Calcul formel pour la combinatoire
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Computer algebra for Combinatorics

Alin Bostan & Bruno Salvy

Algorithms Project, INRIA

ALEA 2012
INTRODUCTION

1. Examples
Problem 6

A flea starts at (0, 0) on the infinite two-dimensional integer lattice and executes a biased random walk: At each step it hops north or south with probability $1/4$, east with probability $1/4 + \epsilon$, and west with probability $1/4 - \epsilon$. The probability that the flea returns to (0, 0) sometime during its wanderings is $1/2$. What is $\epsilon$?

Computer algebra conjectures and proves

$$p(\epsilon) = 1 - \sqrt{\frac{A}{2}} \cdot _2F_1 \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \left| \frac{2\sqrt{1 - 16\epsilon^2}}{A} \right. \right)^{-1}, \quad \text{with} \quad A = 1 + 8\epsilon^2 + \sqrt{1 - 16\epsilon^2}.$$
Theorem [M.Fl11]

The balanced urns class \( \left( \frac{2\alpha}{\alpha}, \frac{\beta}{\alpha + \beta} \right) \), with \( \alpha > 0, \beta \geq 0 \), has an algebraic bivariate generating function.

- Computer algebra conjectures and proves larger classes of algebraic balanced urns.
- More in Basile Morcrette’s talk!
Gessel’s conjecture

- **Gessel walks**: walks in \( \mathbb{N}^2 \) using only steps in \( S = \{ \uparrow, \downarrow, \leftarrow, \rightarrow \} \)

- \( g(i, j, n) = \) number of walks from \((0, 0)\) to \((i, j)\) with \(n\) steps in \( S \)

**Question**: Nature of the generating function

\[
G(x, y, t) = \sum_{i,j,n=0}^{\infty} g(i, j, n) x^i y^j t^n \in \mathbb{Q}[[x, y, t]]
\]

- Computer algebra conjectures and proves:

**Theorem** [B. & Kauers 2010] \( G(x, y, t) \) is an algebraic function† and

\[
G(1, 1, t) = \frac{1}{2t} \cdot _2F_1\left(\begin{array}{c}
-1/12 \\ 2/3
\end{array}; 1/4 \left| - \frac{64t(4t + 1)^2}{(4t - 1)^4} \right) - \frac{1}{2t}.
\]

- A simpler variant as an exercise tomorrow.

---

†Minimal polynomial \( P(x, y, t, G(x, y, t)) = 0 \) has \( > 10^{11} \) monomials; \( \approx 30 \text{Gb} \) (!)
Inverse moment problem for walk sequences [B., Flajolet & Penson 2011]

**Question:** Given \((f_n)\), find \(I \subseteq \mathbb{R}\) and \(w : I \rightarrow \mathbb{R}\) s.t. \(f_n = \int_I w(t) t^n dt \quad (n \geq 0)\).

| Step set and walks sequence | GF | Measure \((w(t))\); | \(t :\) |
|-----------------------------|----------------|-------------------|
| A126087 \((1, 1, 3, 5, 15, 29, 87)\) | \(\frac{2z - 1 + \sqrt{1 - 8z^2}}{2z(1 - 3z)}\) | \(\frac{1}{2\pi} \sqrt{8 - t^2} / 3 - t\) | \([-2\sqrt{2}, 2\sqrt{2}]\) |
| A128356 \((1, 1, 4, 7, 28, 58, 232, 523, 2092)\) | \(\frac{2z - 1 + \sqrt{1 - 12z^2}}{2z(1 - 4z)}\) | \(\frac{1}{2\pi} \sqrt{12 - t^2} / 4 - t\) | \([-2\sqrt{3}, 2\sqrt{3}]\) |
| A151282 \((1, 2, 6, 18, 58, 190, 638)\) | \(\frac{3z - 1 + \sqrt{1 - 2z - 7z^2}}{2z(1 - 4z)}\) | \(\frac{1}{2\pi} \sqrt{7 + 2t - t^2} / 4 - t\) | \([1 - 2\sqrt{2}, 1 + 2\sqrt{2}]\) |
| A151292 \((1, 2, 7, 23, 85, 314, 1207, 4682)\) | \(\frac{3z - 1 + \sqrt{1 - 2z - 11z^2}}{2z(1 - 5z)}\) | \(\frac{1}{2\pi} \sqrt{11 + 2t - t^2} / 5 - t\) | \([1 - 2\sqrt{3}, 1 + 2\sqrt{3}]\) |
| A129400 \((1, 2, 8, 32, 144, 672, 3264)\) | \(\frac{1 - 2z - \sqrt{1 - 4z - 12z^2}}{8z^2}\) | \(\frac{1}{8\pi} \sqrt{(t + 2)(6 - t)}\) | \([-2, 6]\) |
| A151318 \((1, 3, 13, 55, 249, 1131, 5253)\) | \(\frac{5z - 1 + \sqrt{1 - 2z - 15z^2}}{4z(1 - 5z)}\) | \(\frac{1}{4\pi} \sqrt{3 + t} / 5 - t\) | \([-3, 5]\) |
| A60899 \((1, 2, 8, 24, 96, 320, 1280, 4480)\) | \(\frac{4z - 1 + \sqrt{1 - 16z^2}}{4z(1 - 4z)}\) | \(\frac{1}{4\pi} \sqrt{4 + t} / 4 - t\) | \([-4, 4]\) |
| A05773 \((1, 2, 5, 13, 35, 96, 267, 750, 2123)\) | \(\frac{3z - 1 + \sqrt{1 - 2z - 3z^2}}{2z(1 - 3z)}\) | \(\frac{1}{2\pi} \sqrt{1 + t} / 3 - t\) | \([-1, 3]\) |
| A001405 \((1, 1, 2, 3, 6, 10, 20, 35, 70, 126)\) | \(\frac{2z - 1 + \sqrt{1 - 4z^2}}{2z(1 - 2z)}\) | \(\frac{1}{2\pi} \sqrt{2 + t} / 2 - t\) | \([-2, 2]\) |
| A151281 \((1, 2, 6, 16, 48, 136, 408, 1184)\) | \(\frac{4z - 1 + \sqrt{1 - 8z^2}}{4z(1 - 3z)}\) | \(\frac{1}{4\pi} \sqrt{8 - t^2} / 3 - t\) | \([-2\sqrt{2}, 2\sqrt{2}]\) |
| A129637 \((1, 3, 11, 41, 157, 607, 2367, 9277)\) | \(\frac{5z - 1 + \sqrt{1 - 2z - 7z^2}}{4z(1 - 4z)}\) | \(\frac{1}{4\pi} \sqrt{7 + 2t - t^2} / 4 - t\) | \([1 - 2\sqrt{2}, 1 + 2\sqrt{2}]\) |
| A151323 \((1, 3, 14, 67, 342, 1790, 9580)\) | \(\frac{\sqrt{\frac{1 + \sqrt{1 - 6z}}{1 - 6z}} - 1}{2z}\) | \(\frac{1}{2\sqrt{2\pi}} \sqrt{\frac{2 + t}{6 - t}}\) | \([-2, 6]\) |
A SIAM Review combinatorial identity

*Problem 87-8, by John W. Moon* (University of Alberta).
Show that

\[ \sum_{n=1}^{\infty} \frac{56n^2 + 33n - 8}{(n+2)(n+1)} f_n^2 = 1 \]

where

\[ f_n = \frac{4^{-n}}{n} \binom{2n-2}{n-1} \quad \text{for } n \geq 1. \]

**Background.** A branch of a rooted tree \( T_n \) is a maximal subtree that does not contain the root. A branch \( B \) with \( i \) nodes is a primary branch of \( T_n \) if \( n/2 \leq i \leq n - 1 \); if \( T_n \) has a primary branch \( B \) with \( i \) nodes, then a branch \( C \) with \( j \) nodes is a secondary branch if \( (n-i)/2 \leq j \leq n - 1 - i \). For many families \( F \) of rooted trees, the fraction of trees \( T_n \) in \( F \) that have a primary branch tends to \( 1 \) as \( n \to \infty \). (See A. Meir and J.W. Moon, *On major and minor branches of rooted trees*, Canad. J. Math., 39 (1987) 673-693). It can be shown that the fraction of plane trees \( T_n \) that have a secondary branch tends to a limit \( p \) as \( n \to \infty \), where

\[ p = 3 - 12 \sum_{n=1}^{\infty} \frac{13n^2 + 5n - 2}{(n+1)(n+2)} f_n^2. \]

If we appeal to the proposed identity then we obtain the more rapidly converging expression

\[ p = \frac{3}{14} + \frac{3}{14} \sum_{n=1}^{\infty} \frac{149n + 8}{(n+1)(n+2)} f_n^2 \]

from which we find that \( p = .59 \cdots \).
Expansion of a Symmetric Determinant

E 2297 [1971, 543]. Proposed by Richard Stanley, Harvard University

Let $L(n)$ be the total number of distinct monomials appearing in the expansion of the determinant of an $n \times n$ symmetric matrix $A = (a_{ij})$. For instance, $L(3) = 5$.

Show that

$$\sum_{n=0}^{\infty} \frac{L(n)x^n}{n!} = (1 - x)^{-1/2} \exp(\frac{1}{2}x + \frac{4}{3}x^2),$$

where $|x| < 1$, and where we define $L(0) = 1$.

Units of Chains


Let $f(n)$ be the number of nonisomorphic $n$-element partially ordered sets $P$ which do not contain three pairwise incomparable elements. (Equivalently, $P$ is a union of two chains.) Let

$$F(x) = 1 + \sum_{n \geq 1} f(n) x^n = 1 + x + 2x^2 + 4x^3 + 10x^4 + \cdots.$$

Show that

$$F(x) = \frac{4}{2 - 2x + \sqrt{1 - 4x} + \sqrt{1 - 4x^2}}.$$
Noncrossing Trees

E 3170 [1986, 650]. Proposed by The Howard University Group, Washington, D.C.

Construct a graph as follows: Put \( n + 1 \) labeled vertices around a circle and let the edges be the straight line segments connecting any two vertices. A tree is noncrossing if no two edges intersect except at the vertices. Enumerate the number of noncrossing spanning trees for this graph. For \( n = 1, 2, 3 \), the numbers are 1, 3, 12, respectively.

An Unexpected Appearance of the Catalan Numbers

10905 [2001, 871]. Proposed by Richard P. Stanley, Massachusetts Institute of Technology, Cambridge, MA. Let \( f(n) = \sum_P (-1)^{w(P)} \), where \( P \) ranges over all lattice paths in the plane with \( 2n \) steps, starting and ending at the origin, with steps \((1, 0), (0, 1), (-1, 0), (0, -1)\), and where \( w(P) \) denotes the winding number of \( P \) with respect to the point \((1/2, 1/2)\). Show that \( f(n) = 4^n C_n \), where \( C_n = \binom{2n}{n} / (n + 1) \), the \( n \)th Catalan number.

Three-dimensional Lattice Walks in the Upper Half-Space

10795 [2000, 367]. Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY. A 3-dimensional lattice walk of length \( n \) takes \( n \) successive unit steps, each in one of the six coordinate directions. How many 3-dimensional lattice walks of length \( n \) are there that begin at the origin and never go below the horizontal plane?
Another Type of Lattice Path

10658 [1998, 366]. Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY. Consider walks on the integer lattice in the plane that start at (0, 0), that stay in the first quadrant (they may touch the $x$-axis), and such that each step is either (2, 1), (1, 2), or (1, $-1$). For each nonnegative integer $n$, how many paths are there to $(3n, 0)$?

The First Third


Let $f(n)$ be the sum of the first one-third of the coefficients in the expansion of $(1 + x)^{3n}$, i.e.,

$$f(n) = \sum_{k=0}^{n} \binom{3n}{k} \quad (n = 0, 1, 2, \ldots).$$

Prove that

$$\sum_{n=0}^{\infty} f(n) \left( \frac{4u^2}{27} \right)^n = \frac{u}{u - 2 \sin \left( \frac{1}{3} \arcsin u \right)} - \frac{2u}{2u - 3 \sin \left( \frac{1}{3} \arcsin u \right)}.$$

11501. Proposed by Finbarr Holland, University College Cork, Cork, Ireland. (Correction) Let

$$g(z) = 1 - \frac{3}{\frac{1}{1-az} + \frac{1}{1-iz} + \frac{1}{1+iz}}.$$

Show that the coefficients in the Taylor series expansion of $g$ about 0 are all nonnegative if and only if $a \geq \sqrt{3}$. 
11567. Proposed by David Callan, University of Wisconsin-Madison, Madison, WI. How many arrangements \((a_1, \ldots, a_{2n})\) of the multiset \(\{1, 1, 2, 2, \ldots, n, n\}\) satisfy the following two conditions: (i) All entries between the two occurrences of any given value \(i\) exceed \(i\), and (ii) No three entries increase from left to right with the last two adjacent? (When \(n = 3\), one such arrangement is 122133.)

11573. Proposed by Rob Pratt, SAS Institute, Cary, NC. A Sudoku permutation matrix (SPM) of order \(n^2\) is a permutation matrix of order \(n^2\) with exactly one 1 in each of the \(n^2\) submatrices of order \(n\) obtained by partitioning the original matrix into an \(n\)-by-\(n\) array of submatrices. Thus, for \(n = 2\), the permutation 1324 yields an SPM, but the identity permutation 1234 does not. Find the number of SPMs of order \(n^2\).

11610. Proposed by Richard P. Stanley, Massachusetts Institute of Technology, Cambridge, MA. Let \(f(n)\) be the number of binary words \(a_1 \cdots a_n\) of length \(n\) that have the same number of pairs \(a_ia_{i+1}\) equal to 00 as equal to 01. Show that

\[
\sum_{n=0}^{\infty} f(n)t^n = \frac{1}{2} \left( \frac{1}{1-t} + \frac{1+2t}{\sqrt{(1-t)(1-2t)(1+t+2t^2)}} \right).
\]

▶ Last one as an exercise tomorrow.
A money changing problem

Question†: The number of ways one can change any amount of banknotes of 10 €, 20 €, … using coins of 50 cents, 1 € and 2 € is always a perfect square.

†Free adaptation of Pb. 1, Ch. 1, p. 1, vol. 1 of Pólya and Szegö’s Problems Book (1925)
This is equivalent to finding the number $M_{20k}$ of solutions $(a, b, c) \in \mathbb{N}^3$ of

$$a + 2b + 4c = 20k.$$}

**Euler-Comtet's denumerants:**

$$\sum_{n \geq 0} M_n x^n = \frac{1}{(1 - x)(1 - x^2)(1 - x^4)}.$$

```maple
define := 1/(1-x)/(1-x^2)/(1-x^4):
define := series(define, x, 201):
seq(coeff(series(define, x, 20*k), k=1..10));

[36, 121, 256, 441, 676, 961, 1296, 1681, 2116, 2601]
define := subs(n=20*k, gfun[ratpolytocoeff](define, x, n)):

\[
\frac{17}{32} + \frac{(20k + 1)(20k + 2)}{16} + 5k + \frac{(-1)^{-20k}(20k + 1)}{16} + \frac{5(-1)^{-20k}}{32} + \sum_{\alpha^2 + 1 = 0} \left( - \frac{1}{16} - \frac{1}{16} \alpha \right) \alpha^{-20k}
\]
INTRODUCTION

2. Computer Algebra
General framework

Computeralgebra = effective mathematics and algebraic complexity

- Effective mathematics: what can we compute?
- their complexity: how fast?
Computer algebra books
Mathematical Objects

- **Main objects**
  - integers \( \mathbb{Z} \)
  - polynomials \( K[x] \)
  - rational functions \( K(x) \)
  - power series \( K[[x]] \)
  - matrices \( M_r(K) \)
  - linear recurrences with constant, or polynomial, coefficients \( K[n]\langle S_n \rangle \)
  - linear differential equations with polynomial coefficients \( K[x]\langle \partial_x \rangle \)

where \( K \) is a field (generally supposed of characteristic 0 or large)

- **Secondary/auxiliary objects**
  - polynomial matrices \( M_r(K[x]) \)
  - power series matrices \( M_r(K[[x]]) \)
Overview

Today

1. Introduction
2. High Precision Approximations
   – Fast multiplication, binary splitting, Newton iteration
3. Tools for Conjectures
   – Hermite-Padé approximants, $p$-curvature

Tomorrow morning

4. Tools for Proofs
   – Symbolic method, resultants, D-finiteness, creative telescoping

Tomorrow night

– Exercises with Maple
HIGH PRECISION

1. Fast Multiplication
Complexity yardsticks

Important features:

• addition is easy: naive algorithm already optimal
• multiplication is the most basic (non-trivial) problem
• almost all problems can be reduced to multiplication

Are there quasi-optimal algorithms for:

• integer/polynomial/power series multiplication? Yes!
• matrix multiplication? Big open problem!
Complexity yardsticks

\[ \text{M}(n) = \begin{cases} \text{complexity of multiplication in } \mathbb{K}[x]_{<n}, \text{ and of } n\text{-bit integers} \\ = O(n^2) \text{ by the naive algorithm} \\ = O(n^{1.58}) \text{ by Karatsuba’s algorithm} \\ = O(n^{\log_\alpha (2\alpha-1)}) \text{ by the Toom-Cook algorithm (} \alpha \geq 3) \\ = O(n \log n \log \log n) \text{ by the Schönhage-Strassen algorithm} \end{cases} \]

\[ \text{MM}(r) = \begin{cases} \text{complexity of matrix product in } M_r(\mathbb{K}) \\ = O(r^3) \text{ by the naive algorithm} \\ = O(r^{2.81}) \text{ by Strassen’s algorithm} \\ = O(r^{2.38}) \text{ by the Coppersmith-Winograd algorithm} \end{cases} \]

\[ \text{MM}(r, n) = \begin{cases} \text{complexity of polynomial matrix product in } M_r(\mathbb{K}[x]_{<n}) \\ = O(r^3 \text{ M}(n)) \text{ by the naive algorithm} \\ = O(\text{MM}(r) n \log(n) + r^2 n \log n \log \log n) \text{ by the Cantor-Kaltofen algo} \\ = O(\text{MM}(r) n + r^2 \text{ M}(n)) \text{ by the B-Schost algorithm} \end{cases} \]
Fast polynomial multiplication in practice

Practical complexity of Magma's multiplication in $\mathbb{F}_p[x]$, for $p = 29 \times 2^{57} + 1$. 
What can be computed in 1 minute with a CA system*

polynomial product† in degree $14,000,000$ (>1 year with schoolbook)

product of two integers with $500,000,000$ binary digits

factorial of $N = 20,000,000$ (output of $140,000,000$ digits)

gcd of two polynomials of degree $600,000$

resultant of two polynomials of degree $40,000$

factorization of a univariate polynomial of degree $4,000$

factorization of a bivariate polynomial of total degree $500$

resultant of two bivariate polynomials of total degree $100$ (output 10,000)

product/sum of two algebraic numbers of degree $450$ (output 200,000)

determinant (char. polynomial) of a matrix with $4,500$ ($2,000$) rows

determinant of an integer matrix with 32-bit entries and 700 rows

*on a PC, (Intel Xeon X5160, 3GHz processor, with 8GB RAM), running Magma V2.16-7
†in $\mathbb{K}[x]$, for $\mathbb{K} = \mathbb{F}_{67108879}$
Discrete Fourier Transform

[Gauss 1866, Cooley-Tukey 1965]

**DFT Problem:** Given $n = 2^k$, $f \in \mathbb{K}[x]_{<n}$, and $\omega \in \mathbb{K}$ a primitive $n$-th root of unity, compute $(f(1), f(\omega), \ldots, f(\omega^{n-1}))$

**Idea:** Write $f = f_{\text{even}}(x^2) + x f_{\text{odd}}(x^2)$, with $\deg(f_{\text{even}}), \deg(f_{\text{odd}}) < n/2$. Then $f(\omega^j) = f_{\text{even}}(\omega^{2j}) + \omega^j f_{\text{odd}}(\omega^{2j})$, and $(\omega^{2j})_{0 \leq j < n} = \frac{n}{2}$-roots of 1.

**Complexity:** $F(n) = 2 \cdot F(n/2) + O(n) \implies F(n) = O(n \log n)$
Inverse DFT

IDFT Problem: Given $n = 2^k$, $v_0, \ldots, v_{n-1} \in \mathbb{K}$ and $\omega \in \mathbb{K}$ a primitive $n$-th root of unity, compute $f \in \mathbb{K}[x]_{<n}$ such that $f(1) = v_0, \ldots, f(\omega^{n-1}) = v_{n-1}$

- $V_{\omega} \cdot V_{\omega^{-1}} = n \cdot I_n \rightarrow$ performing the inverse DFT in size $n$ amounts to:
  - performing a DFT at
    \[
    \frac{1}{1}, \frac{1}{\omega}, \ldots, \frac{1}{\omega^{n-1}}
    \]
  - dividing the results by $n$.
- this new DFT is the same as before:
  \[
  \frac{1}{\omega^i} = \omega^{n-i},
  \]
  so the outputs are just shuffled.

Consequence: the cost of the inverse DFT is $O(n \log(n))$
FFT polynomial multiplication

Suppose the basefield $\mathbb{K}$ contains enough roots of unity

To multiply two polynomials $f, g$ in $\mathbb{K}[x]$, of degrees $< n$:

- find $N = 2^k$ such that $h = fg$ has degree less than $N$ \[ N \leq 4n \]
- compute $\text{DFT}(f, N)$ and $\text{DFT}(g, N)$ \[ O(N \log(N)) \]
- multiply pointwise these values to get $\text{DFT}(h, N)$ \[ O(N) \]
- recover $h$ by inverse DFT \[ O(N \log(N)) \]

Complexity: $O(N \log(N)) = O(n \log(n))$

General case: Create artificial roots of unity \[ O(n \log(n) \log \log n) \]
HIGH PRECISION

2. Binary Splitting
Example: fast factorial

Problem: Compute $N! = 1 \times \cdots \times N$

Naive iterative way: unbalanced multiplicands $\tilde{O}(N^2)$

- **Binary Splitting:** balance computation sequence so as to take advantage of fast multiplication (operands of same sizes):

  $$N! = \underbrace{1 \times \cdots \times \lfloor N/2 \rfloor}_{\text{size } \frac{1}{2} N \log N} \times \underbrace{(\lceil N/2 \rceil + 1) \times \cdots \times N}_{\text{size } \frac{1}{2} N \log N}$$

  and recurse. Complexity $\tilde{O}(N)$.

- Extends to **matrix factorials** $A(N)A(N - 1) \cdots A(1)$ $\tilde{O}(N)$

  $\longrightarrow$ recurrences of arbitrary order.
Application to recurrences

Problem: Compute the $N$-th term $u_N$ of a $P$-recursive sequence

$$p_r(n)u_{n+r} + \cdots + p_0(n)u_n = 0, \quad (n \in \mathbb{N})$$

Naive algorithm: unroll the recurrence \( \tilde{O}(N^2) \) bit ops.

Binary splitting: $U_n = (u_n, \ldots, u_{n+r-1})^T$ satisfies the 1st order recurrence

$$U_{n+1} = \frac{1}{p_r(n)} A(n) U_n \quad \text{with} \quad A(n) = \begin{bmatrix} p_r(n) & \cdots & \cdots \\ \cdots & \ddots & \ddots \\ -p_0(n) & -p_1(n) & \cdots & -p_{r-1}(n) \end{bmatrix}.$$ 

\( \implies \) $u_N$ reads off the matrix factorial $A(N-1) \cdots A(0)$

[Chudnovsky-Chudnovsky, 1987]: Binary splitting strategy \( \tilde{O}(N) \) bit ops.
Application: fast computation of $e = \exp(1)$

[Brent 1976]

$$e_n = \sum_{k=0}^{n} \frac{1}{k!} \quad \rightarrow \quad \exp(1) = 2.7182818284590452\ldots$$

Recurrence $e_n - e_{n-1} = 1/n! \iff n(e_n - e_{n-1}) = e_{n-1} - e_{n-2}$ rewrites

$$\begin{bmatrix} e_{N-1} \\ e_N \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 0 & N \\ -1 & N + 1 \end{bmatrix} \begin{bmatrix} e_{N-2} \\ e_{N-1} \end{bmatrix} = \frac{1}{N!} C(N)C(N-1) \cdots C(1) \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$ 

▶ $e_N$ in $\tilde{O}(N)$ bit operations [Brent 1976]

▶ generalizes to the evaluation of any D-finite series at an algebraic number [Chudnovsky-Chudnovsky 1987] $\tilde{O}(N)$ bit ops.
Implementation in gfun
[Mezzarobba, S. 2010]

> rec:={n*(e(n) - e(n-1)) = e(n-1) - e(n-2), e(0)=1, e(1)=2};
> pro:=rectoproc(rec,e(n));

pro := proc(n::nonnegint)
local i1, loc0, loc1, loc2, tmp2, tmp1, i2;
if n <= 22 then
    loc0 := 1;
    loc1 := 2;
    if n = 0 then return loc0
    else for i1 to n - 1 do
    loc2 := (-loc0 + loc1 + loc1*(i1 + 1))/(i1 + 1); loc0 := loc1; loc1 := loc2
    end do
    end if; loc1
else
    tmp1 := 'gfun/rectoproc/binsplit'([
        'ndmatrix'(Matrix([[0, i2 + 2], [-1, i2 + 3]]), i2 + 2), i2, 0, n,
        matrix_ring(ad, pr, ze, ndmatrix(Matrix(2, 2, [[...],[...]],
            datatype = anything, storage = empty, shape = [identity]), 1)),
        expected_entry_size], Vector(2, [...], datatype = anything));
    tmp1 := subs({e(0) = 1, e(1) = 2}, tmp1); tmp1
end if
end proc

> tt:=time(): x:=pro(50000): time()-tt, evalf(x-exp(1), 200000);

1.827, 0.
Application: record computation of $\pi$

[Chudnovsky-Chudnovsky 1987] fast convergence hypergeometric identity

$$
\frac{1}{\pi} = \frac{1}{53360 \sqrt{640320}} \sum_{n \geq 0} \frac{(-1)^n (6n)! (13591409 + 545140134n)}{n!^3 (3n)! (8 \cdot 100100025 \cdot 327843840)^n}.
$$

Used in Maple & Mathematica: 1st order recurrence, yields 14 correct digits per iteration $\rightarrow$ 4 billion digits [Chudnovsky-Chudnovsky 1994]

Current record on a PC: 10000 billion digits [Kondo & Yee 2011]
HIGH PRECISION

3. Newton Iteration
Newton’s tangent method: real case

[Newton, 1671]

\[ x_{\kappa+1} = N(x_{\kappa}) = x_{\kappa} - \frac{x_{\kappa}^2 - 2}{2x_{\kappa}}, \quad x_0 = 1 \]

\[ x_1 = 1.5000000000000000000000000000000 \]
\[ x_2 = 1.4166666666666666666666666666667 \]
\[ x_3 = 1.4142156862745098039215686274510 \]
\[ x_4 = 1.4142135623746899106262955788901 \]
\[ x_5 = 1.4142135623730950488016896235025 \]
Newton’s tangent method: power series case

\[ x_{\kappa + 1} = \mathcal{N}(x_{\kappa}) = x_{\kappa} - \frac{x_{\kappa}^2 - (1 - t)}{2x_{\kappa}}, \quad x_0 = 1 \]

\begin{align*}
x_1 &= 1 - \frac{1}{2}t \\
x_2 &= 1 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3 - \frac{1}{32}t^4 - \frac{1}{64}t^5 - \frac{1}{128}t^6 - \frac{1}{256}t^7 - \frac{1}{512}t^8 - \frac{1}{1024}t^9 + \cdots \\
x_3 &= 1 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3 - \frac{5}{128}t^4 - \frac{7}{256}t^5 - \frac{21}{1024}t^6 - \frac{33}{2048}t^7 - \frac{107}{8192}t^8 - \frac{177}{16384}t^9 + \cdots 
\end{align*}
Newton’s tangent method: power series case

In order to solve $\varphi(x, g) = 0$ in $\mathbb{K}[[x]]$ (where $\varphi \in \mathbb{K}[[x, y]]$, $\varphi(0, 0) = 0$ and $\varphi_y(0, 0) \neq 0$), iterate

$$g_{\kappa+1} = g_{\kappa} - \frac{\varphi(g_{\kappa})}{\varphi_y(g_{\kappa})} \mod x^{2\kappa+1}$$

$$g - g_{\kappa+1} = g - g_{\kappa} + \frac{\varphi(g) + (g_{\kappa} - g)\varphi_y(g) + O((g - g_{\kappa})^2)}{\varphi_y(g) + O(g - g_{\kappa})} = O((g - g_{\kappa})^2).$$

- The number of correct coefficients doubles after each iteration
- Total cost $= 2 \times \left(\text{the cost of the last iteration}\right)$

**Theorem** [Cook 1966, Sieveking 1972 & Kung 1974, Brent 1975]
Division, logarithm and exponential of power series in $\mathbb{K}[[x]]$ can be computed at precision $N$ using $O(M(N))$ operations in $\mathbb{K}$
Division, logarithm and exponential of power series

[Sieveking1972, Kung1974, Brent1975]

To compute the reciprocal of \( f \in \mathbb{K}[[x]] \) with \( f(0) \neq 0 \), choose \( \varphi(g) = 1/g - f \):

\[
g_0 = 1/f_0 \quad \text{and} \quad g_{\kappa+1} = g_{\kappa} + g_{\kappa}(1 - fg_{\kappa}) \mod x^{2\kappa+1} \quad \text{for} \quad \kappa \geq 0.
\]

**Complexity:** \( C(N) = C(N/2) + O(M(N)) \implies C(N) = O(M(N)) \)

**Corollary:** division of power series at precision \( N \) in \( O(M(N)) \)
Division, logarithm and exponential of power series
[Sieveking1972, Kung1974, Brent1975]

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\]

**Complexity:** \( C(N) = C(N/2) + O(M(N)) \implies C(N) = O(M(N)) \)

**Corollary:** division of power series at precision \( N \) in \( O(M(N)) \)

**Corollary:** Logarithm \( \log(f) = -\sum_{i \geq 1} \frac{(1 - f)^i}{i} \) of \( f \in 1 + x\mathbb{K}[[x]] \) in \( O(M(N)) \):

- compute the Taylor expansion of \( h = f'/f \mod x^{N-1} \) \( O(M(N)) \)
- take the antiderivative of \( h \) \( O(N) \)
Division, logarithm and exponential of power series
[Sieveking1972, Kung1974, Brent1975]

To compute the reciprocal of $f \in \mathbb{K}[[x]]$, choose $\varphi(g) = 1/g - f$:

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Complexity: $C(N) = C(N/2) + O(M(N)) \implies C(N) = O(M(N))$

Corollary: division of power series at precision $N$ in $O(M(N))$

Corollary: Logarithm $\log(f) = -\sum_{i \geq 1} \frac{(1 - f)^i}{i}$ of $f \in 1 + x\mathbb{K}[[x]]$ in $O(M(N))$:

- compute the Taylor expansion of $h = f'/f \mod x^{N-1}$ $O(M(N))$
- take the antiderivative of $h$ $O(N)$

Corollary: Exponential $\exp(f) = \sum_{i \geq 0} \frac{f^i}{i!}$ of $f \in x\mathbb{K}[[x]]$. Use $\phi(g) = \log(g) - f$:

$$g_0 = 1 \quad \text{and} \quad g_{\kappa+1} = g_\kappa - g_\kappa (\log(g_\kappa) - f) \mod x^{2\kappa+1} \quad \text{for } \kappa \geq 0.$$  

Complexity: $C(N) = C(N/2) + O(M(N)) \implies C(N) = O(M(N))$
Application: Euclidean division for polynomials
[Strassen, 1973]

Pb: Given $F, G \in \mathbb{K}[x]_{\leq N}$, compute $(Q, R)$ in Euclidean division $F = QG + R$

Naive algorithm: $O(N^2)$

Idea: look at $F = QG + R$ from infinity: $Q \sim_\infty F/G$

Let $N = \deg(F)$ and $n = \deg(G)$. Then $\deg(Q) = N - n$, $\deg(R) < n$ and

\[
F(1/x)x^N = \underbrace{G(1/x)x^n}_{\text{rev}(F')} \cdot \underbrace{Q(1/x)x^{N-n}}_{\text{rev}(G)} + \underbrace{R(1/x)x^{\deg(R)}}_{\text{rev}(Q)} \cdot x^{N-\deg(R)}
\]

Algorithm:

- Compute $\text{rev}(Q) = \text{rev}(F')/\text{rev}(G) \mod x^{N-n+1}$ $O(M(N))$
- Recover $Q$ $O(N)$
- Deduce $R = F - QG$ $O(M(N))$
Application: conversion coefficients ↔ power sums

[Schönhage, 1982]

Any polynomial $F = x^n + a_1 x^{n-1} + \cdots + a_n$ in $\mathbb{K}[x]$ can be represented by its first $n$ power sums $S_i = \sum_{F(\alpha)=0} \alpha^i$

Conversions coefficients ↔ power sums can be performed

- either in $O(n^2)$ using Newton identities (naive way):
  $$ia_i + S_1 a_{i-1} + \cdots + S_i = 0, \quad 1 \leq i \leq n$$

- or in $O(M(n))$ using generating series

$$\frac{\text{rev}(F)'}{\text{rev}(F)} = - \sum_{i \geq 0} S_{i+1} x^i \iff \text{rev}(F) = \exp \left( - \sum_{i \geq 1} \frac{S_i}{i} x^i \right)$$
Application: special bivariate resultants

[B-Flajolet-S-Schost, 2006]

Composed products and sums: manipulation of algebraic numbers

\[ F \otimes G = \prod_{F(\alpha)=0, G(\beta)=0} (x - \alpha \beta), \quad F \oplus G = \prod_{F(\alpha)=0, G(\beta)=0} (x - (\alpha + \beta)) \]

Output size:

\[ N = \text{deg}(F) \text{deg}(G) \]

Linear algebra: \( \chi_{xy}, \chi_{x+y} \) in \( \mathbb{K}[x, y]/(F(x), G(y)) \)

Resultants: \( \text{Res}_y (F(y), y^{\text{deg}(G)} G(x/y)), \text{Res}_y (F(y), G(x - y)) \)

Better: \( \otimes \) and \( \oplus \) are easy in Newton representation

\[
\sum \alpha^s \sum \beta^s = \sum (\alpha \beta)^s \quad \text{and} \\
\sum \frac{(\alpha + \beta)^s}{s!} x^s = \left( \sum \frac{\alpha^s}{s!} x^s \right) \left( \sum \frac{\beta^s}{s!} x^s \right)
\]

Corollary: Fast polynomial shift \( P(x + a) = P(x) \oplus (x + a) \)

\[ O(M(\text{deg}(P))) \]
Newton iteration on power series: operators and systems

In order to solve an equation \( \phi(Y) = 0 \), with \( \phi : (\mathbb{K}[[x]])^r \rightarrow (\mathbb{K}[[x]])^r \),

1. **Linearize**: \( \phi(Y_\kappa - U) = \phi(Y_\kappa) - D\phi|_{Y_\kappa} \cdot U + O(U^2) \), where \( D\phi|_Y \) is the differential of \( \phi \) at \( Y \).

2. **Iterate**: \( Y_{\kappa+1} = Y_\kappa - U_{\kappa+1} \), where \( U_{\kappa+1} \) is found by

3. **Solve linear equation**: \( D\phi|_{Y_\kappa} \cdot U = \phi(Y_\kappa) \) with \( \text{val } U > 0 \).

Then, the sequence \( Y_\kappa \) converges quadratically to the solution \( Y \).
Application: inversion of power series matrices

[Schulz, 1933]

To compute the inverse $Z$ of a matrix of power series $Y \in \mathcal{M}_r(k[[x]])$:

- Choose the map $\phi : Z \mapsto I - YZ$ with differential $D\phi|_Y : U \mapsto -YU$
- Equation for $U$: $-YU = I - YZ_\kappa \mod x^{2\kappa+1}$
- Solution: $U = -Y^{-1}(I - Y Z_\kappa) = -Z_\kappa (I - Y Z_\kappa) \mod x^{2\kappa+1}$

This yields the following Newton-type iteration for $Y^{-1}$

$$Z_{\kappa+1} = Z_\kappa + Z_\kappa (I_r - Y Z_\kappa) \mod x^{2\kappa+1}$$

Complexity:

$$C_{\text{inv}}(N) = C_{\text{inv}}(N/2) + O(\text{MM}(r, N)) \implies C_{\text{inv}}(N) = O(\text{MM}(r, N))$$
Application: non-linear systems

In order to solve a system $Y = H(Y) = \phi(Y) + Y$, with $H : (\mathbb{K}[[x]])^r \to (\mathbb{K}[[x]])^r$, such that $I_r - \partial H / \partial Y$ is invertible at 0.

1. **Linearize**: $\phi(Y_\kappa - U) - \phi(Y_\kappa) = U - \partial H / \partial Y(Y_\kappa) \cdot U + O(U^2)$.

2. **Iterate** $Y_{\kappa+1} = Y_\kappa - U_{\kappa+1}$, where $U_{\kappa+1}$ is found by

3. **Solve linear equation**: $(I_r - \partial H / \partial Y(Y_\kappa)) \cdot U = H(Y_\kappa) - Y_\kappa$ with val $U > 0$.

This yields the following Newton-type iteration:

$$
\begin{cases}
Z_{\kappa+1} = Z_\kappa + Z_\kappa \left( I_r - (I_r - \partial H / \partial Y(Y_\kappa))Z_\kappa \right) \mod x^{2\kappa+1} \\
Y_{\kappa+1} = Y_\kappa - Z_{\kappa+1}(H(Y_\kappa) - Y_\kappa) \mod x^{2\kappa+1}
\end{cases}
$$

computing simultaneously a matrix and a vector.
Example: Mappings

> mappings:=\{M=Set(Cycle(Tree)), Tree=Prod(Z,Set(Tree))\}:
> combstruct[gfeqns](mappings,labeled,x);

\[
M(x) = \frac{1}{1 - Tree(x)}, \quad Tree(x) = x \exp(Tree(x))
\]

> countmappings:=SeriesNewtonIteration(mappings,labelled,x):
> countmappings(10);

\[
M = 1 + x + 2 x^2 + \frac{9}{2} x^3 + \frac{32}{3} x^4 + \frac{625}{24} x^5 + \frac{324}{5} x^6 + \frac{117649}{720} x^7 + \frac{131072}{315} x^8 + \frac{4782969}{4480} x^9 + O(x^{10}),
\]

\[
Tree = x + x^2 + \frac{3}{2} x^3 + \frac{8}{3} x^4 + \frac{125}{24} x^5 + \frac{54}{5} x^6 + \frac{16807}{720} x^7 + \frac{16384}{315} x^8 + \frac{531441}{4480} x^9 + O(x^{10})
\]

Code Pivoteau-S-Soria, should end up in combstruct
Application: quasi-exponential of power series matrices
[B-Chyzak-Ollivier-Salvy-Schost-Sedoglavic 2007]

To compute the solution $Y \in \mathcal{M}_r(\mathbb{K}[[x]])$ of the system $Y' = AY$

- choose the map $\phi : Y \mapsto Y' - AY$, with differential $\phi$.
- the equation for $U$ is $U' - AU = Y'_\kappa - AY_\kappa \mod x^{2^\kappa+1}$
- the method of variation of constants yields the solution $U = Y_\kappa V_\kappa \mod x^{2^\kappa+1}$, $Y'_\kappa - AY_\kappa = Y_\kappa V'_\kappa \mod x^{2^\kappa+1}$

This yields the following Newton-type iteration for $Y$:

$$Y_{\kappa+1} = Y_\kappa - Y_\kappa \int Y_{\kappa}^{-1} (Y'_\kappa - AY_\kappa) \mod x^{2^\kappa+1}$$

Complexity:

$$C_{\text{solve}}(N) = C_{\text{solve}}(N/2) + O(\text{MM}(r, N)) \implies C_{\text{solve}}(N) = O(\text{MM}(r, N))$$
TOOLS FOR CONJECTURES

1. Hermite-Padé Approximants
Definition: Given a column vector \( \mathbf{F} = (f_1, \ldots, f_n)^T \in \mathbb{K}[[x]]^n \) and an \( n \)-tuple \( \mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n \), a Hermite-Padé approximant of type \( \mathbf{d} \) for \( \mathbf{F} \) is a row vector \( \mathbf{P} = (P_1, \ldots, P_n) \in \mathbb{K}[x]^n \), \( (\mathbf{P} \neq 0) \), such that:

1. \( \mathbf{P} \cdot \mathbf{F} = P_1 f_1 + \cdots + P_n f_n = O(x^\sigma) \) with \( \sigma = \sum_i (d_i + 1) - 1 \),
2. \( \deg(P_i) \leq d_i \) for all \( i \).

\( \sigma \) is called the order of the approximant \( \mathbf{P} \).

- Very useful concept in number theory (transcendence theory):
  - [Hermite, 1873]: \( e \) is transcendent.
  - [Lindemann, 1882]: \( \pi \) is transcendent, and so does \( e^\alpha \) for any \( \alpha \in \overline{\mathbb{Q}} \setminus \{0\} \).
  - [Beukers, 1981]: reformulate Apéry’s proof that \( \zeta(3) = \sum_n \frac{1}{n^3} \) is irrational.
  - [Rivoal, 2000]: there exist an infinite number of \( k \) such that \( \zeta(2k + 1) \notin \mathbb{Q} \).
Worked example

Let us compute a Hermite-Padé approximant of type \((1, 1, 1)\) for \((1, C, C^2)\), where \(C(x) = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + O(x^6)\).

This boils down to finding \(\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma_0, \gamma_1\) such that

\[
\alpha_0 + \alpha_1 x + (\beta_0 + \beta_1 x)(1+x+2x^2+5x^3+14x^4) + (\gamma_0 + \gamma_1 x)(1+2x+5x^2+14x^3+42x^4) = O(x^5).
\]

By identifying coefficients, this is equivalent to a homogeneous linear system:

\[
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 2 & 1 \\
0 & 0 & 2 & 1 & 5 & 2 \\
0 & 0 & 5 & 2 & 14 & 5 \\
0 & 0 & 14 & 5 & 42 & 14
\end{bmatrix} \times \begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\beta_0 \\
\beta_1 \\
\gamma_0 \\
\gamma_1
\end{bmatrix} = 0 \iff \begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 2 \\
0 & 0 & 2 & 1 & 5 \\
0 & 0 & 5 & 2 & 14 \\
0 & 0 & 14 & 5 & 42
\end{bmatrix} \times \begin{bmatrix}
\alpha_0 \\
\alpha_1 \\
\beta_0 \\
\beta_1 \\
\gamma_0
\end{bmatrix} = -\gamma_1 \begin{bmatrix}
0 \\
1 \\
2 \\
5 \\
14
\end{bmatrix}.
\]

By homogeneity, one can choose \(\gamma_1 = 1\). Then, the violet minor shows that one can take \((\beta_0, \beta_1, \gamma_0) = (-1, 0, 0)\). The other values are \(\alpha_0 = 1, \alpha_1 = 0\).

Thus the approximant is \((1, -1, x)\), which corresponds to \(P = 1 - y + xy^2\) such that \(P(x, C(x)) = 0 \mod x^5\).
Algebraic and differential approximation = guessing

- Hermite-Padé approximants of $n = 2$ power series are related to Padé approximants, i.e. to approximation of series by rational functions.

- **algebraic approximants** = Hermite-Padé approximants for $f_\ell = A^{\ell-1}$, where $A \in \mathbb{K}[[x]]$.

- **differential approximants** = Hermite-Padé approximants for $f_\ell = A^{(\ell-1)}$, where $A \in \mathbb{K}[[x]]$.

\[> \text{listtoalgeq}([1,1,2,5,14,42,132,429],y(x));\]
\[2\]
\[[1 - y(x) + x y(x), \text{ogf}]\]

\[> \text{listtodiffeq}([1,1,2,5,14,42,132,429],y(x));\]
\[\begin{align*}
\begin{array}{c}
\text{d} & \text{d} \\
\text{d} & \text{d} \\
\text{d} & \text{d} \\
\end{array}
\end{align*}\]
\[\begin{align*}
\{-2 y(x) + (2 - 4 x) & \quad \text{-- y(x)} \quad + x \quad \text{--- y(x)} \quad , \quad y(0) = 1, \quad D(y)(0) = 1\}, \text{egf}\]
Existence and naive computation

**Theorem** For any vector $\mathbf{F} = (f_1, \ldots, f_n)^T \in \mathbb{K}[[x]]^n$ and for any $n$-tuple $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, there exists a Hermite-Padé approximant of type $\mathbf{d}$ for $\mathbf{F}$.

**Proof:** The undetermined coefficients of $P_i = \sum_{j=0}^{d_i} p_{i,j} x^j$ satisfy a linear homogeneous system with $\sigma = \sum_i (d_i + 1) - 1$ equations and $\sigma + 1$ unknowns.

**Corollary** Computation in $O(MM(\sigma)) = O(\sigma^\theta)$, for $2 \leq \theta \leq 3$.

▶ There are better algorithms:

- The linear system is **structured** (Sylvester-like / quasi-Toeplitz)
- **Derksen’s algorithm** (Gaussian-like elimination) $O(\sigma^2)$
- **Beckermann-Labahn’s algorithm** (DAC) $\tilde{O}(\sigma) = O(\sigma \log^2 \sigma)$
Quasi-optimal computation

**Theorem** [Beckermann-Labahn, 1994] One can compute a Hermite-Padé approximant of type \((d, \ldots, d)\) for \(F = (f_1, \ldots, f_n)\) in \(O(MM(n, d) \log(nd))\).

**Ideas:**
- Compute a whole matrix of approximants
- Exploit divide-and-conquer

**Algorithm:**

1. If \(\sigma = n(d + 1) - 1 \leq \text{threshold}\), call the naive algorithm
2. Else:
   - (a) recursively compute \(P_1 \in \mathbb{K}[x]^{n \times n}\) s.t. \(P_1 \cdot F = O(x^{\sigma/2})\), \(\deg(P_1) \approx \frac{d}{2}\)
   - (b) compute “residue” \(R\) such that \(P_1 \cdot F = x^{\sigma/2} \cdot (R + O(x^{\sigma/2}))\)
   - (c) recursively compute \(P_2 \in \mathbb{K}[x]^{n \times n}\) s.t. \(P_2 \cdot R = O(x^{\sigma/2})\), \(\deg(P_2) \approx \frac{d}{2}\)
   - (d) return \(P := P_2 \cdot P_1\)

- The precise choices of degrees is a delicate issue
- Gcd, extended gcd, Padé approximants in \(O(M(n) \log n)\)
Example: Flea from SIAM 100-Digit Challenge

\[
\begin{align*}
&> \text{proba}:=\text{proc}(i,j,n,c) \\
&\text{option remember;} \\
&\quad \text{if abs}(i)+abs(j)>n \text{ then 0} \\
&\quad \text{elif } n=0 \text{ then 1} \\
&\quad \text{else} \\
&\quad \quad \text{expand(proba}(i-1,j,n-1,c)\cdot(1/4+c)+\text{proba}(i+1,j,n-1,c)\cdot(1/4-c) \\
&\quad \quad +\text{proba}(i,j+1,n-1,c)\cdot1/4+\text{proba}(i,j-1,n-1,c)\cdot1/4) \\
&\quad \text{fi} \\
&\text{end:} \\
&> \text{seq(proba}(0,0,k,c),k=0..6); \\
&\quad 1,0,\frac{1}{4} - 2c^2,0,\frac{9}{64} - \frac{9}{4}c^2 + 6c^4,0,\frac{25}{256} - \frac{75}{32}c^2 + 15c^4 - 20c^6 \\
&> \text{gfun:-listtodiffeq}([\text{seq(proba}(0,0,2*k,c),k=0..20)],y(x));
\end{align*}
\]
\[
\left\{ (-1 + 8c^2 + 48xc^4)y(x) + (4 - 8x + 64xc^2 + 192x^2c^4) \frac{d}{dx}y(x) \right.
\]
\[
+ (4x + 64x^3c^4 - 4x^2 + 32x^2c^2) \frac{d^2}{dx^2}y(x),
\]
\[
y(0) = 1, \frac{D(y)(0)}{4 - 2c^2} \right\}, \text{ogf}
\]

Next steps: \texttt{dsolve (+ help)} and evaluation at \(x = 1\).
TOOLS FOR CONJECTURES

2. \( p \)-Curvature of Differential Operators
Important classes of power series

Algebraic: $S(x) \in \mathbb{K}[[x]]$ root of a polynomial $P \in \mathbb{K}[x, y]$.

D-finite: $S(x) \in \mathbb{K}[[x]]$ satisfying a linear differential equation with polynomial (or rational function) coefficients $c_r(x)S^{(r)}(x) + \cdots + c_0(x)S(x) = 0$.

Hypergeometric: $S(x) = \sum_n s_n x^n$ such that $\frac{s_{n+1}}{s_n} \in \mathbb{K}(n)$. E.g.

$$\begin{align*}
_2F_1 \left( \begin{array}{c} a & b \\ c \\ \end{array} \right | x \right) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},
\quad (a)_n = a(a+1) \cdots (a+n-1).
\end{align*}$$

Linear differential operators

**Definition:** If $\mathbb{K}$ is a field, $\mathbb{K}\langle x, \partial; \partial x = x\partial + 1 \rangle$, or simply $\mathbb{K}(x)\langle \partial \rangle$, denotes the associative algebra of linear differential operators with coefficients in $\mathbb{K}(x)$. $\mathbb{K}[x]\langle \partial \rangle$ is called the *(rational)* Weyl algebra. It is the algebraic formalization of the notion of linear differential equation with rational function coefficients:

$$a_r(x)y^{(r)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0$$

$$\iff L(y) = 0, \quad \text{where} \quad L = a_r(x)\partial^r + \cdots + a_1(x)\partial + a_0(x)$$

The commutation rule $\partial x = x\partial + 1$ formalizes Leibniz’s rule $(fg)' = f'g + fg'$.

- Implementation in the **DEtools** package: **diffop2de**, **de2diffop**, **mult**

```
DEtools[mult](Dx,x,[Dx,x]);
    x Dx + 1
```
Weyl algebra is Euclidean

**Theorem** [Libri 1833, Brassinne 1864, Wedderburn 1932, Ore 1932]

\( \mathbb{K}(x)\langle \partial \rangle \) is a non-commutative (left and right) **Euclidean domain**: for any \( A, B \in \mathbb{K}(x)\langle \partial \rangle \), there exist unique operators \( Q, R \in \mathbb{K}(x)\langle \partial \rangle \) such that

\[
A = QB + R, \quad \text{and} \quad \deg_\partial(R) < \deg_\partial(B).
\]

This is called the **Euclidean right division** of \( A \) by \( B \).

Moreover, any \( A, B \in \mathbb{K}(x)\langle \partial \rangle \) admit a **greatest common right divisor (GCRD)** and a **least common left multiple (LCLM)**. They can be computed by a non-commutative version of the extended Euclidean algorithm.

▶ rightdivision, GCRD, LCLM from the DEtools package

\[
> \text{rightdivision(Dx}^\text{10}, \text{Dx}^\text{2-x}, [\text{Dx}, \text{x}]) [2];
\]

\[
\begin{array}{ccc}
3 & 2 & 5 \\
(20 x + 80) Dx + 100 x + x
\end{array}
\]

proves that \( \operatorname{Ai}^{(10)}(x) = (20x^3 + 80)\operatorname{Ai}'(x) + (100x^2 + x^5)\operatorname{Ai}(x) \)
1000 terms of a series are enough to guess candidate differential equations below the red curve. GCRD of candidates could jump above the red curve.
The Grothendieck–Katz $p$-curvatures conjecture

**Q:** when does a differential equation possess a basis of algebraic solutions?

E.g. for the Gauss hypergeometric equation $x(1-x)\partial^2 + (\gamma - (\alpha + \beta + 1)x)\partial - \alpha\beta$, Schwarz's list (1873) classifies algebraic $2F_1$'s in terms of $\alpha, \beta, \gamma$

**Conjecture** [Grothendieck, 1960’s, unpublished; Katz, 1972]

Let $A \in \mathbb{Q}(x)^{n \times n}$. The system $(S) : y' = Ay$ has a full set of algebraic solutions if and only if, for almost all prime numbers $p$, the system $(S_p)$ defined by reduction of $(S)$ modulo $p$ has a full set of algebraic solutions over $\mathbb{F}_p(x)$.

**Definition:** The $p$-curvature of $(S)$ is the matrix $A_p$, where

$$A_0 = I_n, \quad \text{and} \quad A_{\ell+1} = A'_\ell + A_\ell A \quad \text{for} \quad \ell \geq 0.$$  

**Theorem** [Cartier, 1957]

The sufficient condition of the G.-K. Conjecture is equivalent to $A_p = 0 \mod p$.

▶ For each $p$, this can be checked algorithmically.
Grothendieck’s conjecture

Q: when does a differential equation possess a basis of algebraic solutions?

For a scalar differential equation, the G.-K. Conjecture can be reformulated:
Grothendieck’s Conjecture: Suppose $L \in \mathbb{K}(x)\langle \partial \rangle$ is irreducible. The equation $(E): L(y) = 0$ has a basis of algebraic solutions if and only if, for almost all prime numbers $p$, the operator $L$ right-divides $\partial^p$ modulo $p$.

▶ For each $p$, this can be checked algorithmically.

▶ Conjecture is proved for Picard-Fuchs equations [Katz 1972] (in particular, for diagonals [Christol 1984]), for $n F_{n-1}$ equations [Beukers & Heckman 1989].
Suppose that we have guessed a linear differential equation \( L(f) = 0 \) (by differential Hermite-Padé approximation) for some power series \( f \in \mathbb{Q}[[x]] \), and that we want to recognize whether \( f \) is algebraic or not.

**Recipe 1:** try algebraic guessing.

**Recipe 2:** For several primes \( p \), compute \( p \)-curvatures mod \( p \), and check whether they are zero; equivalently, test if \( \partial^p \text{ mod } L = 0 \pmod{p} \).

- For many power series coming from counting problems (diagonals, constant terms, integrals of algebraic functions, ...) Grothendieck’s conjecture is true.
Grothendieck’s conjecture at work

Chebychev in his work on the distribution of primes numbers used the following fact

\[ u_n := \frac{(30n)!n!}{(15n)!(10n)!(6n)!} \in \mathbb{Z}, \quad n = 0, 1, 2, \ldots \]

This is not immediately obvious (for example, this ratio of factorials is not a product of multinomial coefficients) but it is not hard to prove. The only proof I know proceeds by checking that the valuations \( v_p(u_n) \) are non-negative for every prime \( p \); an interpretation of \( u_n \) as counting natural objects or being dimensions of natural vector spaces is far from clear.

As it turns out, the generating function

\[ u := \sum_{\nu \geq 1} u_n \lambda^n \]

is algebraic over \( \mathbb{Q}(\lambda) \); i.e. there is a polynomial \( F \in \mathbb{Z}[x, y] \) such that

\[ F(\lambda, u(\lambda)) = 0. \]

However, we are not likely to see this polynomial explicitly any time soon as its degree is 483,840 (!)

(excerpt from Rodriguez-Villegas’s “Integral ratios of factorials”)

- Algebraicity of \( u \) can be however guessed using any prior knowledge, by computing \( p \)-curvatures of the (minimal) order-8 operator \( L \) s.t. \( L(u) = 0 \)

- For \( p < 300 \), they are all zero, except when \( p \in \{11, 13, 17, 19, 23\} \)
**G-series and global nilpotence**

**Definition:** A power series \( \sum_{n \geq 0} \frac{a_n}{b_n} x^n \) in \( \mathbb{Q}[[x]] \) is called a \( G \)-series if it is
(a) D-finite; (b) analytic at \( x = 0 \); (c) \( \exists C > 0, \text{lcm}(b_0, \ldots, b_n) \leq C^n \).

**Basic examples:**
1. algebraic functions [Eisenstein 1852]
2. \( -\log(1 - x) = \sum_{n \geq 1} \frac{x^n}{n} \) ([Chebyshev 1852] \( \text{lcm}(1, 2, \ldots, n) \leq 4^n \))
3. \( _2F_1 \left( \begin{array}{c} \alpha \ 
\beta \\
\gamma \end{array} \right| x \), \( \alpha, \beta, \gamma \in \mathbb{Q} \)
4. OGF of any \( P \)-recursive, integer-valued, exponentially bounded, sequence

**Theorem** [Chudnovsky 1985] The minimal-order linear differential operator annihilating a \( G \)-series is **globally nilpotent**: for almost all prime numbers \( p \), it right-divides \( \partial^\mu p \) modulo \( p \), for some \( \mu \leq \deg \partial L \).

(this condition is equivalent to the **nilpotence** mod \( p \) of the \( p \)-curvature matrix)

**Examples:** algebraic resolvents; Gauss’s \( x(1 - x)\partial^2 + (\gamma - (\alpha + \beta + 1)x)\partial - \alpha \beta x. \)
Global nilpotence for combinatorics

Suppose we have guessed (by differential approximation) a linear differential equation \( L(f) = 0 \) for a power series \( f \in \mathbb{Q}[[x]] \) which is a \( G \)-series (typically, the OGF of a \( P \)-recursive, integer-valued, exponentially bounded, sequence).

A way to empirically certify that \( L \) is very plausible:

**Recipe**: compute \( p \)-curvatures mod \( p \), and check whether they are nilpotent; equivalently, test if \( \partial^{pr} \mod L = 0 \mod p \), where \( r = \deg_\partial L \)

**Example**:

```plaintext
> L:=x^2*(64*x^4+40*x^3-30*x^2-5*x+1)*Dx^3+
x*(576*x^4+200*x^3-252*x^2-33*x+5)*Dx^2+
4*(1+288*x^4+22*x^3-117*x^2-12*x)*Dx+384*x^3-12-144*x-72*x^2:
> p:=7; for j to 3 do N:=rightdivision(Dx^(3*p),L,[Dx,x])[2] mod p;
p:=nextprime(p); print(p, N); od:
  11, 0
  13, 0
  17, 0
```
Overview

Today
1. Introduction
2. High Precision Approximations
   – Fast multiplication, binary splitting, Newton iteration
3. Tools for Conjectures
   – Hermite-Padé approximants, $p$-curvature

Tomorrow morning
4. Tools for Proofs
   – Symbolic method, resultants, D-finiteness, creative telescoping

Tomorrow night
– Exercises with Maple