Interference Characteristics and Success Probability at the Primary User in a Cognitive Radio Network

Prasanna Madhusudhanan, Timothy Brown, Youjian Liu

Interference Characteristics and Success Probability at the Primary User in a Cognitive Radio Network

Prasanna Madhusudhanan*, Timothy X Brown*+, Youjian (Eugene) Liu*
* Department of Electrical, Computer and Energy Engineering, † Interdisciplinary Telecommunications Program
University of Colorado, Boulder, CO 80309-0425 USA
{mprasanna, timxb, eugeneliu}@colorado.edu

***Abstract*—We analyze a cognitive radio network where the primary users (PUs) and cognitive radio (CR) devices are distributed over the two-dimensional plane according to two independent homogeneous Poisson processes. Any CR that lies within the detection region of some PU switches to a different channel in order to prevent causing harmful interference at the PU. Using the concepts of stochastic geometry, we study the characteristics of the interference caused by the PUs and the CRs to a given PU. Further, these results are used to obtain tight upper and lower bounds for the success probability at the PU; defined as the probability that the signal-to-interference-plus-noise ratio (SINR) is beyond a certain operating threshold.**

***Index Terms*—**Cognitive radio, Interference modeling, Beaconing, Poisson point process, Boolean Model, Poisson cluster process, Order Statistics, Stochastic geometry.

I. Introduction

The FCC has allowed the operation of cognitive radio (CR) devices in the ultra high frequency (UHF) television (TV) bands under the condition that they do not cause harmful interference to the operation of the primary users [1], [2]. This paper assesses the effect of the CRs on the primary users in this band, that are the TV transmitter-receiver pairs and the wireless microphone systems.

The CR devices sense the primary user by the signals from beaconing devices or by checking its current position against a database of primary receiver locations to decide whether it should transmit or not. In [3] and references therein, the CR arrangement is modeled as a homogeneous Poisson process and a characteristic function based interference analysis at the primary receiver has been considered. For a similar system model, [4] derives expressions for the primary user outage probability due to the interference by the CRs in the system employing various dynamic spectrum sharing techniques. But, all these studies are restricted to a single primary user case.

Here, the CRs and the primary users are both distributed over the plane according to independent homogeneous Poisson point processes. Such a model has been studied extensively for establishing transmission capacity bounds for the coexistence of the primary users and CRs when the CRs do not perform spectrum sensing (e.g. [5]–[7]). In this paper, as in [8], the CRs perform a location based sensing, and engages in transmission over the channel only when they detect a free channel. While in [8], closed form lower bounds and approximations for the outage probability at the primary receiver and the CR receiver are computed, here, we develop a systematic approach where we first study the various characteristics of the interference to a given primary user due to the presence of the other primary users and the CRs in the network and obtain a series of bounds for the same (see Section III). Then, we use these results to obtain tight upper and lower bounds for the tail probability of the signal-to-interference-plus-noise ratio (SINR) at a typical primary user (see Section IV). Once the tail probability of SINR is known, the success probability at the typical primary user is also characterized. Section V discusses the various bounds derived using a numerical example.

II. System Model

We briefly describe the modeling details for the cognitive radio (CR) and the primary user arrangements that are used in the SINR analysis at a typical primary user.

1) CR and Primary User Relationship: The potential CR transmitters and the primary receivers are distributed according to independent homogeneous Poisson point processes in $\mathbb{R}^2$ with constant densities λ and μ, respectively. All the CR receivers and the primary transmitters are at a fixed distance r_c, r_p, respectively from their corresponding counterparts, in a direction that is independent and identically distributed according to a uniform distribution in $[0, 2\pi]$. Let $\mathcal{C} = \{\xi_i,\ i \geq 1\}$ and $\mathcal{P} = \{\zeta_i,\ i \geq 1\}$ represent the set of locations of the potential CR transmitters and primary receivers from the corresponding homogeneous Poisson processes, respectively.

These potential CRs sense the channel for the primary user. If a CR lies within a given distance D of a primary user, it is in the primary user's so-called *detection region* and will vacate the channel. Otherwise it will transmit and be *active* on the channel. As a result, notice that the set of active CRs is no more a homogeneous Poisson point process as shown in Figure 1. The detection region for the primary receiver located at the origin is $\mathcal{S} = \{x :\ \|x\|_2 \leq D\}$, where $\|\cdot\|_2$ is the Euclidean distance. The set $\mathcal{B}$ is the union of the detection regions around all the primary users in $\mathbb{R}^2$, and the result is the Boolean model [9], [10].

2) Performance Metric: Without loss of generality, we study the SINR at a typical primary receiver located at the origin. Hence, we have the Palm distribution of the primary receivers and the CR transmitters conditioned on this receiver. By Slivnyak's theorem [11], the Palm distribution of the primary receivers is the same as the homogeneous Poisson point process with density μ. The Palm distribution of the Boolean model $\mathcal{B}$ conditioned on the typical primary receiver is the same as the set $\mathcal{B}\bigcup\mathcal{S}$ [10, Page 202]. The received power at the origin from either a primary or CR transmitter

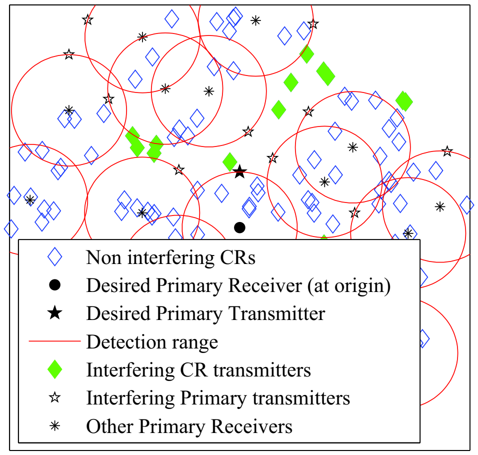

Figure 1. Primary user and cognitive radio arrangement

is modeled as $P = \frac{K\Psi}{R^{\varepsilon}}$, where K captures radio factors including the transmission power of the transmitter, R is the separation between the transmitter and the receiver, ε (> 2) is the path-loss exponent, and Ψ is an exponential random variable with mean 1 for fading. The parameters are distinguished by subscripts p for PU and s for CR as $K_p, K_s, \varepsilon_p, \varepsilon_s$.

Further, SINR $= \frac{K_P \Psi_p r_p^{-\varepsilon_P}}{\eta + I}$, where the numerator is the received power at the primary receiver due to a primary transmitter located at a distance 'r_p' from the primary receiver, η is the noise power, I is the total interference power at the primary receiver due to other primary users and the active CRs in the system, Ψ_p is independent of I.

Successful communication between the primary transmitter-receiver pair is possible only when the SINR at the primary receiver is above a given threshold, say Γ. The probability of this event is called the *success probability* and is denoted by $\mathbb{P}(\{\text{SINR} > \Gamma\})$. The evaluation of the success probability at the primary user can be simplified as follows:

$$\begin{aligned}
&\mathbb{P}(\{\text{SINR} > \Gamma\}) \\
&\overset{(a)}{=} \mathbb{P}\left(\left\{\Psi_P > \frac{\Gamma(\eta + I)}{K_P r_p^{-\varepsilon_P}}\right\}\right) \overset{(b)}{=} \mathbb{E}\left[\exp\left(-\frac{\Gamma(\eta + I)}{K_P r_p^{-\varepsilon_P}}\right)\right] \\
&\overset{(c)}{=} \underbrace{\exp\left(-\frac{\eta\Gamma}{K_P r_p^{-\varepsilon_P}}\right)}_{\text{Factor } 1 = \mathbb{P}_{\text{SNR}}(\Gamma)} \times \underbrace{\mathbb{E}\left[\exp\left(-\frac{\Gamma I}{K_P r_p^{-\varepsilon_P}}\right)\right]}_{\text{Factor } 2 = \mathbb{P}_{\text{SIR}}(\Gamma)}, \qquad (1)
\end{aligned}$$

where (a) is obtained by rearranging the terms in SINR; (b) is obtained by noting that Ψ_P is an exponential random variable with mean 1 and independent of I; $\mathbb{E}[\cdot]$ is the expectation operator, and is with respect to the joint distribution of all the random variables involved in I; *Factor 1* in (c) is the success probability in the absence of CRs and other PUs and the *Factor 2* is success probability in the interference-limited system.

III. Interference Characteristics

The total interference at the primary receiver at the origin is $I = I_{pp} + I_{pc}$, with $I_{pp} = \sum_{i=1}^{\infty} K_p \Psi_{pi} \|\zeta_i + X_i\|_2^{-\varepsilon_p}$, $I_{pc} = \sum_{i=1}^{\infty} K_s \Psi_{si} \|\xi_i\|_2^{-\varepsilon_s} \mathcal{I}(\xi_i \notin (\mathcal{B} \bigcup \mathcal{S}))$, where I_{pp} is the sum of the interferences from all the primary transmitters in the system; $X_i' s$ are the locations of the primary transmitters for the corresponding primary receivers, and are r_p away from the corresponding primary receiver in a direction uniformly distributed in $[0, 2\pi]$, I_{pc} is the sum of the interferences from all the active CRs in the system, obtained by considering the CR transmitters that do not belong to the detection region of any primary receiver; $\mathcal{B}$ and $\mathcal{S}$ are as defined in Section II-1. Notice that I_{pp} and I_{pc} are dependent random variables, and characterizing the total interference I in closed form has not been possible. In the rest of this section, we study the characteristics of I_{pp} and I_{pc}, such as their mean and moment generating functions (Laplace transforms) and developed sufficient machinery by the end of the section to be able to characterize the total interference I as well as the success probability, using upper and lower bounds.

A. Interference due to the Primary Transmitters (I_{pp})

Using Campbell's theorem [12], it can be shown that the expected value of the random variable I_{pp}, $\mathbb{E}[I_{pp}]$ is infinite. Now, we provide closed form expressions for the Laplace transform of I_{pp} (denoted as $\mathcal{L}_{I_{pp}}(s)$) and the variance of the random variable $\mathrm{e}^{-sI_{pp}}$, denoted by $\text{var}\left(\mathrm{e}^{-sI_{pp}}\right)$, the proof for the former can be found in [11], and the latter is obtained by the definition of variance of a random variable.

$$\mathcal{L}_{I_{pp}}(s) = \mathrm{e}^{-\mu G\left((sK_p)^{-1}, \varepsilon_p, 0\right)}, \qquad (2)$$

$$\text{var}\left(\mathrm{e}^{-sI_{pp}}\right) = \mathcal{L}_{I_{pp}}(2s) - \left[\mathcal{L}_{I_{pp}}(s)\right]^2, \qquad (3)$$

where $s > 0$, and the function $G()$ is defined as

$$\begin{aligned}
G(\alpha, \varepsilon, \delta) &\triangleq \int_{r=\delta}^{\infty} \frac{2\pi r dr}{1 + \alpha r^{\varepsilon}} \\
&= \frac{2\pi^2 \alpha^{-\frac{2}{\varepsilon}}}{\varepsilon \sin\left(\frac{2\pi}{\varepsilon}\right)} - \pi\delta^2 {}_2F_1\left(1, \frac{2}{\varepsilon}; 1 + \frac{2}{\varepsilon}; -\alpha\delta^{\varepsilon}\right), \qquad (4)
\end{aligned}$$

where ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is the Gauss hypergeometric function.

B. Interference due to the CR Transmitters (I_{pc})

The average interference caused by the active CRs to the typical primary user can be evaluated in closed form and is presented in the following lemma.

Lemma 1. The expected value of I_{pc}, denoted as, $\mathbb{E}[I_{pc}]$ is

$$\mathbb{E}[I_{pc}] = \frac{\lambda \mathrm{e}^{-\mu\pi D^2} K_s 2\pi D^{2-\varepsilon_s}}{\varepsilon_s - 2}. \qquad (5)$$

Proof: We note

$$\begin{aligned}
\mathbb{E}[I_{pc}] &\overset{(a)}{=} \mathbb{E}\left[\sum_{i=1}^{\infty} K_s \|\xi_i\|_2^{-\varepsilon_s} \Psi_i \mathbb{P}(\{\xi_i \in \mathcal{B}^c\}) \mathcal{I}(\xi_i \notin \mathcal{S})\right] \\
&\overset{(b)}{=} \mathrm{e}^{-\mu\pi D^2} \mathbb{E}\left[\sum_{i=1}^{\infty} K_s \|\xi_i\|_2^{-\varepsilon_s} \mathcal{I}(\xi_i \notin \mathcal{S})\right],
\end{aligned}$$

where (a) is obtained by noting that the random set $\mathcal{B}$ is independent of the other random variables and by taking the expectation w.r.t. $\mathcal{B}$ inside the summation, (b) is obtained after evaluating the expectations w.r.t. the i.i.d. Rayleigh fading factors, and the random set $\mathcal{B}$, respectively, with $\mathbb{P}(\{x_i \notin \mathcal{B}\}) = \mathbb{P}(\{\text{No primary users upto a distance } D \text{ from } x\}) = \mathrm{e}^{-\mu\pi D^2}$, and finally (5) is obtained by applying Campbell's theorem for Poisson point processes [12]. ■

Next, it is of interest to obtain the Laplace transform of I_{pc}, $\mathcal{L}_{I_{pc}}(s) = \mathbb{E}\left[\mathrm{e}^{-sI_{pc}}\right]$. This quantity cannot be characterized in closed form, and hence we find the upper and lower bounds. In this paper, equalities or inequalities between random variables implicitly refer to equivalences or dominances, respectively, in the usual stochastic ordering sense [13].

Let us define $I' \triangleq \sum_{i=1}^{\infty} K_s \Psi_i \|\xi_i\|^{-\varepsilon_s} \mathcal{I}(\xi_i \in \mathcal{S}^c)$, $I_2 \triangleq \sum_{i=1}^{\infty} K_s \Psi_i \|\xi_i\|^{-\varepsilon_s} \mathcal{I}(\xi_i \in \mathcal{B} \bigcap \mathcal{S}^c)$ as the interference due to all the CRs in the system except for those in the detection region of the typical primary receiver (both ♦'s and ♢'s in Figure 1, and beyond D from the origin), and the interference due to the CRs, located beyond D from the origin and within the detection range of one/more primary users (only the ♢'s in Figure 1, and beyond D from the origin), respectively. Notice that $I' = I_{pc} + I_2$. Moreover, I' is independent of $\mathcal{B}$, and conditioned on $\mathcal{B}$, I_{pc} is independent of I_2, since they are the interferences caused by the points that belong to two disjoint sets, namely, $\mathcal{B} \bigcap \mathcal{S}^c$, and $\mathcal{B}^c \bigcap \mathcal{S}^c$, respectively. The following lemma gives a lower bound for $\mathcal{L}_{I_{pc}}(s)$.

Lemma 2. A lower bound for $\mathcal{L}_{I_{pc}}(s)$ is given by

$$\mathcal{L}^{l}_{I_{pc}}(s) = \max\left(\mathbb{E}\left[\mathrm{e}^{-sI'}\right], \ \mathrm{e}^{-s\mathbb{E}[I_{pc}]}\right), \tag{6}$$

where $\mathbb{E}\left[\mathrm{e}^{-sI'}\right] = \mathrm{e}^{-\lambda G\left((sK_s)^{-1}, \varepsilon_s, D\right)}$, and $\mathbb{E}[I_{pc}]$ is derived in Lemma *1*.

Proof: By noting that $I_{pc} = I' - I_2 \leq I'$, we get $\mathcal{L}_{I_{pc}}(s) \geq \mathbb{E}\left[\mathrm{e}^{-sI'}\right] = \mathrm{e}^{-\lambda G\left((sK_s)^{-1}, \varepsilon_s, D\right)}$. This bound is the same as that obtained in [8]. Another lower bound is obtained by applying Jensen's inequality to $\mathcal{L}_{I_{pc}}(s)$, which is a convex function of I_{pc}. $\mathcal{L}_{I_{pc}}(s) = \mathbb{E}\left[\mathrm{e}^{-sI_{pc}}\right] \geq \mathrm{e}^{-s\mathbb{E}[I_{pc}]}$. Since the maximum of the above mentioned two lower bounds also serves as a lower bound for $\mathcal{L}_{I_{pc}}(s)$, we get (6). ■

Next, we provides a series of upper bounds for $\mathbb{E}\left[\mathrm{e}^{-sI_{pc}}\right]$.

Theorem 1. Upper bounds for $\mathcal{L}_{I_{pc}}(s)$ are as follows

$$\mathcal{L}^{u1}_{I_{pc}}(s) = \begin{cases} 1 & , \ sK_s D^{-\varepsilon_s} \geq 1 \\ \mathbb{E}\left[\mathrm{e}^{-sI'}\right] \mathbb{E}\left[\mathrm{e}^{s\hat{I}_2^{(1)}}\right] \mathbb{E}\left[\mathrm{e}^{s\hat{I}_2^{(2)}}\right] & , \ sK_s D^{-\varepsilon_s} < 1 \end{cases}, \tag{7}$$

$$\mathcal{L}^{u2}_{I_{pc}}(s) = \prod_{i=1}^{2}\left(1 - \rho_{NC} H_i\left(\lambda, (2sK_s)^{-1}, \varepsilon_s, D\right)\right)^{\frac{1}{2}} \tag{8}$$

$$\mathcal{L}^{u3}_{I_{pc}}(s) = 1 - \sum_{i=1}^{2} \frac{\rho_{NC} H_i\left(\lambda, (2sK_s)^{-1}, \varepsilon_s, D\right)}{2}, \tag{9}$$

$$\mathcal{L}^{u4}_{I_{pc}}(s) = 1 - \rho_{NC} H_1\left(\lambda, (sK_s)^{-1}, \varepsilon_s, D\right), \tag{10}$$

$$\mathcal{L}^{u}_{I_{pc}}(s) = \min\left(\left\{\mathcal{L}^{ui}_{I_{pc}}(s)\right\}_{i=1}^{4}\right), \tag{11}$$

where $\mathbb{E}\left[\mathrm{e}^{-sI'}\right]$ is as in Lemma 2, $\rho_{NC} = \mathrm{e}^{-\mu\pi D^2}$,

$$\mathbb{E}\left[\mathrm{e}^{s\hat{I}_2^{(1)}}\right] = \mathrm{e}^{\mu\pi\left(D + (2sK_s)^{\frac{1}{\varepsilon_s}}\right)^2\left(\mathrm{e}^{\frac{\lambda\pi D^2}{(sK_s)^{-1} D^{\varepsilon} - 1}} - 1\right)}, \tag{12}$$

$$\mathbb{E}\left[\mathrm{e}^{s\hat{I}_2^{(2)}}\right] = \mathrm{e}^{\mu\int_{r=(2sK_s)^{\frac{1}{\varepsilon_s}}}^{\infty}\left(\mathrm{e}^{\frac{\lambda\pi D^2}{(sK_s)^{-1} r^{\varepsilon_s} - 1}} - 1\right) 2\pi(r+D)dr}, \tag{13}$$

$$H_k\left(\lambda, (2sK_s)^{-1}, \varepsilon_s, D\right) = \int_{r=D}^{\infty} \frac{(\lambda\pi)^k\, 2r \times \left(r^2 - D^2\right)^{k-1} dr}{\left(1 + (2sK_s)^{-1} r^{\varepsilon_s}\right) \mathrm{e}^{\lambda\pi(r^2 - D^2)}}. \tag{14}$$

Proof: See Appendix A. ■

Further, note that $\mathbb{E}\left[\mathrm{e}^{s\hat{I}_2^{(2)}}\right]$ in the above theorem can be easily computed to any desired accuracy by numerical integration. Now, using the lower and upper bounds for $\mathcal{L}_{I_{pc}}(s)$, we find the bounds for the variance of the random variable $\mathrm{e}^{-sI_{pc}}$, denoted by $\mathrm{var}\left(\mathrm{e}^{-sI_{pc}}\right)$, in the following lemma.

Proposition 1. Upper and lower bounds for $\mathrm{var}\left(\mathrm{e}^{-sI_{pc}}\right)$ are

$$\mathrm{var}_u\left(\mathrm{e}^{-sI_{pc}}\right) = \max\left(0, \mathcal{L}^{u}_{I_{pc}}(2s) - \left[\mathcal{L}^{l}_{I_{pc}}(s)\right]^2\right), \tag{15}$$

$$\mathrm{var}_l\left(\mathrm{e}^{-sI_{pc}}\right) = \max\left(0, \mathcal{L}^{l}_{I_{pc}}(2s) - \left[\mathcal{L}^{u}_{I_{pc}}(s)\right]^2\right). \tag{16}$$

The max operation ensures that the bounds are non-negative. Further, the lower (upper) bound is obtained by lower (upper) bounding each term in $\mathrm{var}\left(\mathrm{e}^{-sI_{pc}}\right) = \mathcal{L}_{I_{pc}}(2s) - \left[\mathcal{L}_{I_{pc}}(s)\right]^2$, which is due to the definition of variance of a random variable. Having studied the characteristics of the interference caused by the primary transmitters and the CR transmitters in this section, we are ready to characterize the success probability for a typical primary receiver.

IV. Success Probability at the Primary Receiver

Given the SINR threshold (Γ) for successful communication between a given primary transmitter-receiver pair, the success probability follows from (1) to be

$$\mathbb{P}(\mathrm{SINR} > \Gamma) = \mathbb{P}_{\mathrm{SNR}}(\Gamma) \times \mathbb{P}_{\mathrm{SIR}}(\Gamma), \tag{17}$$

where $\mathbb{P}_{\mathrm{SIR}}(\Gamma) = \mathbb{E}\left[\mathrm{e}^{-s(I_{pp}+I_{pc})}\right]$, with $s = \frac{\Gamma}{K_p r_p^{-\varepsilon_p}}$. Since the above quantity cannot be characterized in closed form, we consider finding their upper and lower bounds. Further, upper and lower bounds for $\mathbb{P}_{\mathrm{SIR}}(\Gamma) = \mathbb{E}\left[\mathrm{e}^{-s(I_{pp}+I_{pc})}\right]$ are derived in this section, which when multiplied by $\mathbb{P}_{\mathrm{SNR}}(\Gamma)$ provide the corresponding bounds on the success probability.

Theorem 2. Upper bounds for $\mathbb{P}_{\mathrm{SIR}}(\Gamma)$ are

$$\mathbb{P}^{u1}_{\mathrm{SIR}}(\Gamma) = \min\left(\mathcal{L}_{I_{pp}}(s), \mathcal{L}^{u}_{I_{pc}}(s), \mathcal{L}_{I_{pp}}(s)\mathcal{L}^{u}_{I_{pc}}(s) + \sqrt{\mathrm{var}\left(\mathrm{e}^{-sI_{pp}}\right)\mathrm{var}_u\left(\mathrm{e}^{-sI_{pc}}\right)}\right), \tag{18}$$

$$\mathbb{P}^{u2}_{\mathrm{SIR}}(\Gamma) = \sqrt{\mathcal{L}_{I_{pp}}(2s) \times \mathcal{L}^{u}_{I_{ps}}(2s)}, \tag{19}$$

$$\mathbb{P}^{u}_{\mathrm{SIR}}(\Gamma) = \min\left(\mathbb{P}^{u1}_{\mathrm{SIR}}(\Gamma), \mathbb{P}^{u2}_{\mathrm{SIR}}(\Gamma)\right), \tag{20}$$

where $s = \frac{\Gamma}{K_p r_p^{-\varepsilon_p}}$, and all the other terms are computed in Section III.

Proof: Firstly, notice that $I_{pp} + I_{pc} \geq \max\left(I_{pp},\ I_{pc}\right)$, and as a result, $\mathbb{P}_{\mathrm{SIR}}\left(\Gamma\right) \leq \min\left(\mathcal{L}_{I_{pp}}\left(s\right), \mathcal{L}^{u}_{I_{pc}}\left(s\right)\right)$. Further, from the definition of the correlation coefficient between two dependent random variables X and Y, we get

$$-1 \leq \frac{\mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]}{\sqrt{\mathrm{var}\left(X\right) \times \mathrm{var}\left(Y\right)}} \quad \leq \quad 1. \tag{21}$$

Hence, using the upper bound in (21), we get

$$\begin{aligned}
&\mathbb{P}_{SIR}\left(\Gamma\right) = \mathbb{E}\left[\mathrm{e}^{-sI_{pp}} \times \mathrm{e}^{-sI_{pc}}\right] \\
&\leq \mathcal{L}_{I_{pp}}\left(s\right)\mathcal{L}_{I_{pc}}\left(s\right) + \sqrt{\mathrm{var}\left(\mathrm{e}^{-sI_{pp}}\right)\mathrm{var}\left(\mathrm{e}^{-sI_{pc}}\right)} \\
&\leq \mathcal{L}_{I_{pp}}\left(s\right)\mathcal{L}^{u}_{I_{pc}}\left(s\right) + \sqrt{\mathrm{var}\left(\mathrm{e}^{-sI_{pp}}\right)\mathrm{var}_{u}\left(\mathrm{e}^{-sI_{pc}}\right)}.
\end{aligned}$$

Combining the above two mentioned bounds, we get (18). Next, by noting that $\mathbb{P}_{SIR}\left(\Gamma\right) \leq \sqrt{\mathbb{E}\left[\mathrm{e}^{-2sI_{pp}}\right]\mathbb{E}\left[\mathrm{e}^{-2sI_{pc}}\right]}$, by Cauchy-Schwartz inequality and by further upper bounding the second expectation term in the product, we get (19). Finally, (20) is obtained since the minimum of (18) and (19) is a tighter upper bound. ■

Theorem 3. Lower bounds for $\mathbb{P}_{\mathrm{SIR}}\left(\Gamma\right)$ are

$$\mathbb{P}^{l1}_{\mathrm{SIR}}\left(\Gamma\right) = \max\left(0, \mathcal{L}_{I_{pp}}\left(s\right)\mathcal{L}^{l}_{I_{pc}}\left(s\right) - \sqrt{\mathrm{var}\left(\mathrm{e}^{-sI_{pp}}\right)\mathrm{var}_{u}\left(\mathrm{e}^{-sI_{pc}}\right)}\right), \tag{22}$$

$$\mathbb{P}^{l2}_{\mathrm{SIR}}\left(\Gamma\right) = \mathcal{L}_{I_{pp}}\left(s\right) \times \mathbb{E}\left[\mathrm{e}^{-sI'}\right], \tag{23}$$

$$\mathbb{P}^{lb}_{\mathrm{SIR}}\left(\Gamma\right) = \max\left(\mathbb{P}^{l1}_{\mathrm{SIR}}\left(\Gamma\right), \mathbb{P}^{l2}_{\mathrm{SIR}}\left(\Gamma\right)\right), \tag{24}$$

where $s = \frac{\Gamma}{K_p r_p^{-\varepsilon_p}}$, and the other terms are in Section III.

Proof: Equation (22) is proved along the same lines as (18), but with the help of the lower bound in (21). A tight lower bound for $\mathbb{P}_{\mathrm{SIR}}\left(\Gamma\right)$ is due to [8] and is obtained by upper bounding I_{pc} with I' defined in Section III-B. By further noting that I_{pp} and I' are independent random variables, we get (23). Further, since the maximum of the lower bounds is also a lower bound, we get (24). ■

Finally, based on Theorem 2 and Theorem 3, the success probability can be bounded as

$$\mathbb{P}_{\mathrm{SNR}}\left(\Gamma\right) \cdot \mathbb{P}^{l}_{\mathrm{SIR}}\left(\Gamma\right) \leq \mathbb{P}\left(\mathrm{SINR} > \Gamma\right) \leq \mathbb{P}_{\mathrm{SNR}}\left(\Gamma\right) \cdot \mathbb{P}^{u}_{\mathrm{SIR}}\left(\Gamma\right).$$

Next, we consider numerical examples to ascertain the efficacy of the success probability bounds shown above.

V. Numerical Example and Discussion

We first discuss the primary receiver success probability bounds obtained through bounds on $\mathcal{L}_{I_{pc}}\left(s\right)$ in Section III-B. The lower bound for $\mathcal{L}_{I_{pc}}\left(s\right)$, based on I' (see Section III-B) ignores the effect of the multiple primary users in the system, and is a good estimate only for small μ, while the other lower bound in (2) dominates in the case of large μ. Next, $\mathcal{L}^{u1}_{I_{pc}}\left(s\right)$ is a good upper bound when μ is small. As μ increases, there are more overlaps between the detection regions of the primary users, the interference from CRs in these regions are counted

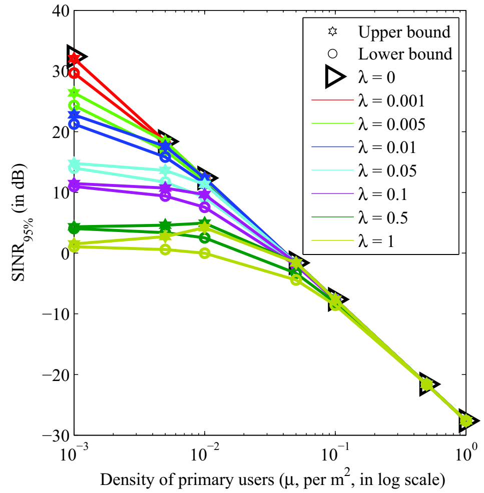

Figure 2. $\mathrm{SINR}_{95\%}$ vs μ for different values of λ

multiple times so that $\hat{I}_2$ is overestimated and $\mathcal{L}^{u1}_{I_{pc}}\left(s\right)$ is a weaker upper bound. On the other hand, as μ increases, more primary users are likely to lie near the origin expanding the effective detection region around the origin. Given that signals decay sharply with distance, the interference caused by a few strong CRs can be a good estimate of I_{pc}. This premise is used to derive $\mathcal{L}^{u2}_{I_{pc}}\left(s\right)$, $\mathcal{L}^{u3}_{I_{pc}}\left(s\right)$, and $\mathcal{L}^{u4}_{I_{pc}}\left(s\right)$. Of the success probability bounds derived in Section IV, $\mathbb{P}^{l2}_{\mathrm{SIR}}\left(\Gamma\right)$ is a tighter lower bound than $\mathbb{P}^{l1}_{\mathrm{SIR}}\left(\Gamma\right)$ in almost all cases of interest, and was already conceived in [8]. The main contribution of this paper are two-folds. Firstly, a systematic study of the interferences at a primary receiver caused by the primary transmitters and the active CRs is conducted. Secondly, a tight upper bound to the success probability at the primary receiver is achieved, which has not been considered previously.

Next, we consider a cognitive radio network consisting of the primary users and CR devices with the following specification: $K_p = 1$, $K_s = .2$, $r_p = .5$, $r_s = .1$, $\varepsilon_p = \varepsilon_s = 4$, $\eta = 0$ and $D = 1$. At $D = 1$ and $\mu = 0.22$ a CR is within the detection region half the time. For a given configuration of (λ, μ), a system designer may be interested to find the SINR that a primary user can expect to see with a high reliability, say with a success probability of 95%. Figure 2 shows such a plot for a wide range of values of λ and μ, obtained using the success probability bounds derived in Section IV. Notice that the bounds are tight and characterize the $\mathrm{SINR}_{95\%}$ within a gap of less than a couple dB for almost all combinations of (λ, μ) considered. An interesting point to note is that, for large μ, irrespective of the CR density (λ), the SINR performance converges to the no CR case $(\lambda = 0)$. Next, Figure 3 shows the plot for the success probability bounds versus the SINR threshold (Γ) for typical values of Γ that is pertinent to the CR and primary user operation. The “Approximation” curves are obtained using [8, Eq. (8)], and are included for reference as they closely match the behavior of system simulations, as shown in [8, Figure 2]. Notice that the bounds derived characterize the success probability with a small gap for all combinations of (λ, μ).

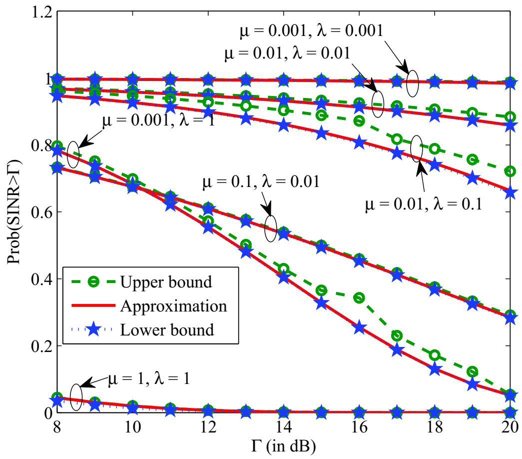

Figure 3. Success Probability bounds for various combinations of (λ, μ)

VI. Conclusions

We consider a cognitive radio network with the primary users and CRs distributed according to independent homogeneous Poisson point processes on the plane. The CRs do not transmit if they are within a detection (protection) region around the primary user. Tight upper and lower bounds for the success probability at a given primary receiver are obtained through a series of bounds developed for the Laplace transform of the interference caused by the CRs. These bounds are derived by employing ideas ranging from exploiting the order statistics of the distances of the CRs, applying Jensen's inequality to the Laplace transform of the CR interference, to suitably exploiting the structure of the Matérn cluster process. In specific, the success probability upper bound obtained in this paper, complements the lower-bound results in [8] and they together characterize the success probability at a given primary receiver within a small gap. Moreover, the techniques developed here can be used to study the success probability at a typical CR receiver, and this will be pursued elsewhere.

References

[1] "Unlicensed Operation in the TV Broadcast Bands," *Federal Communications Commission*, no. ET Docket No. 10-174, 2010.
[2] "Unlicensed Operation in the TV Broadcast Bands," *Federal Communications Commission*, no. ET Docket No. 04-186, 2004.
[3] P. Madhusudhanan, J. G. Restrepo, Y. Liu, T. X. Brown, and K. Baker, "Modeling of Interference from Cooperative Cognitive Radios for Low Power Primary Users," in *IEEE Globecom 2010 Wireless Communications Symposium*, 2010, pp. 1–6.
[4] R. Menon, R. Buehrer, and J. Reed, "On the Impact of Dynamic Spectrum Sharing Techniques on Legacy Radio Systems," *Wireless Communications, IEEE Transactions on*, vol. 7, no. 11, pp. 4198 –4207, november 2008.
[5] J. Lee, J. Andrews, and D. Hong, "Spectrum-Sharing Transmission Capacity," *Wireless Communications, IEEE Transactions on*, vol. 10, no. 9, pp. 3053 –3063, september 2011.
[6] S. Zaidi, M. Ghogho, and D. McLernon, "Transmission Capacity Analysis of Cognitive Radio Networks under Co-existence Constraints," in *Signal Processing Advances in Wireless Communications (SPAWC), 2010 IEEE Eleventh International Workshop on*, june 2010, pp. 1 –5.
[7] C. Yin, L. Gao, T. Liu, and S. Cui, "Transmission Capacities for Overlaid Wireless Ad Hoc Networks with Outage Constraints," in *Communications, 2009. ICC '09. IEEE International Conference on*, june 2009, pp. 1 –5.
[8] C.-H. Lee and M. Haenggi, "Interference and Outage in Poisson Cognitive Networks," *IEEE Transactions on Wireless Communications*, 2012. [Online]. Available: http://www.nd.edu/ mhaenggi/pubs/twc12.pdf
[9] D. Stoyan, W. S. Kendall, and J. Mecke, *Stochastic Geometry and its Applications*, 1995.
[10] P. Hall, *Introduction to the theory of coverage processes*. John Wiley & Sons, Inc., 1988.
[11] M. Haenggi and R. K. Ganti, "Interference in Large Wireless Networks," *Foundations and Trends in Networking*, vol. 3, no. 2, pp. 127–248, 2008, available at http://www.nd.edu/ mhaenggi/pubs/now.pdf. [Online]. Available: http://www.nd.edu/ mhaenggi/pubs/now.pdf
[12] J. F. C. Kingman, *Poisson Processes (Oxford Studies in Probability)*. Oxford University Press, USA, January 1993.
[13] M. Shaked and J. Shanthikumar, *Stochastic Orders*, ser. Springer Series in Statistics. Springer, 2007.
[14] D. Yogeshwaran, "Poisson-Poisson Cluster SINR Coverage Process," INRIA/ENS TREC, Ecole Normale Superieure, Paris, France, Tech. Rep., 2006. [Online]. Available: http://www.di.ens.fr/ yogesh/ppclustersinr.pdf

Appendix

A. Proof for Theorem 1

Proof for (7) : Notice that $\mathbb{E}\left[\mathrm{e}^{-sI'}\right] = \mathbb{E}\left[\mathrm{e}^{-sI_{pc}}\middle|\mathcal{B}\right] \times \mathbb{E}\left[\mathrm{e}^{-sI_2}\middle|\mathcal{B}\right]$, since I' is independent of $\mathcal{B}$, and conditioned on $\mathcal{B}$, I_{pc} and I_2 are independent of each other. So, $\mathcal{L}_{I_{pc}}(s) = \mathbb{E}\left[\mathrm{e}^{-sI'}\right]\mathbb{E}_{\mathcal{B}}\left[1/\mathbb{E}\left[\mathrm{e}^{-sI_2}\middle|\mathcal{B}\right]\right] \leq \min\left(1, \mathbb{E}\left[\mathrm{e}^{-sI'}\right]\mathbb{E}\left[\mathrm{e}^{sI_2}\right]\right)$, by applying Jensen's inequality to the expectation term in the denominator. Here, $\mathcal{L}_{I_{pc}}(s)$ is upper bounded by finding an upper bound for I_2. One approach is to upper bound I_2 with the total interference caused in a Matérn cluster process, where in the parent points are the locations of the primary users, the daughter points are i.i.d. uniformly distributed within the circle of radius D around the origin, and the number of daughter points for a given primary user is a Poisson random variable with mean $\lambda\pi D^2$, which is exactly the mean number of CRs in $\mathcal{S}$. Matérn cluster process is a special case of Poisson cluster processes [9, Section 5.3] and further details in the context of this paper can be found in [14]. Thus, we have, $I_2 \leq \sum_{i=1}^{\infty}\sum_{j=1}^{N_i} K_s\Psi_{ij}\left\|\zeta_i + X_{ij}\right\|^{-\varepsilon_s} \triangleq \tilde{I}_2$, where $\{N_i\}_{i=1}^{\infty} \overset{\text{i.i.d.}}{\sim} \text{Poisson}\left(\lambda\pi D^2\right)$, $\left\{\zeta_i, \{\zeta_i + X_{ij}\}_{j=1}^{N_i}\right\}_{i=1}^{\infty}$ is the set of locations of the primary users and their corresponding CRs, and $\{X_{ij}\}$ is a set of i.i.d. random variables with a uniform distribution in $\mathcal{S}$. [14] derives the expression for the Laplace transform of the above mentioned upper bound, but is not in closed form. Now, consider a certain refinement to $\tilde{I}_2$, denoted as $\hat{I}_2$, whose Laplace transform is easily computed.

$$\hat{I}_2 \triangleq \sum_{i=1}^{\infty}\sum_{j=1}^{N_i} K_s\Psi_{ij}D^{-\varepsilon_s}\mathcal{I}\left(\|\zeta_i\|_2 \leq \Delta\right) + \sum_{i=1}^{\infty}\sum_{j=1}^{N_i} K_s\Psi_{ij}\left\|\zeta_i + X_{ij}\right\|^{-\varepsilon_s}\mathcal{I}\left(\|\zeta_i\|_2 > \Delta\right), \quad (25)$$

where $\Delta = D + (2sK_s)^{\frac{1}{\varepsilon_s}}$. The reason for partitioning $\mathbb{R}^2$ into $\|\zeta_i\|_2 \leq \Delta$ and $\|\zeta_i\|_2 > \Delta$ will be clarified later on. Further, $I_2 \leq \hat{I}_2$, since the interference due to all the CRs within D from the origin have already been captured in I', and the corresponding terms in $\hat{I}_2$ were only double counting them. Moreover, the interference due to CRs lying between D and

Δ are upper bounded by making all points to lie at distance D from the origin. Now, denote the first term of $\hat{I}_2$ as $\hat{I}_2^{(1)}$. It can also be expressed as $\hat{I}_2^{(1)} = \sum_{i=1}^{M}\sum_{j=1}^{N_i} K_s\Psi_{ij}D^{-\varepsilon_s}$, where M is the number of primary users in $\|\zeta_i\|_2 \leq \Delta$; $M \sim \text{Poisson}\left(\mu\pi\Delta^2\right)$ and M is independent of the N_i's defined earlier. $\hat{I}_2$ is further upper bounded as

$$\hat{I}_2 \overset{(a)}{\leq} \hat{I}_2^{(1)} + \sum_{i=1}^{\infty}\sum_{j=1}^{N_i} K_s\Psi_{ij}\left(\|\zeta_i\| - D\right)^{-\varepsilon_s}\mathcal{I}\left(\|\zeta_i\| > \Delta\right)$$

$$= \hat{I}_2^{(1)} + \underbrace{\sum_{i=1}^{\infty}\sum_{j=1}^{N_i} K_s\Psi_{ij}\left(R_i - D\right)^{-\varepsilon_s}\mathcal{I}\left(R_i > \Delta\right)}_{=\hat{I}_2^{(2)}}, \quad (26)$$

where (a) is obtained due to $\|\zeta_i + X_{ij}\| \geq \left|\underbrace{\|\zeta_i\|}_{=R_i} - \|X_{ij}\|\right| \geq |R_i - D|$. As a result, $\mathbb{E}\left[\mathrm{e}^{sI_2}\right] \leq \mathbb{E}\left[\mathrm{e}^{s\left(\hat{I}_2^{(1)}+\hat{I}_2^{(2)}\right)}\right] = \mathbb{E}\left[\mathrm{e}^{s\hat{I}_2^{(1)}}\right]\mathbb{E}\left[\mathrm{e}^{s\hat{I}_2^{(2)}}\right]$, where the last equality is obtained by noting that $\hat{I}_2^{(1)}$, and $\hat{I}_2^{(2)}$ are independent random variables. Next, we compute the Laplace transforms of these random variables. The evaluation of $\mathbb{E}\left[\mathrm{e}^{s\hat{I}_2^{(1)}}\right]$ is easily done and we do not delve into it further. Note that, if $sK_sD^{-\varepsilon} < 1$, this expectation reduces to (12), and is unbounded otherwise. The Laplace transform of $\hat{I}_2^{(2)}$ is computed as follows.

$$\mathbb{E}\left[\mathrm{e}^{s\hat{I}_2^{(2)}}\right] = \mathbb{E}\left[\prod_{i=1}^{\infty}\prod_{j=1}^{N_i}\mathrm{e}^{-sK_s\Psi_{ij}(R_i-D)^{-\varepsilon_s}\mathcal{I}(R_i>\Delta)}\right]$$

$$\overset{(a)}{=} \mathbb{E}\left[\prod_{i=1,\ R_i>\Delta}^{\infty}\mathbb{E}_N\left[\frac{1}{\left(1 - sK_s\left(R_i - D\right)^{-\varepsilon_s}\right)^N}\right]\right]$$

$$\overset{(b)}{=} \mathrm{e}^{-\mu\int_{r=\Delta}^{\infty}\left(1-\mathrm{e}^{\frac{-\lambda\pi D^2}{1-(sK_s)^{-1}(r-D)^{\varepsilon_s}}}\right)2\pi r dr}$$

where (a) is obtained by evaluating the expectation with respect to the i.i.d. $\Psi_{ij}'s$, and by noting that $N_i's$ are i.i.d., and have the same distribution as $N \sim \text{Poisson}\left(\lambda\pi D^2\right)$. Notice that the expectation w.r.t. Ψ_{ij} is bounded since $sK_s\left(R_i - D\right)^{-\varepsilon_s} < 1$ for all the primary receivers that contribute to $\hat{I}_2^{(2)}$. The partition of $\mathbb{R}^2$ in the definition of $\hat{I}_2$ was chosen in such a way that this was possible. Next, (b) is obtained by evaluating the expectation w.r.t. N, and then applying the Campbell's theorem [12, Page 28] for the Poisson point process governing the primary user arrangement, and finally, (13) is obtained by a simple change of the variable of integration. Finally, $\mathcal{L}_{I_{pc}}(s)$ is upper bounded as

$$\mathcal{L}_{I_{pc}}(s) \leq \min\left(1,\ \mathbb{E}\left[\mathrm{e}^{-sI'}\right]\mathbb{E}\left[\mathrm{e}^{sI_2}\right]\right) \leq \min\left(1,\ \mathbb{E}\left[\mathrm{e}^{-sI'}\right]\mathbb{E}\left[\mathrm{e}^{s\hat{I}_2^{(1)}}\right]\mathbb{E}\left[\mathrm{e}^{s\hat{I}_2^{(2)}}\right]\right). \quad (27)$$

This completes the proof for (7).

Proof for $(8)-(10)$: Several upper bounds for $\mathcal{L}_{I_{pc}}(s)$ can be obtained by lower bounding I_{pc} by considering the interference caused by a few active CRs that are closest to the origin (typical primary receiver) and are the dominant interferers. For this we consider the nearest and the next nearest potential CRs in the region $\mathbb{R}^2\bigcap\mathcal{S}^c$, and denote their distances from the origin by R_1 and R_2, respectively. The p.d.f. of R_1 and R_2 denoted by $f_{R_k}(r)$, for $r \geq D$ are

$$f_{R_k}(r) = \frac{\left(\lambda\pi\left(r^2 - D^2\right)\right)^{k-1}\lambda 2\pi r\mathrm{e}^{-\lambda\pi\left(r^2-D^2\right)}}{k!}, \quad (28)$$

for $k = 1,\ 2$. Now, we can obtain the upper bounds (8) and (9) by considering only the interference from the nearest two CRs in the region $\mathbb{R}^2\bigcap\mathcal{S}^c$. The first bound is

$$\mathcal{L}_{I_{pc}}(s) \overset{(a)}{\leq} \mathbb{E}\left[\mathrm{e}^{-\sum_{i=1}^{2}sK_s\Psi_i\|x_i\|_2^{-\varepsilon_s}\mathcal{I}(x_i\notin\mathcal{B}\bigcup\mathcal{S})}\right]$$

$$\overset{(b)}{\leq} \sqrt{\prod_{i=1}^{2}\mathbb{E}\left[\mathrm{e}^{-2sK_s\Psi_i\|x_i\|_2^{-\varepsilon_s}\mathcal{I}(x_i\notin\mathcal{B}\bigcup\mathcal{S})}\right]}$$

$$\overset{(c)}{=} \sqrt{\prod_{i=1}^{2}\mathbb{E}\left[\frac{1}{1 + 2sK_s\|x_i\|_2^{-\varepsilon_s}\mathcal{I}\left(x_i \notin \mathcal{B}\bigcup\mathcal{S}\right)}\right]}$$

$$\overset{(d)}{=} \sqrt{\prod_{i=1}^{2}1 - \rho_{NC} + \rho_{NC}\mathbb{E}\left[\frac{1}{1 + 2sK_s\|x_i\|_2^{-\varepsilon_s}}\right]} \quad (29)$$

where x_i in (a) corresponds to the i^{th} nearest CR in $\mathbb{R}^2\bigcap\mathcal{S}^c$, with $\|x_i\| = R_i$, whose p.d.f. is given in (28), (b) is obtained by applying Cauchy-Schwartz inequality on (a), (c) is obtained by evaluating the expectation with respect to the i.i.d. unit mean exponential random variables Ψ_1 and Ψ_2, respectively, (d) is obtained by evaluating the expectation w.r.t. the random set $\mathcal{B}$, with $\mathbb{P}\left(\{x_i \notin \mathcal{B}\}\right) = \rho_{NC} = \mathrm{e}^{-\mu\pi D^2}$ as seen in $(5)-(b)$, and finally (8) is obtained by evaluating the expectations w.r.t. the random variables R_1 and R_2, and rewriting in terms of the function $H_k(\cdot)$ as defined in (14). The second upper bound based on the same idea is as follows.

$$\mathcal{L}_{I_{pc}}(s) \overset{(a)}{\leq} \mathbb{E}\left[\frac{1}{2}\sum_{i=1}^{2}\mathrm{e}^{-2sK_s\Psi_i\|x_i\|_2^{-\varepsilon_s}\mathcal{I}(x_i\notin\mathcal{B}\bigcup\mathcal{S})}\right], \quad (30)$$

where the expression in $(29)-(a)$ is now upper bounded by applying the Young's inequality to obtain (a), and the upper bound in (9) is obtained by taking the expectation inside the summation, repeating the steps $(29)-(c,\ d,)$, and representing the result in terms of the $H_k(\cdot)$ function.

Next, the upper bound in (10) is obtained by considering the interference caused by only the nearest CR in $\left(\mathcal{B}\bigcup\mathcal{S}\right)^c$.

$$\mathcal{L}_{I_{pc}}(s) \leq \mathbb{E}\left[\mathrm{e}^{-sK_s\Psi_1\|x_1\|_2^{-\varepsilon_s}\mathcal{I}(x_1\notin\mathcal{B}\bigcup\mathcal{S})}\right], \quad (31)$$

and the expression in (10) is obtained by evaluating the above expectation in the same way as shown in (29) and (30).

Finally, minimum of the upper bounds $(7)-(10)$ is also an upper bound which is tighter than all of these, and hence we get (11).