



Equivalent Conditions for an Elasto-Acoustic Problem with a Thin Layer

Victor P  ron

► To cite this version:

Victor P  ron. Equivalent Conditions for an Elasto-Acoustic Problem with a Thin Layer. First Russian-French Conference on Mathematical Geophysics, Mathematical Modeling in Continuum Mechanics and Inverse Problems, Jun 2012, Biarritz, France. hal-00768036

HAL Id: hal-00768036

<https://inria.hal.science/hal-00768036>

Submitted on 25 Mar 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destin  e au d  p  t et ´a la diffusion de documents scientifiques de niveau recherche, publi  s ou non, ´emanant des ´tablissements d'enseignement et de recherche fran  ais ou ´etrangers, des laboratoires publics ou priv  s.

Equivalent Conditions for an Elasto-Acoustic Problem with a Thin Layer

VICTOR PÉRON

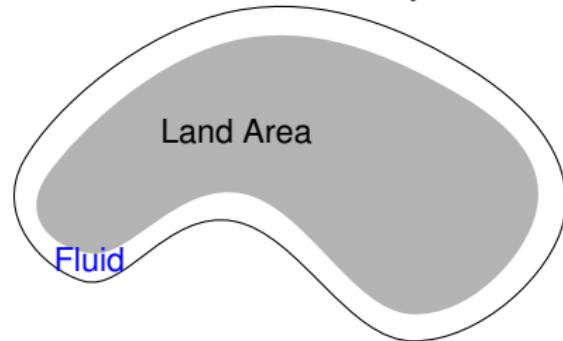
Université de Pau et des Pays de l'Adour
INRIA Bordeaux Sud-Ouest, Team MAGIQUE 3D

First Russian-French Conference on Mathematical Geophysics,
Mathematical Modeling in Continuum Mechanics and Inverse Problems
Biarritz, June 18-22, 2012

Motivations

HPC-GA Project

- **High Performance Computing for Geophysics Applications**
FP7 Project : INRIA, BCAM, UFRGS, UNAM
Earthquakes
- **Numerical Models** (Work Package 2)
 - Elasto-Acoustic Coupling : to reproduce earthquakes
 - Modeling the effects of the ocean on seismic waves
 - The medium : Land Areas surrounded by **thin Fluid zones**



Motivations

Difficulty and Method

- **Difficulty**

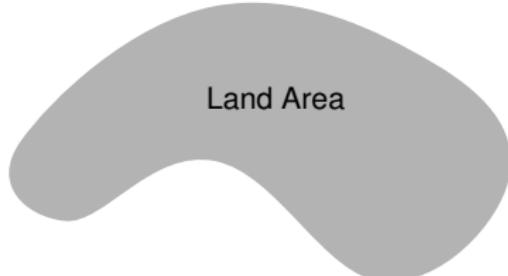
Applying a FEM on a mesh with **fine cells** in the **Fluid** and much larger in the Solid

- **Method**

- ① Replace the **Fluid layer** by a boundary condition

Approach : an asymptotic method to derive Equivalent Conditions

- ② Couple this condition with the elastic wave equation
- ③ Use a FEM



Motivations

Bibliography and Outline

Bibliography

- Modeling Fluid-Solid Interaction Problems
JONES 1983, LUKE-MARTIN 1995, NATROCHVILI ET AL. 2000, ... ,
MONK-SELGAS 2009
- Derivation of Equivalent Conditions for **thin layer** Problems: Laplace,
Helmholtz, Maxwell equations
LAFITTE 1993, ENGQUIST-NÉDÉLEC 1993, VOLAKIS-SENIOR 1995,
BENDALI-LEMRABET 1996, ... , CALOZ ET AL. 2006 ...

Outline

- Asymptotic Analysis for the time-harmonic **Elasto-acoustic** equations in
a domain with a **thin acoustic layer**.

Outline

1 Uniform Estimates

2 Equivalent Conditions

3 Numerical Simulations

An Elasto-Acoustic Problem with a Thin Layer

Framework and Issue

The Problem (\mathbf{P}_ε) set in a smooth bounded domain $\Omega^\varepsilon = \Omega_s \cup \Gamma \cup \Omega_f^\varepsilon$:

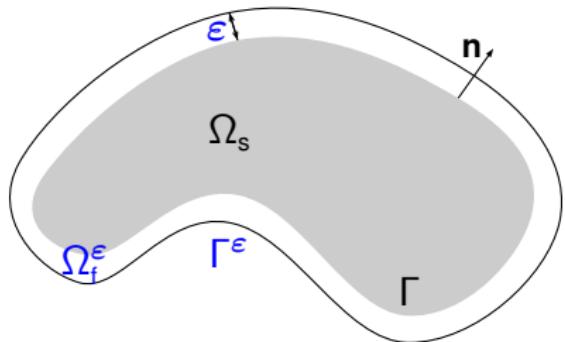
$$\Delta p + \kappa^2 p = 0 \quad \text{in } \Omega_f^\varepsilon$$

$$\nabla \cdot C(x) \nabla u + \omega^2 \rho u = 0 \quad \text{in } \Omega_s$$

$$\partial_n p = \rho_f \omega^2 u \cdot n - \partial_n p_i \quad \text{on } \Gamma$$

$$C(x) \nabla u \cdot n = -p n - p_i n \quad \text{on } \Gamma$$

$$p = 0 \quad \text{on } \Gamma^\varepsilon$$



Assumption

- (i) $C(x) = (C_{ijkl}(x))$ is symmetric : $C_{ijkl} = C_{jikl} = C_{klji}$.
- (ii) $C_{ijkl}(x)$ are real valued smooth functions, up to Γ .

- (iii) $\exists \alpha > 0, \forall \xi = (\xi_{ij})$ sym. tensor, $\sum_{i,j,k,l} C_{ijkl} \xi_{ij} \overline{\xi_{kl}} \geq \alpha \sum_{i,j} |\xi_{ij}|^2$.

Weak Solutions

Variational Problem $(VP)_\varepsilon$

- $V_\varepsilon = \{(\mathbf{u}, p) \in H^1(\Omega_s) \times H^1(\Omega_f^\varepsilon) \mid \gamma_0 p = 0 \text{ on } \Gamma^\varepsilon\}$
- For all $\varepsilon > 0$, $(VP)_\varepsilon$ writes : Find $(\mathbf{u}_\varepsilon, p_\varepsilon) \in V_\varepsilon$ such that

$$\forall (\mathbf{v}, q) \in V_\varepsilon, \quad a_\varepsilon((\mathbf{u}_\varepsilon, p_\varepsilon), (\mathbf{v}, q)) = \langle F, (\mathbf{v}, q) \rangle_{V'_\varepsilon, V_\varepsilon}$$

$$\begin{aligned}
 a_\varepsilon((\mathbf{u}, p), (\mathbf{v}, q)) &:= \int_{\Omega_f^\varepsilon} (\nabla p \cdot \nabla \bar{q} - \kappa^2 p \bar{q}) \, d\mathbf{x} \\
 &+ \int_{\Omega_s} (\underline{C}(x) \nabla \mathbf{u} : \nabla \bar{\mathbf{v}} - \omega^2 \rho \mathbf{u} \bar{\mathbf{v}}) \, d\mathbf{x} + \int_{\Gamma} (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{q} + p \bar{\mathbf{v}} \cdot \mathbf{n}) \, d\Gamma
 \end{aligned}$$

Uniform Estimates

Some resonant frequencies may appear in the solid

Assumption (SA)

The angular frequency ω is not an eigenfrequency of the problem

$$\begin{cases} \nabla \cdot \underline{\mathcal{C}}(x) \underline{\nabla} \mathbf{u} + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \underline{\mathcal{C}}(x) \underline{\nabla} \mathbf{u} \mathbf{n} = 0 & \text{on } \Gamma. \end{cases}$$

Theorem

Under Assumption (SA), $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, the problem $(\mathbf{VP})_\varepsilon$ with data $F \in V'_\varepsilon$ has a unique solution $(\mathbf{u}_\varepsilon, p_\varepsilon) \in V_\varepsilon$ and

$$\|\mathbf{u}_\varepsilon\|_{1, \Omega_s} + \|p_\varepsilon\|_{1, \Omega_s^\varepsilon} \leq C \|F\|_{V'_\varepsilon}.$$

Application : Convergence of an asymptotic expansion as $\varepsilon \rightarrow 0$.

Key for the proof: introduce a "Scaled Problem"

Scaled Problem $(\mathfrak{P})_\varepsilon$

Formulation in a fixed domain

- For any function p defined in Ω_f^ε , define \mathfrak{p} in $\Omega_f := \Gamma \times (0, 1)$:

$$p(\mathbf{x}) = \mathfrak{p}(\mathbf{x}(t), s), \quad s = \frac{s}{\varepsilon} \in (0, 1).$$

- $V = \{(\mathbf{u}, \mathfrak{p}) \in H^1(\Omega_s) \times H^1(\Omega_f) \mid \mathfrak{p}(\cdot, 1) = 0 \text{ on } \Gamma\}$
- $(\mathfrak{P})_\varepsilon$ writes : Find $(\mathbf{u}, \mathfrak{p}) \in V$ such that $\forall (\mathbf{v}, \mathfrak{q}) \in V$

$$\begin{aligned} & \varepsilon \int_0^1 \int_{\Gamma} \left[(1 + \varepsilon s c(t))^{-2} \partial_t \mathfrak{p} \partial_t \mathfrak{q} + \varepsilon^{-2} \partial_s \mathfrak{p} \partial_s \mathfrak{q} - \kappa^2 \mathfrak{p} \bar{\mathfrak{q}} \right] (1 + \varepsilon s c(t)) dt ds \\ & + a_s(\mathbf{u}, \mathbf{v}) + \int_{\Gamma} (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{\mathfrak{q}} + \mathfrak{p} \bar{\mathbf{v}} \cdot \mathbf{n}) dt = \langle \mathfrak{F}_\varepsilon, (\mathbf{v}, \mathfrak{q}) \rangle_{V', V} \end{aligned}$$

Scaled Problem $(\mathfrak{P})_\varepsilon$

Uniform estimates

Lemma

Under Assumption (SA), $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, any solution $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$ of problem $(\mathfrak{P})_\varepsilon$ with a data $\mathfrak{F}_\varepsilon \in V'$ satisfies

$$\|(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_\varepsilon \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_\varepsilon\|_{0, \Gamma} \leq C \|\mathfrak{F}_\varepsilon\|_{V'}.$$

Theorem

Under Assumption (SA), $\exists \varepsilon_0 > 0$, $\forall \varepsilon \in (0, \varepsilon_0)$, the problem $(\mathfrak{P})_\varepsilon$ with data $\mathfrak{F}_\varepsilon \in V'$ has a unique solution $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$ which satisfies

$$\sqrt{\varepsilon} \|\partial_t \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \sqrt{\varepsilon^{-1}} \|\partial_s \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{u}_\varepsilon\|_{1, \Omega_s} \leq C \|\mathfrak{F}_\varepsilon\|_{V'}.$$

Scaled Problem $(\mathfrak{P})_\varepsilon$

Proof of Lemma

Proof by contradiction.

Assume : $\exists (\mathbf{u}_m, \mathbf{p}_m) \in V, m \in \mathbb{N}$, satisfying $(\mathfrak{P})_{\varepsilon_m}$ associated with $\varepsilon_m \rightarrow 0$ and $\mathfrak{F}_m \in V'$ such that

$$\|(\mathbf{u}_m, \mathbf{p}_m)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_m \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_m\|_{0, \Gamma} = 1 \quad \text{and} \quad \|\mathfrak{F}_m\|_{V'} \rightarrow 0$$

- ➊ Prove that $\|(\mathbf{u}_m, \mathbf{p}_m)\|_W \leq C$
with $W = \{(\mathbf{u}, \mathbf{p}) \in V \mid \mathbf{p} \in H^1(0, 1; L^2(\Gamma)), \mathbf{p}(., 1) = 0 \text{ on } \Gamma\}$
- ➋ Compactness argument :
 $\mathbf{u}_m \rightarrow \mathbf{u}$ in $L^2(\Omega_s)$ and $\mathbf{p}_m \rightarrow \mathbf{p} = 0$ in $L^2(\Omega_f)$.
- ➌ Prove that $\|\mathbf{u}\|_{0, \Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0, \Gamma} = 1$
- ➍ Using (SA), we prove $\mathbf{u} = 0$: contradiction

Outline

1 Uniform Estimates

2 Equivalent Conditions

3 Numerical Simulations

Methodology

Framework : $\Gamma, \Gamma^\varepsilon$ smooth curves in \mathbb{R}^2 . Smooth data

- Step 1 : Derive an Asymptotic Expansion for $(\mathbf{u}_\varepsilon, p_\varepsilon)$ when $\varepsilon \rightarrow 0$

$$\mathbf{u}_\varepsilon(x) = \mathbf{u}_0(x) + \varepsilon \mathbf{u}_1(x) + \varepsilon^2 \mathbf{u}_2(x) + \dots,$$

$$p_\varepsilon(x) = p_0(x; \varepsilon) + \varepsilon p_1(x; \varepsilon) + \varepsilon^2 p_2(x; \varepsilon) + \dots, \quad p_j(x; \varepsilon) = p_j(t, \frac{s}{\varepsilon}).$$

- Step 2 : Derive Equivalent Conditions of order $k \in \mathbb{N}$:
 Identify a simpler problem satisfied by

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \dots + \varepsilon^k \mathbf{u}_k \quad \text{up to } \mathcal{O}(\varepsilon^{k+1})$$

- Step 3 : Stability of Equivalent models \mathbf{u}_ε^k
 - Prove well-posedness and estimates
 - Show that \mathbf{u}_ε^k satisfies

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C \varepsilon^{k+1}$$

Step 1 : Multiscale Expansion

First terms

At step $n = 0$, we obtain

$$\mathbf{p}_0 = \mathbf{0} ,$$

and then \mathbf{u}_0 solves the problem

$$\begin{cases} \nabla \cdot \underline{\mathcal{C}}(\mathbf{x}) \nabla \mathbf{u}_0 + \omega^2 \rho \mathbf{u}_0 = \mathbf{0} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_0) = -\mathbf{p}_0 \mathbf{n} & \text{on } \Gamma . \end{cases}$$

At step $n = 1$, we find successively

$$\mathbf{p}_1(., s) = (s - 1) (\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_{\mathbf{n}} \mathbf{p}_0)$$

and that \mathbf{u}_1 solves

$$\begin{cases} \nabla \cdot \underline{\mathcal{C}}(\mathbf{x}) \nabla \mathbf{u}_1 + \omega^2 \rho \mathbf{u}_1 = \mathbf{0} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_1) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} - \partial_{\mathbf{n}} \mathbf{p}_0 \mathbf{n} & \text{on } \Gamma . \end{cases}$$

Step 1 : Multiscale Expansion

First terms

At step $n = 2$,

$$\mathbf{p}_2(., s) = -(s^2 - 1) \frac{c(t)}{2} [\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_{\mathbf{n}} p_i] + (s - 1) \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}$$

and then, \mathbf{u}_2 solves

$$\begin{cases} \nabla \cdot \underline{C}(x) \underline{\nabla} \mathbf{u}_2 + \omega^2 \rho \mathbf{u}_2 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_2) = \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} - \frac{1}{2} c(t) [\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} - \partial_{\mathbf{n}} p_i] \mathbf{n} & \text{on } \Gamma . \end{cases}$$

Step 1 : Validation of the Asymptotic Expansion

Aim : proving Estimates for Remainders

$$\mathbf{r}_\varepsilon^{N+1} := \mathbf{u}_\varepsilon - \sum_{n=0}^N \varepsilon^n \mathbf{u}_n \quad \text{in } \Omega_s, \quad \text{and} \quad r_\varepsilon^{N+1} := p_\varepsilon - \sum_{n=0}^N \varepsilon^n p_n \quad \text{in } \Omega_f^\varepsilon$$

Evaluation of the right hand sides in

$$\left\{ \begin{array}{lcl} \kappa^2 \mathbf{r}_\varepsilon^{N+1} + \Delta \mathbf{r}_\varepsilon^{N+1} & = & f_\varepsilon \quad \text{in } \Omega_f^\varepsilon \\ \omega^2 \rho \mathbf{r}_\varepsilon^{N+1} + \nabla \cdot \underline{C}(x) \nabla \mathbf{r}_\varepsilon^{N+1} & = & 0 \quad \text{in } \Omega_s \\ \partial_{\mathbf{n}} \mathbf{r}_\varepsilon^{N+1} - \rho_f \omega^2 \mathbf{r}_\varepsilon^{N+1} \cdot \mathbf{n} & = & g_\varepsilon \quad \text{on } \Gamma \\ \mathbf{T}(\mathbf{r}_\varepsilon^{N+1}) + r_\varepsilon^{N+1} \mathbf{n} & = & 0 \quad \text{on } \Gamma \\ r_\varepsilon^{N+1} & = & 0 \quad \text{on } \Gamma^\varepsilon. \end{array} \right.$$

The RHS are explicit: $f_\varepsilon = \mathcal{O}(\varepsilon^{N-\frac{1}{2}})$ in Ω_f^ε ; $g_\varepsilon = \mathcal{O}(\varepsilon^N)$ on Γ

Optimal Estimates : $\|\mathbf{r}_\varepsilon^{N+1}\|_{1,\Omega_s} + \sqrt{\varepsilon} \|r_\varepsilon^{N+1}\|_{1,\Omega_f^\varepsilon} \leq C_N \varepsilon^{N+1}$

Step 2 : Equivalent Conditions on Γ

Identify a simpler problem

$$(\mathbf{P}_\varepsilon^k) \quad \begin{cases} \omega^2 \rho \mathbf{u}_\varepsilon^k + \nabla \cdot \underline{\mathcal{C}}(x) \underline{\nabla} \mathbf{u}_\varepsilon^k = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_\varepsilon^k) + \mathbf{B}_{k,\varepsilon}(\mathbf{u}_\varepsilon^k) = \mathbf{h}_{k,\varepsilon} & \text{on } \Gamma, \end{cases}$$

with $\mathbf{B}_{k,\varepsilon}$ a surfacic differential operator.

- $k \in \{0, 1, 2\}$ for a Dirichlet b.c. on Γ^ε
- $k \in \{0, 1\}$ for a Fourier-Robin b.c. on Γ^ε

Step 2 : Equivalent Conditions

Dirichlet b.c.

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_0) = -\mathbf{p}_0 \mathbf{n} \quad \text{on } \Gamma.$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - \varepsilon \omega^2 \rho_f \mathbf{u}_\varepsilon^1 \cdot \mathbf{n} \mathbf{n} = \mathbf{h}_{1,\varepsilon} \quad \text{on } \Gamma.$$

- $k = 2$:

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - \varepsilon \omega^2 \rho_f \left(1 - \frac{\varepsilon}{2} c(t)\right) \mathbf{u}_\varepsilon^2 \cdot \mathbf{n} \mathbf{n} = \mathbf{h}_{2,\varepsilon} \quad \text{on } \Gamma.$$

Step 2 : Equivalent Conditions

Example : Order 1

$\mathbf{u}_{1,\varepsilon} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1$ satisfies the elastic equation in Ω_s and

$$\mathbf{T}(\mathbf{u}_{1,\varepsilon}) - \varepsilon \omega^2 \rho_f \mathbf{u}_{1,\varepsilon} \cdot \mathbf{n} \mathbf{n} = -\mathbf{p}_i \mathbf{n} - \varepsilon \partial_{\mathbf{n}} \mathbf{p}_i \mathbf{n} - \varepsilon^2 \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} \quad \text{on } \Gamma$$

We infer a simpler problem

$$(\mathbf{P}_{\varepsilon}^1) \quad \begin{cases} \omega^2 \rho \mathbf{u}_{\varepsilon}^1 + \nabla \cdot \underline{\mathcal{C}}(x) \underline{\nabla} \mathbf{u}_{\varepsilon}^1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_{\varepsilon}^1) - \varepsilon \omega^2 \rho_f \mathbf{u}_{\varepsilon}^1 \cdot \mathbf{n} \mathbf{n} = \mathbf{h}_{1,\varepsilon} & \text{on } \Gamma, \end{cases}$$

Step 2 : Equivalent Conditions

Fourier-Robin b.c.

Fourier-Robin b.c. : $\partial_{\mathbf{n}} p_{\varepsilon} - i\kappa p_{\varepsilon} = 0$ on Γ^{ε}

- $k = 0$:

$$\mathbf{T}(\mathbf{u}_{\varepsilon}^0) - i\omega c \rho_f \mathbf{u}_{\varepsilon}^0 \cdot \mathbf{n} \mathbf{n} = \mathbf{g}_{0,\varepsilon} \quad \text{on } \Gamma$$

- $k = 1$:

$$\mathbf{T}(\mathbf{u}_{\varepsilon}^1) - i\omega c \rho_f [1 - \varepsilon c(t)] \mathbf{u}_{\varepsilon}^1 \cdot \mathbf{n} \mathbf{n} + \varepsilon c^2 \rho_f \partial_t^2 (\mathbf{u}_{\varepsilon}^1 \cdot \mathbf{n}) \mathbf{n} = \mathbf{g}_{1,\varepsilon} \quad \text{on } \Gamma$$

Step 3 : Stability of Equivalent Conditions

Step 3.1 : Prove that problems (P_ε^k) are well-posed, and uniform estimates.

Proposition

Under Assumption (SA), $\exists \varepsilon_0 > 0$ s.t. $\forall \varepsilon \in (0, \varepsilon_0)$, $\forall k \in \{0, 1, 2\}$, the problem (P_ε^k) with data $h_k \in L^2(\Gamma)$ has a unique solution $u_\varepsilon^k \in H^1(\Omega_s)$

$$\|u_\varepsilon^k\|_{1,\Omega_s} \leq C \|h_k\|_{0,\Gamma}$$

Application : Convergence of an asymptotic expansion of u_ε^k as $\varepsilon \rightarrow 0$.

Step 3 : Order of Equivalent Conditions

Step 3.2 : Prove that the solution \mathbf{u}_ε^k satisfies an error estimate

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C\varepsilon^{k+1}.$$

Proof :

(i) Derive an expansion of \mathbf{u}_ε^k and show that :

$$\begin{aligned}\mathbf{u}_\varepsilon &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \mathbf{r}_\varepsilon^{k+1}, \\ \mathbf{u}_\varepsilon^k &= \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \tilde{\mathbf{r}}_\varepsilon^{k+1}.\end{aligned}$$

Hence,

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} = \|\mathbf{r}_\varepsilon^{k+1} - \tilde{\mathbf{r}}_\varepsilon^{k+1}\|_{1,\Omega_s}$$

(ii) There holds

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^{k+1}) + \mathbf{B}_{k,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^{k+1}) = \mathcal{O}(\varepsilon^{k+1}) \quad \text{on } \Gamma,$$

According to the step 3.1, we infer the uniform error estimates:

$$\|\tilde{\mathbf{r}}_\varepsilon^{k+1}\|_{1,\Omega_s} \leq C\varepsilon^{k+1}.$$

Outline

1 Uniform Estimates

2 Equivalent Conditions

3 Numerical Simulations

Finite Element Method

Joint work with : J. Diaz (INRIA).

We use a Discontinuous Galerkin Method (IPDGM).

- $\omega = 1.5 \times 10^6$
- Ω_s is a disk of radius 0.01
- Source : $p_i(\mathbf{x}) = \exp(i\omega \mathbf{x} \cdot \mathbf{d})$ with $\mathbf{d} = (1, 0)$
- \mathbb{P}_3 -finite elements (Lagrange) available in the Library Hou10ni
- Navier equation : $\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \omega^2 \rho \mathbf{u} = 0$ in Ω_s .

$$\text{Lamé coefficients : } \mu = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

Young's modulus : $E = 70 \times 10^9$ Pa (aluminum)

Poisson ratio : $\nu = 0.33$

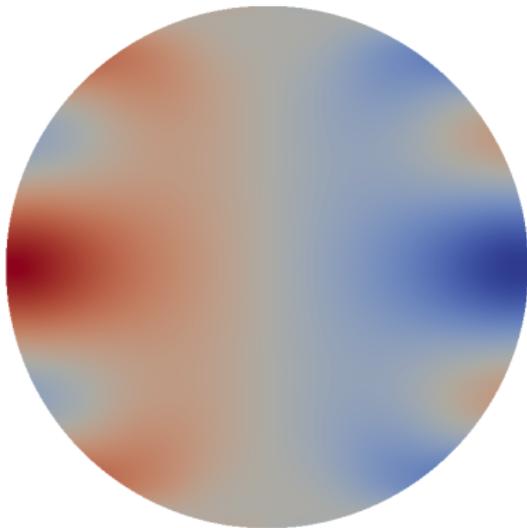
- $c = 1500 \text{ m.s}^{-1}$
 $\rho_f = 1000 \text{ kg.m}^{-3}$
 $\rho = 2700 \text{ kg.m}^{-3}$.

Numerical Tests

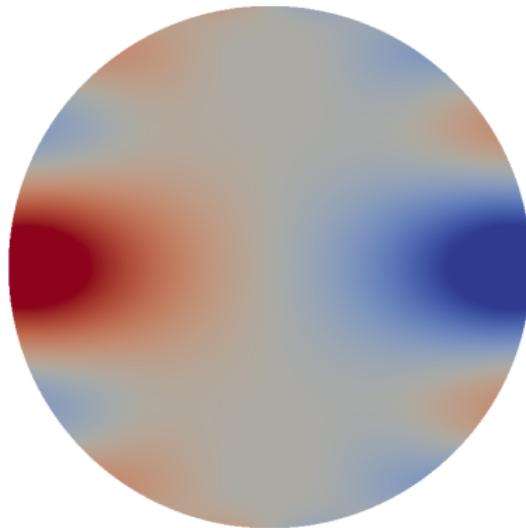
$\varepsilon = 0.001$;

\mathbf{u}_ε : analytical solution;

\mathbf{u}_ε^2 : FE solution, equivalent model of order 2.



$|\operatorname{Re} \mathbf{u}_\varepsilon|$



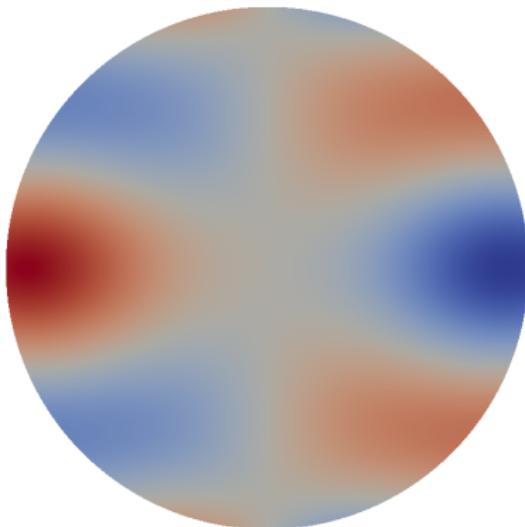
$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$

Numerical Simulations

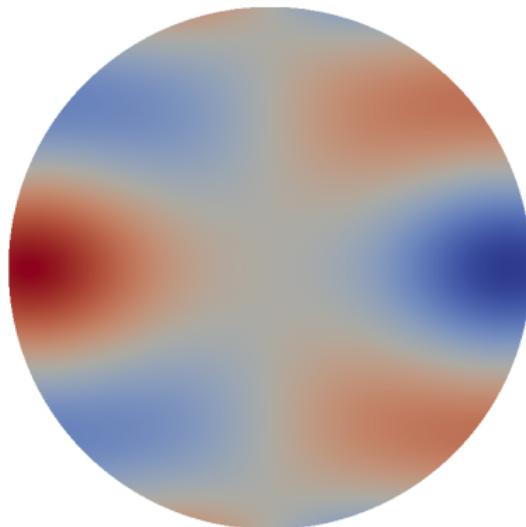
$\varepsilon = 0.0001$;

\mathbf{u}_ε : analytical solution;

\mathbf{u}_ε^2 : FE solution, equivalent model of order 2.



$|\operatorname{Re} \mathbf{u}_\varepsilon|$



$|\operatorname{Re} \mathbf{u}_\varepsilon^2|$

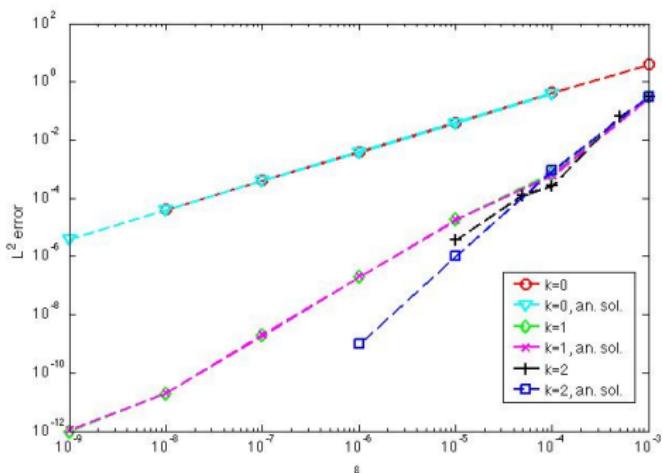
Convergence of the models

Dirichlet b.c.

For $k \in \{0, 1, 2\}$, we plot the L^2 -errors $\|u_\varepsilon - u_\varepsilon^k\|_{0,\Omega_s}$ w.r.t. ε :

u_ε : analytical solution

u_ε^k : FE solution / analytical solution



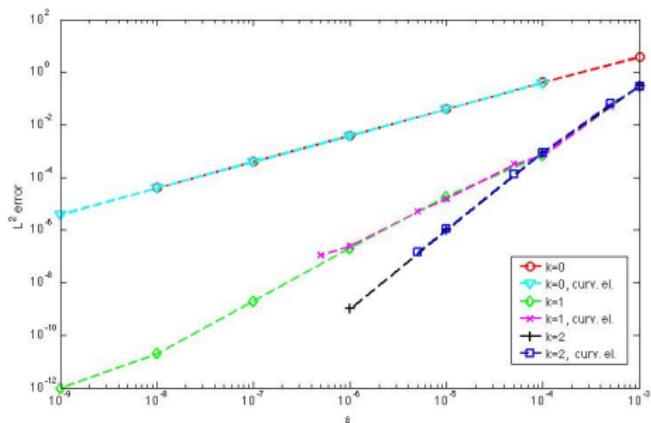
Convergence of the models

Dirichlet b.c.

For $k \in \{0, 1, 2\}$, we plot the L^2 -errors $\|u_\varepsilon - u_\varepsilon^k\|_{0,\Omega_s}$ w.r.t. ε :

u_ε : analytical solution

u_ε^k : analytical solution / FE solutions with Curved elements



The convergence rate coincide with the theory $\|u_\varepsilon - u_\varepsilon^k\|_{0,\Omega_s} = \mathcal{O}(\varepsilon^{k+1})$

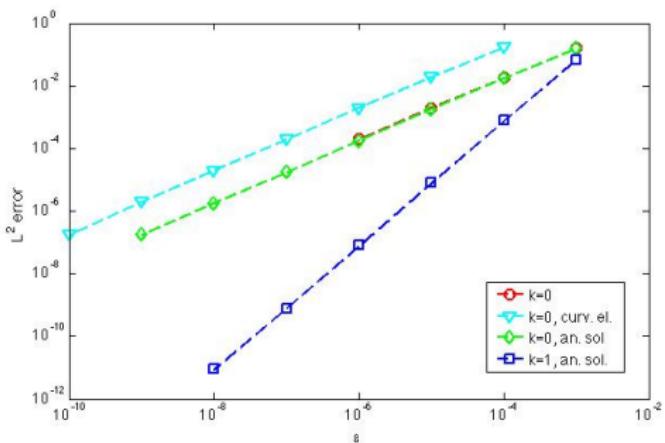
Convergence of the models

Fourier-Robin b.c.

For $k \in \{0, 1\}$, we plot the L^2 -errors $\|u_\varepsilon - u_\varepsilon^k\|_{0,\Omega_s}$ w.r.t. ε ,

u_ε : analytical solution

u_ε^k : analytical solution / FE solutions



Prospects

- ① 3D Modeling
- ② Higher Order Equivalent Conditions
- ③ Thin layer with varying thickness ...

Thank you for attention!