Observavility Brunovsky Normal Form: Multi-Output Linear Dynamical Systems

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***Abstract*— This paper gives the sufficient and necessary conditions to guarantee the existence of a linear change of coordinates to transform a multi-output linear dynamical system (modulo a nonlinear term depending on inputs and outputs) in the observability Brunovsky canonical form.**

I. Introduction

For a single output dynamical linear system the observability rank condition is a necessary and sufficient condition to transform it into the Brunovsky observability normal form. In this last form, it is possible to use classical observer such that [8] observer and, [5] observer. For multi-input linear dynamical systems these condition were given by [3]. For multi-output linear dynamical systems, to our knowledge, there does not exist in literature an equivalent result. However, for multi-output nonlinear dynamical systems there are several works on the topic. All this works dealt with the so-called : Observer linearization error dynamic. This last notion was first addressed by [6] for dynamical systems with a single output. [7] gave, for a multi-output nonlinear system, the sufficient conditions to solve the observer linearization error problem even with a diffeomorphism on the outputs.
[11] gave the sufficient and necessary conditions to solve the linearization problem for multi-output nonlinear systems. After these two important works in the field of observers, other interesting research was followed, we can quote for examples: [1], [2], [4], [10].
Even if the results, obtained in the works quoted above, can be used for multi-output linear dynamical systems. In this paper, we will provide in an algebraic way the necessary and sufficient conditions which ensure existence of a linear change of coordinates to transform a multi-output linear system into the Brunovsky normal form. This paper is outlined as follows. In the next section, we give the notations and the problem statement. In section 3, we recall well-known facts about a matrix and deduce a first preliminary result. In section 4, we will state our main result. In section 5, we provide an algorithm to compute the linear change of coordinates.

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II. Notation and problem statement

Consider the following multi-output dynamical system:

$$\dot{x} = Ax + \gamma(y,u) \quad (1)$$
$$y = Cx \quad (2)$$

where:

- $x \in R^n$ are the state variables, $y = (y_i)_{1\leq i\leq m}$ are the outputs and u is an input,
- A is a $n \times n$ matrix and γ is a vector depending on input and outputs
- $C = \begin{bmatrix} C_1 \\ C_2 \\ .. \\ C_m \end{bmatrix}$

is a $m \times n$ matrix with linearly independent components.
Before, to state the problem to solve here, let us recall the so-called observability indices for dynamical systems (1-2). For this, without loss of generality, we will assume throughout this paper the following.

We assume that the pair (1-2) is observable thus, the following observability rank condition:

$$\text{rank} \begin{bmatrix} C \\ CA \\ .. \\ CA^{n-1} \end{bmatrix} = n$$

is fulfilled.
Now, let us define $s_0 = rank[C] = m$, and by induction:

$$\text{s}_k = rank \begin{bmatrix} C \\ CA \\ .. \\ CA^{k-1} \end{bmatrix} - s_{k-1}.$$

The following definition is drawn from [9].

Definition 1: We call observability indices of the dynamical system (1-2) the integers given for $1 \leq s \leq m$ by:
$\text{r}_i = card\{s_j \geq i \quad for \quad j \geq 0\}$.
By definition, we have $r_1 \geq r_2 \geq \ldots \geq r_m$ and from the observability rank condition II, we have obviously:
$\text{r}_1 + \ldots + r_m = n$.

By reordering the outputs $(y_i)_{1\leq i\leq m}$, we can assume that for $1 \leq i \leq m$ output y_i has index r_i. Thus we have:

- $$rank \begin{bmatrix} \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} \\ C_1A \\ .. \\ C_1A^{r_1-1} \\ C_2A \\ .. \\ C_2A^{r_2-1} \\ .. \\ C_iA \\ .. \\ C_iA^{r_i-1} \end{bmatrix} = r_1+r_2+\ldots+r_i+m-i, \quad (3)$$

- $$C_iA^{r_i} = \sum_{k=1}^{i}\sum_{j=1}^{r_k} p_{i,k,j}C_kA^{j-1} + \sum_{k=i+1}^{m} q_{i,k}C_k. \quad (4)$$

The problem that we will answer here can be stated as follows.

Problem 1: Find the necessary and sufficient algebraic conditions for the existence of a linear change of coordinates $z = Mx$ such that the dynamical system (1-2) becomes:

$$\begin{aligned} \dot{z}_i &= A_iz_i + \beta_i(u, z_{1,r_1}, \cdots, z_{1,r_m}) && (5)\\ z_{1,r_1} &= y_1 && (6)\\ z_{i,r_i} &= y_i + \sum_{j=1}^{i-1}\alpha_{i,j}y_j \quad for \quad 2 \leq i \leq m && (7) \end{aligned}$$

where $A_i = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}_{r_i \times r_i}$,
and $z_i = (z_{i,j})_{1 \leq j \leq r_i}$ for $1 \leq i \leq m$.

We will say that a dynamical system under (5-7) form is in the Brunovsky observability canonical normal form modulo an input-output injection.
In the (5-7) form , we can use the well-known Lunberger observer with nonlinear input and output terms:

$$\dot{\widehat{z}}_i = A_i\widehat{z}_i - K(\overline{y} - \widehat{\overline{y}}) + \beta_i(u, \widehat{z}_{1,r_1}, \cdots, \widehat{z}_{1,r_m}) \quad (8)$$

where for all $1 \leq i \leq m$ we set $\overline{y}_i = z_{i,r_i}$ and K is the gain.

Remark 1: If the case of system (1-2) without noise is perfectly resolved by 8 it is not the case for noisy systems For example, if the output measurement is corrupted by the noise and the output acts quadratically in term then a bias appears.

III. Mathematical Background

In this section, we recall some facts about the structure of a given $n \times n$ matrix $\Im$.
A matrix $\Im$ is said to be cyclic if there exists a vector $v \in R^n$ such that the following family:
$\{v, .., \Im^{n-1}v\}$
is a basis of R^n. This last fact is also equivalent to saying that the minimal polynomial of $\Im$ is equal to its characteristic polynomial.
Therefore, for a cyclic matrix $\Im$, if we denote its transpose by $A = \Im^T$ then, for A we need only to have a single output $y = Cx$ where $C = v^T$ such that the pair (A, C) fulfils the observability rank condition. Thus,

$$rank \begin{bmatrix} C \\ CA \\ .. \\ CA^{n-1} \end{bmatrix} = n. \quad (9)$$

In this case, the following linear change of coordinates transforms the dynamical system (1-2) into the form (5-7) $(m = 1)$:

$$\begin{aligned} z_n &= Cx && (10)\\ z_{n-k} &= CA^kx + \sum_{i=1}^{k} p_{n-i}CA^{k-i}x for 1 \leq k \leq n-1 && (11) \end{aligned}$$

where $(p_i)_{0 \leq i \leq n-1}$ are the coefficients of the characteristic polynomial of A :

$$P_A(s) = p_0 + p_1s + ... + p_{n-1}s^{n-1} + s^n \quad (12)$$

and the vector field $\beta(y, u) = (\beta_i(y, u))_{1 \leq i \leq n}$ is given by:

$$\begin{aligned} \beta_n &= -p_{n-1}y + C\gamma(y, u) \\ \beta_{n-k} &= -p_{n-(k+1)}y \\ &+ \sum_{i=1}^{k} p_{n-i}CA^{i-1}\gamma(y, u) + CA^k\gamma(y, u) for 1 \leq k \leq n-1 \end{aligned}$$

To finish, take the derivative of z_1 given in (11), we obtain:

$$\dot{z}_1 = CA^nx + \sum_{i=1}^{n-1} p_{n-i}CA^{n-i}x + \varphi(y, u). \quad (13)$$

Therefore, by Cayley-Hamilton's theorem, we deduce from (12) expression that:

$$\dot{z_1} = -p_0y + \varphi(y, u) = \beta_1(y, u).$$

Now, if a matrix $\Im$ is not cyclic, then we need more outputs to have the observability rank condition. Precisely, if the Frobenius normal form or rational canonical form of A contains p blocs then we need at least $m \geq p$ outputs to have the observability rank condition. Indeed, assume that the matrix

$$A = bloc\ diag\,[F_1, .., F_m]$$

where F_i for $1 \leq i \leq m$ are the Companion matrices. The characteristic polynomial $\mathcal{M}_1$ of F_1 is the minimal polynomial of A, and for $2 \leq i \leq m$ the characteristic polynomial $\mathcal{M}_i$ of the bloc F_i devised the characteristic polynomial $\mathcal{M}_{i-1}$ of the bloc F_{i-1} such that:

$$P_A = \mathcal{M}_1....\mathcal{M}_p.$$

Moreover, if we denote by r_i the degree of $\mathcal{M}_i$, then $r_1 + .. + r_m = n$.

It is well-known that there exists a family of vectors $\{v_{i,1} \quad for \quad 1 \leq i \leq m\}$ such that the family $\{v_{i,j} = B^j v_{i,1} \quad for \quad 1 \leq i \leq m \quad and \quad 0 \leq j \leq r_i - 1\}$ is a basis of R^n. If we set

$$E_i = span\{v_{i,j} \quad for \quad 0 \leq j \leq r_i - 1\}$$

then it is a A-invariant vector subspace. Thus, $AE_i \subseteq E_i$ and we have:

$$R^n = E_1 \oplus \ldots \oplus E_m.$$

Let us sst

$$C_i = v_{i,1}^T \quad for \quad 1 \leq i \leq m$$

then, we have the following preliminary result:

Claim 1: ideal case
Assume A is cyclic and the output $y_i = C_i = v_{i,1}^T$, then there exists a linear change of coordinates which transforms the dynamical system (1-2) into the form (5-7) with $\alpha_{i,j} = 0$ in the relation (7).

Proof: For $1 \leq i \leq m$ we have $\Im E_i \subseteq E_i$ which implies that:

$$C_i A^{r_i} = \sum_{j=1}^{r_i} p_{i,j} C_k A^{j-1}. \tag{14}$$

This last equation plays the part of Cayley-Hamilton's theorem in the single output $m = 1$. Therefore the linear change of coordinates is given as in the beginning of this section for $m = 1$. For each $1 \leq i \leq m$, we set:

$$\begin{aligned} z_{i,r_i} &= C_i x \quad and, for \quad 1 \leq k \leq r_i - 1 \\ z_{i,r_i-k} &= CA^k x + \sum_{j=1}^{k} p_{i,n-j} CA^{k-j} x \end{aligned}$$

■

IV. General case

In this section, we assume that the number m of the outputs is great or equal to the number p of Frobinus blocs of the matrix A. As above $(r_i)_{1 \leq i \leq m}$ are the observability indices of (1-2). Under assumption II and relations (3-4), we have the following preliminary result.

Lemma 1: There exists a linear change of coordinates which transforms the dynamical systems (1-2) into the following form:

$$\begin{aligned} \dot{z}_i &= A_i z_i + \beta_i(u, z_{1,r_1}, \cdots, z_{1,r_m}) && (15) \\ z_{1,r_1} &= y_1 && (16) \\ z_{i,r_i} &= y_i - \sum_{j=1}^{i-1} \sum_{k=1}^{r_j - r_i} p_{i,j,r_j-k} C_j A^{r_j - r_i - k} x && (17) \\ & \quad for \quad 2 \leq i \leq m \end{aligned}$$

Proof: For sake of clarity, we will give the proof for $m = 2$. Recall from equation (4) that we have:

$$C_1 A^{r_1} = \sum_{j=1}^{r_1 - 1} p_{1,1,j} C_1 A^{j-1} + q_{1,2} C_2$$

$$C_2 A^{r_2} = \sum_{j=1}^{r_1 - 1} p_{2,1,j} C_1 A^{j-1} + \sum_{j=1}^{r_2 - 1} p_{2,2,j} C_2 A^{j-1}.$$

Now, it is easy to see that the following change of coordinates:

$$\begin{aligned} z_{1,r_1} &= C_1 x, \\ z_{1,r_1-k} &= C_1 A^k x - \sum_{j=1}^{k} p_{1,j,r_1-j} C_1 A^{k-j} x, \end{aligned}$$

and

$$\begin{aligned} z_{2,r_2} &= C_2 x - \sum_{j=1}^{r_1 - r2} p_{2,r_1-j} C_1 A^{j-1} x \\ z_{2,r_2-k} &= C_2 A^k x - \sum_{j=1}^{k} p_{2,j,r_i-j} C_2 A^{k-j} x \\ &\quad - \sum_{j=1}^{r_1 - r2} p_{2,1,r_1-j} C_1 A^{k+j-1} x \end{aligned}$$

do the work. ■

Remark 2: The dynamics (15) in lemma 1 are in the form sought in (5). However, the outputs are not in the form of (5). Indeed, for $2 \leq i \leq m$ the i^{th} output y_i depends on the formers output $y_1, \ldots, y_{i-1}$ and some of their derivatives as we saw it in formula (17).
Now, we can deduce the following result from lemma 1.

Corollary 1: Assuming hypothesis II and relation (3) then there exists a linear change of coordinates which solves the problem 1 if and only if equation 4 is in the following form:

$$C_i A^{r_i} = \sum_{k=1}^{i} \sum_{j=1}^{r_i} p_{i,k,j} C_k A^{j-1} + \sum_{k=i+1}^{m} q_{i,k} C_k. \tag{18}$$

Thus, for $2 \leq i \leq m$ we have $p_{i,k,j} = 0$ for all $2 \leq k \leq i$ and $j > r_i$.

V. Change of coordinates

In this section, we will give the algorithm to compute the change of coordinates. Let the following family of vectors $\{\tau_{1,i}\}_{1 \leq i \leq m}$ defined by the following algebraic equations :

$$\begin{aligned} C_j A^{k-1} \tau_{i,1} = 0 \quad for \quad j < i \quad and \quad 1 \leq k \leq r_i && (19) \\ C_i A^{k-1} \tau_{i,1} = 0 \quad for \quad 1 \leq k \leq r_i && (20) \\ C_i A^{r_i - 1} \tau_{i,1} = 1 && (21) \\ C_j A^{k-1} \tau_{i,1} = 0 \quad for \quad j > i \quad and \quad 1 \leq k \leq r_j. && (22) \end{aligned}$$

Now, we set for all: $\tau_{j,i} = C_i A^j \tau_{1,i} \quad for \quad 1 \leq i \leq m \quad and \quad 1 \leq j \leq r_i - 1$. By the observability rank condition the family $\{\tau_{j,i}\}$ is a basis of R^n. Let

$$\theta = \begin{bmatrix} C_1 \\ C_1A \\ .. \\ C_1A^{r_1-1} \\ C_2 \\ .. \\ C_m \\ .. \\ C_mA^{r_m-1} \end{bmatrix} \text{ and let}$$

$\Lambda = \left[\begin{array}{c} \theta\tau_{1,1}, ..., \theta\tau_{r_1,1}, \theta\tau_{1,2}, ..., \theta\tau_{r_m,m} \end{array} \right]$

Now set $M = \Lambda^{-1}\theta$

Lemma 2: The change of coordinates $z = Mx$ transforms the dynamical system (1-2) into the form given in lemma 1.

Proof: Let us set for $1 \leq i \leq m$ for $1 \leq j \leq r_i$: $\mathrm{M}\tau_{i,j} = \frac{\partial}{\partial z_{i,j}}$. Now, take the derivative in the direction $\frac{\partial}{\partial z_{i,j}}$ of $\dot{z} = M\dot{x}$. We obtain for $1 \leq i \leq m$ for $1 \leq j \leq r_i - 1$:

$$\begin{aligned} \frac{\partial}{\partial z_{i,j}} M\dot{x} &= MA\tau_{i,j} \\ &= M\tau_{i,j+1} = \frac{\partial}{\partial z_{i,j+1}}. \end{aligned}$$

Therefore, by integration, the dynamic (1) in the new coordinates has the form (5) and the outputs have the form (17). ■

Now, we will give the equivalent condition (18) of corollary 1 by means of the vectors $\tau_{i,j}$.

Corollary 2: Condition (18) is equivalent to the following: for all $1 \leq i \leq m-1$ such that $j < i$ and $r_j > r_i$

$$C_jA^k\tau_{i,1} = 0 \quad for \quad r_j - r_i < k \leq r_j \tag{23}$$

Remark 3: Condition (23) means that for $2 \leq i \leq m$ output y_i do not depend on the derivatives of outputs y_j for $j < i$.

VI. Discussion on the nonlinear dynamical systems

Consider a multi-output nonlinear dynamical system in the following form:

$$\dot{x} = f(x) \tag{24}$$

$$y = h(x). \tag{25}$$

where f is a smooth vector field and h a smooth function with m-linearly independent components on a neighborhood of 0 such that $f(0) = 0$ and $h(0) = 0$. Let us assume the regular observability rank condition[1]. More precisely, if $h = (h_i)_{1 \leq i \leq m}$ then, as for the linear case, there exist integers $r_1 \geq r_2 \ldots r_m$ (for more details see for example [7]) such that: $\mathrm{rank}\{\mathrm{dh}_1, dL_fh_1, .., dL_f^{r_1-1}h_1, .., dh_m, .., dL_f^{r_m-1}h_1\} = m$. In [11], we can fund corresponding vector fields $\{\tau_{i,1}\}_{1 \leq i \leq m}$ by the same equations as in (19-20). Thus: equations :

$$\begin{aligned} dL_f^{k-1}h_j(\tau_{i,1}) = 0 \quad for \quad j < i \quad and \quad 1 \leq k \leq r_i \\ dL_f^{k-1}h_i(\tau_{i,1}) = 0 \quad for \quad 1 \leq k \leq r_i \\ dL_f^{r_i-1}h_i(\tau_{i,1}) = 1 \\ dL_f^{k-1}{}_j\tau_{i,1} = 0 \quad for \quad j > i \quad and \quad 1 \leq k \leq r_j. \end{aligned}$$

[1]locally weakly observable without singularity, i.e. fixe observability index

And then, we define:

$$\tau_{i,j} = [\tau_{i,j+1}, f] \quad for \quad 1 \leq j \leq r_i - 1.$$

where the $[,]$ means the Lie bracket. Now, the main difference between the nonlinear and the linear case is that for the nonlinear case, we assume in lemma 1 and corollary 2 that vector fields $\tau_{i,j}$ commute:

$$[\tau_{i,j}, \tau_{k,s}] = 0. \tag{26}$$

This last condition is always fulfilled in the linear case because the vector fields $\tau_{i,j}$ are constant. Now, for condition (23) is the same for both case linear one and nonlinear one. To summer, the important thing to keep in mind is:

- Condition (26), which is always fulfilled in the linear case, ensures the existence of the coordinate change in which the dynamic is in form (5),
- Condition (23), ensures that the outputs in the new coordinates are only functions of the outputs in the former coordinates.

To end this discussion, we can say that the corollary (1) provide a necessary condition for the multi-output nonlinear dynamical systems. Indeed, if we set:

$$\begin{aligned} A &= \frac{\partial f}{\partial x}(0) \quad itslinearpartat \quad 0 \\ C &= \frac{\partial h}{\partial x}(0) \quad itslinearpartat \quad 0 \end{aligned}$$

then, the existence of change of coordinates which transforms the nonlinear dynamical system (24-25) into the canonical form (5-6) implies the existence of a change of coordinates which transforms the linear part (A, C) into the canonical form (5-6). Obviously, it is only a necessary condition, because this does not guarantee that the linear part is only function of y and u.

VII. Conclusion

This paper dealt with the Brunovsky observability normal form for a multi-output linear dynamical system. Then, it gave a comparison between with the stated result and the well-known results in the nonlinear case.

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