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Résumé
Cet article présente un cadre algébrique et une mise en œuvre pour le signal analytique 2D+t à l'aide des biquaternions de Clifford et de la Transformée de Fourier Cliffordienne. La contribution de ce travail est de définir et réaliser le signal analytique spatio-temporel pour analyser des séquences vidéo alors que la plupart des travaux précédents ont traité du signal analytique pour les images 2D. À partir du signal analytique 2D+t que nous définissons, plusieurs paramètres physiques liés à la dimension du temps et de l'espace pourront être calculés. Nous détaillons ici la méthode de calcul numérique pour obtenir le signal analytique spatio-temporel. Dans cet article, on considère l'exemple de simulation d'une onde plane progressive et d'une onde modulée en amplitude. À partir de ces exemples, on montre le calcul algébrique et numérique du signal hyperanalytique, puis l'extraction des phases pour l'estimation de paramètres physiques tels que la fréquence instantanée, la vitesse de phase et la vitesse de groupe. Les résultats numériques obtenus permettent de valider l'approche et montrent le potentiel de la méthode pour l'estimation de paramètres physiques.

Abstract
This paper presents an algebraic framework and an implementation for the 2D + t analytic signal by using Clifford biquaternions and Clifford Fourier transform. The contribution of this work is to define and realize the analytical signal for space-time analysis of video sequences, whereas most of the previous works are based on 2D images. It enables us to obtain the physical parameters related to time dimension from some specific cases of 2D + t analytic signal. Here, we describe the numerical calculation method to obtain the space-time analytical signal. Considering the simulation example of a progressive plane wave and an amplitude modulated wave, the calculation of algebraical and numerical hyper-analytic is introduced. The physical parameters of the phase function, such as instantaneous frequency, phase velocity and group velocity are extracted. Numerical results are given for validating the proposed method and show the potential of physical parameters estimating method.

Mots-clés : biquaternion de Clifford, Transformée de Fourier Clifford, signal analytique 2D+t, paramètres physiques

Keywords: Clifford biquaternions, Clifford Fourier transforms, 2D+t analytic signals, physical parameters

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1. Introduction

During the last years, the quaternion-based method has been applied to image analysis and signal processing based on Clifford algebras [5,7]. As a mathematical tool, it has been used for motion

detection, motion estimation and tracking [1,6,9]. It's also used for edge detection of color image thanks to its multicomponent property [8].
Generally, phase information can be obtained with quaternion-based method. Some advantages of phase information are known as that [3]: (1) there is strong evidence that the human visual system makes use of local phase; (2) phase-based processing is to a large extend invariant to changes of lighting conditions; (3) the reconstruction of an image from phase information is much better than that from amplitude information. Furthermore, several works have shown previously the advantage of using phase information to analyse images and image sequences. Bülow and Sommer [2] found that a component of the quaternionic phase is sensitive to the change of a structure like a flaw in a textile. In 2006, using hypercomplex phase, Witten and Shragge [10] proposed an image disparity estimation method to find differences between subtly varying images. Woo *et al.* [11] used both the local phase information and the intensity information for ultrasound image registration. Felsberg [3] used the monogenic phase for optical flow estimation and indicated several advantages of phase information.
However, most of these developments above used the approaches to process only two-dimension images or image sequences. It means that the dimension of time was not involved when calculating the analytic signal and the phase information of analytic signal. In order to deal with this problem and to process an entire video signal at once, a $2D + t$ analytic video signal is defined in this paper. Therefore, it has three components: time, vertical position and horizontal position. With the objective to process an entire video signal at once, a biquaternion algebraic framework is developed to express Clifford Fourier Transforms(CFT) of $2D + t$ video analytic signals in standard form. On the specific case of a progressive plane wave, the physic parameters are found, which could be useful to estimate the motion of the progressive plane wave.

2. Clifford Biquaternion 2D+t Analytic Signal: Implementation

In $2D + t$ dimensions, from the introduction of Clifford biquaternion Algebra in [4], we have the full algebra elements:

$$\begin{bmatrix} 1 & i = e_2 e_3 & j = e_3 e_1 & k = e_1 e_2 \\ \epsilon = i'I = -e_1 e_2 e_3 & e_1 = \epsilon i & e_2 = \epsilon j & e_3 = \epsilon k \end{bmatrix}. \tag{1}$$

2.1. Clifford Fourier Transform

Given a function $f(x)$ having its value in the Clifford algebra with $x = (x_1, x_2, x_3)$, and let $F(u)$ with $u = (u_1, u_2, u_3)$ denotes the biquaternion CFT:

$$F(u) = \int_{R^3} f(x) e^{-\epsilon i 2\pi u_1 x_1} e^{-\epsilon j 2\pi u_2 x_2} e^{-\epsilon k 2\pi u_3 x_3} dx_1 dx_2 dx_3. \tag{2}$$

Thus, here the function $F(u)$ is a biquaternion of eight algebra elements that introduced in Eq. 1. The inverse CFT is given by:

$$f(x) = \int_{R^3} F(u) e^{\epsilon k 2\pi u_3 x_3} e^{\epsilon j 2\pi u_2 x_2} e^{\epsilon 2\pi u_1 x_1} du_1 du_2 du_3. \tag{3}$$

To compute the direct CFT, one proceeds in cascade integrating first with respect to x_1 using a standard FFT. The second FFT (integration with respect to x_2) is then applied on each real component of the previous complex number. Then, the third FFT (integration on x_3) is applied on each of the resulting real components. Finally, all the components are properly displayed as a Clifford biquaternion. For the inverse CFT, one proceeds in the same way on each real component of the Clifford biquaternion using IFFT and reversing the order of integration. Due to the symmetries of the CFT of a scalar function $f(x_1, x_2, x_3)$, only one orthant of the Fourier space is necessary to obtain the entire Fourier space.

2.2. Analytic Signal

The analytic Clifford Fourier transform is defined by:

$$F_A(u) = [1 + \mathrm{sign}(u_1)]\,[1 + \mathrm{sign}(u_2)]\,[1 + \mathrm{sign}(u_3)]\,F(u), \tag{4}$$

and the analytic signal by:

$$f_A(x) = \int_{R^3} F_A(u) e^{\epsilon k 2\pi u_3 x_3} e^{\epsilon j 2\pi u_2 x_2} e^{\epsilon i 2\pi u_1 x_1} du_1 du_2 du_3. \tag{5}$$

3. Application of Analytic Signal

To put the analytic signal into application, a progressive plane wave and an amplitude modulated wave are created by cosine functions. Then their analytic video signals are calculated. Subsequently, some physic parameters from the phase of analytic video signal can be found, such as wave velocity, instantaneous frequency, phase velocity and group velocity.

3.1. Example 1: progressive plane wave and its instantaneous frequency and wave velocity

Consider the progressive plane wave given by:

$$f_1(t, x, y) = \cos(\omega t + k_x x + k_y y), \tag{6}$$

where ω, k_x and k_y are the pulsations of the plane wave. t represents the time axis , x represents the horizontal axis, and y represents the vertical axis of a frame in an image sequence. The analytic signal is given by:

$$f_A(t, x, y) = \begin{bmatrix} \cos(\omega t + k_x x + k_y y) \\ -\cos(\omega t - k_x x + k_y y) \\ \cos(\omega t + k_x x - k_y y) \\ -\cos(\omega t - k_x x - k_y y) \end{bmatrix} + \varepsilon \begin{bmatrix} \sin(\omega t - k_x x + k_y y) \\ \sin(\omega t + k_x x + k_y y) \\ -\sin(\omega t - k_x x - k_y y) \\ -\sin(\omega t + k_x x - k_y y) \end{bmatrix}. \tag{7}$$

Based on the analytic signal given by Eq. 7, the instantaneous frequency and wave velocity can be obtained as follows:

$$\phi_1 = angle[f_A\ (1) + i f_A\ (6)] = atan2\,(FA(1), FA(6)) = \omega t + k_x x + k_y y, \tag{8}$$

where $FA\ (1)$ means the first component of f_A and $FA\ (6)$ the sixth component of f_A. The $atan2()$ function is the four quadrant arctangent that used in [2]. Then, the instantaneous frequency ω, k_x, k_y are given by:

$$\omega = \frac{\partial \phi_1}{\partial t}, \quad k_x = \frac{\partial \phi_1}{\partial x}, \quad k_y = \frac{\partial \phi_1}{\partial y}, \tag{9}$$

next, the wave velocities correspond to horizontal axis x and vertical axis y are given by:

$$v_x = -\frac{\omega}{k_x}, \quad v_y = -\frac{\omega}{k_y}. \tag{10}$$

Thus, from this example, we can get the velocities of this progressive plane wave along horizontal and vertical axis separately by using its analytic signal. Thanks to the 2D+t analytic signal, the equations of these velocities in Eq. 10 are continuous with respect to the time axis.

3.2. Example 2: Amplitude modulated wave and its phase velocity and group velocity

Consider the function

$$f_2(t, x, y) = \cos(\omega_1 t + k_{x1} x + k_{y1} y) + \cos(\omega_2 t + k_{x2} x + k_{y2} y), \tag{11}$$

with $\omega_i = \Omega \pm \frac{\omega}{2}, k_{xi} = K_x \pm \frac{k_x}{2}, k_{yi} = K_y \pm \frac{k_y}{2}$ and $\Omega > \omega, K_x > k_x, K_y > k_y$. The analytic signal is given by:

$$\begin{aligned} f_A(t, x, y) \quad = & \begin{bmatrix} 2\cos(\omega t + k_x x + k_y y)\cos(\Omega t + K_x x + K_y y) \\ -2\cos(\omega t - k_x x + k_y y)\cos(\Omega t - K_x x + K_y y) \\ 2\cos(\omega t + k_x x - k_y y)\cos(\Omega t + K_x x - K_y y) \\ -2\cos(\omega t - k_x x - k_y y)\cos(\Omega t - K_x x - K_y y) \end{bmatrix} \\ & + \epsilon \begin{bmatrix} 2\cos(\omega t - k_x x + k_y y)\sin(\Omega t - K_x x + K_y y) \\ 2\cos(\omega t + k_x x + k_y y)\sin(\Omega t + K_x x + K_y y) \\ -2\cos(\omega t - k_x x - k_y y)\sin(\Omega t - K_x x - K_y y) \\ -2\cos(\omega t + k_x x - k_y y)\sin(\Omega t + K_x x - K_y y) \end{bmatrix}. \end{aligned} \tag{12}$$

Then computes

$$d = [f_A(1)]^2 + [f_A(6)]^2 = 4\cos^2(\omega t + k_x x + k_y y) = 2\{1 + \cos[2(\omega t + k_x x + k_y y)]\}. \tag{13}$$

The phase and group velocities are obtained as follows:

$$\Phi_1 = \Omega t + K_x x + K_y y = \mathrm{atan2}\left(FA(1), FA(6)\right), \tag{14}$$

$$\Phi_2 = \omega t + k_x x + k_y y = \frac{1}{2}\arccos\left(\frac{d}{2} - 1\right), \tag{15}$$

then the phase velocities $v_{\varphi x}, v_{\varphi y}$ and group velocities v_{gx}, v_{gy} are:

$$v_{\varphi x} = -\frac{\frac{\partial \Phi_1}{\partial t}}{\frac{\partial \Phi_1}{\partial x}} = -\frac{\Omega}{K_x}, \quad v_{\varphi y} = -\frac{\frac{\partial \Phi_1}{\partial t}}{\frac{\partial \Phi_1}{\partial y}} = -\frac{\Omega}{K_y}, \tag{16}$$

$$v_{gx} = -\frac{\frac{\partial \Phi_2}{\partial t}}{\frac{\partial \Phi_2}{\partial x}} = -\frac{\omega}{k_x}, \quad v_{gy} = -\frac{\frac{\partial \Phi_2}{\partial t}}{\frac{\partial \Phi_2}{\partial y}} = -\frac{\omega}{k_y}. \tag{17}$$

Thus, from the analytic signal of this amplitude modulated wave, we can get the phase velocities and the group velocities along horizontal and vertical axis separately. Thanks to the 2D+t analytic signal, the equations of these velocities in Eq. 16 and Eq. 17 are continuous with respect to the time axis.

4. Numerical Approximation

The numerical approximation is calculated via MatLab. For the progressive plane wave video signal $f_1(t, x, y) = \cos(\omega t + k_x x + k_y y)$, its analytic video signal f_A is shown in Fig. 1. The range of t, x, y is normalized into $[0, 1]$ and the pulsation values are $\omega = k_x = k_y = 2\pi$. From Eq. 8, there is the numerical presentation in Fig. 2 of the phase function. Next, by Eq. 9, the instantaneous frequencies are obtained from ϕ_1. For a fixed value of horizontal index X and vertical index Y, ω can be found. The value is equal to $6.283 (= 2\pi)$, which corresponds to the pulsation value of ω of input video signal. Meanwhile, the instantaneous frequencies k_x and k_y can be obtained correctly by the same way. Fig. 3 shows the ω value and a profile of ϕ_1 from Fig. 2(a). Hence, the wave velocity can be calculated by Eq. 10, i.e., $V_x = -\frac{\omega}{k_x} = -\frac{6.283}{6.283} = -1 m \cdot s^{-1}$ and $V_y = -\frac{\omega}{k_y} = -\frac{6.283}{6.283} = -1 m \cdot s^{-1}$. The error between the algebraical and numerical result is less than $1 \times 10^{-13} \approx 0$.

Considering another plane wave signal $f_2(t, x, y) = \cos(\omega_1 t + k_{x1} x + k_{y1} y) + \cos(\omega_2 t + k_{x2} x + k_{y2} y)$, the range of x, y is normalized into $[-0.5, 0.5]$ and the range of t is $[0, 2\pi/\omega_1]$. The pulsation values are $\omega_1 = \omega_2 = 2 rad \cdot s^{-1}, k_{x1} = 8 rad \cdot m^{-1}, k_{x2} = 4 rad \cdot m^{-1}, k_{y1} = 2 rad \cdot m^{-1}, k_{y2} = 1 rad \cdot m^{-1}$. As the result, the phase of the analytic signal Φ_1 with unwrap and Φ_2 without unwrap are shown in Fig. 4, which corresponds to Eq. 14. Subsequently, by partial differential, we obtain the frequency parameters Ω, K_x and K_y of phase Φ_1, that are presented in Fig. 5. The values of these parameters correspond correctly to the algebraical results of K_x, K_y and Ω in Eqs. 11 and 12. Hence, the phase velocities are calculated as: $v_{\varphi x} = \Omega/K_x = 1/3 m \cdot s^{-1}$, $v_{\varphi y} = \Omega/K_y = 4/3 m \cdot s^{-1}$. Meanwhile, we obtain also the frequency parameters ω, k_x and k_y of phase Φ_2 shown in Fig. 6.

5. Conclusion

The paper has presented a concrete algebraic framework, i.e., Clifford's biquaternions for the expression of Clifford Fourier transforms and 2D + t analytic signals. Then, we have shown firstly how to extract the instantaneous frequency and wave velocity from a progressive plane wave, and secondly how to extract the phase velocity and group velocity from an amplitude modulated wave. Finally, with the numerical approximation, we have shown that these physical parameters can be recovered correctly. Our next work is to develop the method to estimate the motion from 2D+t medical image sequences based on the phase information of image sequences.

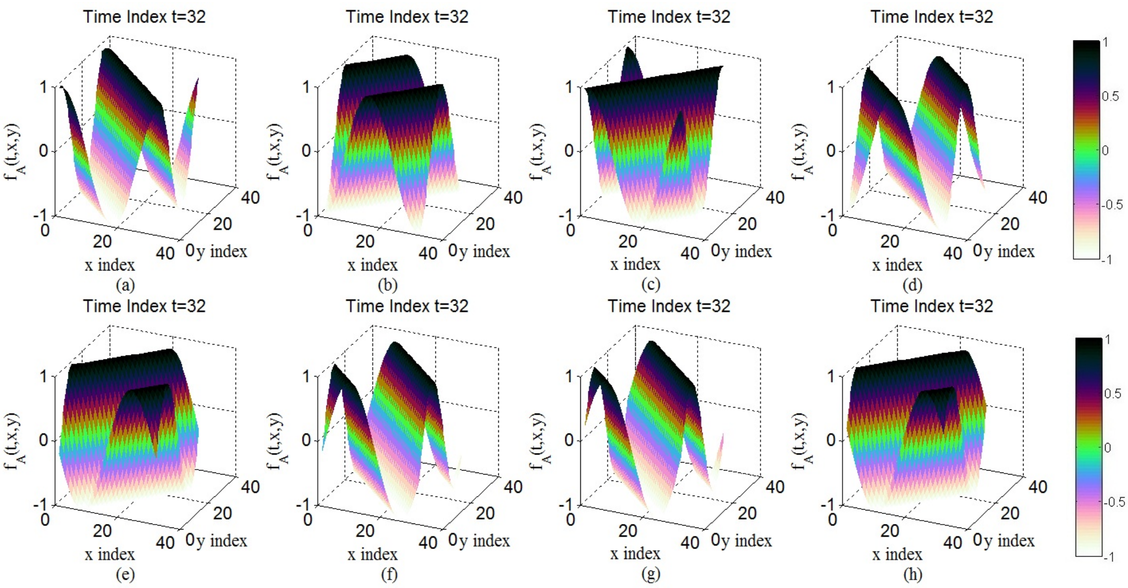

Figure 1: The eight components of the analytic video signal: $f_A = [a, b, c, d] + \epsilon[e, f, g, h]$ at t index= 32, with the 32 sample points for each t, x and y.

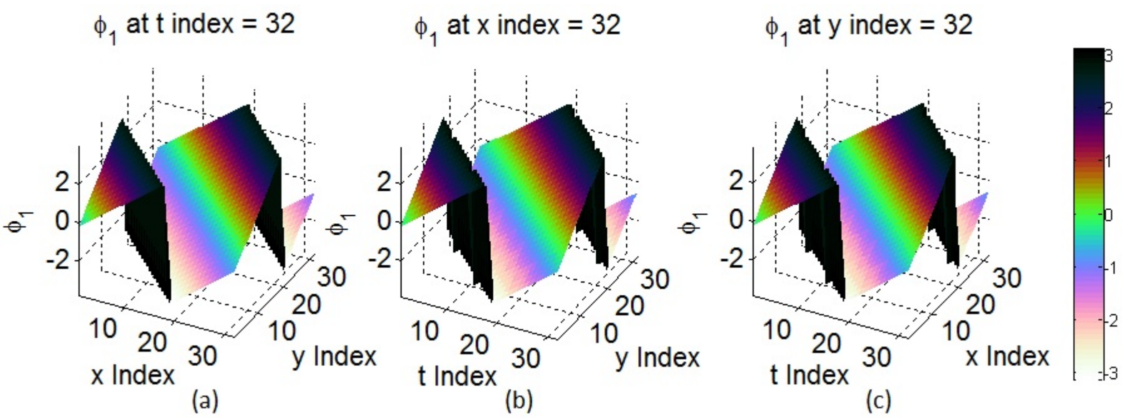

Figure 2: The phase ϕ_1 in 3 point of views at a fixed value of t index=32, x index=32 and y index= 32 separately.

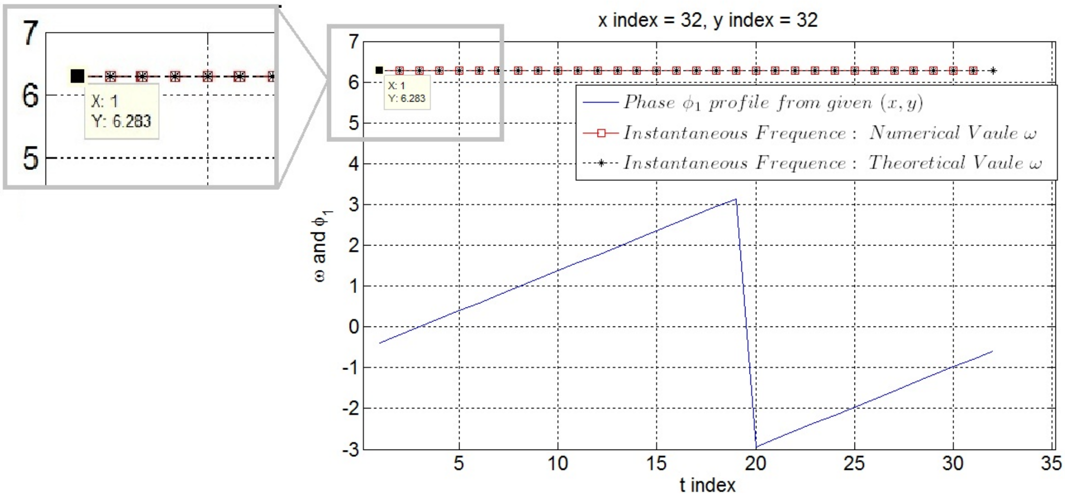

Figure 3: ω value from a profile of ϕ_1 on x index=32 and y index=32. The continuous line represents the profile of ϕ_1, the square symbol line is the numerical value of instantaneous frequency $\frac{\partial \phi_1}{\partial t}$, that equals to the algebraical value $\omega = 2\pi$(star symbol line).

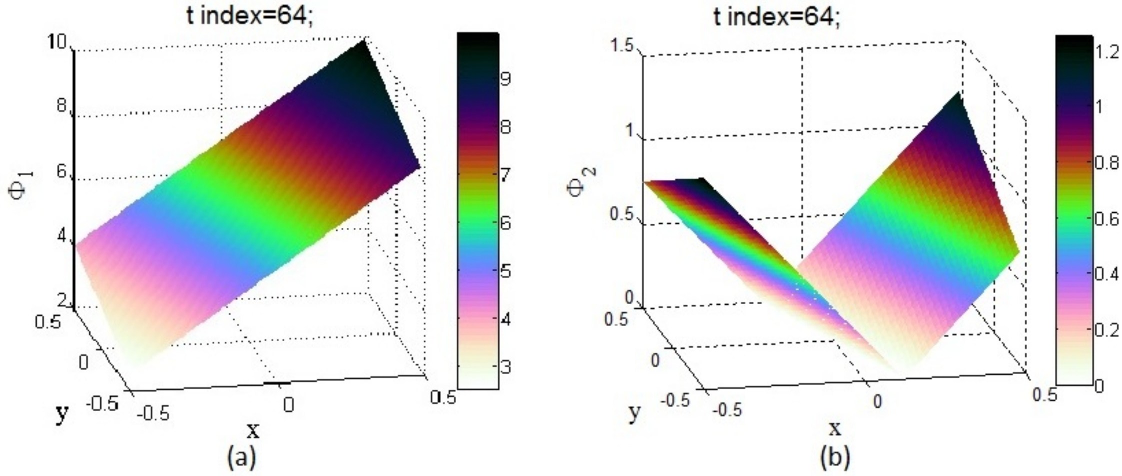

Figure 4: (a) The phase $\Phi_1(t, x, y)$ with unwrap at t index= 64. (b) The phase $\Phi_2(t, x, y)$ without unwrap at t index= 64.

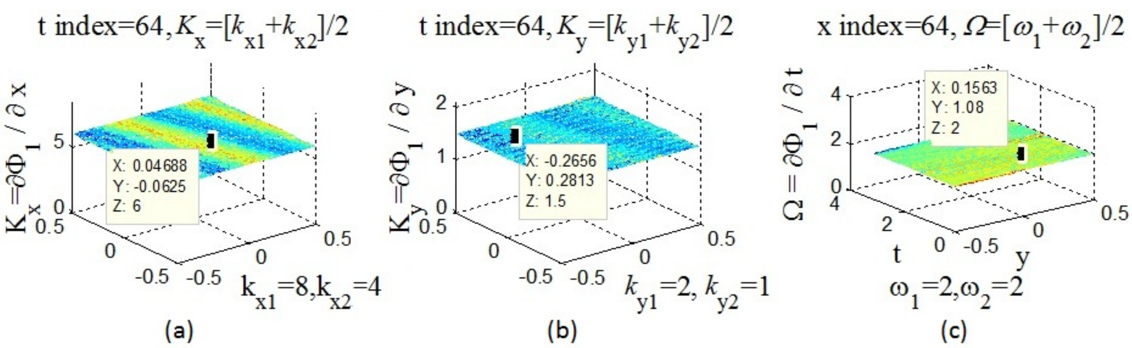

Figure 5: Three frequency parameters of Φ_1 at a given index separately: (a) $K_x = 6rad \cdot m^{-1}$ at t index=64, (b) $K_y = 1.5rad \cdot m^{-1}$ at t index=64, (c) $\Omega = 2rad \cdot s^{-1}$ at x index=64; The values of these parameters correspond correctly to the algebraical results of K_x, K_y and Ω in Eqs. 12.

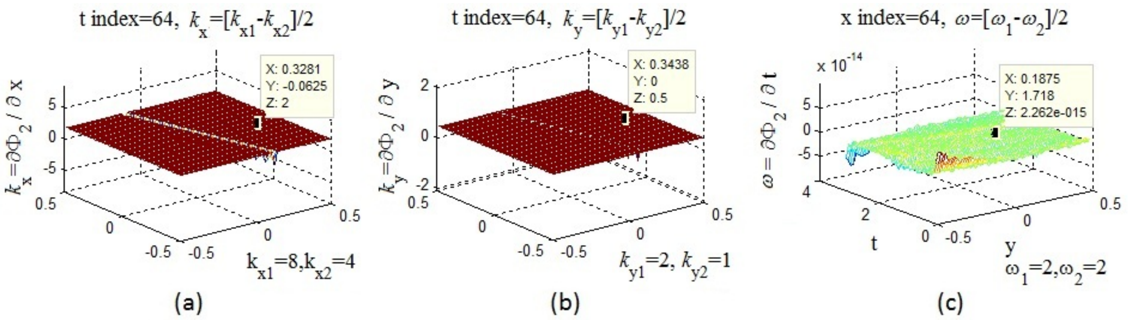

Figure 6: Three frequency parameters of Φ_2 at a given index separately: (a) $k_x = 2rad \cdot m^{-1}$ at t index=64, (b) $k_y = 0.5rad \cdot m^{-1}$ at t index=64, (c) $\omega \in \pm 5 \times 10^{-14} \approx 0rad \cdot s^{-1}$ at x index=64; The values of these parameters correspond correctly to the algebraical results of k_x, k_y and ω in Eqs. 12.

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