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The snapping out Brownian motion

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Abstract

We give a probabilistic representation of a one-dimensional diffusion, equation where the solution is discontinuous at 0 with a jump proportional to its flux. This kind of interface condition is usually seen as a semi-permeable barrier. For this, we use a process called here the snapping out Brownian motion, whose properties are studied. As this construction is motivated by applications, for example in brain imaging or in chemistry, a simulation scheme is also provided.

Keywords: Interface condition, elastic Brownian motion, semi-permeable barrier, thin layer, piecing out a Markov process.

1. Introduction

Many diffusion phenomena have to deal with interface conditions. Let D be a diffusivity coefficient which is smooth away from a regular surface S , but presents some discontinuity there. In this case, the solution to the diffusion equation

$$\partial_t u(t, x) = \frac{1}{2} \nabla(D(x) \nabla u(t, x)) = 0 \text{ with } u(0, x) = f(x) \quad (1)$$

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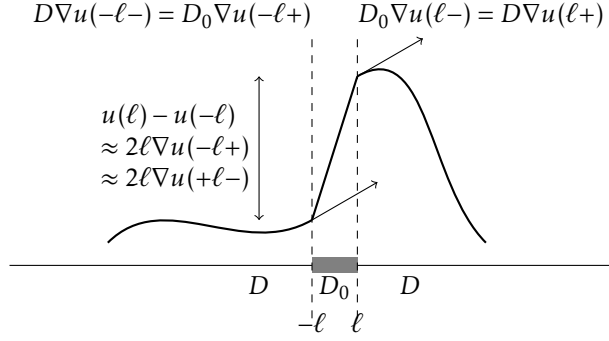


Figure 1: The thin layer problem.

has to be understood as a *weak solution*. However, u is smooth away from S and satisfies

$$u(t, x+) = u(t, x-) \text{ and } D(x+)n^+(x) \cdot \nabla u(t, x+) = D(x-)n^-(x) \cdot \nabla u(t, x-), \quad (2)$$

for $x \in S$, when S is assumed to separate locally \mathbb{R}^d into a “+” and a “-” part and where n^\pm is a vector normal to S at x pointing to the “ \pm ” side. The second condition is called the *continuity of the flux*.

Now, let us assume that D takes scalar values, and is constant away from a thin layer of width 2ℓ enclosed between two parallel surfaces S^+ and S^- . When the width ℓ of the layer tends to 0, S^+ and S^- merge into a single interface located on a surface S .

When the diffusivity D_0 decreases to 0 with ℓ and $D_0/\ell \rightarrow 2\lambda > 0$, then the solution to (1) converges to a function v satisfying (1) away from S with the interface condition for $x \in S$:

$$\nabla v(t, x+) = \nabla v(t, x-) \text{ and } \lambda(v(t, x+) - v(t, x-)) = D(x\pm)\nabla v(t, x\pm). \quad (3)$$

The solution has a continuous flux on S but is discontinuous on S (See for example [36, Chap. 13]). A heuristic explanation is given Figure 1.

If D is smooth on \mathbb{R}^d , it is well known that

$$u(t, x) = \mathbb{E}_x[f(X_t)], \quad (4)$$

where X is the diffusion process generated by $\frac{1}{2}\nabla(D\nabla)$ which is solution under \mathbb{P}_x to the stochastic differential equation (SDE)

$$X_t = x + \int_0^t \sigma(X_s) dB_s + \int_0^t \frac{1}{2} \sum_{i=1}^d \frac{D_{i,\cdot}}{\partial x_i}(X_s) ds \text{ with } \sigma \sigma^T = D \quad (5)$$

for a Brownian motion B .

When D presents some discontinuities, (5) is no longer a meaning. However, a Feller processes $(X, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathbb{R}})$ is associated to $\frac{1}{2} \nabla(D \nabla \cdot)$ for which (4) holds. In particular, the marginal distributions X_t have a density $p(t, x, \cdot)$ under \mathbb{P}_x , where $p(t, x, y)$ is the fundamental solution to (1) (See [40] for example).

Let us now assume that the dimension of the space is equal to 1 and that D is discontinuous at some separated points $\{x_i\}$ with left and right limit there, and smooth elsewhere. The process X is solution to a SDE with local time. The Itô-Tanaka formula is the key tool to manipulate it, and several simulation algorithms have been proposed (See the references in [26] for example). The process called the *Skew Brownian motion* is the basic brick of this construction [24, 25].

Coming back to the thin layer problem, we assume that D is constant and equal to D_1 on $(-\infty, -\ell)$ and (ℓ, ∞) , and to D_0 on $(-\ell, \ell)$, with $D_0/\ell \rightarrow \lambda$. The associated stochastic process is solution to

$$X_t = x + B_t + \frac{D_1 - D_0}{D_1 + D_0} L_t^\ell(X) + \frac{D_0 - D_1}{D_1 + D_0} L_t^{-\ell}(X),$$

where $L_t^{\pm\ell}(X)$ is the local time of X at $\pm\ell$ [25].

Letting D_0/ℓ converging to 2λ with $\ell \rightarrow 0$, one may expect that X converges in distribution to some stochastic process Y such that the solution to (1) with the interface condition (3) is given by $v(t, x) = \mathbb{E}_x[f(Y_t)]$.

The article then aims at constructing and giving several properties related to the process Y , which we call a *snapping out Brownian motion*. In particular, this process is not a Feller process on \mathbb{R} . Indeed, this is a Feller process living on $\mathcal{R} = (-\infty, 0] \cup [0, +\infty)$ where the union of the two intervals is seen as a disjoint union and then 0 corresponds either to $0+$ or $0-$.

The behavior of this process is the following: Assume that its starting point is $x \geq 0$. It behaves as a positively reflected Brownian motion until its local time is greater than an independent exponential random variable of parameter 2λ . Then it decides its sign with probability $1/2$ and starts afresh as a new reflected Brownian motion, until its local time is greater than a new exponential random variable, and so on... Using the properties of the exponential random variable, it is equivalent to assert that the particle changes its sign when its local time is greater than an exponential random variable with parameter λ , and behaves like a positively or negatively reflected Brownian motion between these switching times.

Its name is justified by the following fact: As the time at which the particle possibly changes its sign is the same as for the *elastic Brownian motion* [9, 18, 19], it could also be seen as some elastic Brownian motion which is reborn once killed.

The elastic Brownian motion, also called a *partially reflected Brownian motion*, is associated to the Robin boundary condition and has then many applications [7, 15, 39]. This process is the “basic brick” for constructing the snapping out Brownian motion.

The behavior of the snapping-out Brownian motion justifies also the old heuristic that the interface condition (3) corresponds to a *semi-permeable barrier*, which arises for example in diffusion Magnetic Resonance Imaging [10] or in chemistry [1, 7]. The interface condition (3) is different from (2), to which is associated a Skew Brownian motion and where the particle crosses the interface when it reaches it, and which corresponds to a *permeable barrier* (See references in [24, 26]).

Here, we work under the condition of a single interface at 0. In short time, it is sufficient to describe the behavior of the process even in a more complex media, since other interface or boundary conditions far enough have “exponentially small” influence on the distribution of the process. This is sufficient for simulation purposes, where particles positions are represented by the stochastic process and move according to its dynamic.

Using similar computations, one may generalize our work to the case where $D(x) = D^+$ if $x \geq 0$, D^- if $x \leq 0$ and an interface condition

$$\nabla u(t, 0+) = \beta \nabla u(t, 0-) \text{ and } \lambda u(t, 0+) - \mu u(t, 0-) = \nabla u(t, 0)$$

with $\lambda, \mu > 0$. Diffusions on graphs specified by a condition at each vertex could also be considered, which could be of interest in several applications. This process has been described without proof by R. Bobrowski in [4], which have studied its limit behavior when the diffusion coefficients increase.

Although the snapping-out Brownian motion may be seen as a diffusion on a graph, it is not a diffusion on a metric graph, where the edges are joint by vertices. Such diffusions have been classified by M. Freidlin and A. Wentzell in [11, 12]. The conditions that are required at the vertices of the graphs are some extension of the possible boundary conditions for a Markov process studied by W. Feller [9]. See also [21] for example for the related problem of pasting diffusions¹.

Our interface condition does not fall in these categories. Our process is best thought as a kind of *random evolution process*, that is a collection of processes that switches back and forth randomly among a collection of processes (See e.g. [16, 38]).

Outline. In Section 2, we compute the fundamental solution of the operator. In Section 3, we show that a Feller process may be associated to the Laplace

¹The article [32] defines a notion of *semipermeable membrane* which is different from ours, where the solution is continuous with a discontinuous gradient.

operator with the considered interface condition. In Section 4, we shown that the snapping out Brownian motion may be constructed by letting reborn an elastic Brownian motion. The thin layer problem is studied from a probabilistic point of view in Section 5. Simulation issue is treated in Section 6. Finally, we end by some appendix where various resolvent and Laplace transform related to the elastic or the snapping out Brownian motion are computed.

2. The fundamental solution

For $t > 0$ and $x \in \mathbb{R}$, let us set

$$\gamma(t, x) := \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{-|x|^2}{2t}\right)$$

and $\gamma(t, x, y) := \gamma(t, y - x)$ for $x, y \in \mathbb{R}$. The Gaussian kernel $(t, x, y) \mapsto \gamma(t, x, y)$ is the fundamental solution of the heat equation $\partial_t \gamma(t, x, y) = \frac{1}{2} \Delta_y \gamma(t, x, y)$ with $\gamma(t, x, y) \xrightarrow[t \rightarrow 0]{} \delta_x(y)$.

We also set

$$\bar{\Phi}(x) := \int_x^{+\infty} \frac{1}{\sqrt{2\pi}} \exp(-|\xi|^2/2) d\xi$$

which is the complementary distribution function of the Gaussian law.

2.1. The Robin condition

We consider first the parabolic problem with Robin (or third type) boundary condition given by

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \frac{1}{2} u(t, x), & t > 0, x \in \mathbb{R}_+ \setminus \{0\}, \\ u(0, x) = f(x), & x \in \mathbb{R}_+ \setminus \{0\}, \\ u'(t, 0+) = \kappa u(t, 0+), & t > 0. \end{cases} \quad (6)$$

This problem may be solved thanks to a variational formulation (See *e.g.* [23, § II.5 and § III.4]). Yet, it may also be solved using a *fundamental solution* ([13, Chap. 5]), which we construct.

Let us consider first the operator $\mathcal{L}^r = \frac{1}{2} \Delta$ whose domain $\text{Dom}(\mathcal{L}^r)$ is the closure (w.r.t. to the graph norm operator defined with respect to $L^2(\mathbb{R}_+)$) of

$$\mathcal{D}^r := \{ \phi \in C_c^\infty(\mathbb{R}_+; \mathbb{R}) \mid \phi'(0) = \kappa \phi(0) \}.$$

The operator $(\mathcal{L}^r, \text{Dom}(\mathcal{L}^r))$ is self-adjoint and generates a strongly continuous, contraction semi-group $(P_t^e)_{t \geq 0}$.

Let $p^e(t, x, y)$ be the fundamental solution of \mathcal{L}^r , which is then solution to

$$\begin{cases} \frac{\partial p^e(t, x, y)}{\partial t} = \frac{1}{2} \Delta_y p^e(t, x, y), \\ \kappa p^e(t, x, 0) = \nabla_y p^e(t, x, 0), \\ p^e(t, x, y) \xrightarrow[t \rightarrow 0]{} \delta_x(y). \end{cases} \quad (7)$$

For any $f \in L^2(\mathbb{R}_+)$, the solution u to (6) is $(t, x) \mapsto \int_{\mathbb{R}_+} p^e(t, x, y) f(y) dy$, which is for any $t > 0$ a version of $P_t^e f(x)$. Since $(\mathcal{L}^r, \text{Dom}(\mathcal{L}^r))$ is self-adjoint, $p^e(t, x, y) = p^e(t, y, x)$ for any $x, y \geq 0, t > 0$.

Proposition 1 ([5, 22, 39]). *For $y \geq 0$,*

$$p^e(t, x, y) = \gamma(t, y - x) + \gamma(t, x + y) - 2\kappa \int_0^{+\infty} \exp(-\kappa \xi) \gamma(t, x + \xi + y) d\xi.$$

Proof. First, $p^e(t, x, y)$ is solution to the first line of (7), as γ is. Also, since $\gamma(t, y, z) \xrightarrow[t \rightarrow 0]{} \delta_z(y)$ and $y \geq 0$, $p^e(t, x, y)$ also satisfies the third line of (7).

Since $\nabla_z \gamma(t, z) = -zt^{-1} \gamma(t, z)$,

$$\begin{aligned} \nabla_y p^e(t, x, 0) &= 2\kappa \int_0^{+\infty} \exp(-\kappa \xi) \frac{x + \xi}{t} \gamma(t, x + \xi) d\xi \\ &= -2\kappa \int_0^{+\infty} \exp(-\kappa \xi) \nabla_\xi \gamma(t, x + \xi) d\xi \\ &= 2\kappa \gamma(t, x) - 2\kappa^2 \int_0^{+\infty} \exp(-\kappa \xi) \gamma(t, x + \xi) d\xi. \end{aligned}$$

On the other hand, since $\gamma(t, -x) = \gamma(t, x)$,

$$p^e(t, x, 0) = 2\gamma(t, x) - 2\kappa \int_0^{+\infty} \exp(-\kappa \xi) \gamma(t, x + \xi) d\xi.$$

This leads to the required boundary condition. \square

Finally, the expression of p^e may be simplified since some algebraic manipulations show that

$$\int_0^{+\infty} \exp(-\kappa \xi) \gamma(t, \xi + y) d\xi = \exp\left(y\kappa + \frac{\kappa^2 t}{2}\right) \overline{\Phi}\left(\frac{y}{\sqrt{t}} + \kappa \sqrt{t}\right). \quad (8)$$

2.2. Fundamental solution for the heat equation with a semi-permeable barrier

Let \mathcal{R} be the disjoint union of $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0]$ so that 0 appears in \mathcal{R} as an element $0+$ and an element $0-$ which are distinct. For $x \in \mathbb{R}_+$ and $y \in \mathbb{R}_-$, the distance between x and y is assumed to be $+\infty$. However, there is a natural projection π from \mathcal{R} onto \mathbb{R} in which $0+$ and $0-$ are identified.

Let us consider now $\mathcal{L} = \frac{1}{2}\Delta$ whose domain $\text{Dom}(\mathcal{L})$ is the closure of

$$\mathcal{D} := \left\{ \phi : \mathcal{R} \rightarrow \mathbb{R} \left| \begin{array}{l} \phi|_{\mathbb{R}_\pm} \in C_c^\infty(\mathbb{R}_\pm) \\ \phi'(0-) = \phi'(0+) \\ \lambda(\phi(0+) - \phi(0-)) = \phi'(0) \end{array} \right. \right\}$$

for the graph norm in $L^2(\mathcal{R})$. Again, it is easily seen that $(\mathcal{L}, \text{Dom}(\mathcal{L}))$ is self-adjoint.

A maximum principle holds for $(\mathcal{L}, \text{Dom}(\mathcal{L}))$, so that there exists a unique fundamental solution p to $(\mathcal{L}, \text{Dom}(\mathcal{L}))$ which satisfies

$$\frac{\partial p}{\partial t}(t, x, y) = \frac{1}{2}\Delta_y p(t, x, y) \text{ for } y \neq 0\pm, \quad (9a)$$

$$\nabla p(t, x, 0+) = \nabla p(t, x, 0-), \quad (9b)$$

$$\lambda(p(t, x, 0+) - p(t, x, 0-)) = \nabla p(t, x, 0\pm), \quad (9c)$$

$$p(t, x, y) \xrightarrow[t \rightarrow 0]{} \delta_x(y). \quad (9d)$$

In addition, for any $t, s \geq 0$ and $x, y \in \mathcal{R}$, the Chapman-Kolmogorov equation holds:

$$p(t+s, x, y) = \int_{\mathbb{R}_-} p(t, x, z)p(s, z, y) dz + \int_{\mathbb{R}_+} p(t, x, z)p(s, z, y) dz. \quad (10)$$

Our aim is now to give an analytic expression for p .

Lemma 1. *For any $t > 0$ and any $(x, y) \in \mathcal{R}^2$, and $p(t, x, y) = p(t, y, x)$ and*

$$p(t, x, -y) = p(t, -x, y).$$

From now, we assume that $x \geq 0$, as the expression of $p(t, -x, y)$ is deduced from the one of $p(t, x, -y)$.

We set

$$\Theta(t, y) := 2\lambda \int_0^{+\infty} \exp(-2\lambda\xi) \gamma(t, \xi + y) d\xi = 2\lambda \exp(2\lambda y + 2\lambda^2 t) \overline{\Phi}\left(\frac{y}{\sqrt{t}} + 2\lambda\sqrt{t}\right).$$

Proposition 2. For $x \geq 0$ and $y \geq 0$ (or $x \leq 0$ and $y \leq 0$),

$$p(t, x, y) = \gamma(t, y - x) + \gamma(t, x + y) - \Theta(t, |y| + |x|). \quad (11)$$

For $x \geq 0$ and $y \leq 0$ (or $x \leq 0$ and $y \geq 0$),

$$p(t, x, y) = \Theta(t, |y| + |x|). \quad (12)$$

Proof. Let us set for $x \geq 0, y \geq 0$,

$$\begin{aligned} q_1(t, x, y) &= \frac{1}{2}p(t, x, y) + \frac{1}{2}p(t, -x, y), \\ q_2(t, x, y) &= \frac{1}{2}p(t, x, y) - \frac{1}{2}p(t, -x, y). \end{aligned}$$

Clearly, q_1 and q_2 satisfies (9a) as well as

$$q_i(t, x, y) \xrightarrow{t \rightarrow 0} \frac{1}{2} \delta_x(y), \quad i = 1, 2.$$

Besides,

$$\nabla_y q_1(t, x, y) = \frac{1}{2} \nabla_y p(t, x, y) - \frac{1}{2} \nabla_y p(t, x, -y)$$

so that by the condition (9b) on the continuity of the flux,

$$\nabla_y q_1(t, x, 0) = 0.$$

This proves that q_1 is half the fundamental solution of the Neumann problem for $\frac{1}{2}\Delta$ and then

$$q_1(t, x, y) = \frac{1}{2} \gamma(t, y - x) + \frac{1}{2} \gamma(t, x + y).$$

On the other hand,

$$\nabla_y q_2(t, x, 0+) = \nabla_y p(t, x, 0+) \quad (13)$$

and

$$q_2(t, x, 0+) = \frac{1}{2} (p(t, x, 0+) - p(t, x, 0-)).$$

With (9c) and (13),

$$q_2(t, x, 0+) = \frac{1}{2\lambda} \nabla q_2(t, x, 0+).$$

Hence $q_2(t, x, y)$ is half the fundamental solution of $\frac{1}{2}\Delta$ with a Robin boundary condition and is then solution to (7) with $\kappa = 2\lambda$. Hence

$$q_2(t, x, y) = \frac{1}{2} \gamma(t, y - x) + \frac{1}{2} \gamma(t, x + y) - 2\lambda \int_0^{+\infty} \exp(-2\lambda \xi) \gamma(t, x + \xi + y) d\xi.$$

Eq. (11) follows from $p(t, x, y) = q_1(t, x, y) + q_2(t, x, y)$.

Assume now that $x \leq 0$ and $y \geq 0$. We still consider q_1 and q_2 as above. Remark that

$$\begin{aligned} q_1(t, x, y) &\xrightarrow{t \rightarrow 0} \frac{1}{2} \delta_{-x}(y), \\ q_2(t, x, y) &\xrightarrow{t \rightarrow 0} -\frac{1}{2} \delta_{-x}(y), \end{aligned}$$

so that

$$q_1(t, x, y) = \frac{1}{2} \gamma(t, y - x) + \frac{1}{2} \gamma(t, y + x),$$

while

$$q_2(t, x, y) = -\frac{1}{2} \gamma(t, y - x) - \frac{1}{2} \gamma(t, y + x) + 2\lambda \int_0^{+\infty} \exp(-2\lambda\xi) \gamma(t, -x + \xi + y) d\xi.$$

Eq. (12) follows from $p(t, x, y) = q_1(t, x, y) + q_2(t, x, y)$. \square

Since

$$\int_{-\infty}^{+\infty} p(t, x, y) dy = 1,$$

no mass is lost with the interface condition (9c).

Besides, from the maximum principle, $p(t, x, y) \geq 0$ for any $t > 0$ and $(x, y) \in \mathcal{R}^2$. In addition,

$$p(t, x, y) \leq \begin{cases} 2\gamma(t, x, y) & \text{for } xy \geq 0, \\ \gamma(t, x, y) & \text{for } xy \leq 0, \end{cases} \quad (14)$$

since for $z > 0$, $\gamma(t, \xi + z) \leq \gamma(t, z)$ and $\int_0^{+\infty} 2\lambda \exp(-2\lambda\xi) d\xi = 1$.

3. Probabilistic representation

3.1. The snapping out Brownian motion

Let us start with the definition of the snapping out Brownian motion.

Definition 1. A *snapping out Brownian motion* is a stochastic Markov process $(X, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathcal{R}})$ living in \mathcal{R} whose infinitesimal generator is $(\mathcal{L}, \text{Dom}(\mathcal{L}))$.

Using the fundamental solution $p(t, x, y)$ of $(\mathcal{L}, \text{Dom}(\mathcal{L}))$ and the Chapman-Kolmogorov property (10), it is now possible to consider a stochastic diffusion process X generated by $(\mathcal{L}, \text{Dom}(\mathcal{L}))$.

Proposition 3. *The operator $(\mathcal{L}, \text{Dom}(\mathcal{L}))$ is the infinitesimal generator of a Feller process $((X_t)_{t \geq 0}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathcal{R}})$ with values in \mathcal{R} , which is a snapping out Brownian motion.*

Remark 1. The filtration $(\mathcal{F}_t)_{t \geq 0}$ is the filtration generated by X by $\mathcal{F}_t = \cap_{\mu} \mathcal{F}_t^{\mu}$, where \mathcal{F}_t^{μ} is the completion of $\sigma(X_s, s \leq t)$ under \mathbb{P}_{μ} for any initial probability measure μ on \mathcal{R} . This filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous (See [33, Sect. III.9] or [37] for example) and we refer at it as its *augmented natural filtration*.

Remark 2. Being a Feller process, X has the strong Markov property with respect to $(\mathcal{F}_t)_{t \geq 0}$ constructed as above [33, Theorem III.9.14]. However, as it will be clear from the probabilistic representation, we could consider the process $Y = \pi(X)$ with values in \mathbb{R} obtained by identifying the points $0+$ and $0-$. This process Y is then *not* Markov.

Remark 3. As such, we define a process which has right-continuous with left-limit paths. However, its natural projection from \mathcal{R} onto \mathbb{R} has continuous paths, as seen in Proposition 5 below.

Proof. Let $\mathcal{C}_0(\mathcal{R})$ the space of functions on \mathcal{R} which are continuous on \mathbb{R}_+ and \mathbb{R}_- and decreasing to 0 at infinity. Set for $f \in \mathcal{C}_0(\mathcal{R})$,

$$P_t f(x) = \int_{\mathcal{R}} p(t, x, y) f(y) dy.$$

Using the Gaussian type inequalities (14), $P_t f \in \mathcal{C}_0(\mathcal{R})$. With (10), $P_{t+s} = P_t P_s$ for any $s, t \geq 0$ so that $(P_t)_{t \geq 0}$ generates a Feller process with values in \mathcal{R} [33, Theorem III.7.17]. \square

3.2. Elastic Brownian motion

By definition, the (reflected) *elastic Brownian motion*² is the process associated to the Robin boundary condition, the same way the Reflected Brownian motion is associated to the Neumann boundary condition. This process is constructed in [18, p. 45] and in [19, p. 255] (see also [9, § 11] among other references).

We present briefly the elastic Brownian motion. A more detailed account is given in Appendix A.

Let B be a Brownian motion, L be its symmetric local time at 0 defined by

$$L_t = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{B_s \in [-\epsilon, \epsilon]} ds.$$

Let us add to \mathbb{R}_+ a point \dagger and let us set

$$Y_t = \begin{cases} |B_t| & \text{if } t < \tau \\ \dagger & \text{if } t \geq \tau \end{cases} \text{ with } \mathbb{P}[\tau > t | |B \cdot|] = e^{-\kappa L_t}.$$

²Also called *partially reflected Brownian motion* in the literature [15].

This process is a strong Markov process and its density is

$$p^e(t, x, y) dy = \mathbb{E}_x[e^{-\kappa L_t}, |B_t| \in dy]. \quad (15)$$

Its infinitesimal generator³ is $\mathcal{L}^e = \frac{1}{2}\Delta$ with domain

$$\text{Dom}(\mathcal{L}^e) = \mathcal{C}_b^2(\mathbb{R}_+; \mathbb{R}) \cap \{f : \mathbb{R}_+ \rightarrow \mathbb{R} \mid \kappa f(0) = f'(0)\}$$

with $f'(0) = \lim_{\epsilon \rightarrow 0} \epsilon^{-1}(f(\epsilon) - f(0))$.

The event $\{|B_t| \leq y\}$ may be decomposed as the disjoint union of $\{|B_t| \leq y\} \cap \{\tau \leq t\}$ and of $\{|B_t| \leq y\} \cap \{\tau > t\}$, where $\mathfrak{h}_0 = \inf\{t > 0 \mid B_t = 0\}$. Derived from the reflection principle, the density of \mathfrak{h}_0 is well known (See e.g. [33, (13.5) in Section I.13]). Using the Markov property and the fact that the local time of a Continuous Additive Functional, this gives an alternative representation of p^e [14]:

$$p^e(t, x, y) = \gamma(t, x, y) - \gamma(t, x, -y) + e^{\kappa(x+y)} \int_{x+y}^{+\infty} \frac{\sqrt{2}e^{-\kappa\xi}}{\sqrt{\pi t^3}} \xi e^{-\xi^2/2t} d\xi. \quad (16)$$

It is easily seen that until it hits $0\pm$, the process X behaves like a Brownian motion. The problem is now to describe its behavior at $0\pm$. We will give a stronger statement in Proposition 5 below.

3.3. A probabilistic representation of the density of the snapping out Brownian motion

Proposition 4. *Let B be a Brownian motion on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and \mathbb{P}_x be a family of probabilities such that $\mathbb{P}_x[B_0 = x] = 1$. Let L be the symmetric local time of B at 0. Then for any $t > 0$, any $x \in \mathcal{R}$ and any bounded measurable function $f : \mathcal{R} \rightarrow \mathbb{R}$,*

$$\begin{aligned} & \int_{\mathcal{R}} p(t, x, y) f(y) dy \\ &= \mathbb{E}_x \left[\left(\frac{1 + e^{-2\lambda L_t}}{2} \right); f(\text{sgn}(x)|B_t|) \right] + \mathbb{E}_x \left[\left(\frac{1 - e^{-2\lambda L_t}}{2} \right); f(-\text{sgn}(x)|B_t|) \right]. \end{aligned} \quad (17)$$

Proof. This is an immediate consequence of the construction of $p(t, x, y)$ in the proof of Proposition 2 and (15). \square

The relation (17) could be informally written

$$p(t, x, y) dy = \mathbb{E}_x \left[\frac{1 + \text{sgn}(y) \text{sgn}(x) e^{-2L_t}}{2}; \text{sgn}(y) \text{sgn}(x) |B_t| \in dy \right].$$

³Defined here as the image by the resolvent of the set of continuous bounded functions, see Appendix A, while we have considered first Robin boundary problem in $L^2(\mathbb{R}_+)$.

Let $z = \inf\{t > 0 \mid X_t \in \{0\pm\}\}$. For the sake of simplicity, we assume that $x \geq 0$. Then for a bounded, measurable function f ,

$$\mathbb{E}_x[f(X_t)] = \mathbb{E}_x\left[\left(\frac{1+e^{-2\lambda L_t}}{2}\right)f(|B_t|)\right] + \mathbb{E}_x\left[\left(\frac{1-e^{-2\lambda L_t}}{2}\right)f(-|B_t|)\right].$$

With the strong Markov property, since $L_z = 0$,

$$\mathbb{E}_x\left[\left(\frac{1 \pm e^{-2\lambda L_t}}{2}\right)f(\pm|B_t|); z < t\right] = \int_0^t \mathbb{E}_{0+}\left[\left(\frac{1 \pm e^{-2\lambda L_{t-s}}}{2}\right)f(\pm|B_{t-s}|)\right] \mathbb{P}_x[z \in ds]$$

and a similar inequality holds when $x \leq 0$. Yet, we know that [33, I(13.5)]

$$r(x, s) := \mathbb{P}_x[z \in ds] = \frac{|x|}{\sqrt{2\pi s^3}} \exp\left(\frac{-|x|^2}{2s}\right) ds. \quad (18)$$

On the other hand,

$$\mathbb{E}_x[f(X_t); z > t] = \mathbb{E}_x[f(B_t); z > t] = \int_{\mathbb{R}_+} (\gamma(t, y-x) - \gamma(t, y+x)) f(y) dy.$$

This implies a formula similar to (16) for the elastic Brownian motion: For $x \geq 0$ and $y \geq 0$,

$$p(t, x, y) = \gamma(t, y-x) - \gamma(t, y+x) + \int_0^t \frac{|x|}{\sqrt{2\pi s^3}} e^{-|x|^2/2s} p(t-s, 0+, y) ds \quad (19a)$$

while for $x \geq 0$ and $y \leq 0$,

$$p(t, x, y) = \int_0^t \frac{|x|}{\sqrt{2\pi s^3}} e^{-|x|^2/2s} p(t-s, 0+, y) ds. \quad (19b)$$

Remark 4. One of interest of (19a)-(19b) lies in its interpretation in term of simulation. Since we known how to simulate $(t \wedge z, B_{t \wedge z})$, it is possible to simulate X_t under \mathbb{P}_x from X_t under $\mathbb{P}_{0\pm}$, as seen in Section 6.

Remark 5. The following description of the action of the semi-group on the even and odd functions will be useful. For a bounded, measurable function f , set

$$\hat{f}(x) = \frac{f(x) + f(-x)}{2} \text{ and } \check{f}(x) = \frac{f(x) - f(-x)}{2}$$

so that \hat{f} is even and \check{f} is odd. Let $(P_t^r)_{t>0}$ be the semi-group of the reflected Brownian motion and $(P_t^e)_{t>0}$ the one of the elastic Brownian motion. Then

$$P_t f(x) = P_t^r \hat{f}(|x|) + \text{sgn}(x) P_t^e \check{f}(|x|) \quad (20)$$

for any $x \in \mathcal{R}$. This representation will be used to prove Proposition 6 below.

Proposition 5. Let $|\cdot| : \mathcal{R} \rightarrow \mathbb{R}_+$ be defined as $|x| = x$ for $x \in \mathbb{R}_+$ and $|x| = -x$ for $x \in \mathbb{R}_-$. The processes $|X|$ and $|B|$ have the same distribution under \mathbb{P}_x .

Proof. With the Markov properties, it is sufficient to prove that $|X|$ and $|B|$ have the same marginals.

We assume $x \geq 0$ for the sake of simplicity. Indeed, for a bounded measurable function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ extended as a function on \mathcal{R} with $f(-x) = f(x)$, $\mathbb{E}_x[f(|X_t|)] = \mathbb{E}_x[f(X_t)]$. With (19a)-(19b),

$$\begin{aligned} \int_{\mathcal{R}} p(t, x, y) f(y) dy &= \int_{\mathbb{R}_+} (\gamma(t, y - x) - \gamma(t, y + x)) f(y) dy \\ &+ \int_0^t \int_{\mathcal{R}} \frac{|x|}{\sqrt{2\pi s^3}} e^{-|x|^2/2s} (p(t-s, 0+, y) + p(t-s, 0+, -y)) f(y) dy ds \\ &= \mathbb{E}_x[f(B_t); t < \tau] + \mathbb{E}_x[f(|B_t|); \tau \leq t] = \mathbb{E}_x[f(|B_t|)]. \end{aligned}$$

This concludes the proof. \square

It is also possible to compute the probability to be on a given side of the interface, which gives then the total mass of particles on each side. Let us recall that by the Mills ratio (see e.g. [18, Problem 1, p. 17], $\bar{\Phi}(z) \sim_{z \rightarrow \infty} \gamma(z)/z$ so that $\Theta(t, y)$ is integrable in y . Thus, after an integration by parts and some straightforward computations,

$$\theta(t) := \int_0^{+\infty} \Theta(t, y) dy = \frac{1}{2} - \exp(2\lambda^2 t) \bar{\Phi}(2\lambda\sqrt{t}).$$

By the Mills ratio,

$$\theta(t) \sim_{t \rightarrow \infty} \frac{1}{2} - \frac{1}{\lambda\sqrt{8\pi t}}.$$

This proves that

$$\mathbb{P}_{0+}[X_t \geq 0] = \frac{1}{2} + \exp(2\lambda^2 t) \bar{\Phi}(2\lambda\sqrt{t}) \text{ and } \mathbb{P}_{0+}[X_t \leq 0] = \frac{1}{2} - \exp(2\lambda^2 t) \bar{\Phi}(2\lambda\sqrt{t}).$$

In particular, $\mathbb{P}_{0+}[X_t \geq 0] \xrightarrow{t \rightarrow \infty} \frac{1}{2}$.

On the other hand, using the expansion

$$\bar{\Phi}(2\lambda\sqrt{t}) = \frac{1}{2} - \frac{\sqrt{2t}}{\sqrt{\pi}} \lambda + O(t^{3/2}) \text{ and } \exp(2\lambda^2 t) = 1 + 2\lambda^2 t + O(t^2),$$

we obtain that

$$\theta(t) = \frac{\sqrt{2t}}{\sqrt{\pi}} \lambda + \lambda^2 t + O(t^{3/2})$$

and then that

$$\mathbb{P}_{0+}[X_t < 0] \sim_{t \rightarrow 0} \frac{\sqrt{2t}}{\sqrt{\pi}} \lambda \text{ and } \mathbb{P}_{0-}[X_t > 0] \sim_{t \rightarrow 0} \frac{\sqrt{2t}}{\sqrt{\pi}} \lambda. \quad (21)$$

4. A dynamical description of the snapping out Brownian motion

The elastic Brownian motion corresponds to a reflected Brownian motion which is killed when its local time at 0 is bigger than an independent exponential random variable. Hence, the killing may only take place when the elastic Brownian motion is at 0. Let B be a Brownian motion, let L be its symmetric local time and let \mathfrak{k} be a random variable such that

$$\mathbb{P}[\mathfrak{k} > t | B] = e^{-\kappa L_t}, \quad \kappa = 2\lambda.$$

For a bounded, measurable function f and $x > 0$,

$$\mathbb{E}_x[f(|B_t|); t < \mathfrak{k}] = \mathbb{E}_x[e^{-\kappa L_t} f(|B_t|)] = P_t^e f(x)$$

where $P_t^e f(x) = \int_0^{+\infty} p^e(t, x, y) f(y) dy$ (See Appendix A).

We have a similar result here, excepted that the process is immediately reborn from $0+$ or $0-$.

This justify the classical interpretation of the interface condition at 0 as a *semi-permeable barrier* [4, 10].

Our task is then to kill and let reborn a Markov process. Our main tool will be a result presented in Appendix E.

Proposition 6. *For $x \in \mathcal{R}$, let $(B, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ (with $\mathbb{P}_{0+} = \mathbb{P}_{0-} = \mathbb{P}_0$) be a Brownian motion with τ as its inverse local time. Let ξ_1, ξ_2, \dots be independent exponential random variables of parameter 2λ . Set $\mathfrak{u}_0 := 0$, $\mathfrak{u}_k := \tau_{\xi_1 + \dots + \xi_k}$ for $k \geq 1$. Let $\sigma_0 := \text{sgn}(x)$ and for $k \geq 1$, σ_k be independent random variables with $\mathbb{P}[\sigma_k = 1] = \mathbb{P}[\sigma_k = -1] = 1/2$. Then there exists an appropriate probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with a family $(\widehat{\mathbb{P}}_x)_{x \in \mathcal{R}}$ of probabilities and a filtration $(\mathcal{G}_t)_{t \geq 0}$ such that*

$$Y_t := \sigma_k |B_t| \text{ for } t \in [\mathfrak{u}_k, \mathfrak{u}_{k+1}), \quad Y_0 = x, \quad (22)$$

is a snapping out Brownian motion under $\widehat{\mathbb{P}}_x$ with the strong Markov property with respect to $(\mathcal{G}_t)_{t \geq 0}$.

Remark 6. With this proposition, we may add a new starting point 0 which is inaccessible by setting $\sigma_0 = \pm 0$ with probability 1/2, independently from any other random variables.

From this result, we deduce a result on the resolvent, in the spirit of the Itô excursions approach through resolvent [34, 35]. This will be useful to study the approximation by a thin layer in Section 5 below.

For a measurable bounded function f and $\alpha > 0$, we define

$$G_\alpha f(x) := \mathbb{E}_x \left[\int_0^{+\infty} f(X_s) ds \right]$$

so that $(G_\alpha)_{\alpha>0}$ is the resolvent family associated to $(\mathcal{L}, \text{Dom}(\mathcal{L}))$. We also set for $x \geq 0$,

$$G_\alpha^e f(x) := \mathbb{E}_x \left[\int_0^k f(X_s) ds \right] = \int_0^{+\infty} e^{-\alpha t} P_t^e f(X) dt,$$

so that $(G_\alpha^e)_{\alpha>0}$ is the resolvent family associated to the elastic Brownian motion. We extend G_α^e to \mathcal{R} by $G_\alpha^e f(x) = G_\alpha^e \tilde{f}(-x)$ for $x \geq 0$ with $\tilde{f}(x) = f(-x)$.

Corollary 1. *For any $x \in \mathcal{R}$ and any $\alpha > 0$,*

$$G_\alpha f(x) = G_\alpha^e f(x) + e^{-\sqrt{2\alpha}x} \frac{\kappa}{\sqrt{2\alpha}} \frac{G_\alpha^e f(0+) + G_\alpha^e f(0-)}{2}. \quad (23)$$

The following result is asserted in [4].

Corollary 2. *Let B be a Brownian motion with τ as its inverse local time. Let ζ_1, ζ_2, \dots be independent exponential random variables of parameter λ . Set $\mathfrak{s}_0 := 0$, $\mathfrak{s}_k := \tau_{\zeta_1 + \dots + \zeta_k}$ for $k \geq 1$ and*

$$Y_t := \text{sgn}(x)^{k+1} |B_t| \text{ for } t \in [\mathfrak{s}_k, \mathfrak{s}_{k+1}).$$

Then, on an appropriate probability space, there exists a filtration $(\mathcal{H}_t)_{t \geq 0}$ for which $(Y, (\mathcal{H}_t)_{t \geq 0}, (\widehat{\mathbb{P}}_x)_{x \in \mathcal{R}})$ is a strong Markov, snapping out Brownian motion.

In Figure 2, we represent this construction together with the one of Proposition 6.

Remark 7. We see easily that for any $k \geq 1$, $\mathbb{P}[\mathfrak{s}_{k+1} - \mathfrak{s}_k > t | \mathcal{B}, \mathfrak{s}_k] = \exp(-\lambda L_{t-\mathfrak{s}_k})$.

Remark 8. The filtrations $(\mathcal{G}_t)_{t \geq 0}$ of Proposition 6 and $(\mathcal{H}_t)_{t \geq 0}$ of Proposition 1 differs. In particular, in Theorem 1 of Appendix E, \mathfrak{u}_k (resp. \mathfrak{s}_k) is a $(\mathcal{G}_t)_{t \geq 0}$ (resp. $(\mathcal{H}_t)_{t \geq 0}$)-stopping time. However, \mathfrak{u}_k is not of $(\mathcal{H}_t)_{t \geq 0}$ -stopping time as information is lost about the time at which the local time becomes higher than the ξ_i 's whose sums give the ζ_k 's.

The filtration $(\mathcal{G}_t)_{t \geq 0}$ contains more information than $(\mathcal{H}_t)_{t \geq 0}$.

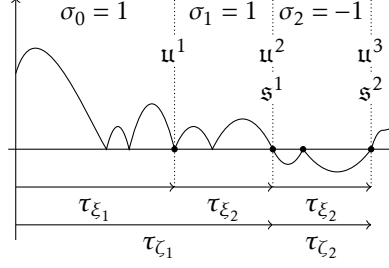


Figure 2: Construction of a snapping out Brownian motion from elastic Brownian motions.

Proof of Proposition 6. Let B^{e1}, B^{e2}, \dots be a sequence of independent elastic Brownian motions⁴ each with lifetime k^i defined as $k^i = \tau_{\zeta_i}^i$, where τ^i is the inverse of the local time of B^{e_i} and ζ_i is an exponential random variable of parameter 2λ . Here, B^{e1} is an elastic Brownian motion starting at $|x|$, while B^{e2}, B^{e3}, \dots start from 0.

The space $\widehat{\Omega}$ is the product of a countable number of copies of $\Omega \times \text{Bor}(\mathcal{R})$, with elements $\widehat{\omega} = ((\omega_0, x_0), (\omega_1, x_1), \dots)$.

Let us construct iteratively a sequence of times $\{u_k\}_k$ and a process Y living in \mathcal{R} by

$$\begin{aligned} u_0(\widehat{\omega}) &= 0, \quad Y_t(\widehat{\omega}) = \sigma_k B_{t-u_k}^{e_k}(\omega_k) \text{ for } t \in [u_k(\widehat{\omega}), u_{k+1}(\widehat{\omega})], \\ x_0 &= x, \quad \sigma_k := \text{sgn}(x_k) \text{ and } u_{k+1}(\widehat{\omega}) = u_k(\widehat{\omega}) + k^k(\omega_k). \end{aligned}$$

Here, we set $\sigma_k \times 0 = 0\pm$ according to the sign of σ_k . Since $B_{k^i}^{e_i} = 0$ for $i = 1, 2, \dots$, it clear that $|Y|$ has the same distribution as $|B|$ and is continuous.

Using the results of Appendix E, when killed, the process is reborn according to the instantaneous distribution $\mu(\omega, \cdot) = \frac{1}{2}\delta_{0+} + \frac{1}{2}\delta_{0-}$.

Using the results of Appendix E, Y is a strong Markov process when defined on the appropriate setting.

With the properties of the Brownian motion and since its local time is a continuous additive functional, it is easily seen that Y may be identified with the process given by (22) constructed on a probability space $\Omega := \mathcal{C}(\mathbb{R}_+; \mathbb{R}) \times \mathbb{R}_+^{\mathbb{N}} \times \{-1, 1\}^{\mathbb{N}}$ encoding a Brownian motion, independent exponential random variables and Bernoulli random variables.

Let f be a bounded measurable function. Then for $x \in \mathcal{R}$, $Q_t f(x) = P_t^r f(|x|)$ if f is even.

⁴Recall that in our setting, an elastic Brownian motion is a reflected Brownian motion killed at rate $-\kappa dL_t$ and is then non-negative.

Assume now that f is odd and choose $\alpha > 0$. As $f(\sigma_k x) = \sigma_k f(|x|)$,

$$\begin{aligned} H_\alpha f(x) &= \int_0^{+\infty} e^{-\alpha t} Q_t f(x) dt = \sum_{k=0}^{+\infty} \mathbb{E}_x \left[e^{-\alpha u_k} \int_{u_k}^{u_{k+1}} e^{-\alpha(t-u_k)} \sigma_k f(|B_t|) dt \right] \\ &= \operatorname{sgn}(x) G_\alpha^e f(|x|) + \sum_{k=1}^{+\infty} \mathbb{E}_x [\sigma_k e^{-\alpha u_k}] G_\alpha^e f(0), \end{aligned}$$

where G_α^e is the resolvent of the elastic Brownian motion defined by $G_\alpha^e f(x) := \int_0^{+\infty} e^{-\alpha t} P_t^e f(x) dt$. By the independence of σ_k and u_k , $H_\alpha f(x) = \operatorname{sgn}(x) G_\alpha^e f(|x|)$ when f is odd.

Using the notation of Remark 5, for a bounded, measurable function f ,

$$H_\alpha f(x) = G_\alpha^r \hat{f}(|x|) + \operatorname{sgn}(x) G_\alpha^e \check{f}(|x|).$$

Inverting the Laplace transform and comparing this expression with (20), this means that the semi-group $(Q_t)_{t>0}$ of Y is equal to $(P_t)_{t>0}$. This is sufficient to identify Y with a snapping out Brownian motion.

Indeed, it is clear also that Y may be realized from a single Brownian motion with inverse local time τ , by letting $u_k = \tau_{\zeta_1 + \dots + \zeta_k}$, since from the properties of the subordinator, $\tau_{t+s} - \tau_t = \tau_s \circ \theta_{\tau_t}$, where $(\theta)_{t>0}$ is the family of shift operators on $\mathcal{C}(\mathbb{R}_+; \mathbb{R})$. \square

Proof of Corollary 1. We have since $k = u_1$,

$$\begin{aligned} G_\alpha f(x) &= \mathbb{E}_x \left[\int_0^{+\infty} e^{-\alpha s} f(X_s) ds \right] \\ &= \mathbb{E}_x \left[\int_0^k e^{-\alpha s} f(X_s) ds \right] + \frac{1}{2} \mathbb{E}_x [e^{-\alpha k}] (G_\alpha f(0+) + G_\alpha f(0-)). \end{aligned}$$

Using (27) of Appendix A and the fact that up to time k , Y behaves like the elastic Brownian motion,

$$G_\alpha f(x) = G_\alpha^e f(x) + \frac{1}{2} \frac{\kappa}{\sqrt{2\alpha + \kappa}} e^{-\sqrt{2\alpha} x} (G_\alpha f(0+) + G_\alpha f(0-)).$$

Summing $G_\alpha f(0+)$ with $G_\alpha f(0-)$ in (4), we obtain easily (23). \square

Proof of Corollary 2. If $\sigma_0 = \pm 1$, then the first index N for which $\sigma_N = \mp 1$ follows a geometric distribution of parameter $1/2$. Hence, the first time at which the sign of Y changes is then $\tau_{\xi_1 + \dots + \xi_N}$. Using for example the Laplace transform of the exponential distribution, it is easily seen that $\zeta = \xi_1 + \dots + \xi_N \sim \exp(\lambda)$. For a family $(\zeta_k)_k$ of independent exponential random variables of parameter λ , we then set $s_{k+1} = s_k + \tau_{\zeta_k} \circ \theta_{s_k}$, where $(\theta_t)_{t \geq 0}$ is the family of shift operators. \square

5. The thin layer

Let us fix a small parameter $\epsilon > 0$. Let us consider the process X^ϵ generated by the divergence form operator $\mathcal{L}^\epsilon = \frac{1}{2}\nabla(a^\epsilon \nabla \cdot)$ with

$$a^\epsilon(x) = \begin{cases} D^\epsilon := \kappa\epsilon & \text{if } x \in [-\epsilon, \epsilon], \\ 1 & \text{otherwise.} \end{cases}$$

The existence of the process X^ϵ is granted for example by the existence and properties of the fundamental solution associated to L^ϵ [40]. Besides, this process is the strong solution to (See [25] for example)

$$X_t^\epsilon = x + \int_0^t \sqrt{a^\epsilon(X_s^\epsilon)} dB_s + \eta_\epsilon L_t^\epsilon(X^\epsilon) - \eta_\epsilon L_t^{-\epsilon}(X^\epsilon) \text{ with } \eta_\epsilon = \frac{1 - D^\epsilon}{1 + D^\epsilon},$$

where B is a Brownian motion and $L_t^x(X^\epsilon)$ is the symmetric local time at x of X^ϵ . We study the limit of the process X^ϵ . The convergence results used for example in [25] cannot be applied here, since a^ϵ decreases to 0.

Let us recall that π is the natural projection from \mathcal{R} onto \mathbb{R} where $0+$ and $0-$ are identified. Let X be a snapping out Brownian motion of parameter 2λ . As noted in Remark 2, $\pi \circ X$ is not a Markov process.

Proposition 7. *For any $t > 0$, the distribution $\mathbb{P}_x \circ (X_t^\epsilon)^{-1}$ converges to $\mathbb{P}_x \circ (\pi \circ X_t)^{-1}$.*

Remark 9. Since $\pi \circ X_t$ is not Markov, this property and the tightness of $(X^\epsilon)_{t>0}$ is not sufficient to prove the convergence of $\mathbb{P}_x \circ (X^\epsilon)^{-1}$ to $\mathbb{P}_x \circ (\pi \circ X)^{-1}$ on $\mathcal{C}([0, T]; \mathbb{R})$.

Remark 10. This result follows from classical results in homogenization theory (See [36] for example) where the convergence holds in Sobolev spaces. Here, we consider a direct computational proof for the convergence of the Green kernel which we need for proving Proposition 9 below.

Proof. We assume that $x > 0$ and we set $\mu := \sqrt{2\alpha}$ for some $\alpha > 0$. The resolvent density g_α^ϵ of X^ϵ has the form, for $x > \epsilon$,

$$g_\alpha^\epsilon(x, y) = \begin{cases} C_\epsilon(x)e^{-\mu y} & \text{for } y > x \\ A_\epsilon(x)e^{-\mu y} + B_\epsilon(x)e^{\mu y} & \text{for } y \in [\epsilon, x], \\ H_\epsilon(x)e^{\mu y/\sqrt{D^\epsilon}} + E_\epsilon(x)e^{-\mu y/\sqrt{D^\epsilon}} & \text{for } y \in [-\epsilon, \epsilon], \\ F_\epsilon(x)e^{\mu y} & \text{for } y < -\epsilon. \end{cases}$$

By this, we mean that

$$\mathbb{E}_x \left[\int_0^{+\infty} e^{-\alpha t} f(X_s^\epsilon) ds \right] = \int_{\mathbb{R}} g_\alpha^\epsilon(x, y) f(y) dy$$

for any bounded, measurable function f . Or equivalently, $g_\alpha^\epsilon(x, y)$ is the Laplace transform of $p^\epsilon(\cdot, x, y)$, where p^ϵ is the density transition function of X^ϵ .

With the conditions

$$\begin{aligned} g_\alpha^\epsilon(x, \epsilon+) &= g_\alpha^\epsilon(x, \epsilon-), \quad g_\alpha^\epsilon(x, \epsilon-) = g_\alpha^\epsilon(x, \epsilon+), \\ \nabla_y g_\alpha^\epsilon(x, -\epsilon-) &= D^\epsilon \nabla_y g_\alpha^\epsilon(x, -\epsilon+), \quad D^\epsilon \nabla_y g_\alpha^\epsilon(x, \epsilon-) = \nabla_y g_\alpha^\epsilon(x, \epsilon+), \\ \nabla_y g_\alpha^\epsilon(x, x+) - \nabla_y g_\alpha^\epsilon(x, x-) &= 2, \end{aligned}$$

we may find an expression for the coefficients A_ϵ , B_ϵ , C_ϵ , H_ϵ and F_ϵ . Let us set

$$G_\epsilon := \left(2e^{4\frac{\mu\epsilon}{\sqrt{D^\epsilon}}} \sqrt{D^\epsilon} + e^{4\frac{\mu\epsilon}{\sqrt{D^\epsilon}}} D^\epsilon + 2\sqrt{D^\epsilon} - D^\epsilon + e^{4\frac{\mu\epsilon}{\sqrt{D^\epsilon}}} - 1 \right) \mu.$$

Let us note that

$$G_\epsilon = 4\sqrt{D^\epsilon} \left(1 + \frac{\mu}{\kappa} + O(D^\epsilon) \right) \mu$$

since $\epsilon \rightarrow 0$, $D^\epsilon \rightarrow 0$ and $D^\epsilon/\epsilon = \kappa$. Then using a symbolic computation system,

$$\begin{aligned} A_\epsilon(x) &= - \left(e^{4\frac{\mu\epsilon}{\sqrt{D^\epsilon}}} D^\epsilon - D^\epsilon - e^{4\frac{\mu\epsilon}{\sqrt{D^\epsilon}}} + 1 \right) e^{\mu(2\epsilon-x)/G_\epsilon} \xrightarrow{\epsilon \rightarrow 0} A_0(x) := -\frac{e^{-\mu x}}{\kappa + \mu}, \\ B_\epsilon(x) &= B_0(x) := -\frac{e^{-\mu x}}{\mu}, \\ C_\epsilon(x) &= -2 \sinh(2\epsilon/\sqrt{D^\epsilon}) \left(d e^{-\mu x + 2\mu\epsilon} - e^{-\mu x + 2\mu\epsilon} + d e^{\mu x} + e^{-\mu x} \right) e^{\frac{2\mu\epsilon}{\sqrt{D^\epsilon}}/G_\epsilon} \\ &\quad + 4\sqrt{D^\epsilon} e^{\mu x} \cosh(2\epsilon/\sqrt{D^\epsilon}) e^{\frac{2\mu\epsilon}{\sqrt{D^\epsilon}}/G_\epsilon} \xrightarrow{\epsilon \rightarrow 0} C_0(x) := \frac{\kappa e^{\mu x}}{\mu(\kappa + \mu)}, \\ H_\epsilon(x) &= -2e^{\frac{\mu(3\epsilon + \epsilon\sqrt{D^\epsilon} - x\sqrt{D^\epsilon})}{\sqrt{D^\epsilon}}} (1 + \sqrt{D^\epsilon}) \sqrt{D^\epsilon}/G_\epsilon \xrightarrow{\epsilon \rightarrow 0} H_0(x) := -\frac{\kappa e^{-\mu x}}{2\mu(\kappa + \mu)}, \\ E_\epsilon(x) &= -2e^{\frac{\mu(\epsilon + \epsilon\sqrt{D^\epsilon} - x\sqrt{D^\epsilon})}{\sqrt{D^\epsilon}}} (1 - \sqrt{D^\epsilon}) \sqrt{D^\epsilon}/G_\epsilon \xrightarrow{\epsilon \rightarrow 0} H_0(x), \\ F_\epsilon(x) &= -4\sqrt{D^\epsilon} e^{\frac{\mu(2\epsilon + 2\epsilon\sqrt{D^\epsilon} - x\sqrt{D^\epsilon})}{\sqrt{D^\epsilon}}} /G_\epsilon \xrightarrow{\epsilon \rightarrow 0} F_0(x) := \frac{-\kappa e^{-\mu x}}{\mu(\kappa + \mu)} = -C_0(-x). \end{aligned}$$

Let g_α be the function

$$g_\alpha(x, y) := \begin{cases} C_0(x) e^{-\mu y} & \text{if } y > x, \\ A_0(x) e^{-\mu y} + B_0(x) e^{\mu y} & \text{if } y \in [0, x], \\ F_0(x) e^{\mu y} & \text{if } y < 0. \end{cases}$$

Thus, we see that

$$\frac{\kappa}{2} (g_\alpha(x, 0+) - g_\alpha(x, 0-)) = g'_\alpha(x, 0+) = g'_\alpha(x, 0-),$$

that $(\alpha - \frac{1}{2}\Delta_y)g_\alpha(x, y) = \delta_y(x)$ and

$$g_\alpha^\epsilon(x, y) \xrightarrow{\epsilon \rightarrow 0} g_\alpha(x, y).$$

Indeed, we recover the resolvent density $g_\alpha(x, y)$ of the snapping out Brownian motion computed in Section C. A similar work may be performed for $x = 0+$ and for $x \leq 0$ using symmetry arguments. The convergence of the resolvent implies the convergence of the density by inverting the Laplace transforms as stated by the Trotter-Kato theorem (See e.g. [20, Theorem IX.2.16, p. 504]). \square

We may now investigate the properties of the time k at which the particle starts “afresh” as a new snapping out Brownian motion on a randomly selected side. This time k corresponds to the lifetime of an elastic Brownian motion of parameter 2λ .

As in [8, 25], we introduce $\Phi^\epsilon(x)$ as the piecewise linear function defined by

$$\frac{d\Phi^\epsilon}{dx}(x) = \begin{cases} 1/\sqrt{D^\epsilon} & \text{if } x \in [-\epsilon, \epsilon], \\ 1 & \text{otherwise.} \end{cases}$$

Set $Y^\epsilon = \Phi^\epsilon(X^\epsilon)$ so that Y^ϵ is solution to the SDE [8, 25]

$$Y_t^\epsilon = \Phi^\epsilon(x) + B_t + \theta^\epsilon L_t^{y^\epsilon}(Y^\epsilon) - \theta^\epsilon L_t^{-y^\epsilon}(Y^\epsilon) \text{ with } \theta^\epsilon = \frac{1 - \sqrt{D^\epsilon}}{1 + \sqrt{D^\epsilon}} \text{ and } y^\epsilon := \Phi^\epsilon(\epsilon) = \sqrt{\epsilon/\kappa}.$$

The infinitesimal generator of Y^ϵ is $\frac{1}{2}\Delta$ with a domain which is the closure of $f \in C_b^2(\mathbb{R} \setminus \{-y^\epsilon, y^\epsilon\})$ with $f(\pm y^\epsilon +) = f(\pm y^\epsilon -)$, $(1 - \theta^\epsilon)f'(y^\epsilon -) = (1 + \theta^\epsilon)f'(y^\epsilon +)$ and $(1 + \theta^\epsilon)f'(-y^\epsilon -) = (1 - \theta^\epsilon)f'(-y^\epsilon +)$.

Let $\mathfrak{h}_0^\epsilon = \inf\{t > 0 \mid Y_t^\epsilon = 0\}$, the first hitting time of zero by Y^ϵ , which is also the first hitting time of zero by X^ϵ .

Proposition 8. *Under \mathbb{P}_x , \mathfrak{h}_0^ϵ converges in distribution to a random variable k whose Laplace transform is*

$$\psi(x, \alpha) := \mathbb{E}_x[\exp(-\alpha k)] = \frac{\kappa}{\kappa + \sqrt{2\alpha}} \exp(-\sqrt{2\alpha}x). \quad (24)$$

The distribution of k is the one of τ_ζ — the lifetime of the elastic Brownian motion —, where ζ is an exponential random variable of parameter κ and $\tau_t = \inf\{s > 0 \mid L_s > t\}$ for the local time L at 0 of the Brownian motion.

Remark 11. Inverting the Laplace transform, the density f of k under \mathbb{P}_0 is then

$$f(y) = \frac{\kappa}{\sqrt{2\pi y}} - \kappa^2 \exp\left(\frac{\kappa^2 y}{2}\right) \overline{\Phi}(\sqrt{y}\kappa), \quad y \geq 0.$$

Proof. From now, we assume that $x > 0$. Since the Feynman-Kac formula is still valid for the process Y^ϵ , $\psi^\epsilon(x, \alpha) := \mathbb{E}_x[e^{-\alpha \mathfrak{h}_0^\epsilon}]$ is solution to

$$\begin{cases} \frac{1}{2} \Delta \psi^\epsilon(x, \alpha) = \alpha \psi^\epsilon(x, \alpha) \text{ for } x \neq y^\epsilon, \\ \psi^\epsilon(0, \alpha) = 1, \\ \psi^\epsilon(y^\epsilon-, \alpha) = \psi^\epsilon(y^\epsilon+, \alpha), \\ (1 - \theta^\epsilon) \nabla_x \psi^\epsilon(y^\epsilon-, \alpha) = (1 + \theta^\epsilon) \nabla_x \psi^\epsilon(y^\epsilon+, \alpha), \end{cases}$$

Hence, $\psi^\epsilon(x, \alpha)$ is sought as

$$\psi^\epsilon(x, \alpha) = \begin{cases} \gamma^\epsilon \exp(-\sqrt{2\alpha}x) & \text{if } x > y^\epsilon, \\ \cos(\sqrt{2\alpha}x) + \beta^\epsilon \sin(\sqrt{2\alpha}x) & \text{if } x \in [0, y^\epsilon]. \end{cases}$$

After some computations,

$$\beta^\epsilon = \frac{-\cos(\sqrt{2\alpha}y^\epsilon) + \sqrt{D^\epsilon} \sin(\sqrt{2\alpha}y^\epsilon)}{\sin(\sqrt{2\alpha}y^\epsilon) + \sqrt{D^\epsilon} \cos(\sqrt{2\alpha}y^\epsilon)} \text{ and } \sqrt{\epsilon} \beta^\epsilon \sim_{\epsilon \rightarrow 0} \frac{-\sqrt{\kappa}}{\kappa + \sqrt{2\alpha}}.$$

Since

$$\gamma^\epsilon = e^{\sqrt{2\alpha}y^\epsilon} \sqrt{D^\epsilon} (\beta^\epsilon \cos(\sqrt{2\alpha}y^\epsilon) - \beta^\epsilon \sin(\sqrt{2\alpha}y^\epsilon))$$

we have that

$$\gamma^\epsilon \sim_{\epsilon \rightarrow 0} \frac{\kappa}{\kappa + \sqrt{2\alpha}}.$$

Hence, for any $x > 0$,

$$\psi^\epsilon(x, \alpha) \xrightarrow{\epsilon \rightarrow 0} \psi(x, \alpha) := \frac{\kappa}{\kappa + \sqrt{2\alpha}} e^{-\sqrt{2\alpha}x}. \quad (25)$$

This proves that under \mathbb{P}_x , \mathfrak{h}_0^ϵ converges to a random variable \mathfrak{k} whose Laplace transform is $\psi(x, \alpha)$ under \mathbb{P}_x . The stopping time \mathfrak{h}_0^ϵ is also the first hitting time of zero by X^ϵ .

The identification of the distribution of \mathfrak{k} follows from the the Laplace transform and the computations done in Appendix A. \square

In [9, § 11], the elastic Brownian motion is constructed as the limit of a process which either jumps at ϵ or is killed with probability $\kappa \cdot \epsilon$ when it arrives at 0.

Here, we propose a somehow similar construction, in the sense that we construct a process Z^ϵ which jumps to $\pm\epsilon$ when it reaches 0.

Let us consider the process $(X^\epsilon, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x^k)_{x \in \mathbb{R}})$ with lifetime \mathfrak{h}_0^ϵ , that is the process generated by the divergence form operator $\frac{1}{2} \nabla(a^\epsilon \nabla \cdot)$ killed when it

reaches 0. Once killed, it is placed at ϵ or $-\epsilon$ with probability 1/2, independently from the past, and so on, ... Using the piecing out procedure described in Appendix E, this gives rise to a right-continuous strong Markov process (with respect to the appropriate filtration) $Z^\epsilon(x)$ when $X_0^\epsilon = x$.

This process $Z^\epsilon(x)$ It is then fully characterized by its semi-group and its resolvent. Besides, as 0 is never reached (when it arrives at 0, the process is instantaneously replaced at $-\epsilon$ or ϵ), $Z^\epsilon(x)$ may be identified with a process living in \mathcal{R} , by setting $\mathbb{P}_{0\pm} \circ (Z^\epsilon(x))^{-1} = \mathbb{P}_{\pm\epsilon} \circ (Z^\epsilon(x))^{-1}$.

Proposition 9. *For any $x \geq 0$, the process $Z^\epsilon(x)$ converges in distribution to the snapping out Brownian motion starting from x .*

Proof. Using the properties of the resolvent, for $\alpha > 0$ and a bounded, measurable function f ,

$$G_\alpha^\epsilon f(x) := \mathbb{E}_x \left[\int_0^{+\infty} e^{-\alpha t} f(X_s^\epsilon) ds \right] = R_\alpha^\epsilon f(x) + \mathbb{E}_x[e^{-\alpha h_0^\epsilon}] \frac{1}{2} (G_\alpha^\epsilon f(\epsilon) + G_\alpha^\epsilon f(-\epsilon)).$$

with

$$R_\alpha^\epsilon f(x) := \mathbb{E}_x \left[\int_0^{h_0^\epsilon} e^{-\alpha t} f(X_s^\epsilon) ds \right].$$

Hence, using the self-adjointness of \mathcal{L}^ϵ so that $\psi^\epsilon(x, \alpha) = \psi^\epsilon(-x, \alpha)$,

$$G_\alpha^\epsilon f(x) = R_\epsilon^\alpha f(x) + \frac{\psi^\epsilon(x, \alpha)}{1 - \psi^\epsilon(\epsilon, \alpha)} \frac{R_\epsilon^\alpha f(\epsilon) + R_\epsilon^\alpha f(-\epsilon)}{2}.$$

For the sake of simplicity, we assume that $x > 0$. Using the symmetry properties of \mathcal{L}^ϵ ,

$$R_\alpha^\epsilon f(x) = \int_0^{+\infty} (g_\alpha^\epsilon(x, y) - g_\alpha^\epsilon(x, -y)) f(y) dy.$$

In the proof of Proposition 2, we have seen that for $x, y > 0$

$$g_\alpha^\epsilon(x, y) - g_\alpha^\epsilon(x, -y) \xrightarrow{\epsilon \rightarrow 0} g_\alpha(x, y) - g_\alpha(x, -y) = g_\alpha^\epsilon(x, y),$$

where $g_\alpha^\epsilon(x, y)$ is the resolvent of the elastic Brownian motion. Thus, $R_\epsilon^\alpha f(x) \xrightarrow{\epsilon \rightarrow 0} G_\alpha^\epsilon f(x)$ for any $x > 0$. It is also easily obtained that

$$R_\epsilon^\alpha f(\epsilon) \xrightarrow{\epsilon \rightarrow 0} G_\alpha^\epsilon f(0+) \text{ and } R_\epsilon^\alpha f(-\epsilon) \xrightarrow{\epsilon \rightarrow 0} G_\alpha^\epsilon f(0-).$$

Using (23) and (25), $G_\alpha^\epsilon f(x) \xrightarrow{\epsilon \rightarrow 0} G_\alpha f(x)$. The Trotter-Kato theorem (See e.g. [20, Theorem IX.2.16, p. 504]) and the Markov property imply the convergence

in finite-dimensional distributions of $Z^\epsilon(x)$ to X under \mathbb{P}_x for $x \geq 0$. By symmetry, this could be extended to $x \leq 0$.

The only remaining point of the tightness. When away from $[-\epsilon, \epsilon]$, X^ϵ behaves like a Brownian motion. Hence, for $0 \leq s \leq t \leq T$, let us set $\bar{f}(s, t) = \inf\{u > s \mid |X_u^\epsilon| = \epsilon\}$ with possibly $\bar{f}(s, t) = +\infty$ and $\bar{l}(s, t) = \sup\{u < t \mid |X_u^\epsilon| = \epsilon\}$ with possibly $\bar{l}(s, t) = -\infty$.

If $\bar{f}(s, t) \geq t$ and $\bar{l}(s, t) \leq s$, then for $\beta < 1/2$ and almost each ω , there exists a $C(\omega)$ such that $|X_t^\epsilon(\omega) - X_s^\epsilon(\omega)| \leq C(\omega)(t-s)^\beta$ for any $0 \leq s \leq t \leq T$.

If $\bar{f}(s, t) \leq t$ and $\bar{l}(s, t) \leq s$, then

$$|X_t^\epsilon - X_s^\epsilon| \leq |X_{\bar{f}(s,t)}^\epsilon - X_s^\epsilon| + |X_t^\epsilon - X_{\bar{f}(s,t)}^\epsilon| \leq C(t-s)^\beta + 2\epsilon$$

since X_t^ϵ belongs to $[-\epsilon, \epsilon]$. A similar analysis could be carried for the other cases, which means that for some random variable C ,

$$\sup_{|t-s| < \delta} |X_t^\epsilon - X_s^\epsilon| \leq C\delta^\beta + 2\epsilon.$$

This proves that $(Z^\epsilon(x))_{\epsilon > 0}$ is tight in the space $\mathcal{D}([0, T]; \mathbb{R})$ [3] and then on $\mathcal{D}([0, T]; \mathcal{R})$. Hence, we easily deduce the convergence of $Z^\epsilon(x)$ to the snapping out Brownian motion in $\mathcal{D}([0, T]; \mathcal{R})$. \square

Now, let τ^ϵ be the first exit time from $[0, 2y^\epsilon]$ for Y^ϵ .

Proposition 10. *The distribution $\mathbb{P}_{y^\epsilon} \circ (L_{\tau^\epsilon}^{y^\epsilon}(Y^\epsilon))^{-1}$ converges weakly to the distribution of an exponential random variable of parameter κ , which is the distribution of $L_{\bar{\kappa}}$ for the snapping out Brownian motion under \mathbb{P}_{0+} .*

Proof. Let us consider excursions of Y^ϵ away from y^ϵ . In order to hit zero, Y^ϵ shall perform a negative excursion with height greater than y^ϵ . Under \mathbb{P}^ϵ , $(Y_{t \wedge \tau^\epsilon}^\epsilon - y^\epsilon)$ behaves like a Skew Brownian motion of parameter θ^ϵ stopped when it leaves $[-y^\epsilon, y^\epsilon]$. Its symmetric local time is the same of for the Brownian motion. Hence, using a well-known property of the Brownian motion, $L_{\tau^\epsilon}^{y^\epsilon}(Y^\epsilon)$ has an exponential distribution of parameter $1/y^\epsilon$ (See e.g. [33, I.14] using the identification between the local time and the supremum of the Brownian motion).

Still using the properties of the Brownian motion, the probability an excursion is below y^ϵ among the excursions away from y^ϵ of height greater than y^ϵ is $(1 - \theta^\epsilon)/2$ (For this, follow the Itô-McKean construction of the Skew Brownian motion [18, 24]). By the Independence of the excursions, the random number N^ϵ of such excursions away from y^ϵ of height greater y^ϵ done by Y^ϵ before reaching

0 is then geometrically distributed with parameter $(1 - \theta^\epsilon)/2$. Since the local time is a continuous additive functional, we have that under \mathbb{P}^{y^ϵ} ,

$$L_{\mathfrak{h}_0^\epsilon}^{y^\epsilon} \stackrel{\text{dist}}{=} \sum_{k=0}^{N^\epsilon} \zeta_i$$

where ζ_i are independent random variables distributed as $L_{\mathfrak{c}^\epsilon}^{y^\epsilon}(Y^\epsilon) \sim \exp(y^\epsilon)$. Using properties of the exponential distribution, we obtain that $L_{\mathfrak{h}_0^\epsilon}$ is exponentially distributed as an exponential of parameter $\nu^\epsilon := (1 - \theta^\epsilon)/2y^\epsilon$ so that

$$\nu^\epsilon \xrightarrow{\epsilon \rightarrow 0} \kappa.$$

This proves the result. □

6. Simulation of a snapping out Brownian motion

It is easy to simulate a discretized process X in the same way it is easy to simulate the Brownian motion. It suffices to simulate $X_{(k+1)\Delta t}$ by using a random variate with density $p(\Delta t, X_{k\Delta t}, \cdot)$ to get $\{X_{k\Delta t}\}_{k=0,1,2,\dots}$. Using the Gaussian upper bound on $p(t, x, y)$, when t is small enough, one may use the Gaussian distribution when X is far enough from 0_\pm , for example when $|X_{k\Delta t}| \geq 4\sqrt{\Delta t}$ (See [26] for a discussion).

Following Proposition 5, we may use the following algorithm to draw a random variate with density $p(t, x, \cdot)$.

We assume that $x \leq 0$. For this, we use a Brownian bridge technique to check if the process reaches 0_- before Δt . This technique is now standard (see for example [2] and [26, Sect. B.2] for an example of application and further references). Let us recall the the inverse Gaussian distribution $\mathcal{IG}(\lambda, \mu)$ has density

$$r(\mu, \lambda, x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left(\frac{-\lambda(x - \mu)^2}{2\mu^2 x}\right).$$

This distribution is easily simulated using the algorithm given in [6, p. 148] following a method proposed in [30].

If 0 was reached, then use the explicit representation of the semi-group of the snapping out Brownian motion, as well as a convenient simulation of the local time.

Let us assume that $x \geq 0$.

- Set $y = x + \sqrt{\Delta t}G$ with G a random variate whose distribution is $\mathcal{N}(0, 1)$.
- If $y > 0$, then decide with probability $\exp(-2xy/\Delta t)$ if the path X has crossed 0.
 - If no crossing occurs, then return y .

- If a crossing occurs, then draw a realization of the first hitting time $z \sim \mathcal{IG}(x/y, x^2/2\Delta t)$ for a Brownian bridge with $B_0 = x$ and $B_{\Delta t} = y$. Then simulate a random variate with density $p(t - z, 0+, y)$.
- If $y < 0$, then draw a realization of the first hitting time $z \sim \mathcal{IG}(-x/y, x^2/2\Delta t)$ for a Brownian bridge with $B_0 = x$ and $B_{\Delta t} = y$. Then simulate a random variate with density $p(t - z, 0+, y)$.

Hence, simulating $X_{\Delta t}$ when $X_0 = x$ is given reduces to the simulation of $X_{\Delta t}$ when $X_0 = 0\pm$. Still for the sake of simplicity, we assume that $x \geq 0$.

Lemma 2 ([27, 28]). *If $U \sim \mathcal{N}(0, 1)$ and $V \sim \exp(1/2t)$ are independent random variables, then*

$$L_t \stackrel{\text{dist.}}{=} \frac{1}{2}(U + \sqrt{V + U^2}).$$

It is then pretty easy to simulate X_t when $X_0 = 0+$ (change σ into $-\sigma$ to simulate X_t when $X_0 = 0-$):

- For three independent random variates $H \sim \mathcal{N}(0, 1)$ and $V \sim \exp(1/2t)$, set $\sigma := 1$ if $U \leq 1/2$ or $U \geq 1/2$ and $\exp(-\lambda(H + \sqrt{V + H^2})) \leq 2U - 1$. Otherwise, set $\sigma := -1$.
- Return $\sigma\sqrt{t}G$ where $G \sim \mathcal{N}(0, 1)$ is independent from U, V and H .

Remark 12. One may simplify this scheme by providing a simple “crossing probability” $p(x)$, which means that if $X_{k\Delta t} = x \geq 0$, then $X_{(k+1)\Delta t} = |x + \xi|$ with probability $1 - p(x)$ and $X_{(k+1)\Delta t} = -|x + \xi|$ with probability $p(x)$, where $\xi \sim \mathcal{N}(0, \Delta t)$.

Integrating (18), $\mathbb{P}_x[z \leq t] = 2\bar{\Phi}(x/\sqrt{t})$. Hence, a crossing occurs if $z \leq \Delta t$ and $X_{t-\Delta z} \leq 0$. Using (19b) and (21), we have that for $x = z\sqrt{t} \geq 0$,

$$\mathbb{P}_x[X_t \leq 0] = \frac{\lambda\sqrt{2}}{\sqrt{\pi}} \int_0^t r(x, s)\sqrt{t-s} ds + O(t)$$

with r given by (18), and the $O(t)$ depends also on z , which is assumed to be of order 1. Since

$$\int_0^t r(x, s)\sqrt{t-s} ds = x\sqrt{2\pi} \left(\frac{\gamma(z)}{z} - \bar{\Phi}(z) \right).$$

With the Mills ratio,

$$\frac{z}{z^2 + 1} \leq \frac{\bar{\Phi}(z)}{\gamma(z)} \leq \frac{1}{z},$$

the previous quantity is (of course) non-negative and bounded by $1/(z^2 + z)$. This shows that

$$\mathbb{P}_x[X_{\Delta t} \leq 0] = 2\lambda x \left(\frac{\gamma(z)}{z} - \bar{\Phi}(z) \right) + O(\Delta t) \text{ with } x = z\sqrt{\Delta t}.$$

This expression has to be compared with the one of [10].

A. The elastic Brownian motion

The elastic Brownian motion [9, 15, 18, 19] is the key process to construct a snapping out Brownian motion. This process is related to the Robin (or third) boundary problem [15, 31].

Let B be a Brownian motion and $(\mathcal{F}_t)_{t \geq 0}$ be its augmented natural filtration (See Remark 1). Let L is its symmetric local time whose right-continuous inverse τ is defined by

$$\tau_t = \inf\{s \geq 0 \mid L_s > t\},$$

so that $\{\tau_s < t\} = \{L_t > s\}$. Each τ_t is a $(\mathcal{F}_t)_{t \geq 0}$ -stopping time.

Using the canonical representation of the Brownian motion, we assume that the probability space is $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \mathcal{C}(\mathbb{R}_+; \mathbb{R})$ and \mathcal{F} is the Borel set of $\mathcal{C}(\mathbb{R}_+; \mathbb{R})$. We denote by $(\theta_t)_{t \geq 0}$ the family of shift operators on Ω .

The local time L is an additive functional, that is $L_{t+s} - L_t = L_s \circ \theta_t$ and τ satisfies $\tau_{t+s} - \tau_t = \tau_s \circ \theta_{\tau_t}$ for any $s, t \geq 0$ (See [33, p. 264]).

Let us enlarge this probability space as $\widehat{\Omega} = \Omega \times \mathbb{R}_+$, $\widehat{\mathcal{F}} = \mathcal{F} \otimes \text{Bor}(\mathbb{R}_+)$ and extend \mathbb{P}_x to $\widehat{\mathbb{P}}_x$ on $(\widehat{\Omega}, \widehat{\mathcal{F}})$ where the canonical process becomes $(\omega, k(\omega))$ with

$$\widehat{\mathbb{P}}_x[k > t \mid B] = e^{-\kappa L_t}.$$

Let \mathbb{P}^κ be the distribution of an exponential random variable ζ of parameter κ . As $\tau_\zeta = \inf\{t \geq 0 \mid L_t > \zeta\}$ and

$$\mathbb{P}^\kappa[\tau_\zeta > t] = \mathbb{P}^\kappa[L_t \leq \zeta] = \int_0^{+\infty} \kappa e^{-\kappa s} \mathbf{1}_{L_t \leq s} ds = e^{-\kappa L_t},$$

the distribution $\widehat{\mathbb{P}}_x$ may be constructed as the distribution of (B, τ_ζ) under $\mathbb{P}_x \otimes \mathbb{P}^\kappa$.

Because the local time increases only on the closure of $\{t \geq 0 \mid B_t = 0\}$, $B_k = 0$.

Let B^e be the process with values in $\mathcal{C}(\mathbb{R}_+ \cup \{\dagger\}; \mathbb{R})$ (where by definition, all the functions are extended to vanish at \dagger) defined by

$$B_t^e = \begin{cases} |B_t| & \text{if } t < k, \\ \dagger & \text{if } t \geq 0. \end{cases}$$

This process, defined by (B, k) on $\widehat{\Omega}$, is called the *elastic Brownian motion*, or *partially reflected Brownian motion*. The random variable k is called its *lifetime*.

Its semi-group is

$$P_t^e f(x) = \mathbb{E}_x[f(B_t^e)] = \widehat{\mathbb{E}}_x[f(|B_t|); t < k] = \mathbb{E}_x[f(|B_t|)e^{-\kappa L_t}]$$

for any bounded measurable function f .

Thanks to the fact that the local time is a continuous additive functional, it can be seen that B^e is a Markov process [18, p. 45].

Defining $G_\alpha^e f(x) = \mathbb{E}_x \left[\int_0^{+\infty} e^{-\alpha s} f(X_s) ds \right]$ for $f \in \mathcal{C}_b(\mathbb{R}_+; \mathbb{R})$, its infinitesimal generator defined as the differential operator \mathcal{L}^e satisfying $G_\alpha = (\alpha - \mathcal{L}^e)^{-1}$ for any $\alpha > 0$ and $\text{Dom}(\mathcal{L}^e) = G_\alpha^e(\mathcal{C}_b(\mathbb{R}_+; \mathbb{R}))$, it may be checked following [18, p. 45] that

$$\mathcal{L}^e u = \frac{1}{2} \Delta u \text{ and } \text{Dom}(\mathcal{L}^e) = \left\{ u \in \mathcal{C}_b^2(\mathbb{R}_+; \mathbb{R}) \mid u'(0) = \kappa u(0) \right\},$$

with $u'(0) = \lim_{\epsilon \rightarrow 0} \epsilon^{-1}(u(\epsilon) - u(0))$.

Indeed, being a Feller process, B^e is even a strong Markov process since it is a Feller process with respect to its augmented natural filtration (See e.g. [33, end of III.22] for a proof relying on a time change. This is also immediate from the expression of p^e given in Proposition 1).

There are several ways to construct and understand the elastic Brownian motion. In [18, p. 45] and in [33, Exercise III.24.6], the elastic Brownian motion is constructed using a killing rate depending on the local time. In [19, § 8B, p. 255], a construction is provided through a time change without local time. The elastic Brownian motion may also be constructed as a limit of a process with a jump or an absorption at 0 [9, § 11]. About multi-dimensional extensions, see [15] for application to physics and biology, and [1, 39] regarding simulation issues.

B. Laplace transform of the killing time of the elastic Brownian motion

We may compute the Laplace transform of \mathbf{k} , the killing time of a elastic Brownian motion. Starting with

$$\int_0^{+\infty} e^{-\alpha t} \gamma(t, y) dt = \frac{\exp(-\sqrt{2\alpha}|y|)}{\sqrt{2\alpha}}, \quad (26)$$

we get for $y \geq 0$,

$$\int_0^{+\infty} e^{-\alpha t} \Theta(t, y) dt = \frac{2\lambda}{\sqrt{2\alpha}} \int_0^{+\infty} \exp(-(2\lambda + \sqrt{2\alpha})\xi - \sqrt{2\alpha}y) d\xi = \frac{2\lambda \exp(-\sqrt{2\alpha}y)}{\sqrt{2\alpha}(2\lambda + \sqrt{2\alpha})}.$$

It follows that

$$\int_0^{+\infty} e^{-\alpha t} \theta(t) dt = \int_0^{+\infty} \frac{2\lambda \exp(-\sqrt{2\alpha}y)}{\sqrt{2\alpha}(2\lambda + \sqrt{2\alpha})} dy = \frac{\lambda}{\alpha(2\lambda + \sqrt{2\alpha})}.$$

For a bounded, measurable function f ,

$$\mathbb{E}_x[f(|B_t|); t < \mathbf{k}] = \mathbb{E}_x[e^{-\kappa L_t} f(|B_t|)] = P_t^e f(x).$$

It follows that

$$\mathbb{E}_x[e^{-\alpha k}] = 1 - \alpha \mathbb{E}_x \left[\int_0^k e^{-\alpha t} ds \right] = 1 - \alpha \mathbb{E}_x \left[\int_0^{+\infty} e^{-\alpha t} \mathbf{1}_{t < k} ds \right] = 1 - \alpha G_\alpha 1(x)$$

with $\kappa = 2\lambda$ and

$$G_\alpha^e 1(0) = \int_0^{+\infty} e^{-\alpha t} P_t^e 1(0) dt \text{ and } P_t^e 1(0) = 1 - 2\theta(t).$$

This gives⁵ the Laplace transform of k as $\mathbb{E}_{0+}[e^{-\alpha k}] = \kappa/(\sqrt{2\alpha} + \kappa)$. Using the strong Markov property and conditioning with respect to the first hitting time \mathfrak{h}_0 of zero for the Brownian motion, we get that

$$\mathbb{E}_x[e^{-\alpha k}] = \frac{\kappa}{\sqrt{2\alpha} + \kappa} e^{-\sqrt{2\alpha}x}, \quad (27)$$

since $\mathbb{E}[e^{-\alpha \mathfrak{h}_0}] = e^{-\sqrt{2\alpha}x}$. We recover (25) of Section 5 and (24) of Proposition 8.

C. Resolvent of the snapping out Brownian motion

The resolvent $G_\alpha f(x) := \int_0^{+\infty} e^{-\alpha t} P_t f(x) dt$ may be written $G_\alpha f(x) = \int_{\mathcal{R}} g_\alpha(x, y) f(y) dy$.

For $xy \geq 0$, we have using (26) and Proposition 2

$$g_\alpha(x, y) = \frac{\exp(-\sqrt{2\alpha}|x+y|)}{\sqrt{2\alpha}} + \frac{\exp(-\sqrt{2\alpha}|x-y|)}{\sqrt{2\alpha}} - \frac{2\lambda \exp(-\sqrt{2\alpha}(|x|+|y|))}{\sqrt{2\alpha}(2\lambda + \sqrt{2\alpha})} \quad (28)$$

while for $xy \leq 0$, we have

$$g_\alpha(x, y) = \frac{2\lambda \exp(-\sqrt{2\alpha}(|x|+|y|))}{\sqrt{2\alpha}(2\lambda + \sqrt{2\alpha})}. \quad (29)$$

In particular, one could check that for $x \geq 0$,

$$g_\alpha(x, 0+) - g_\alpha(x, 0-) = \frac{2e^{-\sqrt{2\alpha}x}}{2\lambda + \sqrt{2\alpha}}$$

and

$$\nabla_y g_\alpha(x, 0+) = \frac{2\lambda e^{-\sqrt{2\alpha}x}}{2\lambda + \sqrt{2\alpha}} = \nabla_y g_\alpha(x, 0-)$$

so that $g_\alpha(x, \cdot)$ also satisfies the interface condition.

⁵This may be computed by other means, as one knows that $\mathbb{E}[\exp(-\alpha \tau_t)] = \exp(-t\sqrt{2\alpha})$.

D. Resolvent of the elastic Brownian motion

From the proof of Proposition 2, we have that for $x, y \geq 0$,

$$p^e(t, x, y) = p(t, x, y) - p(t, x, -y),$$

where $p^e(t, x, y)$ is the density of the elastic Brownian motion with parameter $\kappa = 2\lambda$. We deduce that the density $g_\alpha^e(x, y)$ of the resolvent of the elastic Brownian motion satisfies

$$g_\alpha^e(x, y) = g_\alpha(x, y) - g_\alpha(x, -y).$$

Then $g_\alpha^e(x, y)$ may be computed using (28)-(29).

An alternative way consists in seeking $g_\alpha^e(x, y)$ under the form

$$g_\alpha^e(x, y) = \begin{cases} A(x)e^{-\mu y} + B(x)e^{\mu y} & \text{for } y \in [0, x], \\ C(x)e^{-\mu y} & \text{for } y \geq x \end{cases}$$

with $\mu := \sqrt{2\alpha}$ and the conditions

$$\nabla_y g_\alpha^e(x, 0) = \kappa g_\alpha^e(x, 0), \quad g_\alpha^e(x, x-) = g_\alpha^e(x, x+) \text{ and } \nabla_y g_\alpha^e(x, x+) - \nabla_y g_\alpha^e(x, x-) = 2.$$

Hence,

$$A(x) = \frac{\mu - \kappa}{\mu + \kappa} \frac{e^{-\mu x}}{\mu}, \quad B(x) = \frac{e^{-\mu x}}{\mu} \text{ and } C(x) = \frac{e^{\mu x}}{\mu} + \frac{\mu - \kappa}{\mu + \kappa} \frac{e^{-\mu x}}{\mu}.$$

We may then check that both approaches give the same result.

E. Constructing a Markov process by Piecing Out

We present in this Section a result due to N. Ikeda, M. Nagasawa and S. Watanabe [17]. See also [29] for similar considerations.

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $((X_t)_{t \geq 0}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \bar{S}})$ be a right continuous strong Markov process living in state space $\bar{S} = S \cup \{\dagger\}$ with a death point \dagger and a lifetime \mathbf{k} .

We set $\mathcal{N}_t := \sigma(X_s; s \leq t)$, which is the filtration generated by X . The shift operator associated to X is denoted by $(\theta_t)_{t \geq 0}$.

We also consider a family μ defined on $\Omega \times \bar{S}$ such that $\mu(\omega, \cdot)$ is a probability measure on \bar{S} and for any fixed Borel subset A , $\mu(\cdot, A)$ is \mathcal{N}_∞ -measurable. We assume additionally that $\mu(\omega, dy) = \delta_\dagger(dy)$ when $\mathbf{k}(\omega) = 0$ and

$$\mathbb{P}_x[\mu(\omega, dy) = \mu(\theta_{\mathbf{t}(\omega)}\omega, dy), \mathbf{t}(\omega) < \mathbf{k}(\omega)] = \mathbb{P}_x[\mathbf{t} < \mathbf{k}]$$

for any stopping time \mathbf{t} . The family μ , called an *instantaneous distribution*, describes the way the process is reborn once killed. The previous condition states some time homogeneity in the way it does.

Let $\widetilde{\Omega}$ be the space $\Omega \times \overline{S}$ and $\widehat{\Omega}$ be the product of an infinite, countable, number of copies of $\widetilde{\Omega}$. For $\widetilde{\omega} = (\omega, y) \in \widetilde{\Omega}$, we set

$$\widetilde{x}(\widetilde{\omega}) = \begin{cases} x_t(\widetilde{\omega}) & \text{if } t < k(\omega), \\ y & \text{if } t \geq k(\omega). \end{cases}$$

Then we define X on $\widehat{\Omega}$ by

$$X_t(\widehat{\omega}) = \begin{cases} x_t(\widetilde{\omega}_1) & \text{if } t \in [0, k(\omega_1)), \\ x_{t-k(\omega_1)}(\widetilde{\omega}_2) & \text{if } t \in [k(\omega_1), k(\omega_1) + k(\omega_2)), \\ \dots & \\ \dagger & \text{if } t \geq k(\omega_1) + \dots + k_N(\omega_N) \end{cases}$$

with $\widehat{\omega} = (\widetilde{\omega}_1, \widetilde{\omega}_2, \dots) \in \widehat{\Omega}$, $\widetilde{\omega}_k = (\omega_k, y)$ and $N = \inf\{k \geq 0 \mid k(\omega_k) = 0\}$.

We also set

$$s_0(\widehat{\omega}) = 0, \quad s_k(\widehat{\omega}) = \sum_{j \leq k} k(\omega^j) \text{ and } \widehat{s}(\widehat{\omega}) = \sum_{0 \leq j \leq N} k(\omega^j),$$

when N is as above.

We restrict the process to

$$\widehat{\Omega}_0 = \{\widehat{\omega} \mid t \mapsto X_t(\widehat{\omega}) \text{ is right-continuous}\},$$

which is a set of full measure under $\widehat{\mathbb{P}}_x$ which is the unique probability measure defined by

$$\widehat{\mathbb{P}}_x[d\widetilde{\omega}_1, \dots, d\widetilde{\omega}_n] = \mathbb{P}_x[d\omega^1] \mu(\omega^1, dx^1) \mathbb{P}_{x^1}[d\omega^2] \mu(\omega^1, dx^2) \dots \mathbb{P}_{x^n}[d\omega^n] \mu(\omega^n, dx^n).$$

Here, the points x^i 's denote the positions at which the process is reborn.

In order to define an appropriate filtration, we need the following notion: $\widehat{\omega}$ and $\widehat{\omega}'$ are R_t -equivalent, which is denoted by $\widehat{\omega} \sim_{R_t} \widehat{\omega}'$ when $X_s(\widehat{\omega}) = X_s(\widehat{\omega}')$ for $s \leq t$ and $t \in [s_k(\widehat{\omega}'), s_{k+1}(\widehat{\omega}'))$ as well as $s_j(\widehat{\omega}) = s_j(\widehat{\omega}')$ for $j \leq k$ when $t \in [s_k(\widehat{\omega}), s_{k+1}(\widehat{\omega}))$.

In order to define a filtration, set first

$$\widehat{\mathcal{B}} = \bigotimes_{j=1}^{+\infty} \mathcal{F}_j \cap \widehat{\Omega}_0,$$

where the \mathcal{F}_j 's are copies of $\mathcal{N}_\infty \otimes \text{Bor}(S)$, $\text{Bor}(S)$ begin the Borel σ -algebra associated to S .

Finally, we set

$$\widehat{\mathcal{B}}_t = \left\{ A \mid A \in \widehat{\mathcal{B}} \text{ and } \widehat{\omega} \in A \implies \widehat{\omega}' \in A \text{ when } \widehat{\omega}' \sim_{R_t} \widehat{\omega} \right\}$$

Hence $\widehat{\mathcal{B}}_t$ is a σ -algebra of $\widehat{\Omega}_0$ and $\mathcal{N}_t \cap \widehat{\Omega}_0 \subset \widehat{\mathcal{B}}_t$.

It is important to note that \mathfrak{s}_k is not necessarily a stopping time in the filtration $(\mathcal{N}_t)_{t \geq 0}$ but is one in $(\widehat{\mathcal{B}}_t)_{t \geq 0}$.

Theorem 1 ([17]). *Using the above defined notations, $(X, (\widehat{\mathcal{B}}_t)_{t \geq 0}, \widehat{\mathbf{k}}, (\widehat{\mathbb{P}}_x)_{x \in S})$ is a strong Markov process on \overline{S} with $\mathbb{P}_+[X_t = \dagger, \forall t \geq 0] = 1$.*

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