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# Asymptotic Models for an Elasto-Acoustic Problem with a Thin Layer

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# Asymptotic Models for an Elasto-Acoustic Problem with a Thin Layer

VICTOR PÉRON

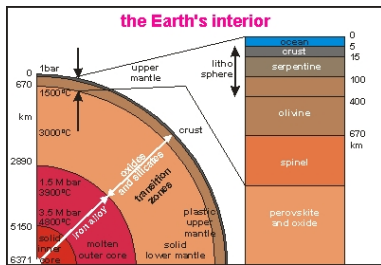
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INRIA Bordeaux Sud-Ouest, Team MAGIQUE 3D

Twelfth International Conference Zaragoza-Pau on Mathematics  
Jaca, September 17, 2012

# Motivations

## HPC-GA Project

- **High Performance Computing for Geophysics Applications :**  
**Earthquakes**
- **Numerical Models :** Time dependent Elasto-Acoustic Coupling
  - Elasto-Acoustic Coupling : to reproduce earthquakes
  - Modeling the effects of the ocean on seismic waves

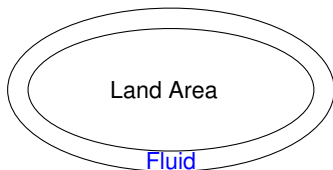


- The medium : a Land Area surrounded by a **thin Fluid zone**

# Motivations

## Configuration, Difficulty and Method

- **Configuration**



- **Difficulty**

Apply a FEM on a mesh with **thin cells** in the **Fluid** and much larger in the solid

- **Method**

- 1 Replace the **Fluid layer** by an Equivalent boundary condition

**Approach** : an asymptotic method

- 2 Couple this condition with the elastic wave equation
- 3 Use a FEM



# Motivations

## Bibliography and Outline

### Bibliography

- Modeling Fluid-Solid Interaction Problems  
JONES 83, LUKE-MARTIN 95, NATROCHVILI ET AL. 00, ... ,  
MONK-SELGAS 09
- Derivation of Equivalent Conditions for **thin layer** Problems  
LAFITTE 93, ENGQUIST-NÉDÉLEC 93, VOLAKIS-SENIOR 95,  
BENDALI-LEMRAËT 96, CALOZ-COSTABEL-DAUGE-VIAL 06 ...

### Outline

- Asymptotic Analysis for the time-harmonic **Elasto-acoustic** equations in a domain with a **thin acoustic layer**.

# Outline

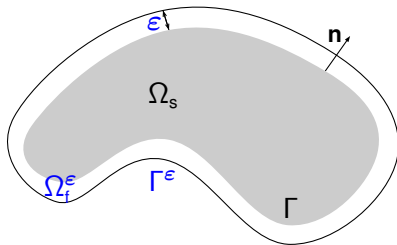
- 1 **Uniform Estimates**
- 2 Equivalent Conditions
- 3 Numerical Simulations

# An Elasto-Acoustic Problem with a Thin Layer

## Framework and Issue

The Problem  $(\mathbf{P}_\epsilon)$  set in a smooth bounded domain  $\Omega^\epsilon = \Omega_s \cup \Gamma \cup \Omega_f^\epsilon$ :

$$\begin{aligned} \Delta p + \kappa^2 p &= 0 && \text{in } \Omega_f^\epsilon \\ \nabla \cdot \underline{C}(x) \nabla \mathbf{u} + \omega^2 \rho \mathbf{u} &= 0 && \text{in } \Omega_s \\ \partial_n p &= \rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} - \partial_n p_i && \text{on } \Gamma \\ \mathbf{n} \underline{C}(x) \nabla \mathbf{u} &= -p \mathbf{n} - p_i \mathbf{n} && \text{on } \Gamma \\ p &= 0 && \text{on } \Gamma^\epsilon \end{aligned}$$



### Assumption

- (i)  $\underline{C}(x) = (C_{ijkl}(x))$  is symmetric :  $C_{ijkl} = C_{jikl} = C_{klij}$ .
- (ii)  $C_{ijkl}(x)$  are real valued smooth functions, up to  $\Gamma$ .
- (iii)  $\exists \alpha > 0, \forall \xi = (\xi_{ij})$  sym. tensor,  $\sum_{i,j,k,l} C_{ijkl} \xi_{ij} \bar{\xi}_{kl} \geq \alpha \sum_{i,j} |\xi_{ij}|^2$ .

## Weak Solutions

### Variational Problem $(VP)_\varepsilon$

- $V_\varepsilon = \{(\mathbf{u}, p) \in H^1(\Omega_s) \times H^1(\Omega_f^\varepsilon) \mid \gamma_0 p = 0 \text{ on } \Gamma^\varepsilon\}$
- For all  $\varepsilon > 0$ ,  $(VP)_\varepsilon$  writes : Find  $(\mathbf{u}_\varepsilon, p_\varepsilon) \in V_\varepsilon$  such that

$$\forall (\mathbf{v}, q) \in V_\varepsilon, \quad a_\varepsilon((\mathbf{u}_\varepsilon, p_\varepsilon), (\mathbf{v}, q)) = \langle F, (\mathbf{v}, q) \rangle_{V_\varepsilon', V_\varepsilon}$$

$$a_\varepsilon((\mathbf{u}, p), (\mathbf{v}, q)) := \int_{\Omega_f^\varepsilon} (\nabla p \cdot \nabla \bar{q} - \kappa^2 p \bar{q}) \, d\mathbf{x}$$

$$+ \int_{\Omega_s} (\underline{C}(\mathbf{x}) \underline{\nabla} \mathbf{u} : \underline{\nabla} \bar{\mathbf{v}} - \omega^2 \rho \mathbf{u} \bar{\mathbf{v}}) \, d\mathbf{x} + \int_{\Gamma} (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{q} + p \bar{\mathbf{v}} \cdot \mathbf{n}) \, ds$$



## Uniform Estimates

Some resonant frequencies may appear in the solid

### Assumption (SA)

*The angular frequency  $\omega$  is not an eigenfrequency of the problem*

$$\begin{cases} \nabla \cdot \underline{C}(x) \underline{\nabla} \mathbf{u} + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \underline{C}(x) \underline{\nabla} \mathbf{u} \mathbf{n} = 0 & \text{on } \Gamma. \end{cases}$$

### Theorem

*Under Assumption (SA),  $\exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0)$ , the problem  $(\mathbf{VP})_\varepsilon$  with data  $F \in V'_\varepsilon$  has a unique solution  $(\mathbf{u}_\varepsilon, p_\varepsilon) \in V_\varepsilon$  and*

$$\|\mathbf{u}_\varepsilon\|_{1, \Omega_s} + \|p_\varepsilon\|_{1, \Omega_f^\varepsilon} \leq C \|F\|_{V'_\varepsilon}.$$

**Application** : Convergence of an asymptotic expansion as  $\varepsilon \rightarrow 0$ .

**Key for the proof**: introduce a "Scaled Problem"

## Scaled Problem $(\mathfrak{P})_\epsilon$

Formulation in a fixed domain

- For any function  $p$  defined in  $\Omega_f^\epsilon$ , define  $\mathfrak{p}$  in  $\Omega_f := \Gamma \times (0, 1)$  :

$$\mathfrak{p}(\mathbf{x}) = p(\Psi(y_\alpha), S), \quad S = \frac{s}{\epsilon} \in (0, 1).$$

- $V = \{(\mathbf{u}, p) \in H^1(\Omega_s) \times H^1(\Omega_f) \mid p(\cdot, 1) = 0 \text{ on } \Gamma\}$
- $(\mathfrak{P})_\epsilon$  writes : Find  $(\mathbf{u}, p) \in V$  such that  $\forall (\mathbf{v}, q) \in V$

$$\begin{aligned} \epsilon \int_0^1 \int_\Gamma [(\mathbb{I} + \epsilon S \mathcal{R})^{-2} \nabla_\Gamma \mathfrak{p} \nabla_\Gamma \bar{q} + \epsilon^{-2} \partial_s \mathfrak{p} \partial_s \bar{q} - \kappa^2 \mathfrak{p} \bar{q}] \det(\mathbb{I} + \epsilon S \mathcal{R}) \, d\Omega_f \\ + a_s(\mathbf{u}, \mathbf{v}) + \int_\Gamma (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{q} + \mathfrak{p} \bar{\mathbf{v}} \cdot \mathbf{n}) \, d\Gamma = \langle \mathfrak{F}_\epsilon, (\mathbf{v}, q) \rangle_{V', V} \end{aligned}$$

## Scaled Problem $(\mathfrak{P})_\epsilon$

### Uniform estimates

#### Lemma

Under Assumption (SA),  $\exists \epsilon_0 > 0$ ,  $\forall \epsilon \in (0, \epsilon_0)$ , any solution  $(\mathbf{u}_\epsilon, \mathbf{p}_\epsilon) \in V$  of problem  $(\mathfrak{P})_\epsilon$  with a data  $\mathfrak{F}_\epsilon \in V'$  satisfies

$$\|(\mathbf{u}_\epsilon, \mathbf{p}_\epsilon)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_\epsilon \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_\epsilon\|_{0, \Gamma} \leq C \|\mathfrak{F}_\epsilon\|_{V'}.$$

#### Theorem

Under Assumption (SA),  $\exists \epsilon_0 > 0$ ,  $\forall \epsilon \in (0, \epsilon_0)$ , the problem  $(\mathfrak{P})_\epsilon$  with data  $\mathfrak{F}_\epsilon \in V'$  has a unique solution  $(\mathbf{u}_\epsilon, \mathbf{p}_\epsilon) \in V$  which satisfies

$$\sqrt{\epsilon} \|\nabla_\Gamma \mathbf{p}_\epsilon\|_{0, \Omega_f} + \sqrt{\epsilon}^{-1} \|\partial_S \mathbf{p}_\epsilon\|_{0, \Omega_f} + \|\mathbf{p}_\epsilon\|_{0, \Omega_f} + \|\mathbf{u}_\epsilon\|_{1, \Omega_s} \leq C \|\mathfrak{F}_\epsilon\|_{V'}.$$

## Sketch of the proof of the Lemma

Proof by contradiction.

Assume :  $\exists(\mathbf{u}_m, \mathbf{p}_m) \in V, m \in \mathbb{N}$ , satisfying  $(\mathfrak{P})_{\epsilon_m}$  associated with  $\epsilon_m \rightarrow 0$  and  $\tilde{\mathfrak{F}}_m \in V'$  such that

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_m \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_m\|_{0, \Gamma} = 1 \quad \text{and} \quad \|\tilde{\mathfrak{F}}_m\|_{V'} \rightarrow 0$$

- 1 Prove that  $\|(\mathbf{u}_m, \mathbf{p}_m)\|_W \leq C$   
 with  $W = \{(\mathbf{u}, \mathbf{p}) \in V \mid \mathbf{p} \in \mathbf{H}^1(0, 1; L^2(\Gamma)), \mathbf{p}(\cdot, 1) = 0 \text{ on } \Gamma\}$
- 2 Compactness argument :  
 $\mathbf{u}_m \rightarrow \mathbf{u}$  in  $L^2(\Omega_s)$  and  $\mathbf{p}_m \rightarrow \mathbf{p} = 0$  in  $L^2(\Omega_f)$ .
- 3 Prove that  $\|\mathbf{u}\|_{0, \Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0, \Gamma} = 1$
- 4 Using (SA), we prove  $\mathbf{u} = 0$  : contradiction

# Outline

- 1 Uniform Estimates
- 2 Equivalent Conditions**
- 3 Numerical Simulations

# Methodology

**Framework** :  $\Gamma, \Gamma^\varepsilon$  smooth surfaces in  $\mathbb{R}^3$ . Smooth data

- Step 1 : Derive an Asymptotic Expansion for  $(\mathbf{u}_\varepsilon, p_\varepsilon)$  when  $\varepsilon \rightarrow 0$

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) + \varepsilon \mathbf{u}_1(x) + \varepsilon^2 \mathbf{u}_2(x) + \dots,$$

$$p_\varepsilon(x) = p_0(x; \varepsilon) + \varepsilon p_1(x; \varepsilon) + \varepsilon^2 p_2(x; \varepsilon) + \dots, \quad p_j(x; \varepsilon) = p_j(y_\alpha, \frac{S}{\varepsilon}).$$

- Step 2 : Derive Equivalent Conditions of order  $k \in \mathbb{N}$  :  
Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \dots + \varepsilon^k \mathbf{u}_k \quad \text{up to } \mathcal{O}(\varepsilon^{k+1})$$

- Step 3 : Prove the Stability of Equivalent models  $\mathbf{u}_\varepsilon^k$  and

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C\varepsilon^{k+1}$$

## Step 1 : Multiscale Expansion

### First terms

- At step  $n = 0$ , we obtain

$$p_0 = 0,$$

and then  $\mathbf{u}_0$  solves the problem

$$\begin{cases} \nabla \cdot \underline{\underline{C}}(x) \nabla \mathbf{u}_0 + \omega^2 \rho \mathbf{u}_0 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_0) = -p_i \mathbf{n} & \text{on } \Gamma. \end{cases}$$

- At step  $n = 1$ , we find successively

$$p_1(\cdot, S) = (S - 1) (\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_n p_i), \quad S \in (0, 1)$$

and that  $\mathbf{u}_1$  solves

$$\begin{cases} \nabla \cdot \underline{\underline{C}}(x) \nabla \mathbf{u}_1 + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_1) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} - \partial_n p_i \mathbf{n} & \text{on } \Gamma. \end{cases}$$

## Step 1 : Multiscale Expansion

### First terms

- At step  $n = 2$ ,

$$p_2(\cdot, Y_3) = (S^2 - 1)\mathcal{H}[\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_n p_i] + (S - 1)\rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}$$

and then,  $\mathbf{u}_2$  solves

$$\begin{cases} \nabla \cdot \underline{C}(x) \nabla \mathbf{u}_2 + \omega^2 \rho \mathbf{u}_2 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_2) = \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} + \mathcal{H}[\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_n p_i] \mathbf{n} & \text{on } \Gamma. \end{cases}$$



## Step 1 : Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$\mathbf{r}_\epsilon^{N+1} := \mathbf{u}_\epsilon - \sum_{n=0}^N \epsilon^n \mathbf{u}_n \quad \text{in } \Omega_s, \quad \text{and} \quad r_\epsilon^{N+1} := p_\epsilon - \sum_{n=0}^N \epsilon^n p_n \quad \text{in } \Omega_f^\epsilon$$

Evaluation of the right hand sides in

$$\left\{ \begin{array}{ll} \kappa^2 r_\epsilon^{N+1} + \Delta r_\epsilon^{N+1} & = f_\epsilon \quad \text{in } \Omega_f^\epsilon \\ \omega^2 \rho r_\epsilon^{N+1} + \nabla \cdot \underline{C}(x) \nabla r_\epsilon^{N+1} & = 0 \quad \text{in } \Omega_s \\ \partial_{\mathbf{n}} r_\epsilon^{N+1} - \rho_f \omega^2 r_\epsilon^{N+1} \cdot \mathbf{n} & = g_\epsilon \quad \text{on } \Gamma \\ \mathbf{T}(r_\epsilon^{N+1}) + r_\epsilon^{N+1} \mathbf{n} & = 0 \quad \text{on } \Gamma \\ r_\epsilon^{N+1} & = 0 \quad \text{on } \Gamma^\epsilon. \end{array} \right.$$

The RHS are explicit:  $f_\epsilon = \mathcal{O}(\epsilon^{N-\frac{1}{2}})$  in  $\Omega_f^\epsilon$ ;  $g_\epsilon = \mathcal{O}(\epsilon^N)$  on  $\Gamma$

**Optimal Estimates :**  $\|\mathbf{r}_\epsilon^{N+1}\|_{1,\Omega_s} + \sqrt{\epsilon} \|r_\epsilon^{N+1}\|_{1,\Omega_f^\epsilon} \leq C_N \epsilon^{N+1}$

## Step 2 : Equivalent Conditions on $\Gamma$

Identify a simpler problem

$$(\mathbf{P}_\varepsilon^k) \quad \begin{cases} \omega^2 \rho \mathbf{u}_\varepsilon^k + \nabla \cdot \underline{C}(x) \underline{\nabla} \mathbf{u}_\varepsilon^k = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_\varepsilon^k) + \mathbf{B}_{k,\varepsilon}(\mathbf{u}_\varepsilon^k) = \mathbf{h}_{k,\varepsilon} & \text{on } \Gamma, \end{cases}$$

with  $\mathbf{B}_{k,\varepsilon}$  a surfacic differential operator.

- $k \in \{0, 1, 2\}$  for a Dirichlet b.c. on  $\Gamma^\varepsilon$
- $k \in \{0, 1\}$  for a Fourier-Robin b.c. on  $\Gamma^\varepsilon$

## Step 2 : Equivalent Conditions

Dirichlet b.c.

- $k = 0$ :

$$\mathbf{T}(\mathbf{u}_0) = -p_i \mathbf{n} \quad \text{on } \Gamma .$$

- $k = 1$ :

$$\mathbf{T}(\mathbf{u}_\epsilon^1) - \epsilon \omega^2 \rho_f \mathbf{u}_\epsilon^1 \cdot \mathbf{n} \mathbf{n} = \mathbf{h}_{1,\epsilon} \quad \text{on } \Gamma .$$

- $k = 2$ :

$$\mathbf{T}(\mathbf{u}_\epsilon^2) - \epsilon \omega^2 \rho_f (1 + \epsilon \mathcal{H}) \mathbf{u}_\epsilon^2 \cdot \mathbf{n} \mathbf{n} = \mathbf{h}_{2,\epsilon} \quad \text{on } \Gamma .$$

## Step 2 : Equivalent Conditions

Example : Order 1

$\mathbf{u}_{1,\epsilon} = \mathbf{u}_0 + \epsilon \mathbf{u}_1$  satisfies the elastic equation in  $\Omega_s$  and

$$\mathbf{T}(\mathbf{u}_{1,\epsilon}) - \epsilon \omega^2 \rho_f \mathbf{u}_{1,\epsilon} \cdot \mathbf{n} \mathbf{n} = -\mathbf{p}_i \mathbf{n} - \epsilon \partial_n \mathbf{p}_i \mathbf{n} - \epsilon^2 \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} \quad \text{on } \Gamma$$

We infer a simpler problem

$$(\mathbf{P}_\epsilon^1) \quad \begin{cases} \omega^2 \rho \mathbf{u}_\epsilon^1 + \nabla \cdot \underline{\mathbf{C}}(x) \underline{\nabla} \mathbf{u}_\epsilon^1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_\epsilon^1) - \epsilon \omega^2 \rho_f \mathbf{u}_\epsilon^1 \cdot \mathbf{n} \mathbf{n} = \mathbf{h}_{1,\epsilon} & \text{on } \Gamma, \end{cases}$$

## Step 2 : Equivalent Conditions

### Fourier-Robin b.c.

Fourier-Robin b.c. :  $\partial_n p_\epsilon - i\kappa p_\epsilon = 0$  on  $\Gamma^\epsilon$

- $k = 0$  :

$$\mathbf{T}(\mathbf{u}_\epsilon^0) - i\omega c \rho_f \mathbf{u}_\epsilon^0 \cdot \mathbf{n} \mathbf{n} = \mathbf{g}_{0,\epsilon} \quad \text{on } \Gamma$$

- $k = 1$  :

$$\mathbf{T}(\mathbf{u}_\epsilon^1) - i\omega c \rho_f \mathbf{u}_\epsilon^1 \cdot \mathbf{n} \mathbf{n} + \epsilon c^2 \rho_f (\Delta_\Gamma + \kappa^2 \mathbb{I} - 2i\kappa \mathcal{H} \mathbb{I})(\mathbf{u}_\epsilon^1 \cdot \mathbf{n}) \mathbf{n} = \mathbf{g}_{1,\epsilon} \quad \text{on } \Gamma$$

## Step 3 : Stability and Convergence of Equivalent Conditions

### Step 3.1 : Stability

**Step 3.1 :** Prove that problems  $(\mathbf{P}_\varepsilon^k)$  are well-posed

#### Proposition

*Under Assumption (SA),  $\exists \varepsilon_0 > 0$  s.t.  $\forall \varepsilon \in (0, \varepsilon_0)$ ,  $\forall k \in \{0, 1, 2\}$ , the problem  $(\mathbf{P}_\varepsilon^k)$  with data  $h_k \in L^2(\Gamma)$  has a unique solution  $\mathbf{u}_\varepsilon^k \in H^1(\Omega_s)$*

$$\|\mathbf{u}_\varepsilon^k\|_{1, \Omega_s} \leq C \|h_k\|_{0, \Gamma}$$

**Application :** Convergence of an asymptotic expansion of  $\mathbf{u}_\varepsilon^k$  as  $\varepsilon \rightarrow 0$ .

## Step 3.2 : Convergence of Equivalent Conditions

**Step 3.2 :** Prove that the solution  $\mathbf{u}_\varepsilon^k$  satisfies an error estimate

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C\varepsilon^{k+1}.$$

**Proof :**

(i) Derive an expansion of  $\mathbf{u}_\varepsilon^k$  and show that :

$$\mathbf{u}_\varepsilon^k = \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \cdots + \varepsilon^k\mathbf{u}_k + \tilde{\mathbf{r}}_\varepsilon^{k+1}.$$

Recall :

$$\mathbf{u}_\varepsilon = \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \cdots + \varepsilon^k\mathbf{u}_k + \mathbf{r}_\varepsilon^{k+1}$$

hence,

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} = \|\mathbf{r}_\varepsilon^{k+1} - \tilde{\mathbf{r}}_\varepsilon^{k+1}\|_{1,\Omega_s}.$$

(ii) There holds

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^{k+1}) + \mathbf{B}_{k,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^{k+1}) = \mathcal{O}(\varepsilon^{k+1}) \quad \text{on } \Gamma.$$

According to the step 3.1, we infer the uniform error estimates:

$$\|\tilde{\mathbf{r}}_\varepsilon^{k+1}\|_{1,\Omega_s} \leq C\varepsilon^{k+1}.$$

# Outline

- 1 Uniform Estimates
- 2 Equivalent Conditions
- 3 Numerical Simulations**



# Finite Element Method

**Joint work with : J. Diaz (INRIA).**

We use a Discontinuous Galerkin Method (IPDGM).

- 2D Computational domain :  $\Omega_s$  is a disk of radius 0.01
- $\omega = 1.5 \times 10^6$
- Source :  $p_i(\mathbf{x}) = \exp(i\omega \mathbf{x} \cdot \mathbf{d})$  with  $\mathbf{d} = (1, 0)$
- $\mathbb{P}_3$ -finite elements (Lagrange) available in the Library [Hou10ni](#)
- Navier equation :

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \omega^2 \rho \mathbf{u} = 0 \quad \text{in } \Omega_s$$

with Lamé coefficients :  $\mu = 26.32 \times 10^9$  and  $\lambda = 51.08 \times 10^9$ .

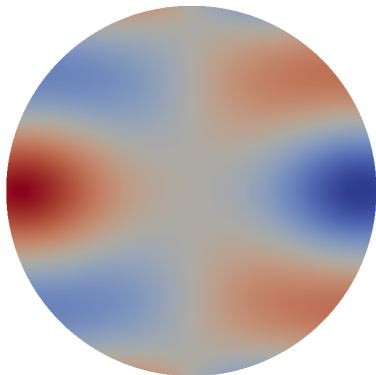
- $c = 1500 \text{ m.s}^{-1}$ ,  $\rho_f = 1000 \text{ kg.m}^{-3}$ ,  $\rho = 2700 \text{ kg.m}^{-3}$ .

## Numerical Simulations

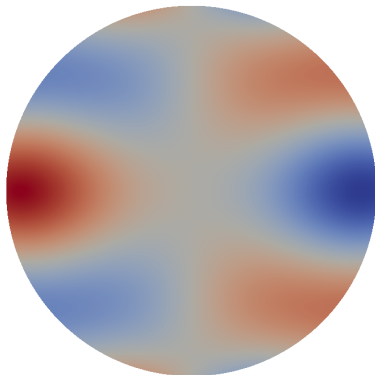
$\varepsilon = 0.0001$ ;

$\mathbf{u}_\varepsilon$ : analytical solution;

$\mathbf{u}_\varepsilon^2$ : FE solution (equivalent model of order 2).



$|\operatorname{Re} \mathbf{u}_\varepsilon|$



$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$

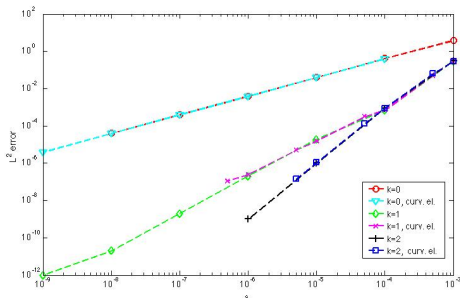
# Convergence of the models

## Dirichlet b.c.

For  $k \in \{0, 1, 2\}$ , we plot the  $L^2$ -errors  $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0, \Omega_s}$  w.r.t.  $\varepsilon$ :

$\mathbf{u}_\varepsilon$ : analytical solution

$\mathbf{u}_\varepsilon^k$ : analytical solution / FE solutions with Curved elements



The convergence rate coincide with the theory  $\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{0, \Omega_s} = \mathcal{O}(\varepsilon^{k+1})$

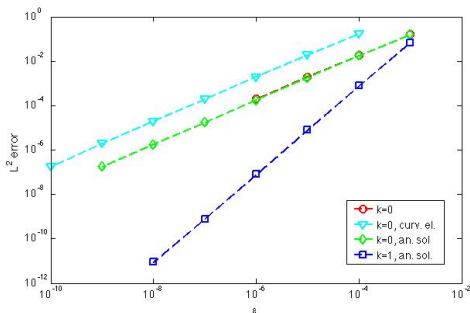
# Convergence of the models

## Fourier-Robin b.c.

For  $k \in \{0, 1\}$ , we plot the  $L^2$ -errors  $\|\mathbf{u}_\epsilon - \mathbf{u}_\epsilon^k\|_{0, \Omega_s}$  w.r.t.  $\epsilon$ ,

$\mathbf{u}_\epsilon$ : analytical solution

$\mathbf{u}_\epsilon^k$ : analytical solution / FE solutions



## Conclusion

- 1 Time dependent Equivalent Conditions
- 2 Higher Order Equivalent Conditions
- 3 Thin layer with variable thickness

Thank you for your attention!