Creative telescoping for rational functions using the Griffiths-Dwork method

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ABSTRACT

Creative telescoping algorithms compute linear differential equations satisfied by multiple integrals with parameters. We describe a precise and elementary algorithmic version of the Griffiths–Dwork method for the creative telescoping of rational functions. This leads to bounds on the order and degree of the coefficients of the differential equation, and to the first complexity result which is simply exponential in the number of variables. One of the important features of the algorithm is that it does not need to compute certificates. The approach is vindicated by a prototype implementation.

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1. INTRODUCTION

In computer algebra, *creative telescoping* is an approach introduced by Zeilberger to address definite summation and integration of a large class of functions and sequences [28, 29, 27]. Its vast scope includes the computation of differential equations for multiple integrals with parameters of rational or algebraic functions. Within this class, creative telescoping is similar to well-studied older approaches whose key notion is the Picard–Fuchs differential equation, see *e.g.* [23].

We study the multivariate rational case: Given a rational function $F(t, x_1, \dots, x_n)$, we aim at finding n other rational functions $A_i(t, x_1, \dots, x_n)$ and a differential operator with polynomial coefficients $\sum_{j=0}^{r} c_j(t)\partial_t^j$, denoted T, such that

$$T(F) \overset{\text{def}}{=} \sum_{j=0}^{r} c_j(t)\partial_t^j F = \sum_{i=1}^{n} \partial_i A_i, \qquad (1)$$

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where ∂_t^j denotes $\frac{\partial^j}{\partial t^j}$ and ∂_i denotes $\frac{\partial}{\partial x_i}$. The integer r is the *order* of T and $\max_j \deg c_j$ is its *degree*.

Throughout the article, the constant field k of F is assumed to be of characteristic zero. Under suitable hypotheses, the operator T is a differential equation satisfied by integrals $\int F \mathrm{d}x$ over a domain γ, without boundaries, where F has no pole. A misbehavior can happen when the certificate has poles outside those of F: it may not be possible to integrate term by term the right-hand side of Eq. (1), see §4.1. The certificate is called *regular* when it does not contain poles other than those of F. For integration, there is no need to compute the actual certificate provided that it is regular.

Several methods are known that can find a telescoper and the corresponding certificate [17, 26, 7, 15]. However, the practical cost of using these methods in multivariate problems remains high and a better understanding of the size or complexity of the objects of creative telescoping is clearly needed. The present work is part of the on-going effort in this direction [2, 3, 4]. The study of the rational case is motivated both by its fundamental nature and by its applications to the computation of diagonals in combinatorics, number theory and physics [17, 6, 20]. The rational case with n variables also includes the algebraic case with $n-1$ variables [4].

Previous works. An obviously related problem is, given a rational function $F(x_1, \dots, x_n)$, to decide whether there exist rational functions $A_1, \dots, A_n$ such that F equals $\sum_{i=1}^{n} \frac{\partial A_i}{\partial x_i}$. When $n = 1$, the latter question is easily solved by Hermite reduction. This is the basis of an algorithm for creative telescoping [3] that we outline in §2.1.

Picard [25, chap. 7] gave methods when $n = 2$ from which he deduced that a telescoping equation can always be found in that case [24]. This too, has led to an algorithm [4]. The Griffiths–Dwork method [8, §3; 9, §8; 12] solves the problem for a general n, in the setting of de Rham cohomology and under a regularity assumption. The method can be viewed as a generalization of Hermite reduction. Independently, Christol used a similar method to prove that diagonals of rational functions, under a regularity hypothesis, are differentially finite [5]; then he noticed that a deformation technique, for which he credits Dwork, can handle singular cases [6]. The Griffiths–Dwork method is also used in point counting applications [1, 11] and the study of mirror maps [20].

In terms of complexity, not much is known. If a rational function $F(t, x_1, \dots, x_n)$ has degree d, a study of Lipshitz's argument [17] shows that there exists a telescoper of order and degree $d^{O(n)}$ with a regular certificate of size $d^{O(n^2)}$.

Most algorithms [17, 28, 26, 7, 2, 15] cannot avoid the computation of the certificate, which impacts their complexity. The complexity of Lipshitz's algorithm is $d^{O(n^2)}$ operations in k; the complexity of no other algorithm is known. Pancratz [22] developed an approach similar to ours, under a restrictive hypothesis, much stronger than Griffiths' regularity assumption. He proceeds to a complexity analysis of his algorithm but in terms of operations in $k(t)$ rather than in the base field k. Algorithms based on non-commutative Gröbner bases and elimination [28, 26] or based on the search of rational solutions to differential equations [7] resist to complexity analysis. The method of Apagodu and Zeilberger [2] requires a generic exponent and specialization seems problematic.

For the restricted class of diagonals of rational functions, there is a heuristic based on series expansion and differential approximation [14]; it does not need to compute a certificate. However, even using the bounds in $d^{O(n)}$, its direct implementation has a complexity of $d^{O(n^2)}$ operations in k.

Contributions. Our main result, obtained with the Griffiths–Dwork method and a deformation technique, is the existence of a telescoper with regular certificate of order at most d^n and degree $d^{O(n)}$ that can be computed in $d^{O(n)}$ arithmetic operations in k. For generic rational functions, the telescoper computed is the minimal order telescoper with regular certificate. Theorems 6, 10 and 11 state precise estimates. To the best of our knowledge, the bounds on the order and degree are better than what was known and it is the first time that a complexity simply exponential in n is reached. Note that we do not know whether a rational function admits a telescoper whose certificate has size polynomial in d^n, but our algorithm does not need to compute the certificate. A prototype implementation shows that this algorithm can lead to a spectacular improvement over previous methods, though the domain of improvement is not satisfactory yet.

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2. OVERVIEW OF THE METHOD

In this section we introduce the basics of the Griffiths–Dwork method. In dimension 1, this method coincides with classical Hermite reduction which we first recall.

2.1 Dimension one: Hermite reduction

Let F be a rational function in x, over a field L, written as a/f^ℓ, with a and f two polynomials, the latter being square-free, *i.e.* the polynomials $\partial_x f$ and f are coprime. In particular a equals $uf + v\partial_x f$ for some polynomials u and v. Then, if $\ell > 1$, the function F rewrites

$$F = \frac{u + \frac{1}{\ell-1}\partial_x v}{f^{\ell-1}} + \partial_x\left(\frac{-v}{(\ell-1)f^{\ell-1}}\right).$$

Iterating this reduction step ℓ times gives F as $\frac{U}{f} + \partial_x \frac{V}{f^{\ell-1}}$ for some polynomials U and V. Next, Euclidean division allows to write U as $r + sf$, with r of degree less than the degree of f, yielding the additive decomposition

$$F = \frac{r}{f} + \partial_x\left(\frac{V}{f^{\ell-1}} + \int s\right).$$

The rational function r/f is the *reduced form* of F and is denoted $[F]$. This reduced form features important properties:

(Linearity) f being fixed, $[F]$ depends linearly on F;

(Soundness) if $[F]$ is zero, then F is a derivative w.r.t. x;

(Confinement) $[F]$ lies in a finite-dimensional vector space over L depending only on f (with dimension $\deg_x f$);

(Normalization) if F is a derivative w.r.t. x, then $[F]$ is zero.

These properties are enough to compute a telescoper: Assume now that L is $k(t)$ for a field k. If for some elements of L, say $a_0, \dots, a_p$, the reduced form $\left[\sum_i a_i \partial_t^i F\right]$ vanishes then the operator $\sum_i a_i \partial_t^i$ is a telescoper, thanks to the soundness property. Thanks to the linearity property, this is equivalent to the vanishing of $\sum_i a_i \left[\partial_t^i F\right]$. Thanks to the confinement property, it is always possible to find such a relation. Thanks to the normalization property, every telescoper arises this way. In particular, so does the telescoper of minimal order.

2.2 Higher dimension: Griffiths–Dwork reduction

Let F be a rational function in n variables $x_1, \dots, x_n$, written as a/f^ℓ, with f a square-free polynomial. If $\ell > 1$ and if a lies in the ideal generated by f and its derivatives $\partial_i f$, then we can write a as $uf + \sum_i v_i \partial_i f$, for some polynomials $u, v_1, \dots, v_n$, and F rewrites

$$F = \frac{u + \frac{1}{\ell-1}\sum_{i=1}^n \partial_i v_i}{f^{\ell-1}} + \sum_{i=1}^n \partial_i \left(\frac{-v_i}{(\ell-1)f^{\ell-1}}\right).$$

Provided that this ideal contains all polynomials, any F can be reduced to a function with simple poles by iteration of this identity. The soundness and linearity properties are naturally satisfied, but extending further the reduction to obtain at least the confinement property is not straightforward and requires stronger assumptions [21]. A difficulty with this approach is that the degrees of the cofactors v_i at each reduction step are poorly controlled: we lack the Euclidean division step and we reduce poles at finite distance at the cost of making worse the pole at infinity. This difficulty is overcome by working in the projective space. The translation between affine and projective is discussed more precisely in Section 7.

Now, assume that a and f are *homogeneous* polynomials in $L[\mathbf{x}] = L[x_0, \dots, x_n]$, with f of degree d. A central role is played by $\operatorname{Jac} f$, the *Jacobian ideal of* f, the ideal generated by the partial derivatives $\partial_0 f, \dots, \partial_n f$. Note that since f is homogeneous, Euler's relation, which asserts that f equals $\frac{1}{d}\sum_{i=0}^n x_i \partial_i f$ implies that $f \in \operatorname{Jac} f$.

We now decompose a as $r + \sum_i v_i \partial_i f$. In contrast with the affine case, each nonzero v_i can be chosen homogeneous of degree precisely $\deg a - \deg \partial_i f$. If $\ell > 1$, we obtain

$$F = \frac{r}{f^\ell} + \underbrace{\frac{\frac{1}{\ell-1}\sum_{i=0}^n \partial_i v_i}{f^{\ell-1}}}_{F_1} + \sum_{i=0}^n \partial_i \left(\frac{-v_i}{(\ell-1)f^{\ell-1}}\right). \qquad (2)$$

If r is not zero, the order of the pole need not decrease, contrary to the affine case, but r is reduced to a normal form modulo $\operatorname{Jac} f$; this will help us obtain the confinement property, see Proposition 2. The reduction process proceeds recursively on F_1, which has pole order $\ell - 1$, and stops when $\ell = 1$. This procedure is summarized in Algorithm 1.

Input $F = a/f^\ell$ a rational function in $x_0,\dots,x_n$
Output $[F]$ such that there exist $A_0,\dots,A_n$ rational functions such that $F=[F]+\sum_i \partial_i A_i$

Precompute a Gröbner basis G for $(\partial_0 f,\dots,\partial_n f)$
procedure $\textsc{Reduce}(a/f^\ell)$
  if $\ell=1$ **then return** a/f^ℓ
  Decompose a as $r+\sum_i v_i\partial_i f$ using G
  $F_1 \leftarrow \frac{1}{\ell-1}\sum_i \frac{\partial_i v_i}{f^{\ell-1}}$
  return $\frac{r}{f^\ell}+\textsc{Reduce}(F_1)$

Algorithm 1. Griffiths–Dwork reduction

3. PROPERTIES OF THE GRIFFITHS–DWORK REDUCTION

Let f in $L[\mathbf{x}]$ be a homogeneous polynomial of degree d, where L is a field of characteristic zero. It is clear that the reduction procedure satisfies the soundness and the linearity properties. Analogues of confinement and normalization hold under the following regularity hypothesis:

$$L[\mathbf{x}]/\operatorname{Jac} f \text{ is finite-dimensional over } L. \tag{H}$$

Geometrically, this hypothesis means that the hypersurface defined by f in $\mathbb{P}^n$ is smooth. In particular f is irreducible.

The ring of rational functions in $L(\mathbf{x})$ whose denominator is a power of f is denoted $L[\mathbf{x},\frac{1}{f}]$. Let $L[\mathbf{x},\frac{1}{f}]_p$ denote the subspace of homogeneous functions of degree p, *i.e.* of F in $L[\mathbf{x},\frac{1}{f}]$ such that $F(\lambda\mathbf{x})$ equals $\lambda^p F(\mathbf{x})$. Note that each derivation ∂_i induces a map from $L[\mathbf{x},\frac{1}{f}]_p$ to $L[\mathbf{x},\frac{1}{f}]_{p-1}$. Let D_f denote the subspace of $L[\mathbf{x},\frac{1}{f}]$ of rational functions $\sum_i \partial_i A_i$ for some A_i in $L[\mathbf{x},\frac{1}{f}]_{-n}$. A major character of this study is the quotient space $L[\mathbf{x},\frac{1}{f}]_{-n-1}/D_f$, denoted H_f^{pr}.

The reduced form of F in $L[\mathbf{x},\frac{1}{f}]_{-n-1}$ is denoted $[F]$. It is by definition the output of the algorithm $\textsc{Reduce}$. It depends on a choice of a Gröbner basis of $\operatorname{Jac} f$, but its vanishing does not, see Theorem 1.

The choice of the space $L[\mathbf{x},\frac{1}{f}]_{-n-1}$ and the degree $-n-1$ may seem arbitrary. It is motivated by it being isomorphic to the space of regular differential n-forms on $\mathbb{P}^n\setminus V(f)$. The evaluation of x_0 to 1 is the restriction map to $\mathbb{A}^n\setminus V(f)$. The space H_f^{pr} is the nth de Rham cohomology space of the algebraic variety $\mathbb{P}^n\setminus V(f)$ over $k(t)$.

THEOREM 1 (GRIFFITHS [12, §4]). *If f satisfies Hypothesis (H), then for all F in $L[\mathbf{x},\frac{1}{f}]_{-n-1}$, the reduced form $[F]$ vanishes if and only if F is in D_f.*

Theorem 1 gives access to the dimension of H_f^{pr}. Let A be the finite dimensional vector space $L[\mathbf{x}]/\operatorname{Jac} f$. For a positive integer ℓ, let A_ℓ denote the linear subspace of A generated by homogeneous polynomials of degree $\ell d-(n+1)$. Let B denote $\oplus_\ell A_\ell$. Finally, for $\ell>0$ let $(g_{\ell,i})_{1\leqslant i\leqslant n_\ell}$ be a basis of A_ℓ, with $n_\ell=\dim_L A_\ell$.

PROPOSITION 2. *Under Hypothesis (H), the family of rational functions $\left(g_{\ell,i}/f^\ell\right)_{0<\ell,i\leqslant n_\ell}$ induces a basis of H_f^{pr}.*

PROOF. Suppose there exists a linear dependency relation between the $g_{\ell i}/f^\ell$ modulo D_f, that is $\sum_{\ell,i} u_{\ell,i}g_{\ell,i}/f^\ell$, denoted F, lies in D_f for some elements $u_{\ell,i}$ of L, not all zero. Let ℓ_0 be the maximum ℓ such that $u_{\ell i}$ is not zero for at least one i. By Theorem 1, $[F]=0$ so that $\sum_{\ell,i}u_{\ell,i}g_{\ell,i}f^{\ell_0-\ell}$, the numerator of F, lies in $\operatorname{Jac} f$. Since f itself is in $\operatorname{Jac} f$, the sum $\sum_i u_{\ell_0,i}g_{\ell_0,i}$ is in $\operatorname{Jac} f$, which contradicts the fact that the $g_{\ell_0,i}$ are a basis of A_{ℓ_0}. Thus the $g_{\ell,i}/f^\ell$ form a free family.

To prove that this family generates H_f^{pr}, we first notice that the family of all the fractions $[F]$, for F in $L[\mathbf{x},\frac{1}{f}]_{-n-1}$, generates H_f^{pr} since $[F]$ equals F modulo D_f. Now we assume for a moment that each $g_{\ell,i}$ is reduced with respect to a Gröbner basis G of $\operatorname{Jac} f$. Then each polynomial of $L[\mathbf{x}]$ of degree $\ell d-n-1$ which is reduced with respect to G is a linear combination of the $g_{\ell,i}$. Thus for all $F=a/f^\ell$ in $L[\mathbf{x},\frac{1}{f}]_{-n-1}$, the reduction $[F]$ is in the span of all the $g_{\ell,i}/f^\ell$. This makes the $g_{\ell,i}/f^\ell$ a system of generators of H_f^{pr} and by the previous paragraph a basis of it. Thus H_f^{pr} has the same dimension as B and any free family of H_f^{pr} of cardinal $\dim_L B$ is a basis of H_f^{pr}. In particular, the $g_{\ell,i}/f^\ell$ form a basis even if the $g_{\ell,i}$ are not reduced with respect to G. $\square$

COROLLARY 3. *Under Hypothesis (H), H_f^{pr} has dimension*

$$\frac{1}{d}\left((d-1)^{n+1}+(-1)^{n+1}(d-1)\right) \quad \left(\leqslant d^n\right).$$

PROOF. It has the dimension of B, see [19, thm. 8.4] for its computation. The inequality is clear. $\square$

4. CREATIVE TELESCOPING

We now introduce an algorithm, based on the Griffiths–Dwork reduction, that computes a telescoper of a rational function under Hypothesis (H).

In Eq. (1), the telescoper T is said to have a *regular certificate* if the irreducible factors of the denominators of the A_i's, as rational function over $k(t)$, divide the denominator of F; in other words, the A_i's have no pole outside those of F, over $k(t)$. Algorithm 2, described in §4.2, returns the telescoper of minimal order having regular certificate. For the application of creative telescoping to integration, this class of telescoper is more interesting than the general one; that is the object of §4.1.

4.1 Telescopers with regular certificate

Back to the affine case, let $F(t,x_1,\dots,x_n)$ be a rational function over $\mathbb{C}$ and γ be a n-cycle in $\mathbb{C}^n$ over which F has no pole for a generic t in $\mathbb{C}$. A common use of creative telescoping is the computation of a differential equation satisfied by the one-parameter integral $I(t)=\int_\gamma F\mathrm{d}\mathbf{x}$. As mentioned in the introduction, it is not always possible to deduce from the telescoping equation (1) that $T(I)$ vanishes. It may happen that the polar locus of the certificate meets γ for all $t\in\mathbb{C}$, and so $\int_\gamma \partial_i A_i \mathrm{d}\mathbf{x}$ may not be zero. An example of this phenomenon is given by Picard [23] for a bivariate algebraic function and translated here into a rational example, using the method in [4, Lemma 4]:

$$\frac{x-y}{z^2-P_t(x)P_t(y)} = \partial_x\frac{2P_t(x)}{(x-y)\left(z^2-P_t(x)P_t(y)\right)}+ \\ \partial_y\frac{2P_t(y)}{(x-y)\left(z^2-P_t(x)P_t(y)\right)}+\partial_z\frac{3\left(x^2+y^2\right)z}{(x-y)\left(z^2-P_t(x)P_t(y)\right)}, \tag{3}$$

where $P_t(u)=u^3+t$. Note the factor $x-y$ in the denominator of the certificate. Let F denote the left-hand side.

Input $F = a/f^\ell$ a rational function in $L[\mathbf{x}, \frac{1}{f}]_{-n-1}$, with f satisfying (H)
Output $T(t, \partial_t)$ an operator such that $T(F) = \sum_i \partial_i A_i$ for some rational functions A_i

```
procedure TELESC(F)
    G_0 ← REDUCE(F)
    i ← 0
    loop
        if rank_L(G_0, ..., G_i) < i + 1 then
            solve Σ_{k=0}^{i-1} a_k G_k = G_i w.r.t. a_0, ..., a_{i-1} in L
            return ∂_t^i − Σ_k a_k ∂_t^k
        else
            G_{i+1} ← REDUCE(∂_t G_i)
            i ← i + 1
```

Algorithm 2. Creative telescoping, regular case

In this case, the operator 1 is a telescoper of F, however there exists a 3-cycle γ on which F has no pole and such that $\int_\gamma F \mathrm{d}\mathbf{x}$ is not zero. It is impossible to find a regular certificate for the telescoper 1.

Nevertheless, a differential equation for $I(t)$ can be obtained in two ways. First, one can carefully study the integral $\sum_i \int_\gamma \partial_i A_i \mathrm{d}\mathbf{x}$ and compute a differential equation for it. Usually this includes the analysis of the poles of the A_i's, and the search of a telescoper for some rational function with one variable less. The second way is to find a telescoper for F such that the certificate does not contains new poles, a *telescoper with regular certificate*. Contrary to the telescoper (3), the operator ∂_t is a telescoper with regular certificate:

$$\partial_t F = \partial_x \left(\tfrac{x}{3t} F\right) + \partial_y \left(\tfrac{y}{3t} F\right) + \partial_z \left(\tfrac{z}{t} F\right).$$

This proves that $\partial_t I = 0$. More generally we have:

PROPOSITION 4. *If $T \in \mathbb{C}(t)\langle\partial_t\rangle$ is a telescoper of F with regular certificate, then $T(I)$ is zero.*

In this case, the certificate itself is not needed to prove the conclusion, its existence and regularity are sufficient. The Griffiths–Dwork method always produces a telescoper with regular certificate, see Eq. (2).

4.2 Algorithm

In this section L is $k(t)$ for some field k and f is a homogeneous polynomial over L of degree d satisfying Hypothesis (H). For F a rational function in $L[\mathbf{x}, \frac{1}{f}]_{-n-1}$ we want to find a nonzero operator T in $L\langle\partial_t\rangle$ such that $T(F)$ lies in D_f. Algorithm 2 describes the procedure TELESC that outputs such a telescoper.

PROPOSITION 5. *Algorithm 2 terminates and outputs the minimal telescoper with regular certificate of F.*

PROOF. The sequence (G_k) is defined by $G_0 = [F]$ and the recurrence relation $G_{k+1} = [\partial_t G_k]$. We show by induction that for all k the fraction G_k equals $[\partial_t^k F]$. It is clear for $k = 0$. Assume that G_k equals $[\partial_t^k F]$. By the soundness of the reduction the operator $G_k - \partial_t^k F$ is in D_f. And then so is $\partial_t G_k - \partial_t^{k+1} F$ since ∂_t commutes with the ∂_i's. By Theorem 1 and linearity, this implies that $[\partial_t G_k]$ equals $[\partial_t^{k+1} F]$.

At the ith step of the loop the algorithm is looking for a linear relation between $[F], \dots, [\partial_t^i F]$. By Theorem 1, there is one if and only if there is a telescoper with regular certificate of order i. If there such a relation, it computes it and returns the corresponding telescoper. By the confinement property of Proposition 2, the algorithm terminates. By construction the telescoper admits a regular certificate, see Identity (2). $\square$

5. EFFECTIVE BOUNDS FOR CREATIVE TELESCOPING

We now review the steps of the algorithm with the aim of bounding the degrees and orders of all polynomials and operators that are constructed. This is then used in the next section to assess the complexity of this approach.

For the needs of Section 7, we track the degrees not only with respect to the parameter t but also to another free variable ε of the base field. In other words, we assume that L is $k(t, \varepsilon)$. For p a polynomial in $k[t, \varepsilon]$, the bidegree $(\deg_t p, \deg_\varepsilon p)$ of p is denoted $\delta(p)$. If $p = \sum_I p_I \mathbf{x}^I$ is a polynomial in t, ε and $\mathbf{x}$, then $\delta(p)$ denotes the supremum of the $\delta(p_I)$'s, component by component.

THEOREM 6. *Let $f \in L[x]$ be homogeneous of degree d satisfying (H). Let a/f^ℓ in $L[\mathbf{x}, \frac{1}{f}]_{-n-1}$ be a rational function, with a polynomial in t and ε. The minimal telescoper with regular certificate of a/f^ℓ has order at most d^n and degree*

$$\mathcal{O}\left(d^n \delta(a) + \left(\ell d^{2n} + d^{3n}\right) e^n \delta(f)\right),$$

uniformly in all the parameters.

The first part of the theorem is a direct consequence of the confinement property of Corollary 3. We now study more precisely the decomposition used in Algorithm 1 in order to control the degree of the certificate and complete the proof.

The notation $a(\mathbf{n}) = \mathcal{O}(b(\mathbf{n}))$, for a tuple $\mathbf{n}$, means that there exists a $C > 0$ such that for all $\mathbf{n} \geqslant 1$, with at most a finite number of exceptions, we have $a(\mathbf{n}) \leqslant Cb(\mathbf{n})$. The notation $a(\mathbf{n}) = \tilde{\mathcal{O}}(b(\mathbf{n}))$ means that $a(\mathbf{n}) = \mathcal{O}(b(\mathbf{n}) \log^k b(\mathbf{n}))$ for some integer k. We emphasize that when there are several parameters in a $\mathcal{O}$, the constant is uniform in all the parameters and there is at most a finite number of exceptions.

5.1 Reduction modulo the Jacobian ideal

An important ingredient of the Griffiths–Dwork reduction is the computation of a decomposition $r + \sum_i u_i \partial_i f$ of a homogeneous polynomial a. This can be done by means of a Gröbner basis of $\operatorname{Jac} f$, but instead of following the steps of a Gröbner basis algorithm, we cast the computation into a linear algebra framework using Macaulay's matrices, for which Cramer's rule and Hadamard's bound can then be used. While not strictly equivalent, both methods ensure that r depends linearly on a and vanishes when a is in $\operatorname{Jac} f$.

For a positive integer q, let φ_q denote the linear map

$$\varphi_q : (u_i) \in L[\mathbf{x}]_{q-d-n}^{n+1} \longrightarrow \sum u_i \partial_i f \in L[\mathbf{x}]_{q-n-1}.$$

Let $\operatorname{Mat} \varphi_q$ be the matrix of φ_q in a monomial basis. It has dimension $R_q \times C_q$, where R_q denotes $\binom{q-1}{n}$ and C_q denotes $(n+1)\binom{q-d}{n}$, and we note for future use that $C_q \leqslant R_q$

for all positive integers n and $d > 2$. Up to a change of ordering of the bases of the domain and codomain, $\operatorname{Mat}\varphi_q$ has the form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A is a square submatrix of maximal rank. Note that D is necessarily $CA^{-1}B$. Then, the endomorphism ψ_q defined by the matrix $\left(\begin{smallmatrix} A^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ satisfies $\varphi_q\psi_q\varphi_q = \varphi_q$; it is called a *split* of φ_q. It depends on the choice of the maximal rank minor. The map $\mathrm{id} - \varphi_q\psi_q$, denoted π_q, performs the reduction in degree $q-n-1$: it is idempotent; if a of degree $q-n-1$ is in $\operatorname{Jac} f$ then it equals $\varphi_q(b)$ for some b and thus $\pi_q(a)$ vanishes; and for all a in $L[\mathbf{x}]_{q-n-1}$ it gives a decomposition

$$a = \pi_q(a) + \sum_i \psi_q(a)_i \partial_i f.$$

Under Hypothesis (H), the map φ_q is surjective when q is at least $(n+1)d - n$. Let D denote this bound, known as *Macaulay's bound* [18, chap. 1; 16, corollaire, p. 169].

For q larger than D, a split of φ_q can be obtained from a split ψ_D of φ_D in the following way. Let S be the set of monomials in $\mathbf{x}$ of total degree $q-D$. Choose a linear map μ from $L[\mathbf{x}]_{q-n-1}$ to $L[\mathbf{x}]^S_{D-n-1}$ such that each a in $L[\mathbf{x}]_{q-n-1}$ equals $\sum_{m\in S} m\mu_m(a)$. Then a split of φ_d is defined by

$$\psi_d(a) = \sum_{m\in S} m\psi_D(\mu_m(a)).$$

Let q be a positive integer and let E_q be the least common multiple of the denominators of the entries of $\operatorname{Mat}\psi_q$. The entries of $\operatorname{Mat}\psi_q$ and $\operatorname{Mat}\pi_q$ are rational functions of the form p/E_q, with p polynomial. Let δ_E denote the supremum of all $\delta(p)$ and all $\delta(E_q)$, for $q \in \mathbb{N}\setminus\{0\}$.

Proposition 7. *The supremum δ_E is finite and bounded above by $e^n d^n \delta(f)$. Moreover, if $q > D$ then E_q equals E_D.*

Proof. Assume first that $q > D$. In this case, the entries of $\operatorname{Mat}\psi_q$ are entries of $\operatorname{Mat}\psi_D$ and π_q is zero. Thus the inequalities will follow from the case where $q \leqslant D$. Let $\operatorname{Mat}\psi_q$ and $\operatorname{Mat}\pi_q$ be written respectively as N/E_q and P/E_q with N and P polynomial matrices. Let r be the rank of ψ_q. The maximal rank minor A in the construction of ψ_q has dimension r. Cramer's rule and Hadamard's bound ensure that $\delta(N)$ is at most $(r-1)\delta(f)$ and that $\delta(E_q)$ is at most $r\delta(f)$. Since P equals $E_q\,\mathrm{id} - (\operatorname{Mat}\varphi_q)N$ and $\delta(\operatorname{Mat}\varphi_d)$ equals $\delta(f)$, the degree $\delta(P)$ is also at most $r\delta(f)$.

Next, r is bounded by R_q, the row dimension of $\operatorname{Mat}\phi_d$. Since $q \leqslant D$, we have $R_q \leqslant R_D$ and we conclude using the following inequality, with $p \geqslant n$ an integer:

$$\binom{p}{n} \leqslant \left(\frac{p\,e}{n+1}\right)^n. \quad \square$$

Algorithm 3 is a slightly modified version of Algorithm 1 which uses the construction above. Its output is in general not equal to the output of the former version, for any monomial order, but of course it satisfies Theorem 1. In particular the output of the algorithm Telesc does not depend on the reduction method in Reduce. From now on the brackets $[\cdot]$ denote the output of Algorithm 3.

5.2 Degree bounds for the reduction

Proposition 8. *Let $a/f^\ell \in L[\mathbf{x}, \frac{1}{f}]_{-n-1}$, with a a polynomial in t and ε. Then*

$$\left[\frac{a}{f^\ell}\right] = \frac{1}{P_\ell}\sum_{k=1}^{n} \frac{b_k}{f^k},$$

where $P_\ell = \prod_{i=1}^{\ell} E_{id}$ and b_k in $L[\mathbf{x}]_{kd-n-1}$ is a polynomial in t and ε such that $\delta(b_k) \leqslant \delta(a) + \ell\delta_E$, for $1 \leqslant k \leqslant n$.

Input $F = a/f^\ell$ a rational function in $x_0, \dots, x_n$, with f of degree d
Output $[F]$ such that there exist $A_0, \dots, A_n$ rational functions such that $F = [F] + \sum_i \partial_i A_i$

For all $1 \leqslant i \leqslant \ell$, precompute a split ψ_{id} of φ_{id} (§5.1)
procedure Reduce(a/f^ℓ)
  if $\ell = 1$ **then return** a/f^ℓ
  $F_1 \leftarrow \dfrac{1}{\ell-1}\sum_i \dfrac{\partial_i \psi_{\ell d}(a)_i}{f^{\ell-1}}$
  return $\frac{\pi_{\ell d}(a)}{f^\ell} + $ Reduce(F_1)

Algorithm 3. Griffiths–Dwork reduction, linear algebra variant

Proof. Using Algorithm 3, we obtain

$$\left[\frac{a}{f^\ell}\right] = \frac{p}{E_{\ell d} f^\ell} + \frac{1}{E_{\ell d}}\left[\frac{g}{f^{\ell-1}}\right],$$

where g and p are polynomials in $\mathbf{x}$, t and ε, with $\delta(p)$ and $\delta(g)$ at most $\delta(a) + \delta_E$. Induction over ℓ yields

$$\left[\frac{a}{f^\ell}\right] = \sum_{k=1}^{\ell} \frac{p_k}{f^k \prod_{j=k}^{\ell} E_{jd}}$$

with p_k polynomials such that $\delta(p_k) \leqslant \delta(a) + (\ell - k + 1)\delta_E$. For $k > n$, and hence $kd > D$, the map π_{kd} is 0 and thus so is p_k. Thus

$$\left[\frac{a}{f^\ell}\right] = \frac{1}{\prod_{j=1}^{\ell} E_{jd}} \sum_{k=1}^{\min(\ell,n)} \frac{p_k \prod_{j=1}^{k-1} E_{jd}}{f^k}. \quad \square$$

This proposition applied to $\partial_t^i(a/f^\ell)$ asserts that

$$\left[\partial_t^i \frac{a}{f^\ell}\right] = \frac{1}{P_{\ell+i}}\sum_{k=1}^{n} \frac{b_{i,k}}{f^k} = \frac{1}{P_{\ell+i}} \frac{b'_i}{f^n} \tag{4}$$

for some polynomials $b_{i,k}$ and b'_i such that

$$\delta(b_{i,k}) \leqslant \delta(a) + i\delta(f) + (i+\ell)\delta_E, \tag{5}$$

$$\text{and} \quad \delta(b'_i) \leqslant \delta(a) + (i+n)\delta(f) + (i+\ell)\delta_E. \tag{6}$$

5.3 Degree bounds for the telescoper

Proposition 9. *Let $T = \sum_{i=0}^{r} c_i \partial_t^i$, with coefficients c_i in $k[t,\varepsilon]$, be the minimal telescoper with regular certificate of a/f^ℓ. Then*

$$\delta(c_i) \leqslant r\delta(a) + \left(r^2 + r\ell\right) e^n d^n \delta(f).$$

Proof. The operator T is the output of Telesc(a/f^ℓ). The rational functions c_i/c_r form the unique solution to the following system of inhomogeneous linear equations over L, with the Y_i's as unknown variables:

$$\sum_{i=0}^{r-1}\left[\partial_t^i \frac{a}{f^\ell}\right] Y_i = -\left[\partial_t^r \frac{a}{f^\ell}\right].$$

We write each $b_{i,k}$ in (4) as $\sum_{m\in S} b_{i,k,m} m$, where S is the set of all monomials in the variables $\mathbf{x}$ of degree at most $nd-n-1$. The previous system rewrites as

$$\forall m \in S, \forall k \in \{1,\dots,n\}\,, \quad \sum_{i=0}^{r-1} Y_i \frac{b_{i,k,m}}{P_{\ell+i}} = -\frac{b_{r,k,m}}{P_{\ell+r}}$$

There is a set I of r indices $\{(k_0,m_0),\dots\}$ such that the square system formed by the corresponding equations admits a unique solution. We apply Cramer's rule to this system. Let B be the square matrix $(b_{i,k_j,m_j})_{i,j}$, for i and j between 0 and $r-1$. Let B_i be the matrix obtained by replacing the row number i of B by the vector $(b_{r,k_j,m_j})_j$. We get, after simplification of the factors $P_{\ell+*}$ by multilinearity of the determinant,

$$\frac{c_i}{c_r} = \frac{\frac{P_{\ell+i}}{P_\ell}\det B_i}{\frac{P_{\ell+r}}{P_\ell}\det B}. \tag{7}$$

Consequently, for all i, the polynomial c_i divides $\frac{P_{\ell+i}}{P_\ell}\det B_i$ and thus

$$\delta(c_i) \leqslant i\delta_E + \sum_{j=0, j\neq i}^{r} \delta(b_j).$$

With the previous bound (5) on $\delta(b_i)$ we get

$$\delta(c_i) \leqslant r\delta(a) + \frac{r(r+1)}{2}(\delta(f)+\delta_E) + r\ell\delta_E,$$

which gives the result with Proposition 7. ☐

6. COMPLEXITY

We assume that L is the field $k(t)$ and we evaluate the algebraic complexity of the steps of REDUCE and TELESC in terms of number of arithmetic operations in k. All the algorithms are deterministic. For univariate polynomial computations, we use the quasi-optimal algorithms in [10]. For simplicity, we assume that $d > 2$ so that several simplifications occur in the inequalities since $C_q \leqslant R_q$ and $d > e \approx 2.72$.

6.1 Primitives of linear algebra

The complexity of Algorithm 2 lies in operations on matrices with polynomial coefficients. Let $A \in k[t]^{n\times m}$ have rank r and coefficients of degree at most d. One can compute r, a basis of $\ker A$ and a maximal rank minor in $\tilde{\mathcal{O}}(nmr^{\omega-2}d)$ operarions in k [30]. A maximal rank minor can be inverted in complexity $\tilde{\mathcal{O}}(r^3 d)$ [13]. In particular, a matrix B such that $ABA = A$ can be computed in $\tilde{\mathcal{O}}(nmr^{\omega-2}d + r^3 d)$ operations in k, or $\tilde{\mathcal{O}}(n^2 md)$, using $r \leqslant n, m$ and $\omega \leqslant 3$.

We implicitly represent a matrix A whose entries are univariate rational functions as the least common denominator g of the entries, together with the polynomial matrix gA.

6.2 Precomputation

Algorithm 2 needs the splits ψ_{id} for i from 1 to the larger of $n+1$ and ℓ. Following §5.1, it is enough to compute ψ_{id} for i between 1 and $n+1$, each for a cost of $\tilde{\mathcal{O}}(R_{id}C_{id}^2\delta(f))$ operations in k, and then ψ_{id} can be obtained with no further arithmetic operation for $i > n+1$. Thus the precomputation needs $\tilde{\mathcal{O}}\left(e^{3n}d^{3n}\delta(f)\right)$ operations in k.

6.3 Reduction

Let $\rho(\ell,\delta(a))$ be the complexity of the linear algebra based variant of the algorithm REDUCE with input a rational function a/f^ℓ. The procedure first computes $\psi_{\ell d}(a)$. Since $\psi_{\ell d}$ is precomputed, it is only the product of a matrix of dimensions $C_{\ell d}$ by $R_{\ell d}$ with the vector of coefficients of a in a monomial basis. The elements of the matrix have degree at most δ_E and the elements of the vector have degree at most $\delta(a)$. Thus the product has complexity $\tilde{\mathcal{O}}(R_{\ell d}C_{\ell d}(\delta(a)+\delta_E))$. Secondly, the procedure computes r as $\pi_{\ell d}(a)$ knowing $\psi_{\ell d}(a)$; this has the same complexity. Thirdly, it computes F_1, computation whose complexity is dominated by the first step. And lastly it computes REDUCE(F_1), which has complexity bounded by $\rho(\ell-1, \delta(a)+\delta_E)$. Unrolling the recurrence leads to

$$\rho(\ell,\delta(a)) = \tilde{\mathcal{O}}\left(\ell\left(\frac{ed\ell}{n+1}\right)^{2n}(\delta(a)+\ell\delta_E)\right).$$

6.4 Main loop

The computation of G_0 has complexity $\rho(\ell,\delta(a))$. Next, G_i has shape given by (4), and is differentiated before being reduced, so that the cost of the computation of G_{i+1} is at most $\rho(n+1, \delta(a)+(i+2n)\delta(f)+(i+\ell)\delta_E)$. Summing up, the computation of $G_0,\dots,G_r$ has a complexity

$$\rho(\ell,\delta(a)) + \tilde{\mathcal{O}}\left((ed)^{2n} r\left(\delta(a)+r\delta(f)+(r+\ell)\delta_E\right)\right). \tag{8}$$

During the ith step, the procedure computes the rank of $i+1$ vectors with $\mathcal{O}(e^n d^n)$ coefficients of degree $\delta(b_i')$ and computes a linear dependence relation if there is one. This is done in complexity $\tilde{\mathcal{O}}\left(i^{\omega-1}e^n d^n \delta(b_i')\right)$. This step is quite expensive and doing it for all i up to r would ruin the complexity. It is sufficient to perform this computation only when i is a power of 2 so that the maximal i which is used is smaller than $2r$. When the rank of the family is not full, we deduce from it the exact order r and perform the computation in that order. Indeed, the rank over L of $G_0,\dots,G_i$ is the least of r and i. This way, finding the rank and solving has cost $\tilde{\mathcal{O}}(r^{\omega-1}e^n d^n(\delta(b_r')+\delta(b_{2r}')))$. In view of (6) and since $r \leqslant d^n$ and $\omega \leqslant 3$, the complexity of that step is bounded by (8). Adding the cost of the precomputation and using the bounds of the previous section leads to the following complexity estimate.

THEOREM 10. *Under Hypothesis (H) and assuming that $d > 2$, Algorithm* TELESC *run with input a/f^ℓ takes*

$$\tilde{\mathcal{O}}\left(\left(d^{5n} + d^{4n}\ell + d^{3n}\ell^2\left(\tfrac{\ell}{n}\right)^{2n}\right)e^{3n}\delta\right)$$

arithmetic operations in k, where δ is the larger of $\delta(a)$ and $\delta(f)$, uniformly in all the parameters. Asymptotically with ℓ and n fixed, this is $\tilde{\mathcal{O}}\left(d^{5n}\delta\right)$.

Note that while this may seem a huge complexity, it is not so bad when compared to the size of the output, which seems to be, empirically, comparable to $d^{3n}\delta$, with n fixed and $\ell = 1$. Note also that for $n = 1$, the complexity improves over that of the algorithm based on Hermite's reduction studied in [3], thanks to our avoiding too many rank computations.

7. AFFINE SINGULAR CASE

Let L denote the field $k(t)$. Let F_{aff} be a rational function in $L(x_1,\dots,x_n)$, written as a/f_{aff}. We do not assume that F_{aff} is homogeneous, nor that f_{aff} satisfies a regularity property. Let d_{aff} be the total degree of f_{aff} w.r.t. $\mathbf{x}$.

degree of f	3	4	5	6
order of telesc.	2	6	12	20
degree of telesc. $\delta = 1$	32 *(68)* 0.4s	153 *(891)* 46s	480 *(5598)* 2h	1175 *(23180)* 150h
— , $\delta = 2$	66 *(136)* 0.6s	336 *(1782)* 140s	1092 *(11196)* 7h	? *(46360)* ∅
— , $\delta = 3$	100 *(204)* 0.9s	519 *(2673)* 270s	1704 *(16794)* 13h	? *(69540)* ∅

Table 1. Empirical order and degree of the minimal telescoper with regular certificate of a random rational function a/f^2 in $\mathbb{Q}(t, x_0, x_1, x_2)$, with f and a homogeneous in $\mathbf{x}$ satisfying $\deg_{\mathbf{x}} a + 3 = 2 \deg_{\mathbf{x}} f$ and $\delta(a)$ and $\delta(f)$ equal to δ; together with a proved upper bound (with a version without simplification of Theorem 9) and mean computation time (CPU time).

In this section we show a deformation technique that regularizes singular cases. In particular, it makes it possible to transfer the previous results to the general case and obtain the following bounds.

Theorem 11. *The function F_{aff} admits a telescoper, with regular certificate, of order at most d^n and degree*

$$\mathcal{O}\left(d_{\text{pr}}^n \delta(a) + d_{\text{pr}}^{3n} e^n \delta(f_{\text{aff}})\right),$$

where d_{pr} is $\max(d_{\text{aff}}, \deg_{\mathbf{x}} a + n + 1)$. This telescoper can be computed in complexity $\tilde{\mathcal{O}}\left(e^{3n} d_{\text{pr}}^{8n} \delta\right)$, with δ the larger of $\delta(a)$ and $\delta(f_{\text{aff}})$.

The algorithm is again based on linear algebra. It is not difficult to see that the *bit* complexity too is polynomial in d_{pr}^n.

7.1 Homogenization and deformation

The regularization proceeds in two steps. First, let F_{pr} be the homogenization of F_{aff} in degree $-n-1$, that is

$$F_{\text{pr}} = x_0^{-n-1} F_{\text{aff}}\left(\tfrac{x_1}{x_0}, \dots, \tfrac{x_n}{x_0}\right),$$

which we write b/f_{pr} for some homogeneous polynomials b and f_{pr}. Let d_{pr} denote the degree of f_{pr}; it is given by Theorem 11. The degrees of b and f_{pr} satisfy the hypothesis of Theorem 6, by construction, but in general f_{pr} does not satisfy Hypothesis (H). (Although it does generically, as long as d_{pr} equals d_{aff}.) We consider a new indeterminate ε, the polynomial f_{reg} defined by

$$f_{\text{reg}} = f_{\text{pr}} + \varepsilon \sum_{i=0}^{n} x_i^d,$$

and the rational function F_{reg} defined by b/f_{reg}.

Lemma 12. *The polynomial f_{reg} satisfies Hypothesis (H) over $L(\varepsilon)$, that is $L(\varepsilon)[\mathbf{x}]/\operatorname{Jac} f_{\text{reg}}$ has finite dimension.*

Proof. It is true for $\varepsilon = \infty$, so it is generically true. □

Now, Theorem 6 gives bounds on the order and degree of a telescoper of F_{reg}, which is in $L(\varepsilon)[\mathbf{x}, \frac{1}{f_{\text{reg}}}]_{-n-1}$. The end of the proof of Theorem 11 follows from the following.

Proposition 13. *If T in $L[\varepsilon]\langle\partial_t\rangle$ is a telescoper of F_{reg} with regular certificate, then so is $T_{|\varepsilon=0}$ for F_{aff}.*

Proof. By assumption, $T(F_{\text{reg}})$ equals $\sum_{i=0}^n \partial_i g_i / f_{\text{reg}}^p$ for some integer p and polynomials g_i in $L(\varepsilon)[\mathbf{x}]$. Each g_i/f_{reg}^p can be expanded in Laurent series in ε as $\sum_{j \geqslant N} h_{ij} \varepsilon^j$ for some possibly negative integer N and rational functions h_{ij} in $L[\mathbf{x}, \frac{1}{f_{\text{pr}}}]_{-n}$. Similarly, we can write the operator $T(F_{\text{reg}})$ as $T_{|\varepsilon=0}(F_{\text{pr}}) + \varepsilon \sum_{j \geqslant 0} b_j \varepsilon^j$ for some rational functions b_j in $L[\mathbf{x}, \frac{1}{f_{\text{pr}}}]$. Since the derivations ∂_i commute with ε, it is clear that $T_{|\varepsilon=0}(F_{\text{pr}})$ equals $\sum_{i=0}^n \partial_i h_{i0}$. Next, in this equality, x_0 can be evaluated to 1 to give

$$T_{|\varepsilon=0}(F_{\text{aff}}) = (\partial_0 h_{00})|_{x_0=1} + \sum_{i=1}^{n} \partial_i (h_{i0}|_{x_0=1}).$$

Euler's relation for h_{00} gives (with the index 00 dropped)

$$(\partial_0 h)|_{x_0=1} = -\sum_{i=1}^{n} \partial_i (x_i h|_{x_0=1}),$$

proving that $(\partial_0 h)|_{x_0=1}$ is in $D_{f_{\text{aff}}}$. Thus, so is $T_{|\varepsilon=0}(F_{\text{aff}})$ and the proof is complete. □

Nevertheless, a telescoper obtained this way does not need to be minimal, even starting from a minimal one for the perturbed function F_{reg}. This is unfortunate because in presence of singularities the dimension of H_f^{pr} can collapse when compared to the generic order given by Corollary 3. Note also that in the proof of the previous Proposition, the integer N can be bounded in $d^{O(n)}$ which implies a bound on the size of the certificate in $d^{O(n^2)}$ similar to Lipshitz's [17].

7.2 Algorithm and complexity

The algorithm is based on Proposition 13. We use an evaluation-interpolation scheme to control the complexity. Let the operator T in $k(t, \varepsilon)\langle\partial_t\rangle$ be the minimal telescoper of F_{reg}, written as $\partial_t^r + \sum_{k=0}^{r-1} \frac{c_k}{c_r} \partial_t^k$. It is the output of Telesc applied to F_{reg}. We aim at computing $(\varepsilon^\alpha T)_{|\varepsilon=0}$, where α is such that this evaluation is finite and not zero.

Proposition 13, slightly adapted, shows that $T_{|\varepsilon=u}$ is a telescoper with regular certificate of $F_{\text{reg}}^{|\varepsilon=u}$ whenever $c_r(t,u)$ is not zero, even if $f_{\text{reg}}^{|\varepsilon=u}$ does not satisfy (H). When it does, the specialization gives the minimal one:

Lemma 14. *If $f_{\text{reg}}^{|\varepsilon=u}$ satisfies hypothesis (H) and if u does not cancel c_r, then $T_{|\varepsilon=u}$ is the minimal telescoper with regular certificate of $F_{\text{reg}}^{|\varepsilon=u}$.*

Proof. We use the notation of Section 5, replacing f by f_{reg} and L by $L(\varepsilon)$. The operator T is the output of Algorithm 2 applied to F_{reg}. Since $f_{\text{reg}}^{|\varepsilon=u}$ satisfies (H), for all d the matrix $\operatorname{Mat}\varphi_d$, with coefficients in $L[\varepsilon]$, has the same rank as its specialization with $\varepsilon = u$ [18, §58]. Thus, to compute the splits ψ_d we can choose maximal rank minors of $\operatorname{Mat}\varphi_d$ that are also maximal rank minors of the specialization. When doing so, the reduction $[\cdot]$ commutes with the evaluation $\cdot|_{\varepsilon=u}$. In particular, the polynomials E_q do not vanish for $\varepsilon = u$.

In the proof of Prop. 9, Eq. (7) shows that c_r, the leading coefficient of T, divides $P_{\ell+r}\det B$. The polynomial $P_{\ell+r}$ is a product of several E_d's, in particular $P_{\ell+r}|_{\varepsilon=u}$ is not zero. Since $c_r|_{\varepsilon=u}\neq 0$, the determinant of $B|_{\varepsilon=u}$ is not zero either. Looking at the definition of B in the proof of Proposition 9, this implies that the $[\partial_t^i F_{\mathrm{reg}}]_{|\varepsilon=u}$, for i between 0 and $r-1$ are free over $L(\varepsilon)$. In particular, a telescoper with regular certificate of $F_{\mathrm{reg}}^{|\varepsilon=u}$ has order at least r. Since $T_{|\varepsilon=u}$ is a telescoper of order is r, it is the minimal one. $\square$

We now present the algorithm. Let N be $e^n(d^{3n}+d^{2n}+d^n)$. By Proposition 9, the polynomials c_k have degree at most N in ε, and at most $N\delta$ in t. Choose U a set of $4N+1$ elements of k. Determine U' the set of elements u of U such that $f_{\mathrm{reg}}^{|\varepsilon=u}$ satisfies (H). This step has complexity $\tilde{\mathcal{O}}\left((ed)^{n\omega}\delta|U|\right)$: The polynomial $f_{\mathrm{reg}}^{|\varepsilon=u}$ satisfies (H) if and only if $(\operatorname{Mat}\varphi_D)|_{\varepsilon=u}$ is full rank. In particular, if $f_{\mathrm{reg}}^{|\varepsilon=u}$ does not satisfy (H), then $E_D|_{\varepsilon=u}$ vanishes. The polynomial E_D has degree at most $e^n d^n$ in ε, by Proposition 7, so $U\setminus U'$ has at most $e^n d^n$ elements. For each u in U', compute $\textsc{Telesc}(f_{\mathrm{reg}}^{|\varepsilon=u})$ with leading coefficient normalized to 1, denoted T_u. This step has complexity $\tilde{\mathcal{O}}(d^{5n}e^{3n}\delta|U'|)$, by Theorem 10. Determine the subset U'' of U' where the order of T_u is maximal. By Lemma 14, the complement $U'\setminus U''$ is formed by u such that $c_r(t,u)=0$. It has at most N elements since c_r has degree at most N in ε. For all u in U'' the operators T_u and $T_{|\varepsilon=u}$ coincide. Thus U'' has at most $2N+1$ elements.

The r rational functions $\frac{c_k(t,0)}{c_r(t,0)}$ can be computed using Lemma 15 in total complexity $\tilde{\mathcal{O}}(N^2 r\delta)$. If $c_r(t,0)$ is zero, we look for the positive integer α such that the functions $\varepsilon^\alpha\frac{c_k(t,\varepsilon)}{c_r(t,\varepsilon)}$ are finite for $\varepsilon=0$ but not zero for at least one k. The integer α is at most N and thus can be found with a binary search, using at most $\log_2 N+1$ times Lemma 15.

Lemma 15. *Let R in $k(x,y)$ be written P/Q, with P and Q polynomials of degree less than d_x in x and d_y in y. Given evaluations $R(x,v)$, for $2d_y+1$ elements v of k, the function $R(x,0)$ (or ∞ if $Q(x,0)$ vanishes) can be computed using $\tilde{\mathcal{O}}(d_xd_y)$ arithmetic operations in k.*

Proof. Let V the set of evaluation points. Choose a set U of $2d_x+1$ points of k. Compute $R(u,v)$ for $u\in U$ and $v\in V$ in $\tilde{\mathcal{O}}(d_xd_y)$ operations. Note that there is no need to check that the elements of U are not poles of the $R(x,v)$: univariate rational reconstruction can handle that. Use univariate rational reconstruction to compute $R(u,y)$, for $u\in U$, in complexity $\tilde{\mathcal{O}}(d_y|U|)$ operations. Reconstruct $R(x,0)$ in complexity $\tilde{\mathcal{O}}(d_x)$ from the evaluations $R(u,0)$. $\square$

8. EXPERIMENTS

A basic implementation of the algorithm Telesc has been written in Maple 16. As it uses only Maple primitives to compute with polynomial matrices, it is certainly too basic to reflect the complexity given in Theorem 10.

Table 1 presents empirical results for some generic rational functions, with $n=2$. The bound on the order are generically exact as expected; however the bound on the degree are not very sharp. For $n=1$ and $\delta(a)$ fixed, a careful study [3] proves that the degree of the minimal telescoper is $\mathcal{O}(d^2\delta)$, which is smaller than the $\mathcal{O}(d^3\delta)$ given by Theorem 6. Analogy, as well as numerical evidence and theoretical clues, lead us to think that for general n, the asymptotic behavior can be improved from $\mathcal{O}(d^{3n}\delta)$ to $\mathcal{O}(d^{2n}\delta)$.

The relative cost of each step of Algorithm 2 in the computation of telescopers of Table 1, on the example of the telescoper of degree 12 and degree 1092 of a generic function a/f^2 as described in Table 1, that is computed in about 7 hours breaks down as follows: The computation of splits of Macaulay matrices takes about 1% of the time, the reduction steps about 40%, and the final solving about 60% of the time. More efficient matrix multiplication and system resolution over univariate polynomials could improve speed dramatically. We have not been able to compute more than the first column of Table 1 with methods and programs in [15,4].

On the other hand, the regularity hypothesis (H) is restrictive in applications: Even though generic polynomials satisfy this hypothesis, examples with physical or combinatorial meaning usually do not. The method shown in Section 7 is only of a theoretical interest. By contrast, the algorithm for the regular case is very efficient in practice.

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