

# Asymptotic Description of the Magnetic Potential near a Corner Singularity in the Bidimensional Eddy-Current Problem

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### **Motivations**

A usual model for electrical engineering applications

Dielectric relaxation vs magnetic diffusion:

$$au_{\mathrm{e}} = rac{\epsilon}{\sigma} \ll au_{\mathrm{m}} = \mu \sigma L^2$$

L,  $\epsilon$ ,  $\sigma$ ,  $\mu$ : characteristic length, permittivity, conductivity and permeability. ⇒ magneto-quasistatic (eddy-current) model.

$$\begin{aligned} & \operatorname{curl} \mathbf{E} = -\frac{\partial(\mu \mathbf{H})}{\partial t} & \operatorname{div}(\mu \mathbf{H}) = \mathbf{0}, \\ & \operatorname{curl} \mathbf{H} = \sigma \mathbf{E} + \frac{\partial(\epsilon \mathbf{E})}{\partial t} & \operatorname{div}(\epsilon \mathbf{E}) = \mathbf{0}. \end{aligned}$$

### **Motivations**

Usage of the magneto-quasistatic model

• Electrothermics



(a) Position of the inductor.



(b) Induced current density.

**Figure :** Surface steel hardening of a shift fork. PhD thesis Sven Wanser ('93 - Renault+EDF/ECL).

• Others: non-destructive testing, most of electromechanical systems.

### A computational simplification Skin Effect

• Skin effect: "exponential decrease" of the electromagnetic field in a volume conductor, *i.e.* 

$$\delta = \sqrt{\frac{1}{\mu \pi f \sigma}} \ll L.$$

 $\delta:$  penetration (skin) depth.

• Instead of strongly meshing close to the surface of the volume conductors: surface impedance conditions. Most usual:

$$Z_{\delta}(\mathbf{H} imes \mathbf{n}) = \mathbf{E}_t, ext{ with } Z_{\delta} = rac{1+\mathrm{i}}{\sigma\delta}.$$

**n**: unitary normal to the conductor,  $E_t$ : tangential component of the electric field.

• For smooth enough surface, the impedance can even be "more accurate".

#### A computational simplification Difficulties in the corner

• Example in a domain decomposition process.



global solution = surf. imp. solution + iterative corrections *P. Dular et al. (Compel, 2007)* 

# A computational simplification

- Few authors have considered impedance modifications : "Surface impedance boundary conditions near corners and edges: rigorous consideration". [IEEE Trans. on Mag., 2002]
- Application of the same asymptotic expansion as in the smooth surface case. Impedance obtained:



$$Z_x = -\frac{\tan(\pi/6)}{
ho\sigma}, \ Z_y = \frac{\tan(\pi/6)}{
ho\sigma}.$$

 $\implies$  blows up in the corner for any  $\sigma < +\infty.$ 

It is not valid for non-magnetic materials with  $\sigma < +\infty$  [Buret *et al*, IEEE Trans. on Mag., 2012]

## **Statement of the Problem**

The magnetic vector potential  $\mathcal{A}$  satisfies

$$\begin{cases} -\Delta \mathcal{A}^{+} = \mu_{0} J \text{ in } \Omega_{+}, & [\mathcal{A}]_{\Sigma} = 0, \text{ on } \Sigma, \\ -\Delta \mathcal{A}^{-} + 4i \zeta^{2} \mathcal{A}^{-} = 0 \text{ in } \Omega_{-}, & [\partial_{n} \mathcal{A}]_{\Sigma} = 0, \text{ on } \Sigma. \end{cases}$$
(1)  
$$\mathcal{A}^{+} = 0 \text{ on } \Gamma, & [\partial_{n} \mathcal{A}]_{\Sigma} = 0, \text{ on } \Sigma. \end{cases}$$

with  $\zeta^2 = \kappa \mu_0 \sigma / 4$ , and a smooth data J.



Aim : Description of the magnetic potential  $\mathcal A$  near the corner **c** in 2D

# Statement of the Problem

#### **Variational Formulation**

Find 
$$\mathcal{A}\in H^1_0(\Omega)$$
 such that  $orall \mathcal{A}_*\in H^1_0(\Omega)$ 

$$\int_{\Omega_{+}} \nabla \mathcal{A}^{+} \cdot \nabla \overline{\mathcal{A}_{*}^{+}} \, \mathrm{d}x + \int_{\Omega_{-}} \nabla \mathcal{A}^{-} \cdot \nabla \overline{\mathcal{A}_{*}^{-}} + 4\mathrm{i}\zeta^{2} \mathcal{A}^{-} \overline{\mathcal{A}_{*}^{-}} \, \mathrm{d}x = \mu_{0} \int_{\Omega} J \overline{\mathcal{A}_{*}}$$

#### Proposition

There exists a unique solution  $\mathcal{A}$  in  $H^1_0(\Omega)$  to problem (1). Moreover,  $\mathcal{A}$  belongs to  $H^{\frac{5}{2}-\varepsilon}(\Omega)$  for any  $\varepsilon > 0$ .

In particular, by Sobolev embedding, we obtain that  $\mathcal{A}$  belongs to  $\mathcal{C}^1(\overline{\Omega})$ .

## **Corner asymptotics**

In general A does not belong to C<sup>2</sup>(\$\overline{\Omega\$}\$) but A possesses a corner asymptotic expansion as the distance r to c goes to zero

$$\mathcal{A}(r,\theta) \underset{r \to 0}{\sim} \Lambda^{0,0} \mathfrak{S}^{0,0}(r,\theta) + \sum_{k \ge 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} \mathfrak{S}^{k,p}(r,\theta)$$

- Key points for the behavior of  ${\cal A}$  in the vicinity of  ${\bf c}$  :
  - Derivation of the singular functions  $\mathfrak{S}^{k,p}$
  - Determination of the singular coefficients  $\Lambda^{k,p}$
- We can provide a constructive procedure to determine  $\mathfrak{S}^{k,p}$

$$\mathfrak{S}^{k,p}(r,\theta) = \underbrace{r^k \cos(k\theta - p\pi/2)}_{kernel \ functions \ of \ \Delta} + \underbrace{r^k \sum_{j \ge 1} (i\zeta^2 r^2)^j \sum_{n=0}^{j} \log^n r \ \Phi_{j,n}^{k,p}(\theta)}_{shadow \ terms}$$

## Outline

- The case  $\zeta = 0$ : For the determination of coefficients  $\Lambda^{k,p}$  we introduce the method of dual singular functions
- 2 The case  $\zeta \neq 0$ : we introduce the method of quasi-dual singular functions. This method is an approximate method. We prove estimates for its convergence.
- Numerical simulations illustrate the theoretical results.

# Laplace Operator ( $\zeta = 0$ )

 $\bullet\,$  We consider the solution  ${\cal A}$  to

$$\left\{ \begin{aligned} -\Delta \mathcal{A} &= \mu_0 J \text{ in } \Omega, \\ \mathcal{A} &= 0 \text{ on } \Gamma, \end{aligned} \right.$$

with a smooth right hand side with support outside the interior point  $\boldsymbol{c}.$ 

•  ${\cal A}$  admits the Taylor expansion at  ${\bf c}$  :

$$\mathcal{A}(r,\theta) \underset{r \to 0}{\sim} \Lambda^{0,0} + \sum_{k \ge 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} r^k \cos(k\theta - p\pi/2)$$

# Methods to extract the coefficients $\Lambda^{k,p}$

• The method of moments :

It uses the orthogonality of the angular parts  $\cos(k\theta - p\pi/2)$ 

• The dual function method :

It is the application to the present smooth case of a general method [Mazya-Plamenevskii, '78] valid for corner coefficient extraction

# **The Dual Function Method**

• Dual harmonic functions singular at r = 0:

$$\mathfrak{k}_{0}^{k,p}(r,\theta) = \begin{cases} -\frac{1}{2\pi} \log r, & \text{if } k = 0, \ p = 0, \\ \frac{1}{2k\pi} r^{-k} \cos(k\theta - p\pi/2), & \text{if } k \ge 1, \ p = 0, 1. \end{cases}$$

• For R > 0, let us introduce the bilinear form

$$\mathcal{J}_{R}(K, A) = \int_{r=R} (K \partial_{r} A - \partial_{r} K A) R \mathrm{d}\theta.$$

#### Proposition

Let A be the solution to the Laplace equation with a smooth right hand side with support outside the ball  $\mathcal{B}(\mathbf{c}, R)$ . Then

$$\mathcal{J}_R(\mathfrak{k}^{0,0}_0,\mathcal{A})=\Lambda^{0,0}$$
 and  $\mathcal{J}_R(\mathfrak{k}^{k,p}_0,\mathcal{A})=\Lambda^{k,p},\ k\geqslant 1,\ p=0,1.$ 

# The Quasi-Dual Function Method ( $\zeta \neq 0$ )

- [Costabel et al, '04; '12] : The QDFM for straight edges, circular edges and homogeneous operators with constant coefficients.
- We use guasi-dual functions  $\Re_m^{k,p}$ :

$$\mathfrak{K}_{m}^{k,p}(r,\theta) = \mathfrak{k}_{0}^{k,p}(r,\theta) + \sum_{j=1}^{m} (\mathrm{i}\zeta^{2})^{j} \underbrace{\mathfrak{t}_{j}^{k,p}(r,\theta)}_{shadow \ term}$$

where  $\mathfrak{k}_{i}^{k,p}$  are solution to

$$\begin{cases} \Delta \mathfrak{k}_{j}^{k,p+} = 0, & \text{in } \mathcal{S}_{+} , \\ \Delta \mathfrak{k}_{j}^{k,p-} = 4 \mathfrak{k}_{j-1}^{k,p-}, & \text{in } \mathcal{S}_{-} \end{cases}$$

in the space :

$$r^{-k+2j}$$
Span  $\left\{\log^q r \, \Phi(\theta), \ q \leqslant j+1 \ , \ \Phi \in \mathcal{C}^1(\mathbb{T}), \ \Phi^{\pm} \in \mathcal{C}^{\infty}(\overline{\mathbb{T}}_{\pm}) \right\}$ 

# **Extraction of coefficients**

The extraction of  $\Lambda^{k,\rho}$  is performed through the evaluation of quantities

```
\mathcal{J}_R(\mathbf{\mathfrak{K}}_m^{k,p},\mathcal{A}), \quad k=0,1,2,\ldots
```

#### Theorem

Let  $\mathcal{A}$  be the solution to problem (1), under the assumptions of the introduction. Let  $k \in \mathbb{N}$  and  $p \in \{0, 1\}$  (p = 0 if k = 0). Let m such that 2m + 2 > k. There exist coefficients  $\mathcal{J}^{k,p;k',p'}$  independent of R and  $\mathcal{A}$  such that

$$\mathcal{J}_{R}(\mathfrak{K}^{k,p}_{m},\mathcal{A}) \underset{R \to 0}{=} \Lambda^{k,p} + \sum_{\ell=1}^{[k/2]} \mathcal{J}^{k,p;\,k-2\ell,p} \Lambda^{k-2\ell,p} + \mathcal{O}(R^{-k}R_{0}^{2m+2}\log R),$$

where  $R_0 = \zeta R (1 + \sqrt{|\log R|})$ .

# **Extraction of coefficients**

### Example

• For k = 0

$$\Lambda^{0,0} \underset{R \to 0}{=} \mathcal{J}_{R}(\mathfrak{K}^{0,0}_{m}, \mathcal{A}) + \mathcal{O}(R_{0}^{2+2m} \log R)$$

• For 
$$k = 1$$
  
$$\Lambda^{1,0} =_{R \to 0} \mathcal{J}_{R}(\mathfrak{K}_{m}^{1,0}, \mathcal{A}) + \mathcal{O}(R^{-1}R_{0}^{2m+2})$$

• For 
$$k = 2$$
, we need  $m \ge 1$ 

$$\Lambda^{2,0} =_{R \to 0} \mathcal{J}_{R}(\mathfrak{K}^{2,0}_{1}, \mathcal{A}) - \mathcal{J}^{2,0;0,0} \Lambda^{0,0} + \mathcal{O}(R^{-2} R_{0}^{4} \log R)$$

## **The Problem**

$$\begin{cases} -\Delta \mathcal{A}^{+} = 0 \text{ in } \Omega_{+}, \\ -\Delta \mathcal{A}^{-} + 4i\zeta^{2}\mathcal{A}^{-} = 0 \text{ in } \Omega_{-}, & [\mathcal{A}]_{\Sigma} = 0, \text{ on } \Sigma, \\ \mathcal{A}^{+} = \frac{|\theta|}{\pi} \text{ on } \Gamma, & [\partial_{n}\mathcal{A}]_{\Sigma} = 0, \text{ on } \Sigma. \end{cases}$$

- $\Omega$  : disk of radius 50 mm.
- Conducting sector :  $\omega = \pi/4$
- $\zeta = 1/(5\sqrt{2}) \, \mathrm{mm^{-1}}$  corresponds to a physical skin depth of 5 mm.



## **Finite Element Solution**

We use  $P_2$  finite elementss



# Accuracy for the computation of $\Lambda^{0,0}$ as a function of R

Recall  $\Lambda^{0,0} = \mathcal{J}_R(\mathfrak{K}_m^{0,0}, \mathcal{A}) + \mathcal{O}(R_0^{2+2m} \log R).$ We plot  $|\mathcal{J}_{R}(\mathfrak{K}_{m}^{0,0},\mathcal{A}) - \mathcal{A}|_{c}|$  w.r.t. the radius R, for m = 0, 1



# Computation of the coefficient $\Lambda^{1,0}$

Recall  $\Lambda^{1,0} = \mathcal{J}_R(\mathfrak{K}_m^{1,0}, \mathcal{A}) + \mathcal{O}(R^{-1}R_0^{2m+2}).$ We plot  $|\mathcal{J}_R(\mathfrak{K}_m^{1,0}, \mathcal{A}) - \mathcal{J}_{R_{\min}}(\mathfrak{K}_m^{1,0}, \mathcal{A})|.$ 



# Computation of the coefficient $\Lambda^{2,0}$

 $\begin{aligned} & \operatorname{Recall} \Lambda^{2,0} = \mathcal{J}_{R}(\mathfrak{K}^{2,0}_{1},\mathcal{A}) - \mathcal{J}^{2,0;0,0}\Lambda^{0,0} + \mathcal{O}(R^{-2}R^{4}_{0}\log R). \\ & \operatorname{We plot} |\mathcal{J}_{R}(\mathfrak{K}^{2,0}_{1},\mathcal{A}) - \mathcal{J}_{R_{\min}}(\mathfrak{K}^{2,0}_{1},\mathcal{A})|. \end{aligned}$ 



## Comparison of the FE solution and of the local expansion

Expansion restricted to order 1:

$$\mathcal{J}_{\textit{R}_{min}}(\mathfrak{K}_{1}^{0,0},\mathcal{A})(1+\mathrm{i}\zeta^{2}\mathfrak{s}_{1}^{0,0})+\mathcal{J}_{\textit{R}_{min}}(\mathfrak{K}_{1}^{1,0},\mathcal{A})\mathfrak{s}_{0}^{1,0},$$

Expansion restricted to order 2: adding to the above expression the term

$$(\mathcal{J}_{\mathit{R_{\mathsf{min}}}}(\mathfrak{K}^{2,0}_{1},\mathcal{A})-\mathcal{J}^{2,0;0,0}\mathcal{J}_{\mathit{R_{\mathsf{min}}}}(\mathfrak{K}^{0,0}_{1},\mathcal{A}))\mathfrak{s}^{2,0}_{0}.$$



(d) Expansion restricted to order 1. Real part. (e) Expansion restricted to order 2. Real part.

# Thank you for your attention!