gamma) \right\rangle + O(u^2 + v^2) \end{aligned}$$

Due to relation (3.25) we obtain

$$\left\langle \nabla \varphi^{(1)}(\gamma) - \nabla \varphi^{(0)}(\gamma), u\mathbf{u}(\gamma) + v\mathbf{v}(\gamma) \right\rangle + O(u^2 + v^2)$$

that is to say

$$\left\langle \nabla \varphi^{(1)}(\gamma) - \nabla \varphi^{(0)}(\gamma), \mathbf{u}(\gamma) \right\rangle = \left\langle \nabla \varphi^{(1)}(\gamma) - \nabla \varphi^{(0)}(\gamma), \mathbf{v}(\gamma) \right\rangle = 0$$

So, exists  $\lambda \in \mathbb{R}$  such that

$$\nabla \varphi^{(1)}(\gamma) = \nabla \varphi^{(0)}(\gamma) + \lambda \mathbf{n}(\gamma)$$

Taking the norm of previous relation and using eikonal equation (3.6) give

$$\lambda = -2 \left\langle \nabla \varphi^{(0)}(\gamma), \mathbf{n}(\gamma) \right\rangle.$$

A similar proof will act in dimension 2.  $\square$

### 3.2.2. Computation of $a_0^{(1)}$ in $\gamma$ .

**Lemma 6.** Let  $\gamma \in \Gamma^e$ . If we know  $a_0^{(0)}(\gamma)$  and  $\nabla \varphi^{(0)}(\gamma)$  then

$$a_0^{(1)}(\gamma) = b_{\nabla \varphi^{(0)}(\gamma)}(\gamma) a_0^{(0)}(\gamma). \quad (3.30)$$

where the function  $b_{\nabla \varphi^{(0)}(\gamma)}$  is given in definition 5.

*Proof.* • If  $\gamma \in \Gamma_D \cap \Gamma^e$ , formula (3.26) give

$$a^{(1)}(\gamma) = -a^{(0)}(\gamma)$$

Using asymptotic expansions of  $a^{(0)}$  and  $a^{(1)}$

$$\sum_{j \in \mathbb{N}} k^{-j} a_j^{(1)}(\gamma) = - \sum_{j \in \mathbb{N}} k^{-j} a_j^{(0)}(\gamma)$$

With high frequency hypothesis, we obtain the leading term in power of  $k$ :

$$a_0^{(1)}(\gamma) = -a_0^{(0)}(\gamma)$$

- If  $\gamma \in \Gamma_N \cap \Gamma^e$ , formula (3.27) give

$$\frac{\partial a^{(1)}}{\partial n}(\gamma) - \imath k a^{(1)}(\gamma) \frac{\partial \varphi^{(1)}}{\partial n}(\gamma) = - \left( \frac{\partial a^{(0)}}{\partial n}(\gamma) - \imath k a^{(0)}(\gamma) \frac{\partial \varphi^{(0)}}{\partial n}(\gamma) \right)$$

Using asymptotic expansions of  $a^{(0)}$  and  $a^{(1)}$

$$\begin{aligned} & \sum_{j \in \mathbb{N}} \left( k^{-j} \frac{\partial a_j^{(1)}}{\partial n}(\gamma) - \imath k^{-j+1} a_j^{(1)}(\gamma) \frac{\partial \varphi^{(1)}}{\partial n}(\gamma) \right) \\ &= \\ & - \sum_{j \in \mathbb{N}} \left( k^{-j} \frac{\partial a_j^{(0)}}{\partial n}(\gamma) - \imath k^{-j+1} a_j^{(0)}(\gamma) \frac{\partial \varphi^{(0)}}{\partial n}(\gamma) \right) \end{aligned}$$

The leading term in power of  $k$  is

$$a_0^{(1)}(\gamma) \frac{\partial \varphi^{(1)}}{\partial n}(\gamma) = -a_0^{(0)}(\gamma) \frac{\partial \varphi^{(0)}}{\partial n}(\gamma)$$

But lemma 5 give  $\frac{\partial \varphi^{(1)}}{\partial n}(\gamma) = -\frac{\partial \varphi^{(0)}}{\partial n}(\gamma)$ . Like  $\gamma \in \Gamma^e$ , we have  $\langle \nabla \varphi^{(0)}(\gamma), \mathbf{n}(\gamma) \rangle \neq 0$ . So we obtain

$$a_0^{(1)}(\gamma) = a_0^{(0)}(\gamma)$$

- If  $\gamma \in \Gamma_R \cap \Gamma^e$ , formula (3.28) give

$$\begin{aligned} & \alpha(k) \left( \frac{\partial a^{(1)}}{\partial n}(\gamma) - \imath k a^{(1)}(\gamma) \frac{\partial \varphi^{(1)}}{\partial n}(\gamma) \right) + \beta(k) a^{(1)}(\gamma) \\ &= - \left[ \alpha(k) \left( \frac{\partial a^{(0)}}{\partial n}(\gamma) - \imath k a^{(0)}(\gamma) \frac{\partial \varphi^{(0)}}{\partial n}(\gamma) \right) + \beta(k) a^{(0)}(\gamma) \right] \end{aligned}$$

Using asymptotic expansions of  $a^{(0)}$ ,  $a^{(1)}$ ,  $\alpha(k)$  and  $\beta(k)$

$$\begin{aligned} & \left( \sum_{j \in \mathbb{N}} k^{-j} \alpha_j \right) \times \sum_{j \in \mathbb{N}} \left( k^j \frac{\partial a_j^{(1)}}{\partial n}(\gamma) - \imath k^{-j+1} a_j^{(1)}(\gamma) \frac{\partial \varphi^{(1)}}{\partial n}(\gamma) \right) \\ &+ \left( \sum_{j \in \mathbb{N}} k^{-j+1} \beta_{j-1} \right) \times \sum_{j \in \mathbb{N}} k^{-j} a_j^{(1)}(\gamma) \\ &= - \left( \sum_{j \in \mathbb{N}} k^{-j} \alpha_j \right) \times \sum_{j \in \mathbb{N}} \left( k^j \frac{\partial a_j^{(0)}}{\partial n}(\gamma) - \imath k^{-j+1} a_j^{(0)}(\gamma) \frac{\partial \varphi^{(0)}}{\partial n}(\gamma) \right) \\ &- \left( \sum_{j \in \mathbb{N}} k^{-j+1} \beta_{j-1} \right) \times \sum_{j \in \mathbb{N}} k^{-j} a_j^{(0)}(\gamma) \end{aligned}$$

The leading term in power of  $k$  is

$$-\imath \alpha_0 a_0^{(1)}(\gamma) \frac{\partial \varphi^{(1)}}{\partial n}(\gamma) + \beta_{-1} a_0^{(1)}(\gamma) = - \left( -\imath \alpha_0 a_0^{(0)}(\gamma) \frac{\partial \varphi^{(0)}}{\partial n}(\gamma) + \beta_{-1} a_0^{(0)}(\gamma) \right)$$

But  $\frac{\partial \varphi^{(1)}}{\partial n}(\gamma) = -\frac{\partial \varphi^{(0)}}{\partial n}(\gamma)$ , so

$$a_0^{(1)}(\gamma) \left( \beta_{-1} + \imath \alpha_0 \frac{\partial \varphi^{(0)}}{\partial n} \right) = -a_0^{(0)}(\gamma) \left( \beta_{-1} - \imath \alpha_0 \frac{\partial \varphi^{(0)}}{\partial n} \right)$$

with hypothesis  $|\beta_{-1}| > |\alpha_0|$  we have

$$\beta_{-1} + \imath \alpha_0 \frac{\partial \varphi^{(0)}}{\partial n} \neq 0$$

So we obtain

$$a_0^{(1)}(\gamma) = - \frac{\beta_{-1} - \imath \alpha_0 \frac{\partial \varphi^{(0)}}{\partial n}}{\beta_{-1} + \imath \alpha_0 \frac{\partial \varphi^{(0)}}{\partial n}} a_0^{(0)}(\gamma)$$

□

We notice that

$$\forall \gamma \in \Gamma^e \quad |b_{\nabla \varphi^{(0)}(\gamma)}(\gamma)| = 1. \quad (3.31)$$

This is the case when there is no absorption in the boundary condition.

3.2.3. *Calculus of Hess  $\varphi^{(1)}$  at  $\gamma \in \Gamma^e$ .*

**Lemma 7.** *Let  $\gamma \in \Gamma^e$ . We note  $\mathbb{B}(\gamma)$  the curvature matrix of  $\Gamma$  at  $\gamma$ . Then, in dimension  $d = 2$  or  $3$ , we have*

$$\text{Hess } \varphi^{(1)}(\gamma) = \mathcal{T}_{\mathbb{B}(\gamma), \mathbf{n}(\gamma), \nabla \varphi^{(0)}(\gamma)}^d(\text{Hess } \varphi^{(0)}(\gamma)). \quad (3.32)$$

*Proof.* We have seen (lemma 4) that

$$\varphi^{(1)}(\gamma) = \varphi^{(0)}(\gamma).$$

Let  $\mathcal{V}_\Gamma(\gamma)$  be a neighborhood of  $\gamma$  in  $\Gamma$ . Taylor's expansion at order 1 of  $\nabla \varphi^{(0)}$  and  $\nabla \varphi^{(1)}$  are, for all  $\gamma' \in \mathcal{V}_\Gamma(\gamma)$ ,

$$\nabla \varphi^{(0)}(\gamma') = \nabla \varphi^{(0)}(\gamma) + \text{Hess } \varphi^{(0)}(\gamma)(\gamma' - \gamma) + o(|\gamma' - \gamma|) \quad (3.33)$$

and

$$\nabla \varphi^{(1)}(\gamma') = \nabla \varphi^{(1)}(\gamma) + \text{Hess } \varphi^{(1)}(\gamma)(\gamma' - \gamma) + o(|\gamma' - \gamma|) \quad (3.34)$$

Let  $\gamma(u, v)$  be the local parametrization of  $\Gamma$  at  $\gamma$  define in section ?? We compute now, Taylor's expansion at order 1 of  $\mathbf{n}(\gamma(u, v))$  in dimension 2 and 3. For that, we first evaluate

$$\begin{aligned} \left( \frac{\partial \gamma}{\partial u} \wedge \frac{\partial \gamma}{\partial v} \right) (u, v) &= \begin{pmatrix} -k_1^\Gamma(\gamma)u \\ -k_2^\Gamma(\gamma)v \\ 1 \end{pmatrix}_{\mathcal{B}_\Gamma(\gamma)} + O(u^2 + v^2), \\ &= \mathbf{n}(\gamma) + \mathbb{B}(\gamma)(\gamma(u, v) - \gamma) + O(u^2 + v^2). \end{aligned}$$

We obtain

$$\mathbf{n}(\gamma(u, v)) = \frac{\mathbf{n}(\gamma) + \mathbb{B}(\gamma)(\gamma(u, v) - \gamma)}{(1 + (k_1^\Gamma(\gamma)u)^2 + (k_2^\Gamma(\gamma)v)^2)^{\frac{1}{2}}} + O(u^2 + v^2).$$

But

$$(1 + (k_1^\Gamma(\gamma)u)^2 + (k_2^\Gamma(\gamma)v)^2)^{-\frac{1}{2}} = 1 + O(u^2 + v^2)$$

so, we have

$$\mathbf{n}(\gamma') = \mathbf{n}(\gamma) + \mathbb{B}(\gamma)(\gamma' - \gamma) + o(|\gamma' - \gamma|) \quad (3.35)$$

In similar way, we obtain the same formula in dimension  $d = 2$ .

Using formulas (3.33) and (3.35) in equation (3.29) gives

$$\begin{aligned} \nabla \varphi^{(1)}(\gamma') &= \nabla \varphi^{(1)}(\gamma) + (\text{Hess } \varphi^{(0)}(\gamma) - 2 \langle \nabla \varphi^{(0)}(\gamma), \mathbf{n}(\gamma) \rangle \mathbb{B}(\gamma))(\gamma' - \gamma) \\ &\quad - 2 \langle \nabla \varphi^{(0)}(\gamma), \mathbb{B}(\gamma)(\gamma' - \gamma) \rangle \mathbf{n}(\gamma) \\ &\quad - 2 \langle \text{Hess } \varphi^{(0)}(\gamma)(\gamma' - \gamma), \mathbf{n}(\gamma) \rangle \mathbf{n}(\gamma) + o(|\gamma' - \gamma|) \end{aligned}$$

So we obtain with formula (3.34), for all  $\gamma' \in \mathcal{V}_\Gamma(\gamma)$ ,

$$\begin{aligned} \text{Hess } \varphi^{(1)}(\gamma)(\gamma' - \gamma) &= (\text{Hess } \varphi^{(0)}(\gamma) - 2 \langle \nabla \varphi^{(0)}(\gamma), \mathbf{n}(\gamma) \rangle \mathbb{B}(\gamma))(\gamma' - \gamma) \\ &\quad - 2 \langle \nabla \varphi^{(0)}(\gamma), \mathbb{B}(\gamma)(\gamma' - \gamma) \rangle \mathbf{n}(\gamma) \\ &\quad - 2 \langle \text{Hess } \varphi^{(0)}(\gamma)(\gamma' - \gamma), \mathbf{n}(\gamma) \rangle \mathbf{n}(\gamma) + o(|\gamma' - \gamma|) \end{aligned}$$

Now, we want to extend the previous formula for all  $x$  in  $\mathcal{V}(\gamma) \subset \mathbb{R}^d$ , a neighborhood of  $\gamma$ . So we must add to previous formula a function from  $\mathbb{R}^d$  to  $\mathbb{R}^d$  which vanishes on  $\Gamma$ . That is to say at order 2, there exists a constant vector  $C(\gamma) \in \mathbb{R}^d$  such that :

for all  $x \in \mathcal{V}(\gamma)$

$$\begin{aligned} \text{Hess } \varphi^{(1)}(\gamma)(x - \gamma) &= (\text{Hess } \varphi^{(0)}(\gamma) - 2 \langle \nabla \varphi^{(0)}(\gamma), \mathbf{n}(\gamma) \rangle \mathbb{B}(\gamma))(x - \gamma) \\ &\quad - 2 \langle \nabla \varphi^{(0)}(\gamma), \mathbb{B}(\gamma)(x - \gamma) \rangle \mathbf{n}(\gamma) \\ &\quad - 2 \langle \text{Hess } \varphi^{(0)}(\gamma)(x - \gamma), \mathbf{n}(\gamma) \rangle \mathbf{n}(\gamma) \\ &\quad + \langle x - \gamma, \mathbf{n}(\gamma) \rangle C(\gamma) + o(|x - \gamma|) \end{aligned} \quad (3.36)$$

To compute  $C(\gamma)$ , we take  $x \in \mathcal{V}(\gamma)$  such that  $x - \gamma = \varepsilon \nabla \varphi^{(1)}(\gamma)$  with  $\varepsilon > 0$  and use (3.6) for  $\varphi^{(0)}$  and  $\varphi^{(1)}$

$$\text{Hess } \varphi^{(0)}(x) \nabla \varphi^{(0)}(x) = 0$$

and

$$\text{Hess } \varphi^{(1)}(x) \nabla \varphi^{(1)}(x) = 0$$

So, formula (3.36) become

$$\begin{aligned}
& (\text{Hess } \varphi^{(0)}(\gamma) - 2 \langle \nabla \varphi^{(0)}(\gamma), \mathbf{n}(\gamma) \rangle \mathbb{B}(\gamma)) \nabla \varphi^{(1)}(\gamma) \\
& - 2 \langle \nabla \varphi^{(0)}(\gamma), \mathbb{B}(\gamma) (\nabla \varphi^{(1)}(\gamma)) \rangle \mathbf{n}(\gamma) \\
& - 2 \langle \text{Hess } \varphi^{(0)}(\gamma) (\nabla \varphi^{(1)}(\gamma)), \mathbf{n}(\gamma) \rangle \mathbf{n}(\gamma) \\
& + \langle \nabla \varphi^{(1)}(\gamma), \mathbf{n}(\gamma) \rangle C(\gamma) \qquad \qquad \qquad = 0
\end{aligned}$$

Using (3.29) and  $\mathbb{B}(\gamma) \mathbf{n}(\gamma) = 0$  we obtain

$$\mathbb{B}(\gamma) \nabla \varphi^{(1)}(\gamma) = \mathbb{B}(\gamma) \nabla \varphi^{(0)}(\gamma)$$

and so

$$\begin{aligned}
C(\gamma) &= 2 \left[ 2 \langle \text{Hess } \varphi^{(0)}(\gamma) \mathbf{n}(\gamma), \mathbf{n}(\gamma) \rangle - \frac{\langle \mathbb{B}(\gamma) \nabla \varphi^{(0)}(\gamma), \nabla \varphi^{(0)}(\gamma) \rangle}{\langle \nabla \varphi^{(0)}(\gamma), \mathbf{n}(\gamma) \rangle} \right] \mathbf{n}(\gamma) \\
&\quad - 2 [\text{Hess } \varphi^{(0)}(\gamma) \mathbf{n}(\gamma) + \mathbb{B}(\gamma) \nabla \varphi^{(0)}(\gamma)]
\end{aligned}$$

Replacing  $C(\gamma)$  by previous formula in (3.36) immediately give (3.32). □

These results are given in [eC89]

### 3.2.4. Properties of Hess $\varphi^{(1)}(\gamma)$ .

**Lemma 8.** *The matrix Hess  $\varphi^{(1)}(\gamma)$  is symmetric and*

$$\text{Hess } \varphi^{(1)}(\gamma) \nabla \varphi^{(1)}(\gamma) = 0. \tag{3.37}$$

In dimension  $d = 3$ , eigenvalues of Hess  $\varphi^{(1)}(\gamma)$  are  $(-k_1^{(1)}(\gamma), -k_2^{(1)}(\gamma), 0)$  and

$$k_2^{(1)}(\gamma) \leq k_1^{(1)}(\gamma) < 0, \quad k_1^{(1)}(\gamma) + k_2^{(1)}(\gamma) < k_1^{(0)}(\gamma) + k_2^{(0)}(\gamma) \quad \text{and} \quad k_1^{(1)}(\gamma)k_2^{(1)}(\gamma) < k_1^{(0)}(\gamma)k_2^{(0)}(\gamma).$$

In dimension  $d = 2$ , eigenvalues of Hess  $\varphi^{(1)}(\gamma)$  are  $(-k^{(1)}(\gamma), 0)$  and

$$k^{(1)}(\gamma) < k^{(0)}(\gamma) \leq 0.$$

*Proof.* As we have seen in section 3.1.2, in dimension  $d = 3$  (resp.  $d = 2$ ), eigenvalues of Hess  $\varphi^{(0)}(\gamma)$  are  $(-k_1^{(0)}(\gamma), -k_2^{(0)}(\gamma), 0)$  (resp.  $(-k^{(0)}(\gamma), 0)$ ) where  $k_2^{(0)}(\gamma) \leq k_1^{(0)}(\gamma) \leq 0$  (resp.  $k^{(0)}(\gamma) \leq 0$ ). So we only have to apply lemma 12 in dimension 2 or lemma 13 in dimension 3 to end the proof. □

### 3.2.5. Conclusion.

Outside a strictly convex compact, we proved following result

**Theorem 3.** *Let  $x \in \Sigma_+ \cap \Omega$ , such that  $\mathcal{G}(x) = \emptyset$ . Then  $\#\mathcal{R}(x) = \#\mathcal{R}_1(x) \leq 1$ ,  $\#\mathcal{R}_0(x) \leq 1$ , and*

$$u^{inc}(x) = \sum_{\rho \in \mathcal{R}_0(x)} u_\rho^{G.O.}(x) + O\left(\frac{1}{k}\right)$$

and

$$u(x) = \sum_{\rho \in \mathcal{R}_1(x)} u_\rho^{G.O.}(x) + O\left(\frac{1}{k}\right)$$

**3.3. Outside an union of strictly convex compacts.** To prove theorem 1, we just have to verify that we can use propagation and reflection lemmas along each ray coming through  $x$ . For that, we have

**Lemma 9.** *Let  $x \in \Sigma_+ \cap \Omega$ , such that  $\mathcal{G}(x) = \emptyset$ . Let  $l \in \mathbb{N}^*$  such that  $\mathcal{R}_l(x) \neq \emptyset$  and  $\rho = (\gamma_0, \gamma_1, \dots, \gamma_l) \in \mathcal{R}_l(x)$ . In dimension  $d = 3$  (resp.  $d = 2$ ), if eigenvalues of the symmetric matrix  $\mathbb{A}_\rho^{(0)}(\gamma_0)$  are  $(\lambda_1^{(0)}(\gamma_0), \lambda_2^{(0)}(\gamma_0), 0)$  (resp.  $(\lambda^{(0)}(\gamma_0), 0)$ ) where*

$$0 \leq \lambda_1^{(0)}(\gamma_0) \leq \lambda_2^{(0)}(\gamma_0) \quad (\text{resp. } 0 \leq \lambda^{(0)}(\gamma_0))$$

then  $\forall j \in \{1, \dots, l\}$  eigenvalues of  $\mathbb{A}_\rho^{(j)}(\gamma_j)$  are  $(\lambda_1^{(j)}(\gamma_j), \lambda_2^{(j)}(\gamma_j), 0)$  (resp.  $(\lambda^{(j)}(\gamma_j), 0)$ ) where

$$0 < \lambda_1^{(j)}(\gamma_j) \leq \lambda_2^{(j)}(\gamma_j) \quad (\text{resp. } 0 < \lambda^{(j)}(\gamma_j))$$

*Proof.* Due to hypothesis we can apply propagation lemma 2 to obtain that

$$\mathbb{A}_\rho^{(0)}(\gamma_1) = \mathcal{S}_{\|\gamma_1 - \gamma_0\|}(\mathbb{A}_\rho^{(0)}(\gamma_0))$$

is well defined. and its eigenvalues are  $(\lambda_1^{(0)}(\gamma_1), \lambda_2^{(0)}(\gamma_1), 0)$  where

$$0 \leq \lambda_1^{(0)}(\gamma_1) \leq \lambda_2^{(0)}(\gamma_1).$$

By hypothesis  $\rho \in \mathcal{R}_l(x)$  so  $\langle \gamma_1 - \gamma_0, \mathbf{n}(\gamma_1) \rangle < 0$  and we can apply reflection lemma 7 to obtain that

$$\mathbb{A}_\rho^{(1)}(\gamma_1) = \mathcal{T}_{\mathbb{B}(\gamma_1), \mathbf{n}(\gamma_1), \frac{\gamma_1 - \gamma_0}{\|\gamma_1 - \gamma_0\|}}(\mathbb{A}_\rho^{(0)}(\gamma_1))$$

is well defined.

Furthermore, lemma 8 give us that matrix  $\mathbb{A}_\rho^{(1)}(\gamma_1)$  is symmetric and its eigenvalues are  $(\lambda_1^{(1)}(\gamma_1), \lambda_2^{(1)}(\gamma_1), 0)$  where

$$0 < \lambda_1^{(1)}(\gamma_1) \leq \lambda_2^{(1)}(\gamma_1).$$

Then a simply recurrence proof give us lemma :

Let  $j \in \{1, \dots, l-1\}$  and suppose that  $\mathbb{A}_\rho^{(j)}(\gamma_j)$  is symmetric and its eigenvalues are  $(\lambda_1^{(j)}(\gamma_j), \lambda_2^{(j)}(\gamma_j), 0)$  where

$$0 < \lambda_1^{(j)}(\gamma_j) \leq \lambda_2^{(j)}(\gamma_j)$$

we can apply propagation lemma 2 to obtain that

$$\mathbb{A}_\rho^{(j)}(\gamma_{j+1}) = \mathcal{S}_{\|\gamma_{j+1} - \gamma_j\|}(\mathbb{A}_\rho^{(j)}(\gamma_j))$$

is well defined. and its eigenvalues are  $(\lambda_1^{(j)}(\gamma_{j+1}), \lambda_2^{(j)}(\gamma_{j+1}), 0)$  where

$$0 < \lambda_1^{(j)}(\gamma_{j+1}) \leq \lambda_2^{(j)}(\gamma_{j+1}).$$

By hypothesis  $\rho \in \mathcal{R}_l(x)$  so  $\langle \gamma_{j+1} - \gamma_j, \mathbf{n}(\gamma_{j+1}) \rangle < 0$  and we can apply reflection lemma 7 to obtain that

$$\mathbb{A}_\rho^{(j+1)}(\gamma_{j+1}) = \mathcal{T}_{\mathbb{B}(\gamma_{j+1}), \mathbf{n}(\gamma_{j+1}), \frac{\gamma_{j+1} - \gamma_j}{\|\gamma_{j+1} - \gamma_j\|}}(\mathbb{A}_\rho^{(j)}(\gamma_{j+1}))$$

is well defined.

Furthermore, lemma 8 give us that matrix  $\mathbb{A}_\rho^{(j+1)}(\gamma_{j+1})$  is symmetric and its eigenvalues are  $(\lambda_1^{(j+1)}(\gamma_{j+1}), \lambda_2^{(j+1)}(\gamma_{j+1}), 0)$  where

$$0 < \lambda_1^{(j+1)}(\gamma_{j+1}) \leq \lambda_2^{(j+1)}(\gamma_{j+1}).$$

A similar proof will act in dimension  $d = 2$ . □

With theorem 1 hypothesis and this lemma, we immediatly have

$$\forall j \in \{1, \dots, l\}, \det\left(\mathbb{I} + \|\gamma_j - \gamma_{j-1}\| \mathbb{A}_\rho^{(j-1)}(\gamma_{j-1})\right) > 0$$

and

$$\det\left(\mathbb{I} + \|x - \gamma_l\| \mathbb{A}_\rho^{(l)}(\gamma_l)\right) > 0.$$

So along each ray comming through  $x$  we can apply propagation and reflection lemmas. Then, by adding the contribution of each ray we obtain theorem 1 results.

**Lemma 10.** *Let  $x \in \Sigma_+ \cap \Omega$ , such that  $\mathcal{G}(x) = \emptyset$ . Let  $l > 1$  such that  $\mathcal{R}_l(x) \neq \emptyset$  and  $\rho = (\gamma_0, \gamma_1, \dots, \gamma_l) \in \mathcal{R}_l(x)$ . We have*

$$|a_\rho(x)| < \frac{|a_\rho^{(0)}(\gamma_0)|}{\left[(1 + 2d(x)\lambda_{\min}^\Gamma) (1 + 2d_{\min}\lambda_{\min}^\Gamma)^{l-1}\right]^{\frac{d-1}{2}}} \quad (3.38)$$

where

$$0 < d(x) = \min_{\gamma \in \Gamma} \|x - \gamma\|,$$

$$0 < \lambda_{\min}^\Gamma = \begin{cases} \min_{\gamma \in \Gamma} \lambda_1(\gamma) & \text{if } d = 3, \\ \min_{\gamma \in \Gamma} \lambda(\gamma) & \text{if } d = 2, \end{cases},$$

and

$$0 < d_{\min} = \min_{i \neq j} \min_{\gamma_i \in \Gamma_i, \gamma_j \in \Gamma_j} \|\gamma_i - \gamma_j\|.$$

*Proof.* Using (3.31) and definition of  $a_\rho^{(j)}(\gamma_j)$ , we obtain

$$\forall j \in \{1, \dots, l\}, |a_\rho^{(j+1)}(\gamma_{j+1})| = \frac{|a_\rho^{(j)}(\gamma_j)|}{\sqrt{\det \left( \mathbb{I} + \|\gamma_{j+1} - \gamma_j\| \mathbb{A}_\rho^{(j)}(\gamma_j) \right)}} \quad (3.39)$$

where  $t_j = \|\gamma_{j+1} - \gamma_j\|$  and  $\gamma_{j+1} = x$ .

**In dimension  $d = 2$ ,** we can use equation (4.1) of lemma 12 to obtain

$$\lambda^{(j)}(\gamma_j) = \lambda^{(j-1)}(\gamma_j) - 2t_{j-1} \frac{\lambda^\Gamma(\gamma_j)}{\langle \mathbf{n}(\gamma_j), \gamma_j - \gamma_{j-1} \rangle}$$

But we have  $\frac{1}{t_{j-1}} \langle \mathbf{n}(\gamma_j), \gamma_j - \gamma_{j-1} \rangle \in [-1; 0[$  and then

$$\lambda^{(j)}(\gamma_j) \geq \lambda^{(j-1)}(\gamma_j) + 2\lambda_{\min}^\Gamma > 0.$$

From proposition (1) we get

$$\lambda^{(j-1)}(\gamma_j) = \frac{\lambda^{(j-1)}(\gamma_{j-1})}{1 + t_{j-1} \lambda^{(j-1)}(\gamma_{j-1})}$$

and so

$$\begin{aligned} \lambda^{(j-1)}(\gamma_j) &> 0 \quad \forall j \in \{2, \dots, l\} \\ \lambda^{(0)}(\gamma_1) &\geq 0 \end{aligned} .$$

From equation (3.39), we immediately obtain

$$\begin{aligned} |a_\rho^{(1)}(\gamma_1)| &\leq |a_\rho^{(0)}(\gamma_0)| \\ |a_\rho^{(j+1)}(\gamma_{j+1})| &< |a_\rho^{(j)}(\gamma_j)| (1 + 2d_{\min} \lambda_{\min}^\Gamma)^{-1/2} \quad \forall j \in \{1, \dots, l-1\} \\ |a_\rho^{(l)}(x)| = |a_\rho(x)| &< |a_\rho^{(l)}(\gamma_l)| (1 + 2d(x) \lambda_{\min}^\Gamma)^{-1/2} \end{aligned}$$

and thus

$$|a_\rho(x)| < \frac{|a_\rho^{(0)}(\gamma_0)|}{\left[ (1 + 2d(x) \lambda_{\min}^\Gamma) (1 + 2d_{\min} \lambda_{\min}^\Gamma)^{l-1} \right]^{\frac{1}{2}}}.$$

**In dimension  $d = 3$ ,** We have

$$\det \left( \mathbb{I} + t_j \mathbb{A}_\rho^{(j)}(\gamma_j) \right) = 1 + (\lambda_1^{(j)}(\gamma_j) + \lambda_2^{(j)}(\gamma_j)) t_j + \lambda_1^{(j)}(\gamma_j) \lambda_2^{(j)}(\gamma_j) t_j^2$$

and we can use equations (4.5) and (4.6) of lemma 13 to obtain

$$\begin{aligned} \lambda_1^{(j)}(\gamma_j) + \lambda_2^{(j)}(\gamma_j) &\geq \lambda_1^{(j-1)}(\gamma_j) + \lambda_2^{(j-1)}(\gamma_j) + 4\lambda_{\min}^\Gamma > 0, \\ \lambda_1^{(j)}(\gamma_j) \lambda_2^{(j)}(\gamma_j) &\geq \lambda_1^{(j-1)}(\gamma_j) \lambda_2^{(j-1)}(\gamma_j) + 4(\lambda_{\min}^\Gamma)^2 > 0. \end{aligned}$$

So, with these inequalities, we have

$$\begin{aligned} \det \left( \mathbb{I} + t_j \mathbb{A}_\rho^{(j)}(\gamma_j) \right) &\geq 1 + 4\lambda_{\min}^\Gamma t_j + 4(\lambda_{\min}^\Gamma t_j)^2 + \left( \lambda_1^{(j-1)}(\gamma_j) + \lambda_2^{(j-1)}(\gamma_j) \right) t_j \\ &\quad + \left( \lambda_1^{(j-1)}(\gamma_j) \lambda_2^{(j-1)}(\gamma_j) \right) t_j^2 \end{aligned}$$

From proposition (1) we get

$$\lambda_\alpha^{(j-1)}(\gamma_j) = \frac{\lambda_\alpha^{(j-1)}(\gamma_{j-1})}{1 + t_{j-1} \lambda_\alpha^{(j-1)}(\gamma_{j-1})}, \quad \alpha \in \{1, 2\}$$

and, as  $\lambda_2^{(j-1)}(\gamma_{j-1}) \geq \lambda_1^{(j-1)}(\gamma_{j-1}) > 0$  for  $j \in \{2, \dots, l+1\}$  and  $\lambda_2^{(0)}(\gamma_0) \geq \lambda_1^{(0)}(\gamma_0) \geq 0$  (see lemma 13) we have

$$\begin{aligned} \lambda_2^{(j-1)}(\gamma_j) &\geq \lambda_1^{(j-1)}(\gamma_j) > 0 \quad \forall j \in \{2, \dots, l\} \\ \lambda_2^{(0)}(\gamma_1) &\geq \lambda_1^{(0)}(\gamma_1) \geq 0 \end{aligned} .$$

With these inequalities, we have

$$\begin{aligned} \det \left( \mathbb{I} + t_j \mathbb{A}_\rho^{(j)}(\gamma_j) \right) &> (1 + 2\lambda_{\min}^\Gamma t_j)^2 \quad \text{if } j > 1 \\ \det \left( \mathbb{I} + t_1 \mathbb{A}_\rho^{(1)}(\gamma_1) \right) &\geq (1 + 2\lambda_{\min}^\Gamma t_1)^2 \end{aligned}$$

From equation (3.39), we immediately obtain

$$\begin{aligned} |a_\rho^{(1)}(\gamma_1)| &\leq |a_\rho^{(0)}(\gamma_0)| \\ |a_\rho^{(j+1)}(\gamma_{j+1})| &< |a_\rho^{(j)}(\gamma_j)| (1 + 2d_{\min} \lambda_{\min}^\Gamma)^{-1} \quad \forall j \in \{1, \dots, l-1\} \\ |a_\rho^{(l)}(x)| = |a_\rho(x)| &< |a_\rho^{(l)}(\gamma_l)| (1 + 2d(x) \lambda_{\min}^\Gamma)^{-1} \end{aligned}$$

and thus

$$|a_\rho(x)| < \frac{|a_\rho^{(0)}(\gamma_0)|}{(1 + 2d(x) \lambda_{\min}^\Gamma) (1 + 2d_{\min} \lambda_{\min}^\Gamma)^{l-1}}.$$

□

#### 4. TECHNICAL RESULTS

**Proposition 1.** *Let  $t > 0$  and  $\mathbb{A} \in S_t^d$ . Assume that  $\lambda$  is an eigenvalue of  $\mathbb{A}$  with corresponding eigenvector  $\zeta$  then  $\frac{\lambda}{1+t\lambda}$  is an eigenvalue of  $\mathcal{S}_t(\mathbb{A})$  with corresponding eigenvector  $\zeta$ .*

*Proof.* By definition  $\mathcal{S}_t(\mathbb{A}) = \mathbb{A}(\mathbb{I} + t\mathbb{A})^{-1}$  and by hypothesis

$$(\mathbb{I} + t\mathbb{A})\zeta = (1 + t\lambda)\zeta$$

with  $1 + t\lambda \neq 0$ . So  $(1 + t\lambda)(\mathbb{I} + t\mathbb{A})^{-1}\zeta = \zeta$  and we obtain

$$(1 + t\lambda)\mathbb{A}(\mathbb{I} + t\mathbb{A})^{-1}\zeta = \mathbb{A}\zeta.$$

Thus

$$\mathbb{A}(\mathbb{I} + t\mathbb{A})^{-1}\zeta = \frac{\lambda}{1 + t\lambda}\zeta.$$

□

**Lemma 11.** *Let  $\eta \in \mathbb{R}^d$  and  $\zeta \in \mathbb{R}^d$  such that  $\|\eta\| = 1$  and  $\langle \eta, \zeta \rangle \neq 0$ . Let  $\mathbb{A}$  and  $\mathbb{B}$  symmetric matrices in  $\mathbb{R}^{d \times d}$ . Assume  $\mathbb{A}\zeta = 0$  and  $\mathbb{B}\eta = 0$  then*

$$(\mathcal{T}_{\mathbb{B}, \eta, \zeta}^d(\mathbb{A}))(\zeta - 2\langle \zeta, \eta \rangle \eta) = 0$$

*Proof.* Note  $\xi = \zeta - 2\langle \zeta, \eta \rangle \eta$  then, by definition,

$$\begin{aligned} (\mathcal{T}_{\mathbb{B}, \eta, \zeta}^d(\mathbb{A}))\xi &= \mathbb{A}\xi - 2\langle \zeta, \eta \rangle \mathbb{B}\xi - 2\langle \eta, \xi \rangle (\mathbb{A}\eta + \mathbb{B}\zeta) \\ &\quad - 2\langle \mathbb{A}\eta + \mathbb{B}\zeta, \xi \rangle \eta + 2 \left[ 2\langle \mathbb{A}\eta, \eta \rangle - \frac{\langle \mathbb{B}\zeta, \zeta \rangle}{\langle \zeta, \eta \rangle} \right] \langle \eta, \xi \rangle \eta \end{aligned}$$

Due to hypothesis, we have

$$\mathbb{A}\xi = -2\langle \zeta, \eta \rangle \mathbb{A}\eta, \quad \mathbb{B}\xi = \mathbb{B}\zeta, \quad \langle \eta, \xi \rangle = -\langle \zeta, \eta \rangle$$

and

$$\langle \mathbb{A}\eta + \mathbb{B}\zeta, \xi \rangle = -2\langle \zeta, \eta \rangle \langle \mathbb{A}\eta, \eta \rangle + \langle \mathbb{B}\zeta, \zeta \rangle.$$

So, we obtain

$$\begin{aligned} (\mathcal{T}_{\mathbb{B}, \eta, \zeta}^d(\mathbb{A}))\xi &= -2\langle \zeta, \eta \rangle (\mathbb{A}\eta + \mathbb{B}\zeta) + 2\langle \zeta, \eta \rangle (\mathbb{A}\eta + \mathbb{B}\zeta) + 4\langle \zeta, \eta \rangle \langle \mathbb{A}\eta, \eta \rangle \eta \\ &\quad - 2\langle \mathbb{B}\zeta, \zeta \rangle \eta - 2 \left[ 2\langle \mathbb{A}\eta, \eta \rangle - \frac{\langle \mathbb{B}\zeta, \zeta \rangle}{\langle \zeta, \eta \rangle} \right] \langle \eta, \zeta \rangle \eta \\ &= 0. \end{aligned}$$

□

**Lemma 12.** *Let  $\mathcal{B} = (\mathbf{u}, \mathbf{w})$  and  $\mathcal{B}^{(I)} = (\mathbf{u}^{(I)}, \mathbf{w}^{(I)})$  two direct orthonormal basis of  $\mathbb{R}^2$  such that  $\langle \mathbf{w}, \mathbf{w}^{(I)} \rangle < 0$ . Let  $\mathbb{B} = \text{diag}(\lambda, 0)$  in  $\mathcal{B}$  basis and  $\mathbb{A}^{(I)} = \text{diag}(\lambda^{(I)}, 0)$  in  $\mathcal{B}^{(I)}$  basis. Then  $\mathbb{A}^{(R)} = \mathcal{T}_{\mathbb{B}, \mathbf{w}, \mathbf{w}^{(I)}}^2(\mathbb{A}^{(I)})$  is a symmetric matrix having for eigenvalues  $(\lambda^{(R)}, 0)$  where*

$$\lambda^{(R)} = \lambda^{(I)} - 2 \frac{\lambda}{\langle \mathbf{w}, \mathbf{w}^{(I)} \rangle} \quad (4.1)$$

and

$$\mathbb{A}^{(R)} \left( \mathbf{w}^{(I)} - 2 \langle \mathbf{w}, \mathbf{w}^{(I)} \rangle \mathbf{w} \right) = 0 \quad (4.2)$$

If  $\lambda > 0$  and  $\lambda^{(I)} \geq 0$  then

$$\lambda^{(R)} > \lambda^{(I)}. \quad (4.3)$$



*Proof.* In basis  $\mathcal{B}$ , we note  $\mathbf{u}^{(I)} = (u_j)_{j=1}^2$ ,  $\mathbf{w}^{(I)} = (w_j)_{j=1}^2$  and  $\mathbb{A}^{(I)} = (a_{ij})_{i,j=1}^2$ . By hypothesis,  $\mathbb{A}^{(I)}$  is symmetric and  $\mathbb{A}^{(I)}\mathbf{w}^{(I)} = 0$ .

As  $\langle \mathbf{w}, \mathbf{w}^{(I)} \rangle \neq 0$ , we can compute  $\mathbb{A}^{(R)} : \forall x \in \mathbb{R}^2$

$$\begin{aligned} \mathbb{A}^{(R)}x &= (\mathbb{A}^{(I)} - 2\langle \mathbf{w}, \mathbf{w}^{(I)} \rangle \mathbb{B})x \\ &\quad - 2\langle \mathbf{w}, x \rangle (\mathbb{A}^{(I)}\mathbf{w} + \mathbb{B}\mathbf{w}^{(I)}) \\ &\quad - 2\langle \mathbb{A}^{(I)}\mathbf{w} + \mathbb{B}\mathbf{w}^{(I)}, x \rangle \mathbf{w} \\ &\quad + 2 \left[ 2\langle \mathbb{A}^{(I)}\mathbf{w}, \mathbf{w} \rangle - \frac{\langle \mathbb{B}\mathbf{w}^{(I)}, \mathbf{w}^{(I)} \rangle}{\langle \mathbf{w}, \mathbf{w}^{(I)} \rangle} \right] \langle \mathbf{w}, x \rangle \mathbf{w} \end{aligned}$$

in basis  $\mathcal{B}$ ,  $x = (x_j)_{j=1}^2$  and

$$\begin{aligned} \mathbb{A}^{(R)}x &= \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 2w_2 \begin{pmatrix} \lambda x_1 \\ 0 \end{pmatrix} - 2x_2 \begin{pmatrix} a_{12} + \lambda w_1 \\ a_{22} \end{pmatrix} \\ &\quad - 2 \left\langle \begin{pmatrix} a_{12} + \lambda w_1 \\ a_{22} \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 2 \left[ 2a_{22} - \frac{\lambda w_1^2}{w_2} \right] x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

That is to say,

$$\mathbb{A}^{(R)} = \begin{pmatrix} a_{11} - 2\lambda w_2 & -a_{12} - 2\lambda w_1 \\ -a_{12} - 2\lambda w_1 & a_{22} - 2\lambda \frac{w_1^2}{w_2} \end{pmatrix}$$

So we have  $\mathbb{A}^{(R)}$  symmetry. We can remark that formula (4.2) is a direct application of Lemma 11.

Now, we study eigenvalues of  $\mathbb{A}^{(R)}$ . Let  $\mathbb{H}$  be the normal matrix given by

$$\mathbb{H} = \begin{pmatrix} u_1 & w_1 \\ -u_2 & -w_2 \end{pmatrix}$$

Then  $\mathbb{H}\mathbb{A}^{(R)}\mathbb{H}^t$  is similar to  $\mathbb{A}^{(R)}$ .

Using  $\mathbb{A}^{(I)}\mathbf{u}^{(I)} = \lambda^{(I)}\mathbf{u}^{(I)}$  and  $\mathbb{A}^{(I)}\mathbf{w}^{(I)} = 0$  give

$$\begin{aligned} \mathbb{A}^{(R)}\mathbb{H} &= \begin{pmatrix} (a_{11} - 2\lambda w_2)u_1 + (a_{12} + 2\lambda w_1)u_2 & (a_{11} - 2\lambda w_2)w_1 + (a_{12} + 2\lambda w_1)w_2 \\ (a_{12} + 2\lambda w_1)u_1 - (a_{22} - 2\lambda \frac{w_1^2}{w_2})u_2 & (a_{12} + 2\lambda w_1)w_1 - (a_{22} - 2\lambda \frac{w_1^2}{w_2})w_2 \end{pmatrix} \\ &= \begin{pmatrix} \lambda^{(I)}u_1 + 2\lambda(w_1u_2 - w_2u_1) & 0 \\ -\lambda^{(I)}u_2 + 2\lambda(\frac{w_1^2}{w_2}u_2 - w_1u_1) & 0 \end{pmatrix} \end{aligned}$$

But  $(\mathbf{u}^{(I)}, \mathbf{w}^{(I)})$  is a direct and orthonormal basis, so

$$w_2u_1 - w_1u_2 = 1, \quad w_1u_1 + w_2u_2 = 0 \quad \text{and} \quad u_1^2 + u_2^2 = 1.$$

Then

$$\begin{aligned} \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H} &= \begin{pmatrix} \lambda^{(I)}(u_1^2 + u_2^2) + 2\lambda \left[ (w_1u_2 - w_2u_1)u_1 - \frac{w_1^2}{w_2}u_2^2 + w_1u_1u_2 \right] & 0 \\ \lambda^{(I)}(u_1w_1 + u_2w_2) + 2\lambda(w_1^2u_2 - w_2u_1w_1 - w_1^2u_2 + w_2u_1w_1) & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda^{(I)} - 2\frac{\lambda}{w_2} [w_2u_1 - w_1u_2]^2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda^{(I)} - 2\frac{\lambda}{w_2} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

□

**Lemma 13.** Let  $\mathcal{B} = (\mathbf{u}, \mathbf{v}, \mathbf{w})$  and  $\mathcal{B}^{(I)} = (\mathbf{u}^{(I)}, \mathbf{v}^{(I)}, \mathbf{w}^{(I)})$  two direct orthonormal basis of  $\mathbb{R}^3$  such that  $\langle \mathbf{w}, \mathbf{w}^{(I)} \rangle < 0$ . Let  $\mathbb{B} = \text{diag}(\lambda_1, \lambda_2, 0)$  in  $\mathcal{B}$  basis and the symmetric matrix  $\mathbb{A}^{(I)} = \text{diag}(\lambda_1^{(I)}, \lambda_2^{(I)}, 0)$  in  $\mathcal{B}^{(I)}$  basis. Then  $\mathbb{A}^{(R)} = \mathcal{T}_{\mathbb{B}, \mathbf{w}, \mathbf{w}^{(I)}}^3(\mathbb{A}^{(I)})$  is a symmetric matrix having for eigenvalues  $(\lambda_1^{(R)}, \lambda_2^{(R)}, 0)$  and

$$\mathbb{A}^{(R)} \left( \mathbf{w}^{(I)} - 2\langle \mathbf{w}, \mathbf{w}^{(I)} \rangle \mathbf{w} \right) = 0. \quad (4.4)$$

Under the hypothesis

$$0 < \lambda_1 \leq \lambda_2 \quad \text{and} \quad 0 \leq \lambda_1^{(I)} \leq \lambda_2^{(I)} \quad (H) \quad (4.5)$$

we have

$$\lambda_1^{(R)} + \lambda_2^{(R)} \geq \lambda_1^{(I)} + \lambda_2^{(I)} + 4\lambda_1 > 0, \quad (4.5)$$

$$\lambda_1^{(R)}\lambda_2^{(R)} \geq \lambda_1^{(I)}\lambda_2^{(I)} + 4(\lambda_1)^2 > 0 \quad (4.6)$$

and thus

$$\lambda_1^{(R)} > 0, \lambda_2^{(R)} > 0 \quad (4.7)$$

*Proof.* In basis  $\mathcal{B}$ ,  $\mathbf{u}^{(I)} = (u_j)_{j=1}^3$ ,  $\mathbf{v}^{(I)} = (v_j)_{j=1}^3$ ,  $\mathbf{w}^{(I)} = (w_j)_{j=1}^3$  and  $\mathbb{A}^{(I)} = (a_{ij})_{i,j=1}^3$ . By hypothesis,  $\mathbb{A}^{(I)}$  is symmetric and  $\mathbb{A}^{(I)}\mathbf{w}^{(I)} = 0$ .

As  $\langle \mathbf{w}, \mathbf{w}^{(I)} \rangle \neq 0$ , we can compute  $\mathbb{A}^{(R)} : \forall x \in \mathbb{R}^3$

$$\begin{aligned} \mathbb{A}^{(R)}x &= (\mathbb{A}^{(I)} - 2\langle \mathbf{w}, \mathbf{w}^{(I)} \rangle \mathbb{B})x \\ &\quad - 2\langle \mathbf{w}, x \rangle (\mathbb{A}^{(I)}\mathbf{w} + \mathbb{B}\mathbf{w}^{(I)}) \\ &\quad - 2\langle \mathbb{A}^{(I)}\mathbf{w} + \mathbb{B}\mathbf{w}^{(I)}, x \rangle \mathbf{w} \\ &\quad + 2 \left[ 2\langle \mathbb{A}^{(I)}\mathbf{w}, \mathbf{w} \rangle - \frac{\langle \mathbb{B}\mathbf{w}^{(I)}, \mathbf{w}^{(I)} \rangle}{\langle \mathbf{w}, \mathbf{w}^{(I)} \rangle} \right] \langle \mathbf{w}, x \rangle \mathbf{w} \end{aligned}$$

in basis  $\mathcal{B}$ ,  $x = (x_j)_{j=1}^3$  and

$$\begin{aligned} \mathbb{A}^{(R)}x &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - 2w_3 \begin{pmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ 0 \end{pmatrix} \\ &\quad - 2x_3 \begin{pmatrix} a_{13} + \lambda_1 w_1 \\ a_{23} + \lambda_2 w_2 \\ a_{33} \end{pmatrix} \\ &\quad - 2 \left\langle \begin{pmatrix} a_{13} + \lambda_1 w_1 \\ a_{23} + \lambda_2 w_2 \\ a_{33} \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &\quad + 2 \left[ 2a_{33} - \frac{\lambda_1 w_1^2 + \lambda_2 w_2^2}{w_3} \right] x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

That is to say,

$$\mathbb{A}^{(R)} = \begin{pmatrix} a_{11} - 2\lambda_1 w_3 & a_{12} & -a_{13} - 2\lambda_1 w_1 \\ a_{12} & a_{22} - 2\lambda_2 w_3 & -a_{23} - 2\lambda_2 w_2 \\ -a_{13} - 2\lambda_1 w_1 & -a_{23} - 2\lambda_2 w_2 & a_{33} - \frac{2}{w_3}(\lambda_1 w_1^2 + \lambda_2 w_2^2) \end{pmatrix}$$

So we have  $\mathbb{A}^{(R)}$  symmetry.

To prove formula (4.4), we make computation in basis  $\mathcal{B}$  where

$$\mathbf{w}^{(I)} - 2\langle \mathbf{w}, \mathbf{w}^{(I)} \rangle \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ -w_3 \end{pmatrix}$$

So

$$\mathbb{A}^{(R)}\mathbf{w}^{(R)} = \begin{pmatrix} a_{11}w_1 - 2\lambda_1 w_3 w_1 + a_{12}w_2 + a_{13}w_3 + 2\lambda_1 w_1 w_3 \\ a_{12}w_1 + a_{22}w_2 - 2\lambda_2 w_3 w_2 + a_{23}w_3 + 2\lambda_2 w_2 w_3 \\ -a_{13}w_1 - 2\lambda_1 w_1^2 - a_{23}w_2 - 2\lambda_2 w_2^2 - a_{33}w_3 + 2(\lambda_1 w_1^2 + \lambda_2 w_2^2) \end{pmatrix}$$

Then, using  $\mathbb{A}^{(I)}\mathbf{w}^{(I)} = 0$  give immediatly formula (4.4).

To establish the formulas (4.5)-(4.6)-(4.7) under the assumptions (H), we will study the eigenvalues of matrix  $\mathbb{A}^{(R)}$ . For that, let  $\mathbb{H}$  be the normal matrix given by

$$\mathbb{H} = \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ -u_3 & -v_3 & -w_3 \end{pmatrix}$$

Then  $\mathbb{H}\mathbb{A}^{(R)}\mathbb{H}^t$  is similar to  $\mathbb{A}^{(R)}$ . Using that  $(\mathbf{u}^{(I)}, \mathbf{v}^{(I)}, \mathbf{w}^{(I)})$  is a direct and orthonormal basis, we have

$$\mathbf{u}^{(I)} \wedge \mathbf{v}^{(I)} = \mathbf{w}^{(I)} \Leftrightarrow \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ -u_1 v_3 + u_3 v_1 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \quad (4.8)$$

$$\mathbf{v}^{(I)} \wedge \mathbf{w}^{(I)} = \mathbf{u}^{(I)} \Leftrightarrow \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ -v_1 w_3 + v_3 w_1 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad (4.9)$$

$$\mathbf{w}^{(I)} \wedge \mathbf{u}^{(I)} = \mathbf{v}^{(I)} \Leftrightarrow \begin{pmatrix} w_2 u_3 - w_3 u_2 \\ -w_1 u_3 + w_3 u_1 \\ w_1 u_2 - w_2 u_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad (4.10)$$

$$\|\mathbf{u}^{(I)}\| = 1 \Leftrightarrow u_1^2 + u_2^2 + u_3^2 = 1 \quad (4.11)$$

$$\|\mathbf{v}^{(I)}\| = 1 \Leftrightarrow v_1^2 + v_2^2 + v_3^2 = 1 \quad (4.12)$$

$$\|\mathbf{w}^{(I)}\| = 1 \Leftrightarrow w_1^2 + w_2^2 + w_3^2 = 1 \quad (4.13)$$

Using  $\mathbb{A}^{(I)}\mathbf{u}^{(I)} = \lambda_1^{(I)}\mathbf{u}^{(I)}$ ,  $\mathbb{A}^{(I)}\mathbf{v}^{(I)} = \lambda_1^{(I)}\mathbf{v}^{(I)}$  and  $\mathbb{A}^{(I)}\mathbf{w}^{(I)} = 0$  give

$$\begin{aligned} & \mathbb{A}^{(R)}\mathbb{H} \\ & = \\ & \begin{pmatrix} \lambda_1^{(I)}u_1 + 2\lambda_1(w_1u_3 - w_3u_1) & \lambda_2^{(I)}v_1 + 2\lambda_1(w_1v_3 - w_3v_1) & 0 \\ \lambda_1^{(I)}u_2 + 2\lambda_2(w_2u_3 - w_3u_2) & \lambda_2^{(I)}v_2 + 2\lambda_2(w_2v_3 - w_3v_2) & 0 \\ \left\{ -\lambda_1^{(I)}u_3 - 2\lambda_1\left(w_1u_1 - \frac{u_3w_1^2}{w_3}\right) \right. & \left. \left\{ -\lambda_2^{(I)}v_3 - 2\lambda_1\left(w_1v_1 - \frac{v_3w_1^2}{w_3}\right) \right. \right. & 0 \\ & \left. \left. -2\lambda_2\left(w_2u_2 - \frac{u_3w_2^2}{w_3}\right) \right\} \right. & \left. \left. -2\lambda_2\left(w_2v_2 - \frac{v_3w_2^2}{w_3}\right) \right\} \right. & 0 \end{pmatrix} \end{aligned}$$

To obtain  $\mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}$ , we compute  $\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\mathbf{x}, \mathbf{y} \rangle \quad \forall x, y \in \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  We can remark that

$$\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\mathbf{x}, \mathbf{w} \rangle = 0 \quad \forall x \in \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$$

Now, we compute the six leading terms of matrix  $\mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}$

- Calculus of  $\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\mathbf{u}, \mathbf{u} \rangle$

$$\begin{aligned} \langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\mathbf{u}, \mathbf{u} \rangle &= \left( \lambda_1^{(I)}u_1 + 2\lambda_1(w_1u_3 - w_3u_1) \right) u_1 \\ &+ \left( \lambda_1^{(I)}u_2 + 2\lambda_2(w_2u_3 - w_3u_2) \right) u_2 \\ &- \left( -\lambda_1^{(I)}u_3 - 2\lambda_1\left(w_1u_1 - \frac{u_3w_1^2}{w_3}\right) - 2\lambda_2\left(w_2u_2 - \frac{u_3w_2^2}{w_3}\right) \right) u_3 \end{aligned}$$

By hypothesis  $\|\mathbf{u}^{(I)}\| = 1$  so

$$\begin{aligned} \langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\mathbf{u}, \mathbf{u} \rangle &= \lambda_1^{(I)} + 2\frac{\lambda_1}{w_3} (2w_3w_1u_3u_1 - w_3^2u_1^2 - w_1^2u_2^2) \\ &+ 2\frac{\lambda_2}{w_3} (2w_3w_2u_2u_3 - w_3^2u_2^2 - w_2^2u_3^2) \\ &= \lambda_1^{(I)} - 2\frac{\lambda_1}{w_3}(w_3u_1 - w_1u_3)^2 - 2\frac{\lambda_2}{w_3}(w_3u_2 - w_2u_3)^2 \end{aligned}$$

Using formula (4.10) give

$$\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\mathbf{u}, \mathbf{u} \rangle = \lambda_1^{(I)} - \frac{2}{w_3} (\lambda_1v_2^2 + \lambda_2v_1^2)$$

- Calculus of  $\langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\mathbf{u}, \mathbf{v} \rangle$

$$\begin{aligned} \langle \mathbb{H}^t\mathbb{A}^{(R)}\mathbb{H}\mathbf{u}, \mathbf{v} \rangle &= \left( \lambda_1^{(I)}u_1 + 2\lambda_1(w_1u_3 - w_3u_1) \right) v_1 \\ &+ \left( \lambda_1^{(I)}u_2 + 2\lambda_2(w_2u_3 - w_3u_2) \right) v_2 \\ &- \left( -\lambda_1^{(I)}u_3 - 2\lambda_1\left(w_1u_1 - \frac{u_3w_1^2}{w_3}\right) - 2\lambda_2\left(w_2u_2 - \frac{u_3w_2^2}{w_3}\right) \right) v_3 \end{aligned}$$

By hypothesis  $\langle \mathbf{u}^{(I)}, \mathbf{v}^{(I)} \rangle = 0$  so

$$\begin{aligned} \langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{u}, \mathbf{v} \rangle &= 2 \frac{\lambda_1}{w_3} (w_3 w_1 u_3 v_1 - w_3^2 u_1 v_1 + w_3 w_1 u_1 v_3 - w_1^2 u_3 v_3) \\ &\quad + 2 \frac{\lambda_2}{w_3} (w_3 w_2 u_3 v_2 - w_3^2 u_2 v_2 + w_3 w_2 u_2 v_3 - w_2^2 u_3 v_3) \\ &= 2 \frac{\lambda_1}{w_3} (w_3 u_1 - w_1 u_3) (w_1 v_3 - w_3 v_1) + 2 \frac{\lambda_2}{w_3} (w_3 u_2 - w_2 u_3) (w_2 v_3 - w_3 v_2) \end{aligned}$$

Using formulas (4.9) and (4.10) give

$$\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{u}, \mathbf{v} \rangle = \frac{2}{w_3} (\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1)$$

- Calculus of  $\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{u}, \mathbf{w} \rangle$

$$\begin{aligned} \langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{u}, \mathbf{w} \rangle &= \left( \lambda_1^{(I)} u_1 + 2 \lambda_1 (w_1 u_3 - w_3 u_1) \right) w_1 \\ &\quad + \left( \lambda_1^{(I)} u_2 + 2 \lambda_2 (w_2 u_3 - w_3 u_2) \right) w_2 \\ &\quad - \left( -\lambda_1^{(I)} u_3 - 2 \lambda_1 \left( w_1 u_1 - \frac{u_3 w_1^2}{w_3} \right) - 2 \lambda_2 \left( w_2 u_2 - \frac{u_3 w_2^2}{w_3} \right) \right) w_3 \end{aligned}$$

By hypothesis  $\langle \mathbf{u}^{(I)}, \mathbf{w}^{(I)} \rangle = 0$  so

$$\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{u}, \mathbf{w} \rangle = 0$$

- Calculus of  $\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{v}, \mathbf{u} \rangle$

$$\begin{aligned} \langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{v}, \mathbf{u} \rangle &= \left( \lambda_2^{(I)} v_1 + 2 \lambda_1 (w_1 v_3 - w_3 v_1) \right) u_1 \\ &\quad + \left( \lambda_2^{(I)} v_2 + 2 \lambda_2 (w_2 v_3 - w_3 v_2) \right) u_2 \\ &\quad - \left( -\lambda_2^{(I)} v_3 - 2 \lambda_1 \left( w_1 v_1 - \frac{v_3 w_1^2}{w_3} \right) - 2 \lambda_2 \left( w_2 v_2 - \frac{v_3 w_2^2}{w_3} \right) \right) u_3 \end{aligned}$$

By hypothesis  $\langle \mathbf{u}^{(I)}, \mathbf{v}^{(I)} \rangle = 0$  so

$$\begin{aligned} \langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{v}, \mathbf{u} \rangle &= 2 \frac{\lambda_1}{w_3} (w_1 w_3 u_1 v_3 - w_3^2 u_1 v_1 + w_1 w_3 u_3 v_1 - w_1^2 u_3 v_3) \\ &\quad + 2 \frac{\lambda_2}{w_3} (w_2 w_3 u_2 v_3 - w_3^2 u_2 v_2 + w_2 w_3 u_3 v_2 - w_2^2 u_3 v_3) \\ &= 2 \frac{\lambda_1}{w_3} (w_3 u_1 - w_1 u_3) (w_1 v_3 - w_3 v_1) + 2 \frac{\lambda_2}{w_3} (w_3 u_2 - w_2 u_3) (w_2 v_3 - w_3 v_2) \end{aligned}$$

Using formulas (4.9) and (4.10) give

$$\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{v}, \mathbf{u} \rangle = \frac{2}{w_3} (\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1)$$

- Calculus of  $\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{v}, \mathbf{v} \rangle$

$$\begin{aligned} \langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{v}, \mathbf{v} \rangle &= \left( \lambda_2^{(I)} v_1 + 2 \lambda_1 (w_1 v_3 - w_3 v_1) \right) v_1 \\ &\quad + \left( \lambda_2^{(I)} v_2 + 2 \lambda_2 (w_2 v_3 - w_3 v_2) \right) v_2 \\ &\quad - \left( -\lambda_2^{(I)} v_3 - 2 \lambda_1 \left( w_1 v_1 - \frac{v_3 w_1^2}{w_3} \right) - 2 \lambda_2 \left( w_2 v_2 - \frac{v_3 w_2^2}{w_3} \right) \right) v_3 \end{aligned}$$

By hypothesis  $\|\mathbf{v}^{(I)}\| = 1$  so

$$\begin{aligned}\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{v}, \mathbf{v} \rangle &= \lambda_2^{(I)} + 2 \frac{\lambda_1}{w_3} (2w_1 w_3 v_1 v_3 - w_3^2 v_1^2 - w_1^2 v_3^2) \\ &\quad + 2 \frac{\lambda_2}{w_3} (2w_2 w_3 v_2 v_3 - w_3^2 v_2^2 - w_2^2 v_3^2) \\ &= \lambda_2^{(I)} - 2 \frac{\lambda_1}{w_3} (w_3 v_1 - w_1 v_3)^2 - 2 \frac{\lambda_2}{w_3} (w_3 v_2 - w_2 v_3)^2\end{aligned}$$

Using formula (4.9) give

$$\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{v}, \mathbf{v} \rangle = \lambda_2^{(I)} - \frac{2}{w_3} (\lambda_1 u_2^2 + \lambda_2 u_1^2)$$

- Calculus of  $\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{v}, \mathbf{w} \rangle$

$$\begin{aligned}\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{v}, \mathbf{w} \rangle &= \left( \lambda_2^{(I)} v_1 + 2\lambda_1 (w_1 v_3 - w_3 v_1) \right) w_1 \\ &\quad + \left( \lambda_2^{(I)} v_2 + 2\lambda_2 (w_2 v_3 - w_3 v_2) \right) w_2 \\ &\quad - \left( -\lambda_2^{(I)} v_3 - 2\lambda_1 (w_1 v_1 - \frac{v_3 w_1^2}{w_3}) - 2\lambda_2 (w_2 v_2 - \frac{v_3 w_2^2}{w_3}) \right) w_3\end{aligned}$$

By hypothesis  $\langle \mathbf{v}^{(I)}, \mathbf{w}^{(I)} \rangle = 0$  so

$$\langle \mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} \mathbf{u}, \mathbf{w} \rangle = 0$$

With all these formulas we obtain :

$$\mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H} = \begin{pmatrix} \lambda_1^{(I)} - \frac{2}{w_3} (\lambda_1 v_2^2 + \lambda_2 v_1^2) & \frac{2}{w_3} (\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1) & 0 \\ \frac{2}{w_3} (\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1) & \lambda_2^{(I)} - \frac{2}{w_3} (\lambda_1 u_2^2 + \lambda_2 u_1^2) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4.14)$$

As  $\mathbb{H}^t \mathbb{A}^{(R)} \mathbb{H}$  is similar to  $\mathbb{A}^{(R)}$ , we rekind that 0 is an eigenvalue of  $\mathbb{A}^{(R)}$  and the two others eigenvalues  $\lambda_1^{(R)}$  and  $\lambda_2^{(R)}$  are also eigenvalues of

$$\mathbb{A} = \begin{pmatrix} \lambda_1^{(I)} - \frac{2}{w_3} (\lambda_1 v_2^2 + \lambda_2 v_1^2) & \frac{2}{w_3} (\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1) \\ \frac{2}{w_3} (\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1) & \lambda_2^{(I)} - \frac{2}{w_3} (\lambda_1 u_2^2 + \lambda_2 u_1^2) \end{pmatrix}$$

Furthermore, we have that

$$\text{Tr}(\mathbb{A}) = \lambda_1^{(R)} + \lambda_2^{(R)} \quad (4.15)$$

$$\det(\mathbb{A}) = \lambda_1^{(R)} \lambda_2^{(R)} \quad (4.16)$$

To establish inequality 4.5, we use equation (4.15) under hypothesis (H) and  $w_3 < 0$  to get

$$\lambda_1^{(R)} + \lambda_2^{(R)} \geq \lambda_1^{(I)} + \lambda_2^{(I)} - \frac{2}{w_3} \lambda_1 ((u_1^2 + u_2^2) + (v_1^2 + v_2^2)).$$

We remark that

$$\begin{aligned}((u_1^2 + u_2^2) + (v_1^2 + v_2^2)) &= (u_1 + v_2)^2 + (u_2 - v_1)^2 - 2(u_1 v_2 - u_2 v_1) \\ &= (u_1 + v_2)^2 + (u_2 - v_1)^2 - 2w_3\end{aligned}$$

and so, we obtain

$$\begin{aligned}\lambda_1^{(R)} + \lambda_2^{(R)} &\geq \lambda_1^{(I)} + \lambda_2^{(I)} - \frac{2}{w_3} \lambda_1 ((u_1 + v_2)^2 + (u_2 - v_1)^2 - 2w_3) \\ &\geq \lambda_1^{(I)} + \lambda_2^{(I)} + 4\lambda_1 + 4\lambda_1 ((u_1 + v_2)^2 + (u_2 - v_1)^2) \\ &\geq \lambda_1^{(I)} + \lambda_2^{(I)} + 4\lambda_1.\end{aligned}$$

To establish inequality 4.6, we use equation (4.16) to get

$$\begin{aligned}
\lambda_1^{(R)}\lambda_2^{(R)} &= \left( \lambda_1^{(I)} - \frac{2}{w_3} (\lambda_1 v_2^2 + \lambda_2 v_1^2) \right) \left( \lambda_2^{(I)} - \frac{2}{w_3} (\lambda_1 u_2^2 + \lambda_2 u_1^2) \right) \\
&\quad - \frac{4}{w_3^2} (\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1)^2 \\
&= \lambda_1^{(I)}\lambda_2^{(I)} + \frac{4}{w_3^2} \left[ (\lambda_1 v_2^2 + \lambda_2 v_1^2) (\lambda_1 u_2^2 + \lambda_2 u_1^2) - (\lambda_1 u_2 v_2 + \lambda_2 u_1 v_1)^2 \right] \\
&\quad - \frac{2}{w_3} \left( (\lambda_1 v_2^2 + \lambda_2 v_1^2) \lambda_2^{(I)} + (\lambda_1 u_2^2 + \lambda_2 u_1^2) \lambda_1^{(I)} \right) \\
&= \lambda_1^{(I)}\lambda_2^{(I)} - \frac{2}{w_3} \left( (\lambda_1 v_2^2 + \lambda_2 v_1^2) \lambda_2^{(I)} + (\lambda_1 u_2^2 + \lambda_2 u_1^2) \lambda_1^{(I)} \right) \\
&\quad + \frac{4}{w_3^2} \left[ \lambda_1^2 u_2^2 v_2^2 + \lambda_1 \lambda_2 u_1^2 v_2^2 + \lambda_1 \lambda_2 v_1^2 u_2^2 - \lambda_1^2 u_2^2 v_2^2 - 2\lambda_1 \lambda_2 u_1 v_1 u_2 v_2 - \lambda_1 \lambda_2 v_1^2 u_2^2 \right] \\
&= \lambda_1^{(I)}\lambda_2^{(I)} - \frac{2}{w_3} \left( (\lambda_1 v_2^2 + \lambda_2 v_1^2) \lambda_2^{(I)} + (\lambda_1 u_2^2 + \lambda_2 u_1^2) \lambda_1^{(I)} \right) \\
&\quad + \frac{4}{w_3^2} \lambda_1 \lambda_2 \left[ u_1^2 v_2^2 - 2u_1 v_1 u_2 v_2 + v_1^2 u_2^2 \right] \\
&= \lambda_1^{(I)}\lambda_2^{(I)} - \frac{2}{w_3} \left( (\lambda_1 v_2^2 + \lambda_2 v_1^2) \lambda_2^{(I)} + (\lambda_1 u_2^2 + \lambda_2 u_1^2) \lambda_1^{(I)} \right) \\
&\quad + \frac{4}{w_3^2} \lambda_1 \lambda_2 (u_1 v_2 - v_1 u_2)^2
\end{aligned}$$

From formula (4.8), we have  $u_1 v_2 - v_1 u_2 = w_3$  and so we obtain

$$\begin{aligned}
\lambda_1^{(R)}\lambda_2^{(R)} &= \lambda_1^{(I)}\lambda_2^{(I)} - \frac{2}{w_3} \left( (\lambda_1 v_2^2 + \lambda_2 v_1^2) \lambda_2^{(I)} + (\lambda_1 u_2^2 + \lambda_2 u_1^2) \lambda_1^{(I)} \right) \\
&\quad + 4\lambda_1\lambda_2
\end{aligned}$$

Under hypothesis (H) and  $w_3 < 0$ , we clearly have

$$\lambda_1^{(R)}\lambda_2^{(R)} \geq \lambda_1^{(I)}\lambda_2^{(I)} + 4\lambda_1^2 > 0$$

and we obtain formula (4.6). □

## REFERENCES

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