

Asymptotic Description of the Magnetic Potential near a Corner Singularity in the Bidimensional Eddy-Current Problem

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Motivations

A usual model for electrical engineering applications

- Dielectric relaxation vs magnetic diffusion:

$$\tau_e = \frac{\epsilon}{\sigma} \ll \tau_m = \mu\sigma L^2$$

L, ϵ, σ, μ : characteristic length, permittivity, conductivity and permeability.

\implies **magneto-quasistatic** (eddy-current) model.

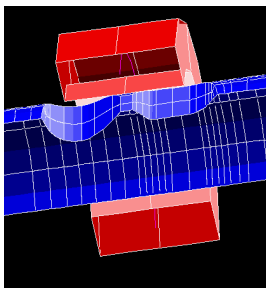
$$\mathbf{curl} \mathbf{E} = -\frac{\partial(\mu\mathbf{H})}{\partial t} \quad \text{div}(\mu\mathbf{H}) = 0,$$

$$\mathbf{curl} \mathbf{H} = \sigma\mathbf{E} + \cancel{\frac{\partial(\epsilon\mathbf{E})}{\partial t}} \quad \text{div}(\epsilon\mathbf{E}) = 0.$$

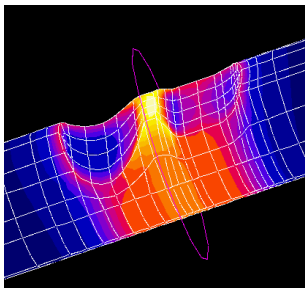
Motivations

Usage of the magneto-quasistatic model

- Electrothermics



(a) Position of the inductor.



(b) Induced current density.

Figure : Surface steel hardening of a shift fork. PhD thesis Sven Wanser ('93 - Renault+EDF/ECL).

- Others: non-destructive testing, most of electromechanical systems.

A computational simplification

Skin Effect

- Skin effect: “exponential decrease” of the electromagnetic field in a volume conductor, *i.e.*

$$\delta = \sqrt{\frac{1}{\mu\pi f\sigma}} \ll L.$$

δ : penetration (skin) depth.

- Instead of strongly meshing close to the surface of the volume conductors: [surface impedance conditions](#). Most usual:

$$Z_\delta(\mathbf{H} \times \mathbf{n}) = \mathbf{E}_t, \text{ with } Z_\delta = \frac{1 + i}{\sigma\delta}.$$

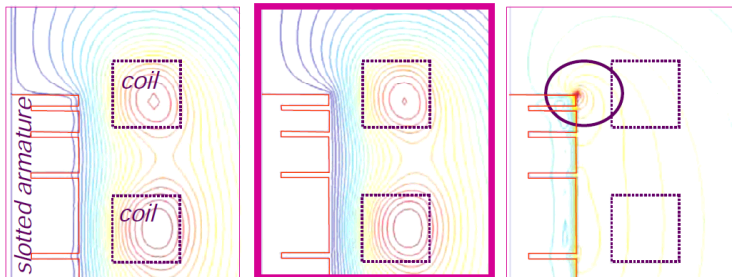
\mathbf{n} : unitary normal to the conductor, \mathbf{E}_t : tangential component of the electric field.

- For smooth enough surface, the impedance can even be “more accurate”.

A computational simplification

Difficulties in the corner

- Example in a domain decomposition process.



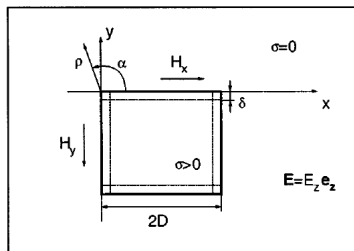
global solution = surf. imp. solution + iterative corrections

P. Dular et al. (Compel, 2007)

A computational simplification

Difficulties in the corner

- Few authors have considered impedance modifications :
“Surface impedance boundary conditions near corners and edges: **rigorous consideration**”. [IEEE Trans. on Mag., 2002]
- Application of the same asymptotic expansion as in the smooth surface case. Impedance obtained:



$$Z_x = -\frac{\tan(\pi/6)}{\rho\sigma}, \quad Z_y = \frac{\tan(\pi/6)}{\rho\sigma}.$$

\implies **blows up in the corner** for any $\sigma < +\infty$.

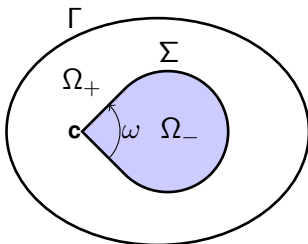
It is not valid for non-magnetic materials with $\sigma < +\infty$ [Buret *et al*, IEEE Trans. on Mag., 2012]

Statement of the Problem

The magnetic vector potential \mathcal{A} satisfies

$$\left\{ \begin{array}{l} -\Delta \mathcal{A}^+ = \mu_0 J \text{ in } \Omega_+, \\ -\Delta \mathcal{A}^- + 4i\zeta^2 \mathcal{A}^- = 0 \text{ in } \Omega_-, \\ \mathcal{A}^+ = 0 \text{ on } \Gamma, \end{array} \right. \quad \begin{array}{l} [\mathcal{A}]_\Sigma = 0, \text{ on } \Sigma, \\ [\partial_n \mathcal{A}]_\Sigma = 0, \text{ on } \Sigma. \end{array} \quad (1)$$

with $\zeta^2 = \kappa\mu_0\sigma/4$, and a smooth data J .



Aim : Description of the magnetic potential \mathcal{A} near the corner \mathbf{c} in 2D

Statement of the Problem

Weak solutions

Variational Formulation

Find $\mathcal{A} \in H_0^1(\Omega)$ such that $\forall \mathcal{A}_* \in H_0^1(\Omega)$

$$\int_{\Omega_+} \nabla \mathcal{A}^+ \cdot \nabla \overline{\mathcal{A}_*^+} \, dx + \int_{\Omega_-} \nabla \mathcal{A}^- \cdot \nabla \overline{\mathcal{A}_*^-} + 4i\zeta^2 \mathcal{A}^- \overline{\mathcal{A}_*^-} \, dx = \mu_0 \int_{\Omega} J \overline{\mathcal{A}_*}$$

Proposition

There exists a unique solution \mathcal{A} in $H_0^1(\Omega)$ to problem (1). Moreover, \mathcal{A} belongs to $H^{\frac{5}{2}-\varepsilon}(\Omega)$ for any $\varepsilon > 0$.

In particular, by **Sobolev embedding**, we obtain that \mathcal{A} belongs to $\mathcal{C}^1(\overline{\Omega})$.

Corner asymptotics

- In general \mathcal{A} does not belong to $C^2(\overline{\Omega})$ but \mathcal{A} possesses a corner asymptotic expansion as the distance r to \mathbf{c} goes to zero

$$\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} \mathfrak{S}^{0,0}(r, \theta) + \sum_{k \geq 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} \mathfrak{S}^{k,p}(r, \theta)$$

- Key points for the behavior of \mathcal{A} in the vicinity of \mathbf{c} :
 - Derivation of the **singular functions** $\mathfrak{S}^{k,p}$
 - Determination of the **singular coefficients** $\Lambda^{k,p}$
- We can provide a constructive procedure to determine $\mathfrak{S}^{k,p}$

$$\mathfrak{S}^{k,p}(r, \theta) = \underbrace{r^k \cos(k\theta - p\pi/2)}_{\text{kernel functions of } \Delta} + \underbrace{r^k \sum_{j \geq 1} (i\zeta^2 r^2)^j \sum_{n=0}^j \log^n r \Phi_{j,n}^{k,p}(\theta)}_{\text{shadow terms}}$$

Outline

- 1 **The case $\zeta = 0$** : For the determination of coefficients $\Lambda^{k,p}$ we introduce the method of **dual singular functions**
- 2 **The case $\zeta \neq 0$** : we introduce the method of **quasi-dual singular functions**. This method is an approximate method. We prove estimates for its convergence.
- 3 Numerical simulations illustrate the theoretical results.

Laplace Operator ($\zeta = 0$)

- We consider the solution \mathcal{A} to

$$\begin{cases} -\Delta \mathcal{A} = \mu_0 J \text{ in } \Omega, \\ \mathcal{A} = 0 \text{ on } \Gamma, \end{cases}$$

with a smooth right hand side with support outside the interior point \mathbf{c} .

- \mathcal{A} admits the Taylor expansion at \mathbf{c} :

$$\mathcal{A}(r, \theta) \underset{r \rightarrow 0}{\sim} \Lambda^{0,0} + \sum_{k \geq 1} \sum_{p \in \{0,1\}} \Lambda^{k,p} r^k \cos(k\theta - p\pi/2)$$

Methods to extract the coefficients $\Lambda^{k,p}$

- The method of moments :
It uses the orthogonality of the angular parts $\cos(k\theta - p\pi/2)$
- The dual function method :
It is the application to the present smooth case of a general method [Mazyra-Plamenevskii, '78] valid for corner coefficient extraction

The Dual Function Method

- Dual harmonic functions singular at $r = 0$:

$$\mathfrak{k}_0^{k,p}(r, \theta) = \begin{cases} -\frac{1}{2\pi} \log r, & \text{if } k = 0, p = 0, \\ \frac{1}{2k\pi} r^{-k} \cos(k\theta - p\pi/2), & \text{if } k \geq 1, p = 0, 1. \end{cases}$$

- For $R > 0$, let us introduce the bilinear form

$$\mathcal{J}_R(K, A) = \int_{r=R} (K \partial_r A - \partial_r K A) R d\theta.$$

Proposition

Let \mathcal{A} be the solution to the Laplace equation with a smooth right hand side with support outside the ball $\mathcal{B}(\mathbf{c}, R)$. Then

$$\mathcal{J}_R(\mathfrak{k}_0^{0,0}, \mathcal{A}) = \Lambda^{0,0} \quad \text{and} \quad \mathcal{J}_R(\mathfrak{k}_0^{k,p}, \mathcal{A}) = \Lambda^{k,p}, \quad k \geq 1, p = 0, 1.$$

The Quasi-Dual Function Method ($\zeta \neq 0$)

- [Costabel *et al*, '04; '12]: The QDFM for straight edges, circular edges and homogeneous operators with constant coefficients.
- We use quasi-dual functions $\mathfrak{R}_m^{k,p}$:

$$\mathfrak{R}_m^{k,p}(r, \theta) = \mathfrak{t}_0^{k,p}(r, \theta) + \sum_{j=1}^m (i\zeta^2)^j \underbrace{\mathfrak{t}_j^{k,p}(r, \theta)}_{\text{shadow term}}$$

where $\mathfrak{t}_j^{k,p}$ are solution to

$$\begin{cases} \Delta \mathfrak{t}_j^{k,p+} = 0, & \text{in } \mathcal{S}_+, \\ \Delta \mathfrak{t}_j^{k,p-} = 4\mathfrak{t}_{j-1}^{k,p-}, & \text{in } \mathcal{S}_- \end{cases}$$

in the space :

$$r^{-k+2j} \text{Span} \{ \log^q r \Phi(\theta), \quad q \leq j+1, \quad \Phi \in \mathcal{C}^1(\mathbb{T}), \quad \Phi^\pm \in \mathcal{C}^\infty(\overline{\mathbb{T}}_\pm) \}$$

Extraction of coefficients

The extraction of $\Lambda^{k,p}$ is performed through the evaluation of quantities

$$\mathcal{J}_R(\mathfrak{R}_m^{k,p}, \mathcal{A}), \quad k = 0, 1, 2, \dots$$

Theorem

Let \mathcal{A} be the solution to problem (1), under the assumptions of the introduction. Let $k \in \mathbb{N}$ and $p \in \{0, 1\}$ ($p = 0$ if $k = 0$). Let m such that $2m + 2 > k$. There exist coefficients $\mathcal{J}^{k,p;k',p'}$ independent of R and \mathcal{A} such that

$$\mathcal{J}_R(\mathfrak{R}_m^{k,p}, \mathcal{A}) \underset{R \rightarrow 0}{=} \Lambda^{k,p} + \sum_{\ell=1}^{\lfloor k/2 \rfloor} \mathcal{J}^{k,p;k-2\ell,p} \Lambda^{k-2\ell,p} + \mathcal{O}(R^{-k} R_0^{2m+2} \log R),$$

where $R_0 = \zeta R (1 + \sqrt{|\log R|})$.

Extraction of coefficients

Example

- For $k = 0$

$$\Lambda^{0,0} \underset{R \rightarrow 0}{=} \mathcal{J}_R(\mathcal{R}_m^{0,0}, \mathcal{A}) + \mathcal{O}(R_0^{2+2m} \log R)$$

- For $k = 1$

$$\Lambda^{1,0} \underset{R \rightarrow 0}{=} \mathcal{J}_R(\mathcal{R}_m^{1,0}, \mathcal{A}) + \mathcal{O}(R^{-1} R_0^{2m+2})$$

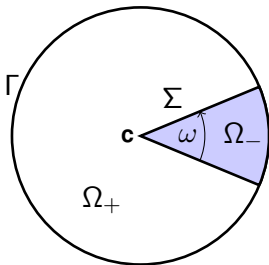
- For $k = 2$, we need $m \geq 1$

$$\Lambda^{2,0} \underset{R \rightarrow 0}{=} \mathcal{J}_R(\mathcal{R}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0;0,0} \Lambda^{0,0} + \mathcal{O}(R^{-2} R_0^4 \log R)$$

The Problem

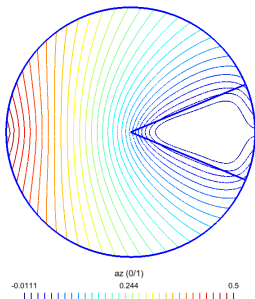
$$\left\{ \begin{array}{l} -\Delta \mathcal{A}^+ = 0 \text{ in } \Omega_+, \\ -\Delta \mathcal{A}^- + 4i\zeta^2 \mathcal{A}^- = 0 \text{ in } \Omega_-, \\ \mathcal{A}^+ = \frac{|\theta|}{\pi} \text{ on } \Gamma, \end{array} \right. \quad \begin{array}{l} [\mathcal{A}]_{\Sigma} = 0, \text{ on } \Sigma, \\ [\partial_n \mathcal{A}]_{\Sigma} = 0, \text{ on } \Sigma. \end{array}$$

- Ω : disk of radius 50 mm.
- Conducting sector : $\omega = \pi/4$
- $\zeta = 1/(5\sqrt{2}) \text{ mm}^{-1}$ corresponds to a physical skin depth of 5 mm.

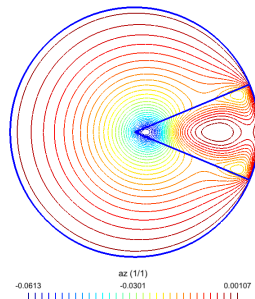


Finite Element Solution

We use P_2 finite elements



(b) Real part of the solution \mathcal{A} .

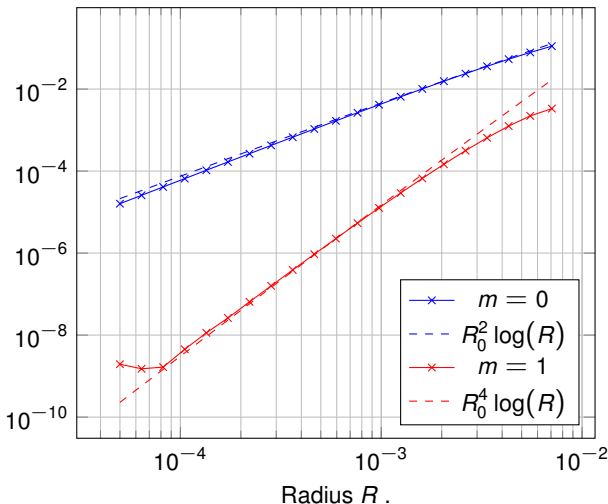


(c) Imaginary part of the solution \mathcal{A} .

Accuracy for the computation of $\Lambda^{0,0}$ as a function of R

Recall $\Lambda^{0,0} = \mathcal{J}_R(\mathcal{R}_m^{0,0}, \mathcal{A}) + \mathcal{O}(R_0^{2+2m} \log R)$.

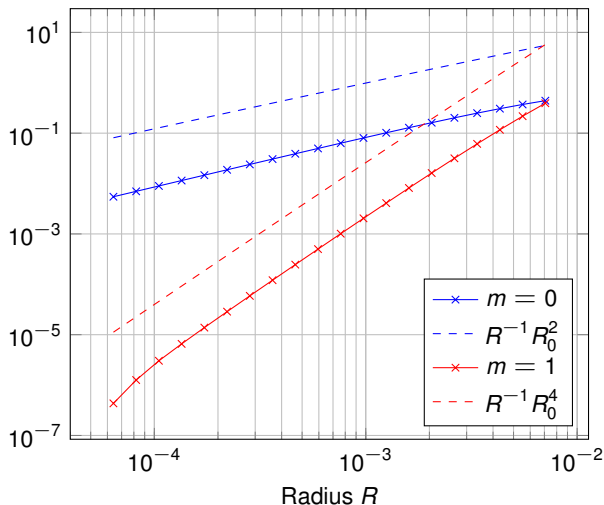
We plot $|\mathcal{J}_R(\mathcal{R}_m^{0,0}, \mathcal{A}) - \Lambda|_c|$ w.r.t. the radius R , for $m = 0, 1$



Computation of the coefficient $\Lambda^{1,0}$

Recall $\Lambda^{1,0} = \mathcal{J}_R(\mathcal{R}_m^{1,0}, \mathcal{A}) + \mathcal{O}(R^{-1}R_0^{2m+2})$.

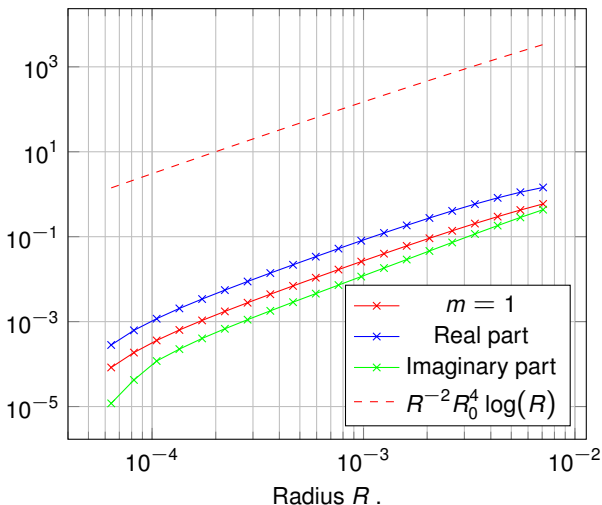
We plot $|\mathcal{J}_R(\mathcal{R}_m^{1,0}, \mathcal{A}) - \mathcal{J}_{R_{\min}}(\mathcal{R}_m^{1,0}, \mathcal{A})|$.



Computation of the coefficient $\Lambda^{2,0}$

Recall $\Lambda^{2,0} = \mathcal{J}_R(\mathfrak{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0;0,0}\Lambda^{0,0} + \mathcal{O}(R^{-2}R_0^4 \log R)$.

We plot $|\mathcal{J}_R(\mathfrak{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}_{R_{\min}}(\mathfrak{K}_1^{2,0}, \mathcal{A})|$.



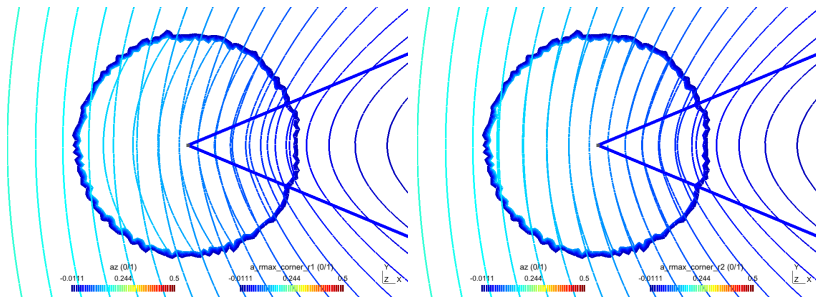
Comparison of the FE solution and of the local expansion

Expansion restricted to order 1:

$$\mathcal{J}_{R_{\min}}(\mathcal{K}_1^{0,0}, \mathcal{A})(1 + i\zeta^2 \mathbf{s}_1^{0,0}) + \mathcal{J}_{R_{\min}}(\mathcal{K}_1^{1,0}, \mathcal{A})\mathbf{s}_0^{1,0},$$

Expansion restricted to order 2: adding to the above expression the term

$$(\mathcal{J}_{R_{\min}}(\mathcal{K}_1^{2,0}, \mathcal{A}) - \mathcal{J}^{2,0;0,0} \mathcal{J}_{R_{\min}}(\mathcal{K}_1^{0,0}, \mathcal{A}))\mathbf{s}_0^{2,0}.$$



(d) Expansion restricted to order 1. Real part.

(e) Expansion restricted to order 2. Real part.

Thank you for your attention!