



HAL
open science

Asymptotic Description of the Magnetic Potential near a Corner Singularity in the Bidimensional Eddy-Current Problem

Monique Dauge, Patrick Dular, Laurent Krähenbühl, Victor Péron, Ronan Perrussel, Clair Poignard

► **To cite this version:**

Monique Dauge, Patrick Dular, Laurent Krähenbühl, Victor Péron, Ronan Perrussel, et al.. Asymptotic Description of the Magnetic Potential near a Corner Singularity in the Bidimensional Eddy-Current Problem. WCCM 2012, Jul 2012, Sao Paulo, Brazil. hal-00768026v1

HAL Id: hal-00768026

<https://inria.hal.science/hal-00768026v1>

Submitted on 8 Dec 2016 (v1), last revised 9 Dec 2016 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Asymptotic Description of the Magnetic Potential near a Corner Singularity in the Bidimensional Eddy-Current Problem

Monique Dauge^{*} Patrick Dular[♡] Laurent Krähenbühl[◇]
Victor Péron[♠] Ronan Perrussel[†] Clair Poignard[♣]

^{*}IRMAR (Rennes), [♡]ACE (Liège), [◇]Ampère (Lyon), [♠]EPI Magique3D & LMAP, UPPA (Pau),
[†]LAPLACE (Toulouse), [♣]EPI MC2 (Bordeaux)

December 8, 2016

Motivations

A usual model for electrical engineering applications

- Dielectric relaxation vs magnetic diffusion:

$$\tau_e = \frac{\epsilon}{\sigma} \ll \tau_m = \mu\sigma L^2$$

L, ϵ, σ, μ : characteristic length, permittivity, conductivity and permeability.

\implies **magneto-quasistatic** (eddy-current) model.

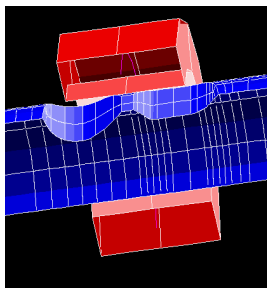
$$\mathbf{curl} \mathbf{E} = -\frac{\partial(\mu\mathbf{H})}{\partial t} \quad \text{div}(\mu\mathbf{H}) = 0,$$

$$\mathbf{curl} \mathbf{H} = \sigma\mathbf{E} + \frac{\partial(\epsilon\mathbf{E})}{\partial t} \quad \text{div}(\epsilon\mathbf{E}) = 0.$$

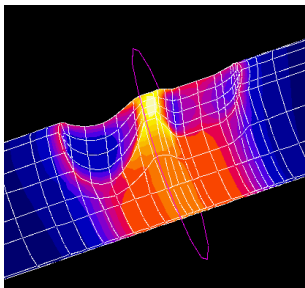
Motivations

Usage of the magneto-quasistatic model

- Electrothermics



(a) Position of the inductor.



(b) Induced current density.

Figure : Surface steel hardening of a shift fork. PhD thesis Sven Wanser ('93 - Renault+EDF/ECL).

- Others: non-destructive testing, most of electromechanical systems.

A computational simplification

Skin Effect

- Skin effect: “exponential decrease” of the electromagnetic field in a volume conductor, *i.e.*

$$\delta = \sqrt{\frac{1}{\mu\pi f\sigma}} \ll L.$$

δ : penetration (skin) depth.

- Instead of strongly meshing close to the surface of the volume conductors: **surface impedance conditions**. Most usual:

$$Z_\delta(\mathbf{H} \times \mathbf{n}) = \mathbf{E}_t, \text{ with } Z_\delta = \frac{1 + i}{\sigma\delta}.$$

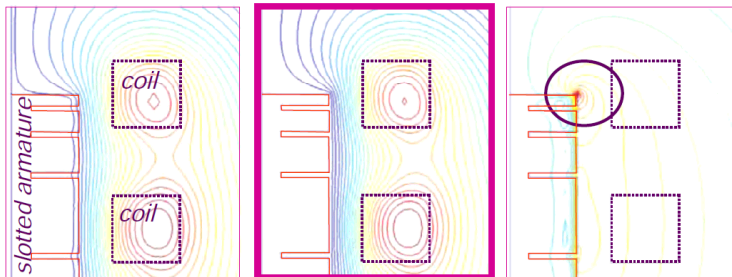
\mathbf{n} : unitary normal to the conductor, \mathbf{E}_t : tangential component of the electric field.

- For smooth enough surface, the impedance can even be “more accurate”.

A computational simplification

Difficulties in the corner

- Example in a domain decomposition process.



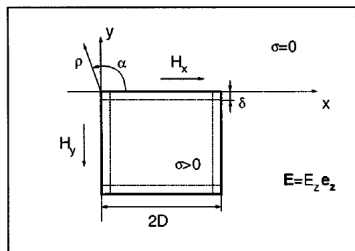
global solution = surf. imp. solution + iterative corrections

P. Dular et al. (Compel, 2007)

A computational simplification

Difficulties in the corner

- Few authors have considered impedance modifications :
“Surface impedance boundary conditions near corners and edges:
rigorous consideration”. [IEEE Trans. on Mag., 2002]
- Application of the same asymptotic expansion as in the smooth surface case. Impedance obtained:



$$Z_x = -\frac{\tan(\pi/6)}{\rho\sigma}, \quad Z_y = \frac{\tan(\pi/6)}{\rho\sigma}.$$

\implies **blows up in the corner** for any $\sigma < +\infty$.

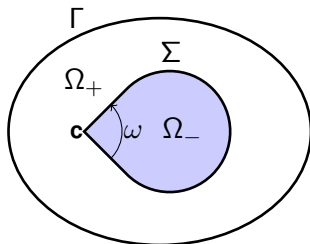
It is not valid for non-magnetic materials with $\sigma < +\infty$ [Buret *et al*, IEEE Trans. on Mag., 2012]

Statement of the Problem

The magnetic vector potential \mathcal{A} satisfies

$$\left\{ \begin{array}{l} -\Delta \mathcal{A}^+ = \mu_0 J \text{ in } \Omega_+, \\ -\Delta \mathcal{A}^- + 4i\zeta^2 \mathcal{A}^- = 0 \text{ in } \Omega_-, \\ \mathcal{A}^+ = 0 \text{ on } \Gamma, \end{array} \right. \quad \begin{array}{l} [\mathcal{A}]_\Sigma = 0, \text{ on } \Sigma, \\ [\partial_n \mathcal{A}]_\Sigma = 0, \text{ on } \Sigma. \end{array} \quad (1)$$

with $\zeta^2 = \kappa\mu_0\sigma/4$, and a smooth data J .



Aim : Description of the magnetic potential \mathcal{A} near the corner \mathbf{c} in 2D

Statement of the Problem

Weak solutions

Variational Formulation

Find $\mathcal{A} \in H_0^1(\Omega)$ such that $\forall \mathcal{A}_* \in H_0^1(\Omega)$

$$\int_{\Omega_+} \nabla \mathcal{A}^+ \cdot \nabla \overline{\mathcal{A}_*^+} \, dx + \int_{\Omega_-} \nabla \mathcal{A}^- \cdot \nabla \overline{\mathcal{A}_*^-} + 4i\zeta^2 \mathcal{A}^- \overline{\mathcal{A}_*^-} \, dx = \mu_0 \int_{\Omega} J \overline{\mathcal{A}_*}$$

Proposition

There exists a unique solution \mathcal{A} in $H_0^1(\Omega)$ to problem (1). Moreover, \mathcal{A} belongs to $H^{\frac{5}{2}-\varepsilon}(\Omega)$ for any $\varepsilon > 0$.

In particular, by **Sobolev embedding**, we obtain that \mathcal{A} belongs to $\mathcal{C}^1(\overline{\Omega})$.

Corner asymptotics

- In general \mathcal{A} does not belong to $C^2(\overline{\Omega})$ but \mathcal{A} possesses a corner asymptotic expansion as the distance r to \mathbf{c} goes to zero

$$\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} \mathfrak{S}^{0,0}(r, \theta) + \sum_{k \geq 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} \mathfrak{S}^{k,p}(r, \theta)$$

- Key points for the behavior of \mathcal{A} in the vicinity of \mathbf{c} :
 - Derivation of the **singular functions** $\mathfrak{S}^{k,p}$
 - Determination of the **singular coefficients** $\Lambda^{k,p}$
- We can provide a constructive procedure to determine $\mathfrak{S}^{k,p}$

$$\mathfrak{S}^{k,p}(r, \theta) = \underbrace{r^k \cos(k\theta - p\pi/2)}_{\text{kernel functions of } \Delta} + \underbrace{r^k \sum_{j \geq 1} (i\zeta^2 r^2)^j \sum_{n=0}^j \log^n r \Phi_{j,n}^{k,p}(\theta)}_{\text{shadow terms}}$$

Outline

- 1 **The case $\zeta = 0$** : For the determination of coefficients $\Lambda^{k,p}$ we introduce the method of **dual singular functions**
- 2 **The case $\zeta \neq 0$** : we introduce the method of **quasi-dual singular functions**. This method is an approximate method. We prove estimates for its convergence.
- 3 Numerical simulations illustrate the theoretical results.

Laplace Operator ($\zeta = 0$)

- We consider the solution \mathcal{A} to

$$\begin{cases} -\Delta \mathcal{A} = \mu_0 J \text{ in } \Omega, \\ \mathcal{A} = 0 \text{ on } \Gamma, \end{cases}$$

with a smooth right hand side with support outside the interior point \mathbf{c} .

- \mathcal{A} admits the Taylor expansion at \mathbf{c} :

$$\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} + \sum_{k \geq 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} r^k \cos(k\theta - p\pi/2)$$

Methods to extract the coefficients $\Lambda^{k,p}$

- The method of moments :
It uses the orthogonality of the angular parts $\cos(k\theta - p\pi/2)$
- The dual function method :
It is the application to the present smooth case of a general method [Mazya-Plamenevskii, '78] valid for corner coefficient extraction

The Dual Function Method

- Dual harmonic functions singular at $r = 0$:

$$\mathfrak{k}_0^{k,p}(r, \theta) = \begin{cases} -\frac{1}{2\pi} \log r, & \text{if } k = 0, p = 0, \\ \frac{1}{2k\pi} r^{-k} \cos(k\theta - p\pi/2), & \text{if } k \geq 1, p = 0, 1. \end{cases}$$

- For $R > 0$, let us introduce the bilinear form

$$\mathcal{J}_R(K, A) = \int_{r=R} (K \partial_r A - \partial_r K A) R d\theta.$$

Proposition

Let \mathcal{A} be the solution to the Laplace equation with a smooth right hand side with support outside the ball $\mathcal{B}(\mathbf{c}, R)$. Then

$$\mathcal{J}_R(\mathfrak{k}_0^{0,0}, \mathcal{A}) = \Lambda^{0,0} \quad \text{and} \quad \mathcal{J}_R(\mathfrak{k}_0^{k,p}, \mathcal{A}) = \Lambda^{k,p}, \quad k \geq 1, p = 0, 1.$$

The Quasi-Dual Function Method ($\zeta \neq 0$)

- [Costabel *et al*, '04; '12]: The QDFM for straight edges, circular edges and homogeneous operators with constant coefficients.
- We use quasi-dual functions $\mathfrak{R}_m^{k,p}$:

$$\mathfrak{R}_m^{k,p}(r, \theta) = \mathfrak{t}_0^{k,p}(r, \theta) + \sum_{j=1}^m (i\zeta^2)^j \underbrace{\mathfrak{t}_j^{k,p}(r, \theta)}_{\text{shadow term}}$$

where $\mathfrak{t}_j^{k,p}$ are solution to

$$\begin{cases} \Delta \mathfrak{t}_j^{k,p+} = 0, & \text{in } \mathcal{S}_+, \\ \Delta \mathfrak{t}_j^{k,p-} = 4\mathfrak{t}_{j-1}^{k,p-}, & \text{in } \mathcal{S}_- \end{cases}$$

in the space :

$$r^{-k+2j} \text{Span} \{ \log^q r \Phi(\theta), \quad q \leq j+1, \quad \Phi \in \mathcal{C}^1(\mathbb{T}), \quad \Phi^\pm \in \mathcal{C}^\infty(\overline{\mathbb{T}}_\pm) \}$$

Extraction of coefficients

The extraction of $\Lambda^{k,p}$ is performed through the evaluation of quantities

$$\mathcal{J}_R(\mathfrak{R}_m^{k,p}, \mathcal{A}), \quad k = 0, 1, 2, \dots$$

Theorem

Let \mathcal{A} be the solution to problem (1), under the assumptions of the introduction. Let $k \in \mathbb{N}$ and $p \in \{0, 1\}$ ($p = 0$ if $k = 0$). Let m such that $2m + 2 > k$. There exist coefficients $\mathcal{J}^{k,p;k',p'}$ independent of R and \mathcal{A} such that

$$\mathcal{J}_R(\mathfrak{R}_m^{k,p}, \mathcal{A}) \underset{R \rightarrow 0}{=} \Lambda^{k,p} + \sum_{\ell=1}^{\lfloor k/2 \rfloor} \mathcal{J}^{k,p;k-2\ell,p} \Lambda^{k-2\ell,p} + \mathcal{O}(R^{-k} R_0^{2m+2} \log R),$$

where $R_0 = \zeta R (1 + \sqrt{|\log R|})$.

Extraction of coefficients

Example

- For $k = 0$

$$\Lambda^{0,0} \underset{R \rightarrow 0}{=} \mathcal{J}_R(\mathcal{R}_m^{0,0}, \mathcal{A}) + \mathcal{O}(R_0^{2+2m} \log R)$$

- For $k = 1$

$$\Lambda^{1,0} \underset{R \rightarrow 0}{=} \mathcal{J}_R(\mathcal{R}_m^{1,0}, \mathcal{A}) + \mathcal{O}(R^{-1} R_0^{2m+2})$$

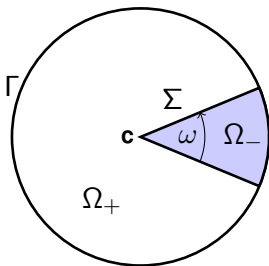
- For $k = 2$, we need $m \geq 1$

$$\Lambda^{2,0} \underset{R \rightarrow 0}{=} \mathcal{J}_R(\mathcal{R}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0;0,0} \Lambda^{0,0} + \mathcal{O}(R^{-2} R_0^4 \log R)$$

The Problem

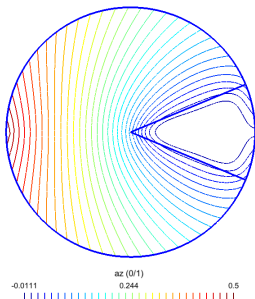
$$\left\{ \begin{array}{l} -\Delta \mathcal{A}^+ = 0 \text{ in } \Omega_+, \\ -\Delta \mathcal{A}^- + 4i\zeta^2 \mathcal{A}^- = 0 \text{ in } \Omega_-, \\ \mathcal{A}^+ = \frac{|\theta|}{\pi} \text{ on } \Gamma, \end{array} \right. \quad \begin{array}{l} [\mathcal{A}]_{\Sigma} = 0, \text{ on } \Sigma, \\ [\partial_n \mathcal{A}]_{\Sigma} = 0, \text{ on } \Sigma. \end{array}$$

- Ω : disk of radius 50 mm.
- Conducting sector : $\omega = \pi/4$
- $\zeta = 1/(5\sqrt{2}) \text{ mm}^{-1}$ corresponds to a physical skin depth of 5 mm.

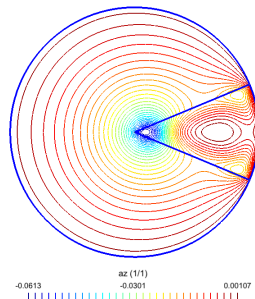


Finite Element Solution

We use P_2 finite elements



(b) Real part of the solution \mathcal{A} .

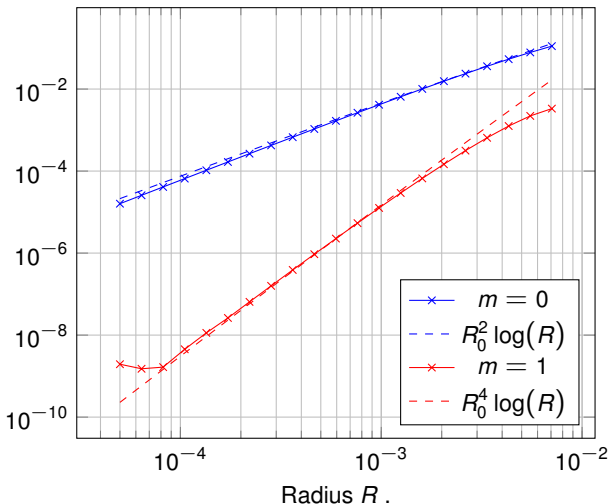


(c) Imaginary part of the solution \mathcal{A} .

Accuracy for the computation of $\Lambda^{0,0}$ as a function of R

Recall $\Lambda^{0,0} = \mathcal{J}_R(\mathcal{R}_m^{0,0}, \mathcal{A}) + \mathcal{O}(R_0^{2+2m} \log R)$.

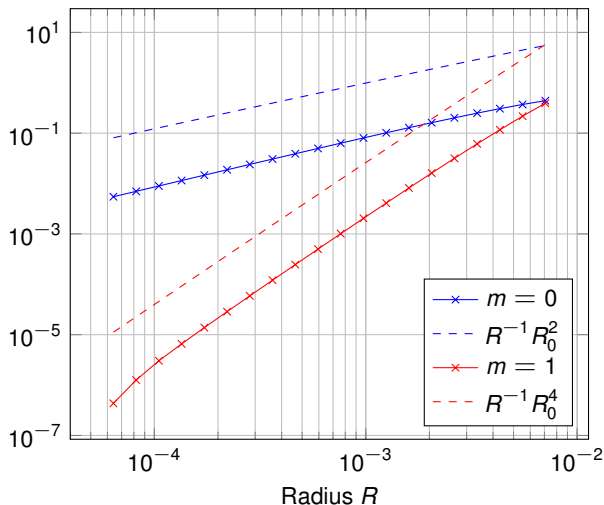
We plot $|\mathcal{J}_R(\mathcal{R}_m^{0,0}, \mathcal{A}) - \Lambda|_c|$ w.r.t. the radius R , for $m = 0, 1$



Computation of the coefficient $\Lambda^{1,0}$

Recall $\Lambda^{1,0} = \mathcal{J}_R(\mathcal{R}_m^{1,0}, \mathcal{A}) + \mathcal{O}(R^{-1}R_0^{2m+2})$.

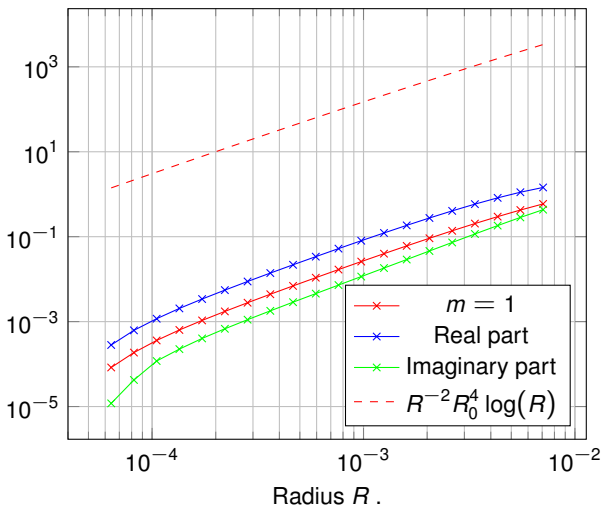
We plot $|\mathcal{J}_R(\mathcal{R}_m^{1,0}, \mathcal{A}) - \mathcal{J}_{R_{\min}}(\mathcal{R}_m^{1,0}, \mathcal{A})|$.



Computation of the coefficient $\Lambda^{2,0}$

Recall $\Lambda^{2,0} = \mathcal{J}_R(\mathfrak{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0;0,0}\Lambda^{0,0} + \mathcal{O}(R^{-2}R_0^4 \log R)$.

We plot $|\mathcal{J}_R(\mathfrak{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}_{R_{\min}}(\mathfrak{K}_1^{2,0}, \mathcal{A})|$.



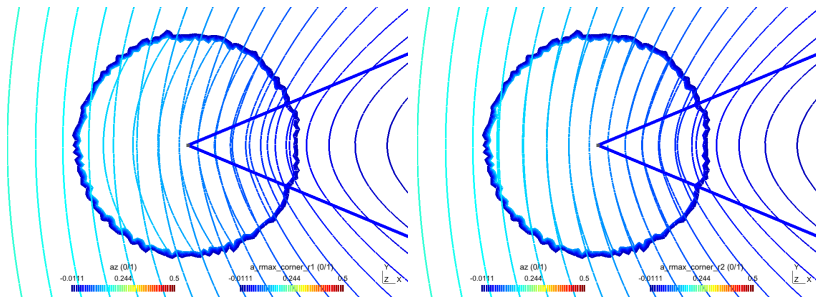
Comparison of the FE solution and of the local expansion

Expansion restricted to order 1:

$$\mathcal{J}_{R_{\min}}(\mathcal{K}_1^{0,0}, \mathcal{A})(1 + i\zeta^2 \mathbf{s}_1^{0,0}) + \mathcal{J}_{R_{\min}}(\mathcal{K}_1^{1,0}, \mathcal{A})\mathbf{s}_0^{1,0},$$

Expansion restricted to order 2: adding to the above expression the term

$$(\mathcal{J}_{R_{\min}}(\mathcal{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0;0,0} \mathcal{J}_{R_{\min}}(\mathcal{K}_1^{0,0}, \mathcal{A}))\mathbf{s}_0^{2,0}.$$



(d) Expansion restricted to order 1. Real part.

(e) Expansion restricted to order 2. Real part.

Thank you for your attention!