



# Equivalent Boundary Conditions for an Elasto-Acoustic Problem set in a Domain with a Thin Layer

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**Abstract:** In this paper we present equivalent conditions and asymptotic models for the diffraction problem of elasto-acoustic waves in a solid medium surrounded by a thin layer of fluid medium. This problem is well suited for the notion of equivalent conditions : since the thickness of the layer is small with respect to the wavelength, the effect of the fluid medium on the solid is as a first approximation local. We derive and validate equivalent conditions up to the third order for the elastic displacement. These conditions approximate the acoustic waves which propagate in the fluid region. This approach leads us to solve only elastic equations. The construction of equivalent conditions is based on a multiscale expansion in power series of the thickness of the layer for the solution of the transmission problem.

**Key-words:** equivalent boundary conditions, thin layer, asymptotic expansion, elastic waves, acoustic waves, geophysics

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## **Conditions équivalentes pour un problème de transmission élasto-acoustique avec une couche mince**

**Résumé :** Dans ce document, on présente des conditions d'impédances ainsi que des modèles asymptotiques adaptés à des problèmes de transmission pour des couplages élasto-acoustiques dans des milieux avec couche mince acoustique. Ce type de problème est bien posé pour la notion de conditions équivalentes car la faible épaisseur de la couche assure que l'effet acoustique sur la déformation élastique est en première approximation locale. On introduit et justifie rigoureusement des conditions équivalentes adaptées à une telle couche mince. Cette méthode permet d'approcher la solution du problème couplé par celle d'un problème aux limites avec une condition d'impédance, l'erreur d'approximation est contrôlée par rapport à l'épaisseur de la couche.

**Mots-clés :** condition d'impédance, couche mince, couplage élasto-acoustique, développement asymptotique

## 1. INTRODUCTION

The concept of Equivalent Boundary Conditions (also called approximate, effective, or impedance conditions) is classical in the mathematical modeling of wave propagation phenomena. Equivalent Conditions (ECs) are usually introduced to reduce the computational domain of interest. The main idea consists to replace an “exact” model inside a part of the domain (for instance a thin layer of dielectric material or absorbing medium) by an approximate boundary condition. This idea is pertinent when the Equivalent Condition can be readily handled for numerical computations, for instance when this condition is local [11, 27, 3, 13]. In the 1990’s Engquist–Nedelec [11], Bendali–Lemrabet [3], and Lafitte [17] derived equivalent conditions for acoustic and electromagnetic scattering problems to approximate an obstacle coated by a thin layer of dielectric (absorbing) material inside the domain of interest. Impedance conditions are also used to reduce the computational domain when considering scattering problems from highly absorbing obstacles. Taking advantage of the fast decrease of the electromagnetic field inside an absorbing medium, approximate conditions were first proposed by Rytov [25] and by Leontovich [20] in the 1940’s and then extended by Lafitte–Lebeau [18], Senior–Volakis [27] and by Antoine–Barucq–Vernhet [2]. We refer to Haddar–Joly–Nguyen [13] for the mathematical justification of generalized impedance boundary conditions of order 1, 2, and 3 adapted to a conductor with a smooth surface.

Elasto-acoustic coupling problems have become rather classical in the mathematical modeling of wave propagation phenomena. We refer to several works [16, 21, 9, 23, 15, 22] and the monography [7, Chap. 5, §5.4.e] which concern the direct problem for fluid-structure interaction systems and acoustic scattering by smooth elastic obstacles.

In this paper, the main application concern the mathematical modeling of earthquake on the Earth’s surface. The simulation of large-scale geophysics phenomena represents a main challenge for our society. Seismic activities worldwide have shown how crucial it is to enhance our understanding of the impact of earthquakes. In this context, the coupling of elastic and acoustic waves equations is essential if we want to reproduce real physical phenomena such as an earthquake. Acoustic waves propagate into the ocean whereas elastic waves propagate into the Earth subsurfaces. We can thus take into account the effects of the ocean on the propagation of seismic waves.

We intend to work in the context of this application for which we consider that the medium consists of land areas surrounded by fluid zones whose thickness is very small, typically with respect to the wavelength. This raises the difficulty of applying a finite element method on a mesh that combines fine cells in the fluid zone and much larger cells in the solid zone. To overcome this difficulty and to solve this problem, we adopt an asymptotic method which consists to “approximate” the fluid portion by an equivalent boundary condition. This boundary condition is then coupled with the elastic wave equation and a finite element method can be applied to solve the resulting boundary value problem.

In this paper, our aim is to present elements of derivation together with mathematical justifications for equivalent boundary conditions of order 0, 1, 2 and 3 (when considering a Dirichlet external boundary condition for the acoustic pressure) satisfied by the elastic displacement when the thickness of the fluid layer is small enough. These conditions are of “ $\mathbf{u} \cdot \mathbf{n}$ –to– $\mathbf{T}(\mathbf{u}) \cdot \mathbf{n}$ ” nature since a local impedance operator links the normal traces of the elastic displacement  $\mathbf{u}$  and the stress vector  $\mathbf{T}(\mathbf{u})$ . Likewise we derive and analyze equivalent conditions up to the first order when the pressure satisfied a Fourier-Robin external boundary condition. The consideration of this absorbing boundary condition is motivated as well by geophysical applications.

The main tool to construct ECs is a two-scale asymptotic expansion in power series of a small parameter (the thickness of the thin layer). The multiscale analysis to describe a thin layer in a transmission problem is rather classical, see for instance [3, 4, 10, 26]. One difficulty to validate the equivalent conditions lies in the proof of uniform energy estimates (with respect to the size of the thin layer) for the elasto-acoustic coupling. In this paper, we overcome this difficulty for the elasto-acoustic coupling provided a discret set of resonant frequencies are avoided. These frequencies may appear in the solid part of the domain. For this purpose, we revisit and adapt a proof of uniform estimates given by Bendali–Lemrabet [3] (for

the scattering problem of an acoustic wave by a thin sheet) to the elasto-acoustic coupling. We prove well-posedness and convergence results for ECs up to the third order (or a Dirichlet external boundary condition. One difficulty appears for the well-posedness of the first order condition adapted to a Fourier-Robin external boundary condition since the EC appears with a lack of coerciveness. We modify and regularize this condition applying a “ $\mathbf{T}(\mathbf{u}) \cdot \mathbf{n}$ -to- $-\mathbf{u} \cdot \mathbf{n}$ ” formulation.

The outline of the paper proceeds as follows. In Section 2, we introduce the mathematical model and the framework for the elasto-acoustic problem and equivalent conditions. Then we present uniform estimates for the solution of the transmission problem. In Section 3, we present equivalent conditions and asymptotic models associated with the solution of the exact problem. In Section 4, we prove uniform estimates for the solution of the elasto-acoustic problem. In Section 5, we derive and validate a two-scale asymptotic expansion at any order for the solution of the problem, and we construct formally ECs. In Section 6, we prove stability results for ECs and the convergence of ECs towards the exact model. In Appendix A, we derive and prove stability results for ECs of order 0 and 1, when considering a Fourier-Robin external boundary condition. In Appendix B, we explicit the Helmholtz operator in a *normal coordinate* system, and then we expand this operator in power series of the small parameter.

## 2. THE MATHEMATICAL MODEL

**2.1. The model problem.** Our interest lies in an elasto-acoustic wave propagation problem set in a domain with a thin layer. We consider the following fluid-solid transmission problem in time-harmonic regime

$$(1) \quad \begin{cases} \Delta p_\varepsilon + \kappa^2 p_\varepsilon = 0 & \text{in } \Omega_f^\varepsilon \\ \nabla \cdot \underline{\sigma}(\mathbf{u}_\varepsilon) + \omega^2 \rho \mathbf{u}_\varepsilon = \mathbf{f} & \text{in } \Omega_s \\ \partial_n p_\varepsilon = \rho_f \omega^2 \mathbf{u}_\varepsilon \cdot \mathbf{n} & \text{on } \Gamma \\ \mathbf{T}(\mathbf{u}_\varepsilon) = -p_\varepsilon \mathbf{n} & \text{on } \Gamma \\ p_\varepsilon = 0 & \text{on } \Gamma^\varepsilon, \end{cases}$$

set in a smooth bounded simply connected domain  $\Omega^\varepsilon$  in  $\mathbb{R}^3$  made of a solid, elastic object occupying a smooth connected subdomain  $\Omega_s$  entirely immersed in a fluid region occupying the subdomain  $\Omega_f^\varepsilon$ . The domain  $\Omega_f^\varepsilon$  is a thin layer of uniform thickness  $\varepsilon$ , see figure 1. We denote by  $\Gamma^\varepsilon$  the boundary of the domain  $\Omega^\varepsilon$ , and by  $\Gamma$  the interface between the subdomains  $\Omega_f^\varepsilon$  and  $\Omega_s$ . We denote by  $\mathbf{n}$  the unit normal to  $\Gamma$  oriented from  $\Omega_s$  to  $\Omega_f^\varepsilon$ .

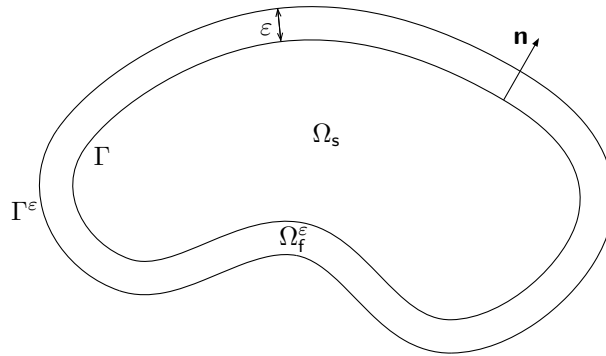


FIGURE 1. A cross-section of the domain  $\Omega^\varepsilon$  and its subdomains  $\Omega_s$  and  $\Omega_f^\varepsilon$

In the elasto-acoustic system (1), we denote the unknowns by  $\mathbf{u}_\varepsilon$  for the elastic displacement and by  $p_\varepsilon$  for the acoustic pressure. The time-harmonic wave field with angular frequency  $\omega$  is characterized by using the Helmholtz equation for the pressure  $p_\varepsilon$ , and by using an anisotropic discontinuous linear elasticity system for the displacement  $\mathbf{u}_\varepsilon$ . These equations contain several physical constants:  $\kappa = \omega/c$  is the acoustic wave number,  $c$  is the speed of the sound,  $\rho$  is the density of the solid, and  $\rho_f$  is the density of the fluid.

In the linear elastic equation,  $\nabla \cdot$  denotes the divergence operator for tensors,  $\underline{\underline{\sigma}}(\mathbf{u})$  is the stress tensor given by Hooke's law

$$\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\varepsilon}}(\mathbf{u}) .$$

Here  $\underline{\underline{\varepsilon}}(\mathbf{u}) = \frac{1}{2}(\underline{\underline{\nabla}}\mathbf{u} + \underline{\underline{\nabla}}\mathbf{u}^T)$  is the strain tensor where  $\underline{\underline{\nabla}}$  denotes the gradient operator for tensors, and  $\underline{\underline{C}} = \underline{\underline{C}}(\mathbf{x})$  is the elasticity tensor, where  $\mathbf{x} \in \mathbb{R}^3$  are the cartesian coordinates. The components of  $\underline{\underline{C}}$  are the elasticity moduli  $C_{ijkl}$  :  $\underline{\underline{C}} = (C_{ijkl}(\mathbf{x}))$ . The *traction operator*  $\mathbf{T}$  is a surfacic differential operator defined on  $\Gamma$  as

$$\mathbf{T}(\mathbf{u}) = \underline{\underline{\sigma}}(\mathbf{u})\mathbf{n} .$$

The right-hand side  $\mathbf{f}$  is a data with support in  $\Omega_s$ . The first transmission condition set on  $\Gamma$  is a kinematic interface condition whereas the second one is a dynamic interface condition. The kinematic condition requires that the normal velocity of the fluid match the normal velocity of the solid on the interface  $\Gamma$ . The dynamic condition results from the equilibrium of forces on the interface  $\Gamma$ . The transmission conditions are natural.

**Remark 2.1.** We consider in this paper mainly Dirichlet external boundary conditions. In Appendix A, we present also equivalent conditions of order 0 and 1 for the elato-acoustic transmission problem complemented with a Fourier-Robin external boundary condition :

$$\partial_n p_\varepsilon - i\kappa p_\varepsilon = 0 \quad \text{on} \quad \Gamma^\varepsilon .$$

In the framework above we address two issues :

- (1) The issue of Equivalent Conditions (ECs) for the elastic displacement  $\mathbf{u}_\varepsilon$  as  $\varepsilon \rightarrow 0$ , see Section 2.3,
- (2) The issue of Uniform Estimates for the displacement  $\mathbf{u}_\varepsilon$  and the pressure  $p_\varepsilon$  solutions of the problem (1) as  $\varepsilon \rightarrow 0$ , see Section 2.4.

The two issues are linked, since the issue of Uniform Estimates is a main ingredient in the mathematical justification of ECs. To answer these questions, we make hereafter several assumptions on the data and on the regularity of the interface  $\Gamma$ . These assumptions simplify the asymptotic modeling.

**2.2. Assumptions.** We will work under usual assumptions (symmetry and positiveness) on the elasticity tensor  $\underline{\underline{C}}$ .

**Assumption 2.2.** (i) The elasticity moduli  $C_{ijkl}(\mathbf{x})$  are real valued smooth functions in  $\overline{\Omega_s}$ .

(ii) The tensor  $\underline{\underline{C}}$  is symmetric :

$$C_{ijkl} = C_{jikl} = C_{klij} \quad \text{almost everywhere in } \Omega_s .$$

(iii) The tensor  $\underline{\underline{C}}$  is positive :

$$\exists \alpha > 0, \quad \forall \xi = (\xi_{ij}) \text{ symmetric tensor}, \quad \sum_{i,j,k,l} C_{ijkl} \xi_{ij} \overline{\xi_{kl}} \geq \alpha \sum_{i,j} |\xi_{ij}|^2 .$$

**Remark 2.3.** According to the assumption 2.2 (ii), the Hooke's law writes also

$$\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{C}} \underline{\underline{\nabla}}\mathbf{u} .$$

The assumption 2.2 (iii) ensures that the matrix differential operator  $\nabla \cdot \underline{\underline{\sigma}} + \omega^2 \rho \mathbb{I}$  is strongly elliptic.

Some resonant frequencies may appear in the solid domain. However, we prove uniform estimates for the elasto-acoustic field  $(\mathbf{u}_\varepsilon, p_\varepsilon)$  as well as ECs for  $\mathbf{u}_\varepsilon$  when  $\varepsilon \rightarrow 0$  under the following spectral assumption on the limit problem set in the solid part  $\Omega_s$ , and when  $\mathbf{f} = 0$ .

**Assumption 2.4.** *The angular frequency  $\omega$  is not an eigenfrequency of the problem*

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}) = 0 & \text{on } \Gamma. \end{cases}$$

Our whole analysis is valid under the following assumption on the surfaces  $\Gamma$  and  $\Gamma_\varepsilon$ .

**Assumption 2.5.** *The fluid-solid interface  $\Gamma$  and the surface  $\Gamma_\varepsilon$  are smooth.*

For the sake of simplicity in the asymptotic modeling, we will work under the following assumption on the data  $\mathbf{f}$ .

**Assumption 2.6.** *The right-hand side  $\mathbf{f}$  in (1) is a smooth  $\varepsilon$ -independent data.*

In the framework above, it is possible to replace the fluid region  $\Omega_f^\varepsilon$  by appropriate boundary conditions called equivalent conditions and set on  $\Gamma$ .

**2.3. Equivalent conditions.** In this paper we derive and prove equivalent conditions (ECs) : we derive surfacic differential operators  $B_\varepsilon$

$$B_\varepsilon : \mathcal{C}^\infty(\Gamma) \rightarrow \mathcal{C}^\infty(\Gamma),$$

together with  $\tilde{\mathbf{u}}_\varepsilon$  which is a solution of the boundary value problem :

$$(2) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\tilde{\mathbf{u}}_\varepsilon) + \omega^2 \rho \tilde{\mathbf{u}}_\varepsilon = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\tilde{\mathbf{u}}_\varepsilon) + B_\varepsilon(\tilde{\mathbf{u}}_\varepsilon \cdot \mathbf{n})\mathbf{n} = 0 & \text{on } \Gamma. \end{cases}$$

Then in the framework of Sec. 2.2, we prove uniform estimates for the error between  $\mathbf{u}_\varepsilon$  (which satisfies (1)) and  $\tilde{\mathbf{u}}_\varepsilon$  provided  $\varepsilon$  is small enough:

$$(3) \quad \|\mathbf{u}_\varepsilon - \tilde{\mathbf{u}}_\varepsilon\|_{1, \Omega_s} \leq C \varepsilon^{k+1},$$

with  $k \in \mathbb{N}$ , see Th. 3.2 for the main result and precise estimates. Here, we denote by  $\|\cdot\|_{1, \Omega_s}$  the norm in the Sobolev space  $\mathbf{H}^1(\Omega_s) = \mathbf{H}^1(\Omega_s)^3$ . We say that the equivalent condition is of order  $k$  when such an a priori estimate (3) holds. Then we define  $\mathbf{u}_\varepsilon^k = \tilde{\mathbf{u}}_\varepsilon$  and we denote by  $B_{k, \varepsilon}$  the operator  $B_\varepsilon$  corresponding to the order  $k$ , see Sec. 3.

The validation of ECs relies on uniform estimates for solutions  $(\mathbf{u}_\varepsilon, p_\varepsilon)$  of (1) as  $\varepsilon \rightarrow 0$ . This issue is developped in Sec. 2.4.

**2.4. Uniform estimates.** We first introduce a suitable variational framework for the solution of the problem (1) with more general right-hand sides. This framework is useful to prove error estimates (3).

**2.4.1. Weak solutions.** For given data  $(\mathbf{f}, \mathbf{f}, \mathbf{g})$  we consider the boundary value problem

$$(4) \quad \begin{cases} \Delta p_\varepsilon + \kappa^2 p_\varepsilon = \mathbf{f} & \text{in } \Omega_f^\varepsilon \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon) + \omega^2 \rho \mathbf{u}_\varepsilon = \mathbf{f} & \text{in } \Omega_s \\ \partial_n p_\varepsilon = \rho_f \omega^2 \mathbf{u}_\varepsilon \cdot \mathbf{n} + \mathbf{g} & \text{on } \Gamma \\ \mathbf{T}(\mathbf{u}_\varepsilon) = -p_\varepsilon \mathbf{n} & \text{on } \Gamma \\ p_\varepsilon = 0 & \text{on } \Gamma^\varepsilon. \end{cases}$$

Hereafter, we explicit a weak formulation of the problem (4). We first introduce the functional space adapted to a variational formulation

$$V_\varepsilon = \{(\mathbf{u}, p) \in \mathbf{H}^1(\Omega_s) \times H^1(\Omega_f^\varepsilon) \mid \gamma_0 p = 0 \text{ on } \Gamma^\varepsilon\}.$$



Here,  $\gamma_0 \mathbf{p}$  is the Dirichlet trace of  $\mathbf{p}$  on  $\Gamma^\varepsilon$ . The space  $V_\varepsilon$  is endowed with the piecewise  $H^1$  norm in  $\Omega_\varepsilon$  and  $\Omega_f^\varepsilon$ . Then the variational problem writes : Find  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V_\varepsilon$  such that

$$(5) \quad \forall (\mathbf{v}, \mathbf{q}) \in V_\varepsilon, \quad a_\varepsilon((\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon), (\mathbf{v}, \mathbf{q})) = \langle F, (\mathbf{v}, \mathbf{q}) \rangle_{V_\varepsilon', V_\varepsilon},$$

where the sesquilinear form  $a_\varepsilon$  is defined as

$$\begin{aligned} a_\varepsilon((\mathbf{u}, \mathbf{p}), (\mathbf{v}, \mathbf{q})) &= \int_{\Omega_f^\varepsilon} (\nabla \mathbf{p} \cdot \nabla \bar{\mathbf{q}} - \kappa^2 \mathbf{p} \bar{\mathbf{q}}) \, d\mathbf{x} \\ &\quad + \int_{\Omega_s} (\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\varepsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \bar{\mathbf{v}}) \, d\mathbf{x} + \int_{\Gamma} (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{\mathbf{q}} + \mathbf{p} \bar{\mathbf{v}} \cdot \mathbf{n}) \, d\sigma \end{aligned}$$

and the right-hand side  $F$  is defined as

$$\langle F, (\mathbf{v}, \mathbf{q}) \rangle_{V_\varepsilon', V_\varepsilon} = - \int_{\Omega_f^\varepsilon} f \bar{\mathbf{q}} \, d\mathbf{x} - \int_{\Omega_s} \mathbf{f} \cdot \bar{\mathbf{v}} \, d\mathbf{x} - \int_{\Gamma} g \bar{\mathbf{q}} \, d\sigma.$$

We assume that the data  $(\mathbf{f}, f, g)$  are smooth enough such that the right-hand side  $F$  belongs to the space  $V_\varepsilon'$ .

**2.4.2. Statement of uniform estimates.** In the framework of assumptions quoted in Section 2.2 we can prove  $\varepsilon$ -uniform a priori estimate for the solution of problem (5). Precisely, the following theorem is the main result in this section.

**Theorem 2.7.** *Under Assumptions 2.2-2.4-2.5, there exists constants  $\varepsilon_0, C > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$ , the problem (5) with data  $F \in V_\varepsilon'$  has a unique solution  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V_\varepsilon$  which satisfies*

$$(6) \quad \|\mathbf{p}_\varepsilon\|_{1, \Omega_f^\varepsilon} + \|\mathbf{u}_\varepsilon\|_{1, \Omega_s} \leq C \|F\|_{V_\varepsilon'}.$$

This result is proved in Sec. 4. The proof is based on a formulation of the problem in a fixed domain. This formulation is obtained through a scaling along the thickness of the layer.

As an application of uniform estimates (6), we will prove in Sec. 6 the convergence result (3). This result means that the Equivalent Condition in the asymptotic model (2) is of order  $k$ .

### 3. STATEMENT OF EQUIVALENT CONDITIONS

In the framework of Section 2, for all  $k \in \{0, 1, 2, 3\}$ , we can derive an equivalent boundary condition set on  $\Gamma$  which is associated with the exact problem (1) and satisfied by  $\mathbf{u}_\varepsilon^k$ . It means that  $\mathbf{u}_\varepsilon^k$  solves the problem

$$(7) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon^k) + \omega^2 \rho \mathbf{u}_\varepsilon^k = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_\varepsilon^k) + \mathbf{B}_{k, \varepsilon}(\mathbf{u}_\varepsilon^k \cdot \mathbf{n}) \mathbf{n} = 0 & \text{on } \Gamma. \end{cases}$$

Here  $\mathbf{B}_{k, \varepsilon}$  is a surfacic differential operator acting on functions defined on  $\Gamma$  and which depends on  $\varepsilon$ . We present hereafter Equivalent Conditions (ECs) of order  $k \in \{0, 1, 2, 3\}$ . Formal calculus adapted to derive these ECs are presented in Sec. 5. The well-posedness and convergence of ECs are proved in Sec. 6.

**3.1. Equivalent conditions.** We obtain a hierarchy of boundary-value problems. Each one gives a model with a different order of accuracy in  $\varepsilon$  and reflects the effect of the thin layer. We derive in Section 5.3 the following differential operators  $\mathbf{B}_{k, \varepsilon}$  for all  $k \in \{0, 1, 2, 3\}$  in problem (7) :

$$\mathbf{B}_{0, \varepsilon} = 0,$$

$$\mathbf{B}_{1, \varepsilon} = -\varepsilon \omega^2 \rho_f \mathbb{I},$$

$$\mathbf{B}_{2, \varepsilon} = -\varepsilon \omega^2 \rho_f (1 - \varepsilon \mathcal{H}(y_\alpha)) \mathbb{I},$$

$$\mathbf{B}_{3,\varepsilon} = -\varepsilon\omega^2\rho_f \left( (1 - \varepsilon\mathcal{H}(y_\alpha))\mathbb{I} + \frac{\varepsilon^2}{3} [\Delta_\Gamma + \kappa^2\mathbb{I} + (4\mathcal{H}^2(y_\alpha) - \mathcal{K}(y_\alpha))\mathbb{I}] \right).$$

Here,  $(y_\alpha)$ ,  $\alpha = 1, 2$ , are tangential coordinates on  $\Gamma$ ,  $\mathcal{H}$  and  $\mathcal{K}$  denote the *mean curvature* and the *Gaussian curvature* of the surface  $\Gamma$ , see Sec. 5.1, and  $\Delta_\Gamma$  is the Laplace-Beltrami operator along  $\Gamma$ . Hence, Equivalent conditions of order  $k \in \{0, 1, 2, 3\}$  write

*Order 0.*

$$\mathbf{T}(\mathbf{u}_0) = 0 \quad \text{on } \Gamma$$

*Order 1.*

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - \varepsilon\omega^2\rho_f\mathbf{u}_\varepsilon^1 \cdot \mathbf{n}\mathbf{n} = 0 \quad \text{on } \Gamma$$

*Order 2.*

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - \varepsilon\omega^2\rho_f(1 - \varepsilon\mathcal{H}(y_\alpha))\mathbf{u}_\varepsilon^2 \cdot \mathbf{n}\mathbf{n} = 0 \quad \text{on } \Gamma$$

*Order 3.*

$$\mathbf{T}(\mathbf{u}_\varepsilon^3) - \varepsilon\omega^2\rho_f \left( 1 - \varepsilon\mathcal{H}(y_\alpha) + \frac{\varepsilon^2}{3} [\Delta_\Gamma + \kappa^2\mathbb{I} + 4\mathcal{H}^2(y_\alpha) - \mathcal{K}(y_\alpha)] \right) (\mathbf{u}_\varepsilon^3 \cdot \mathbf{n})\mathbf{n} = 0 \quad \text{on } \Gamma$$

These conditions highlight the successive corrections brought when increasing the order. The conditions of order  $k \in \{0, 1, 2\}$  involves only partial derivatives of order 1 in the operator  $\mathbf{T}$ , whereas the Equivalent condition of order  $k = 3$  is a Ventcel's condition [1, 19] since it involves partial derivatives of order 2.

**Remark 3.1.** *In the equivalent condition of order 0,  $\mathbf{u}_0 := \mathbf{u}_\varepsilon^0$  is independent of  $\varepsilon$ . It corresponds to a model where the effect of the thin layer is completely neglected. The effect of the fluid part appears at the order 1 through the density  $\rho_f$ . The influence of the geometry of the interface  $\Gamma$  appears from the order 2 through the mean curvature of  $\Gamma$ .*

**3.2. Stability and convergence of Equivalent conditions.** Towards the theoretical justification of Equivalent Conditions, our goal in the next sections is to validate ECs set on  $\Gamma$  (Sec. 3.1) proving estimates for  $\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k$  for all  $k \in \{0, 1, 2, 3\}$ , where  $\mathbf{u}_\varepsilon^k$  is the solution of the approximate model (7), and  $\mathbf{u}_\varepsilon$  satisfies the problem (1).

**Theorem 3.2.** *Under Assumptions 2.2-2.4-2.5-2.6, for all  $k \in \{0, 1, 2, 3\}$  there exists constants  $\varepsilon_k, C_k > 0$  such that for all  $\varepsilon \in (0, \varepsilon_k)$ , the problem (7) with data  $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$  has a unique solution  $\mathbf{u}_\varepsilon^k \in \mathbf{V}^k$  which satisfies uniform estimates*

$$(8) \quad \|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C_k \varepsilon^{k+1}.$$

Here  $\mathbf{V}^k$  denotes the space  $\mathbf{H}^1(\Omega_s)$  when  $k \in \{0, 1, 2\}$ , and  $\{\mathbf{u} \in \mathbf{H}^1(\Omega_s) \mid \mathbf{u} \cdot \mathbf{n}|_\Gamma \in \mathbf{H}^1(\Gamma)\}$  when  $k = 3$ .

The well-posedness result for the problem (7) is stated in Prop. 5.3, Sec. 5.5 and is proved in Sec. 6.1. It appears nontrivial to work directly with the difference  $\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k$ , see [13] for a similar context in electromagnetism. Hence, the usual trick consists to use the truncates series  $\mathbf{u}_{k,\varepsilon}$  introduced in Sec. 5.3 as intermediate quantities. Therefore, the error analysis is split into two steps detailed in the next sections :

- (1) We prove uniform estimates for the difference  $\mathbf{u}_\varepsilon - \mathbf{u}_{k,\varepsilon}$ , see Thm. 5.2 in Section 5.4,
- (2) We prove uniform estimates for the difference  $\mathbf{u}_{k,\varepsilon} - \mathbf{u}_\varepsilon^k$  : this is done in Section 6.2 and more precisely in Sec. 6.2.2.

**Remark 3.3.** *Note that the first step of the proof is independent of ECs and will be valid for any integer  $k$ . The second step for  $k = 0$  is useless since  $\mathbf{u}_{0,\varepsilon} = \mathbf{u}_\varepsilon^0$ .*

## 4. UNIFORM ESTIMATES

Since the functional setting of the variational problem (5) depends on the thickness parameter  $\varepsilon$ , it is not well suited to give a precise sense for an asymptotic analysis of solutions  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V_\varepsilon$ . To overcome this difficulty, we follow an idea developed in [3] : we write equivalently the variational problem (5) in a common functional framework as  $\varepsilon$  varying, section 4.2. We state uniform estimates in this new framework, Th. 4.2. In Sec. 4.3 we prove some preliminary results (Poincaré and trace inequalities). Then, we prove uniform estimates in Sec. 4.4. We first introduce geometrical tools and notations.

**4.1. Geometrical tools - Notations.** We parameterize the smooth surface  $\Gamma$  through a local coordinates system on  $\Gamma$  defined by a local map  $(\mathcal{U}, \Psi)$

$$\Psi : (y_\alpha) \in \mathcal{U} \mapsto \mathbf{x}_\Gamma = \Psi(y_\alpha) \in \Gamma .$$

We then define the unit tangent vectors  $\tau_\alpha = \partial_{y_\alpha} \Psi$  on  $\Gamma$ .

Then, there exists a constant  $\varepsilon_0 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$  the thin layer  $\Omega_f^\varepsilon$  can be parameterized with coordinates  $(y_\alpha, s)$  :

$$\Omega_f^\varepsilon = \{ \mathbf{x} = \mathbf{x}_\Gamma + s\mathbf{n}(\mathbf{x}_\Gamma) \in \mathbb{R}^2 \mid \mathbf{x}_\Gamma \in \Gamma, \quad s \in (0, \varepsilon) \} .$$

Here  $s$  is the distance to the surface  $\Gamma$  and  $\mathbf{n}(\mathbf{x}_\Gamma)$  denotes the normal vector on  $\Gamma$  at the point  $\mathbf{x}_\Gamma$  :  $\mathbf{n} = \tau_1 \times \tau_2 / |\tau_1 \times \tau_2|$ .

Hence, we can define a parameterization of the thin shell  $\Omega_f^\varepsilon$  by the *scaled domain*  $\Omega_f = \Gamma \times (0, 1)$  through the mapping

$$(\mathbf{x}_\Gamma, S) \in \Omega_f \mapsto \mathbf{x} = \mathbf{x}_\Gamma + \varepsilon S \mathbf{n}(\mathbf{x}_\Gamma) \in \Omega_f^\varepsilon .$$

For any function  $\mathbf{p}$  defined in  $\Omega_f^\varepsilon$ , we denote  $\mathbf{p}$  the function defined in the scaled domain  $\Omega_f$  such that :

$$\mathbf{p}(\mathbf{x}) = \mathbf{p}(\mathbf{x}_\Gamma, S) , \quad (\mathbf{x}_\Gamma, S = \frac{s}{\varepsilon}) \in \Gamma \times (0, 1) .$$

We introduce an intrinsic symmetric linear operator  $\mathcal{R}$  defined on the tangent plane  $\mathbf{T}_{\mathbf{x}_\Gamma}(\Gamma)$  to  $\Gamma$  at the point  $\mathbf{x}_\Gamma \in \Gamma$  as follow:

$$\frac{\partial \mathbf{n}}{\partial y_\alpha} = \mathcal{R} \tau_\alpha .$$

This operator  $\mathcal{R}$  characterizes the curvature of  $\Gamma$  at the point  $\mathbf{x}_\Gamma$ . Using the projection operator  $\Pi_{\mathbf{x}_\Gamma}$  from  $\mathbb{R}^3$  onto the tangent plane  $\mathbf{T}_{\mathbf{x}_\Gamma}(\Gamma)$ , there holds :

$$(9) \quad \Pi_{\mathbf{x}_\Gamma} \nabla \mathbf{p} = (\mathbb{I} + \varepsilon S \mathcal{R})^{-1} \nabla_\Gamma \mathbf{p} ,$$

[3]. In the framework above we obtain

$$(10) \quad \forall \mathbf{p}, \mathbf{q} \in L^2(\Omega_f^\varepsilon) , \quad \int_{\Omega_f^\varepsilon} \mathbf{p} \bar{\mathbf{q}} \, d\mathbf{x} = \varepsilon \int_0^1 \int_\Gamma \mathbf{p} \bar{\mathbf{q}} \det(\mathbb{I} + \varepsilon S \mathcal{R}) \, d\Gamma \, dS ,$$

and according to (9), there holds for all  $\mathbf{p}, \mathbf{q} \in H^1(\Omega_f^\varepsilon)$

$$(11) \quad \int_{\Omega_f^\varepsilon} \nabla \mathbf{p} \cdot \nabla \bar{\mathbf{q}} \, d\mathbf{x} = \varepsilon \int_0^1 \int_\Gamma \left\{ (\mathbb{I} + \varepsilon S \mathcal{R})^{-2} \nabla_\Gamma \mathbf{p} \nabla_\Gamma \bar{\mathbf{q}} + \varepsilon^{-2} \partial_S \mathbf{p} \partial_S \bar{\mathbf{q}} \right\} \det(\mathbb{I} + \varepsilon S \mathcal{R}) \, d\Gamma \, dS ,$$

and

$$(12) \quad \int_\Gamma (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{\mathbf{q}} + \mathbf{p} \bar{\mathbf{v}} \cdot \mathbf{n}) \, d\sigma = \int_\Gamma (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{\mathbf{q}} + \mathbf{p} \bar{\mathbf{v}} \cdot \mathbf{n}) \, d\Gamma .$$

**Remark 4.1.** According to (10) and (11), there holds for all  $\varepsilon \in (0, \varepsilon_0)$

$$(13a) \quad \forall \mathbf{p} \in L^2(\Omega_f^\varepsilon) , \quad \|\mathbf{p}\|_{0, \Omega_f^\varepsilon} \simeq \sqrt{\varepsilon} \|\mathbf{p}\|_{0, \Omega_f} ,$$

$$(13b) \quad \forall \mathbf{p} \in H^1(\Omega_f^\varepsilon) , \quad \|\nabla \mathbf{p}\|_{0, \Omega_f^\varepsilon} \simeq \sqrt{\varepsilon} \|\nabla_\Gamma \mathbf{p}\|_{0, \Omega_f} + \sqrt{\varepsilon}^{-1} \|\partial_S \mathbf{p}\|_{0, \Omega_f} .$$

**4.2. The scaled problem.** We can write now the variational problem (5) in the fixed domain  $\Omega_s \times \Omega_f$ . We first introduce the ad-hoc functional space: it writes

$$V = \{(\mathbf{u}, \mathbf{p}) \in \mathbf{H}^1(\Omega_s) \times H^1(\Omega_f) \mid \mathbf{p}(\cdot, 1) = 0 \text{ on } \Gamma\}.$$

Then the variational problem writes : Find  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$  such that

$$(14) \quad \forall (\mathbf{v}, \mathbf{q}) \in V, \quad \varepsilon a_f(\varepsilon; \mathbf{p}_\varepsilon, \mathbf{q}) + a_s(\mathbf{u}_\varepsilon, \mathbf{v}) + \int_{\Gamma} (\omega^2 \rho_f \mathbf{u} \cdot \mathbf{n} \bar{\mathbf{q}} + \mathbf{p} \bar{\mathbf{v}} \cdot \mathbf{n}) \, d\Gamma = \langle \mathfrak{F}_\varepsilon, (\mathbf{v}, \mathbf{q}) \rangle_{V', V},$$

where

$$(15) \quad a_f(\varepsilon; \mathbf{p}, \mathbf{q}) = \int_0^1 \int_{\Gamma} \left\{ (\mathbb{I} + \varepsilon S\mathcal{R})^{-2} \nabla_{\Gamma} \mathbf{p} \nabla_{\Gamma} \bar{\mathbf{q}} + \varepsilon^{-2} \partial_S \mathbf{p} \partial_S \bar{\mathbf{q}} \right\} \det(\mathbb{I} + \varepsilon S\mathcal{R}) \, d\Gamma \, dS \\ - \kappa^2 \int_0^1 \int_{\Gamma} \mathbf{p} \bar{\mathbf{q}} \det(\mathbb{I} + \varepsilon S\mathcal{R}) \, d\Gamma \, dS, \\ a_s(\mathbf{u}, \mathbf{v}) = \int_{\Omega_s} (\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\varepsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \bar{\mathbf{v}}) \, d\mathbf{x},$$

and

$$\langle \mathfrak{F}_\varepsilon, (\mathbf{v}, \mathbf{q}) \rangle_{V', V} = -\varepsilon \int_{\Omega_f} \bar{\mathbf{f}} \bar{\mathbf{q}} \det(\mathbb{I} + \varepsilon S\mathcal{R}) \, d\Gamma \, dS - \int_{\Omega_s} \mathbf{f} \cdot \bar{\mathbf{v}} \, d\mathbf{x} - \int_{\Gamma} \mathbf{g} \bar{\mathbf{q}} \, d\Gamma.$$

Our main result for the problem (14) is the following a priori estimate, uniform as  $\varepsilon \rightarrow 0$ .

**Theorem 4.2.** *Under Assumptions 2.2-2.4-2.5, there exist constants  $\varepsilon_0, C > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$ , the problem (14) with data  $\mathfrak{F}_\varepsilon \in V'$  has a unique solution  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$  which satisfies the uniform estimates*

$$(16) \quad \sqrt{\varepsilon} \|\nabla_{\Gamma} \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \sqrt{\varepsilon}^{-1} \|\partial_S \mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{p}_\varepsilon\|_{0, \Omega_f} + \|\mathbf{p}_\varepsilon\|_{0, \Gamma} + \|\mathbf{u}_\varepsilon\|_{1, \Omega_s} \leq C \|\mathfrak{F}_\varepsilon\|_{V'}.$$

This theorem is the key for the proof of Thm. 6 : as a consequence of estimates (13a)-(13b), we obtain estimates (6). The proof of Thm. 4.2 is based on the following statement.

**Lemma 4.3.** *Under Assumptions 2.2-2.4-2.5, there exist constants  $\varepsilon_0, C > 0$  such that for all  $\varepsilon \in (0, \varepsilon_0)$ , any solution  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$  of problem (14) with a data  $\mathfrak{F}_\varepsilon \in V'$  satisfies the uniform estimates*

$$(17) \quad \|(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_\varepsilon \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_\varepsilon\|_{0, \Gamma} \leq C \|\mathfrak{F}_\varepsilon\|_{V'}.$$

This Lemma is going to be proved in Sec. 4. As a consequence of estimates (17), we infer estimates (16). Since the problem (14) is of Fredholm type, the Thm. 4.2 is then obtained as a consequence of the Fredholm alternative. This argument is rather standard, we refer for instance to the proof of [6, Cor. 4.3] for a similar argument.

The term  $\varepsilon$  weighting the form  $a_f(\varepsilon; \mathbf{p}, \mathbf{q})$  in formulation (14) may lead to a solution  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$  such that the surface gradient  $\nabla_{\Gamma} \mathbf{p}_\varepsilon$  can be unbounded as  $\varepsilon \rightarrow 0$ . We refer to [3] where the authors exhibit this kind of singular perturbation term for a diffraction problem of acoustic waves. Furthermore, the sign of the left-hand side of the problem (14) for  $(\mathbf{v}, \mathbf{q}) = (\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)$  cannot be controlled.

Due to the lack of coerciveness of the variational formulation (14) one cannot get straightforwardly estimates : the formulation is not sufficient by itself to ensure the stability with respect to  $\varepsilon$  on which the asymptotic analysis is based. Hence the proof of this stability result constitutes a main part of the asymptotic analysis. The proof of this result involves a compactness argument and the spectral assumption 2.4 of the related problem obtained when completely neglecting the effect of the layer. We first prove some preliminary results (Poincaré and trace inequalities) useful for the proof of Lemma 4.3.

**4.3. Poincaré and trace inequalities.** We prove hereafter that for suitable right-hand sides, the solution  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon) \in V$  of problem (14) will be bounded but only in a weakened norm related to the Hilbert space

$$W = \{(\mathbf{u}, \mathbf{p}) \in \mathbf{H}^1(\Omega_s) \times H^1(0, 1; L^2(\Gamma)) \mid \mathbf{p}(\cdot, 1) = 0 \text{ on } \Gamma\}.$$

Recall that  $H^1(0, 1; L^2(\Gamma))$  is the space of distributions  $\mathbf{p} \in \mathcal{D}'(0, 1; L^2(\Gamma))$  such that  $\mathbf{p}$  and  $\mathbf{p}'$  belong to  $L^2(0, 1; L^2(\Gamma))$ . Subsequently, we identify the space  $L^2(0, 1; L^2(\Gamma))$  and  $L^2(\Omega_f)$ . The space  $H^1(0, 1; L^2(\Gamma))$  endowed with its canonical norm is a Hilbert space. For any  $S \in (0, 1)$ , the trace  $\mathbf{p}(\cdot, S)$  is well defined since  $S \mapsto \mathbf{p}(\cdot, S)$  is then a continuous function of  $S$  with values in  $L^2(\Gamma)$ . We introduce the Hilbert space  $H$  characterizing  $\mathbf{p}$ :

$$H = \{\mathbf{p} \in H^1(0, 1; L^2(\Gamma)) \mid \mathbf{p}(\cdot, 1) = 0 \text{ on } \Gamma\}.$$

We first prove a Poincaré inequality and a trace inequality in  $H$ .

**Proposition 4.4** (Poincaré inequality in  $H$ ). *There exists  $C_P > 0$  such that  $\forall \mathbf{p} \in H$ ,  $\|\mathbf{p}\|_{0, \Omega_f} \leq C_P \|\partial_S \mathbf{p}\|_{0, \Omega_f}$ .*

*Proof.* For any  $\mathbf{p} \in H$ , and  $\mathbf{x}_\Gamma \in \Gamma$ , we denote  $f$  the function defined in  $(0, 1)$  as  $f(S) = \mathbf{p}(\mathbf{x}_\Gamma, S)$ . There holds  $f \in H^1(0, 1)$  and  $f(1) = 0$ . Hence, using a Poincaré inequality in  $(0, 1)$ : There exists  $C_P > 0$  such that

$$\forall f \in H^1(0, 1) \text{ s.t. } f(1) = 0, \quad \int_0^1 |f(S)|^2 dS \leq C_P \int_0^1 |f'(S)|^2 dS;$$

we infer the proposition.  $\square$

Hereafter, we denote by  $\partial_S$  the partial derivative with respect to the coordinate  $S$ .

**Proposition 4.5** (Trace inequality in  $H$ ). *There exists  $C > 0$  such that*

$$\forall \mathbf{p} \in H, \quad \|\mathbf{p}\|_{0, \Gamma} \leq C \|\partial_S \mathbf{p}\|_{0, \Omega_f}.$$

*Proof.* For all  $\mathbf{p} \in H$ , there holds

$$\mathbf{p}(\mathbf{x}_\Gamma, Y) = \mathbf{p}(\mathbf{x}_\Gamma, 0) + \int_0^Y \partial_S \mathbf{p}(\mathbf{x}_\Gamma, S) dS.$$

Note that since  $\mathbf{p}(\mathbf{x}_\Gamma, \cdot)$  is a continuous function, the trace  $\mathbf{p}(\mathbf{x}_\Gamma, 0)$  on  $\Gamma$  is well defined. Using the boundary condition  $\mathbf{p}(\cdot, 1) = 0$ , we infer

$$\mathbf{p}(\mathbf{x}_\Gamma, 0) = - \int_0^1 \partial_S \mathbf{p}(\mathbf{x}_\Gamma, S) dS.$$

We conclude the proof using a Jensen inequality.  $\square$

**4.4. Proof of Lemma 4.3: Uniform estimate of  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)$ .** We prove this lemma by contradiction. Reductio ad absurdum: We assume that there exists a sequence  $(\mathbf{u}_m, \mathbf{p}_m) \in V$ ,  $m \in \mathbb{N}$ , of solutions of problem (14) associated with a parameter  $\varepsilon_m$  and a right-hand side  $\mathfrak{F}_m \in V'$ :

$$(18) \quad \forall (\mathbf{v}, \mathbf{q}) \in V, \quad \varepsilon_m a_f(\varepsilon_m; \mathbf{p}_m, \mathbf{q}) + a_s(\mathbf{u}_m, \mathbf{v}) + \int_\Gamma (\omega^2 \rho_f \mathbf{u}_m \cdot \mathbf{n} \bar{\mathbf{q}} + \mathbf{p}_m \bar{\mathbf{v}} \cdot \mathbf{n}) d\Gamma = \langle \mathfrak{F}_m, (\mathbf{v}, \mathbf{q}) \rangle_{V', V},$$

satisfying the following conditions

$$\begin{aligned} (19a) \quad & \varepsilon_m \rightarrow 0 & \text{as } m \rightarrow \infty, \\ (19b) \quad & \|(\mathbf{u}_m, \mathbf{p}_m)\|_{0, \Omega_s \times \Omega_f} + \|\mathbf{u}_m \cdot \mathbf{n}\|_{0, \Gamma} + \|\mathbf{p}_m\|_{0, \Gamma} = 1 & \text{for all } m \in \mathbb{N}, \\ (19c) \quad & \|\mathfrak{F}_m\|_{V'} \rightarrow 0 & \text{as } m \rightarrow \infty. \end{aligned}$$

Choosing tests functions  $(\mathbf{v}, \mathbf{q}) = (\mathbf{u}_m, \mathbf{p}_m)$  in (18), we obtain with the help of conditions (19a)-(19b)-(19c) the following uniform bounds:

(i) The sequence  $\{\mathbf{u}_m, \mathbf{p}_m\}$  is bounded in  $W$  :

$$(20) \quad \|(\mathbf{u}_m, \mathbf{p}_m)\|_W \leq C.$$

(ii) The sequence  $\{\sqrt{\varepsilon_m}^{-1} \partial_S \mathbf{p}_m\}$  is bounded in  $L^2(\Omega_f)$ , hence the sequence  $\{\partial_S \mathbf{p}_m\}$  converges to 0 in  $L^2(\Omega_f)$ .

4.4.1. *Limit of the sequences  $\{\mathbf{u}_m, \mathbf{p}_m\}$ .* As a consequence of (ii) and propositions 4.4 and 4.5, we infer that sequences  $\{\mathbf{p}_m\}$  and  $\{\gamma_0 \mathbf{p}_m\}$  converge to 0 in  $L^2(\Omega_f)$  and to 0 in  $L^2(\Gamma)$  respectively

$$(21) \quad \begin{cases} \mathbf{p}_m \rightarrow 0 & \text{in } L^2(\Omega_f) \\ \gamma_0 \mathbf{p}_m \rightarrow 0 & \text{in } L^2(\Gamma). \end{cases}$$

Since the domain  $\Omega_s$  is bounded, the embedding of  $\mathbf{H}^1(\Omega_s)$  into  $\mathbf{L}^2(\Omega_s)$  is compact. Hence as a consequence of (20), using the Rellich Lemma we can extract a subsequence of  $\{\mathbf{u}_m, \mathbf{p}_m\}$  (still denoted by  $\{\mathbf{u}_m, \mathbf{p}_m\}$ ) which is converging in  $W$ , and we can assume that the sequence  $\{\underline{\nabla} \mathbf{u}_m\}$  is weakly converging in  $\mathbf{L}^2(\Omega_s)$ . Another consequence of (20) is that the sequence  $\{\mathbf{u}_m \cdot \mathbf{n}\}$  is bounded in  $H^{\frac{1}{2}}(\Gamma)$ . Therefore (up to the extraction of a subsequence) we can assume that the sequence  $\{\mathbf{u}_m \cdot \mathbf{n}\}$  is strongly converging in  $L^2(\Gamma)$ . Summarizing these convergence results, we deduce that there exists  $\mathbf{u} \in \mathbf{L}^2(\Omega_s)$  such that

$$(22) \quad \begin{cases} \underline{\nabla} \mathbf{u}_m \rightharpoonup \underline{\nabla} \mathbf{u} & \text{in } \mathbf{L}^2(\Omega_s) \\ \mathbf{u}_m \rightarrow \mathbf{u} & \text{in } \mathbf{L}^2(\Omega_s) \\ \mathbf{u}_m \cdot \mathbf{n} \rightarrow \mathbf{u} \cdot \mathbf{n} & \text{in } L^2(\Gamma). \end{cases}$$

As a consequence of the strong convergence of sequences  $\{\mathbf{u}_m\}$  in  $\mathbf{L}^2(\Omega_s)$  and  $\{\mathbf{u}_m \cdot \mathbf{n}\}$  in  $L^2(\Gamma)$ , and the strong convergence of  $\mathbf{p}_m$  and  $\gamma_0 \mathbf{p}_m$  (21) together with (19b), we infer

$$\|\mathbf{u}\|_{0,\Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0,\Gamma} = 1.$$

4.4.2. *Conclusion.* Using Assumption 2.4, we are going to prove hereafter that  $\mathbf{u} = 0$ , which will contradict  $\|\mathbf{u}\|_{0,\Omega_s} + \|\mathbf{u} \cdot \mathbf{n}\|_{0,\Gamma} = 1$ , and finally prove estimate (17).

We use  $(\mathbf{v}, \mathbf{q} = 0) \in V$  as test functions in (18): there holds

$$\int_{\Omega_s} (\underline{\sigma}(\mathbf{u}_m) : \underline{\varepsilon}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u}_m \bar{\mathbf{v}}) \, d\mathbf{x} + \int_{\Gamma} \mathbf{p}_m \bar{\mathbf{v}} \cdot \mathbf{n} \, d\Gamma = \langle \mathfrak{F}_m, (\mathbf{v}, 0) \rangle_{V',V}.$$

According to (22), (21) and (19c), taking limits as  $m \rightarrow +\infty$ , we deduce from the previous equalities  $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$  satisfies for all  $\mathbf{v} \in \mathbf{H}^1(\Omega_s)$ :

$$(23) \quad \int_{\Omega_s} (\underline{\sigma}(\mathbf{u}) : \underline{\varepsilon}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \bar{\mathbf{v}}) \, d\mathbf{x} = 0.$$

Integrating by parts the first term of the sesquilinear form (23), there holds

$$(24) \quad -(\underline{\sigma}(\mathbf{u}), \underline{\varepsilon}(\bar{\mathbf{v}}))_{0,\Omega_s} = (\nabla \cdot \underline{\sigma}(\mathbf{u}), \bar{\mathbf{v}})_{0,\Omega_s} - (\mathbf{T}(\mathbf{u}), \bar{\mathbf{v}})_{|\Gamma}.$$

Thus  $\mathbf{u}$  satisfies the problem

$$\begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}) = 0 & \text{on } \Gamma. \end{cases}$$

By Assumption 2.4, we deduce

$$\mathbf{u} = 0 \quad \text{in } \Omega_s,$$

which contradicts  $\|\mathbf{u}\|_{0,\Omega} + \|\mathbf{u} \cdot \mathbf{n}\|_{0,\Gamma} = 1$  and ends the proof of Lemma 4.3.

## 5. DERIVATION OF EQUIVALENT CONDITIONS

In this section, we first present formal calculations to derive an asymptotic expansion for the elastic displacement  $\mathbf{u}_\varepsilon$  together with the acoustic pressure  $p_\varepsilon$ , §5.1. Then we construct formally equivalent conditions, §5.3. In §5.4 we validate the asymptotic expansion by proving error estimates for the remainders. The main result of this section is the Proposition 5.3 in §5.5 which proves the stability of equivalent conditions.

**5.1. Multiscale expansion.** We can exhibit series expansions in powers of  $\varepsilon$  for the elastic displacement  $\mathbf{u}_\varepsilon$  and for the acoustic pressure  $p_\varepsilon$  :

$$(25) \quad \mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) + \varepsilon \mathbf{u}_1(\mathbf{x}) + \varepsilon^2 \mathbf{u}_2(\mathbf{x}) + \cdots ,$$

$$(26) \quad p_\varepsilon(\mathbf{x}) = p_0(\mathbf{x}; \varepsilon) + \varepsilon p_1(\mathbf{x}; \varepsilon) + \varepsilon^2 p_2(\mathbf{x}; \varepsilon) + \cdots , \quad p_j(\mathbf{x}; \varepsilon) = \mathbf{p}_j(y_\alpha, \frac{y_3}{\varepsilon}) ,$$

see Sec. 5.4 for precise estimates. Here  $(y_\alpha, y_3)$  is a “normal coordinate system” in  $\Omega_f^\varepsilon$ , see Appendix B. The term  $\mathbf{p}_j$  is a “profile” defined on  $\Gamma \times (0, 1)$ . The formal calculus concerning the problem are presented in Sec. 5.1.2 and the first terms  $(\mathbf{p}_j, \mathbf{u}_j)$  for  $j = 0, 1, 2, 3$  are explicited in Sec. 5.2.

**5.1.1. Notations.** We first write the three dimensional Helmholtz operator in the layer  $\Omega_f^\varepsilon$  through the local coordinates  $(y_\alpha, y_3)$ : there holds for all  $h = y_3 \in (0, \varepsilon)$

$$(27) \quad \Delta + \kappa^2 \text{Id} = (\partial_3^h)^2 - b_\alpha^\alpha(h) \partial_3^h + a^{\alpha\beta}(h) D_\alpha^h \partial_\beta + \kappa^2 \text{Id} \quad \text{on } \Gamma_h ,$$

see Prop. B.1. Here  $\partial_3^h$  is the partial derivative with respect to the normal coordinate  $y_3 = h$ ,  $D_\alpha^h$  is the covariant derivative on the manifold  $\Gamma_h$ ,  $a^{\alpha\beta}(h)$  is the inverse of the metric tensor in  $\Gamma_h$ ,  $b_\alpha^\alpha(h) = a^{\alpha\beta}(h) b_{\beta\alpha}(h)$  and  $b_{\alpha\beta}(h)$  is the curvature tensor in  $\Gamma_h$ .

We make the scaling  $Y_3 = \varepsilon^{-1} h \in (0, 1)$  into the local coordinates. It maps  $\Gamma \times (0, \varepsilon)$  onto  $\Gamma \times (0, 1)$ . The small parameter  $\varepsilon$  does not appear anymore in the geometry but in the equations (1) written through the expression of the Helmholtz operator into power of  $\varepsilon$  in the thin layer :

$$\kappa^2 \text{Id} + \Delta = \varepsilon^{-2} \left( \sum_{n=0}^{N-1} \varepsilon^n L^n + \varepsilon^N R_\varepsilon^N \right) \quad \text{for all } N \in \mathbb{N}^* .$$

The operators  $L^n$ ,  $n = 0, 1, 2$ , are explicited in Prop. B.3 :

$$L^0 = \partial_3^2 , \quad L^1 = 2\mathcal{H}(y_\alpha) \partial_3 , \quad L^2 = \Delta_\Gamma + \kappa^2 \text{Id} - 2(2\mathcal{H}^2 - \mathcal{K})(y_\alpha) Y_3 \partial_3 .$$

Here  $\partial_3$  is the partial derivative with respect to  $Y_3$ . We remind that  $\Delta_\Gamma$  is the Laplace-Beltrami operator along  $\Gamma$ ,  $\mathcal{H}$  and  $\mathcal{K}$  are the *mean curvature* and the *Gaussian curvature* of the surface  $\Gamma$ . The generic operator writes  $L^n = L^n(y_\alpha, Y_3; D_\alpha, \partial_3)$  and has smooth coefficients in  $y_\alpha$  and polynomial in  $Y_3$  of degree  $n - 1$ ; it contains at most one differentiation with respect to  $Y_3$  and  $D_\alpha$  is the covariant derivative in  $\Gamma$ . The remainder  $R_\varepsilon^N$  has smooth coefficients in  $y_\alpha$ , and  $Y_3$ , and bounded in  $\varepsilon$ .

**Remark 5.1.** The sign of  $\mathcal{H}$  depends on the orientation of the surface  $\Gamma$ . As a convention, the unit normal vector  $\mathbf{n}$  on  $\Gamma$  is outwardly oriented to  $\Omega_s$ , see Figure 1.

In the thin layer, the normal derivative writes

$$\partial_{\mathbf{n}} = \varepsilon^{-1} \partial_3 .$$

For any function  $p$  defined in  $\Omega_f^\varepsilon$ , we denote by  $\mathbf{p}$  the function such that :

$$\mathbf{p}(\mathbf{x}) = \mathbf{p}(y_\alpha, Y_3) , \quad (y_\alpha, Y_3 = \frac{h}{\varepsilon}) \in \Gamma \times (0, 1) .$$

5.1.2. *Elementary problems.* After the change of variables  $h \mapsto Y_3 = \varepsilon^{-1}h$  in the thin layer  $\Omega_f^\varepsilon$ , problem (1) becomes :

$$(28) \quad \begin{cases} \varepsilon^{-2}[\partial_3^2 \mathbf{p}_\varepsilon + \sum_{n \geq 1} \varepsilon^n L^n \mathbf{p}_\varepsilon] = 0 & \text{in } \Gamma \times (0, 1) \\ \varepsilon^{-1} \partial_3 \mathbf{p}_\varepsilon = \rho_f \omega^2 \mathbf{u}_\varepsilon \cdot \mathbf{n} & \text{on } \Gamma \times \{0\} \\ \mathbf{p}_\varepsilon = 0 & \text{on } \Gamma \times \{1\} \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon) + \omega^2 \rho \mathbf{u}_\varepsilon = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_\varepsilon) = -\mathbf{p}_\varepsilon \mathbf{n} & \text{on } \Gamma. \end{cases}$$

Inserting the Ansatz (25)-(26) in equations (28), we get the following two families of problems, coupled by their boundary conditions on  $\Gamma$  (i.e. when  $Y_3 = 0$ ):

$$(29) \quad \begin{cases} \partial_3^2 \mathbf{p}_n = - \sum_{l+p=n, l \geq 1} L^l \mathbf{p}_p & \text{for } Y_3 \in (0, 1) \\ \partial_3 \mathbf{p}_n = \rho_f \omega^2 \mathbf{u}_{n-1} \cdot \mathbf{n} & \text{for } Y_3 = 0 \\ \mathbf{p}_n = 0 & \text{for } Y_3 = 1 \end{cases}$$

$$(30) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_n) + \omega^2 \rho \mathbf{u}_n = \mathbf{f} \delta_0^n & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_n) = -\mathbf{p}_n \mathbf{n} & \text{on } \Gamma. \end{cases}$$

In the second equation in (29), we use the convention  $\mathbf{u}_{-1} = 0$ , and in (30)  $\delta_0^n$  is the Kronecker symbol.

5.2. **First terms.** In the case  $n = 0$ , we obtain from (29)

$$\mathbf{p}_0 = 0,$$

and then (30) yields  $\mathbf{u}_0$  solves the problem

$$(31) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_0) = 0 & \text{on } \Gamma. \end{cases}$$

At step  $n = 1$ , we find successively

$$\mathbf{p}_1(\cdot, Y_3) = (Y_3 - 1) \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}|_\Gamma$$

and that  $\mathbf{u}_1$  solves

$$(32) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_1) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} & \text{on } \Gamma. \end{cases}$$

At step  $n = 2$ , since  $\mathbf{p}_0 = 0$ , we find

$$(33) \quad \mathbf{p}_2(\cdot, Y_3) = -(Y_3^2 - 1) \mathcal{H} \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}|_\Gamma + (Y_3 - 1) \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}|_\Gamma$$

and then,  $\mathbf{u}_2$  solves

$$(34) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_2) + \omega^2 \rho \mathbf{u}_2 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_2) = \rho_f \omega^2 (\mathbf{u}_1 - \mathcal{H} \mathbf{u}_0) \cdot \mathbf{n} \mathbf{n} & \text{on } \Gamma. \end{cases}$$

At step  $n = 3$ , we find

$$(35) \quad \mathbf{p}_3(\cdot, Y_3) = a Y_3^3 + b Y_3^2 + c Y_3 + d$$



where the functions  $a, b, c, d$  are defined on  $\Gamma$  as follow

$$\begin{aligned} a &= \frac{1}{6}\rho_f\omega^2 (8\mathcal{H}^2 - 2\mathcal{K} - (\Delta_\Gamma + \kappa^2\mathbb{I})) \mathbf{u}_0 \cdot \mathbf{n}, \\ b &= \frac{1}{2}\rho_f\omega^2 (\Delta_\Gamma + \kappa^2\mathbb{I}) \mathbf{u}_0 \cdot \mathbf{n} - \mathcal{H}\rho_f\omega^2 \mathbf{u}_1 \cdot \mathbf{n}, \\ c &= \rho_f\omega^2 \mathbf{u}_2 \cdot \mathbf{n}, \\ d &= -a - b - c. \end{aligned}$$

Then,  $\mathbf{u}_3$  solves the problem

$$(36) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_3) + \omega^2 \rho \mathbf{u}_3 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_3) = \rho_f\omega^2 \left( \frac{1}{3} (4\mathcal{H}^2 - \mathcal{K} + \Delta_\Gamma + \kappa^2\mathbb{I}) \mathbf{u}_0 + (-\mathcal{H}\mathbf{u}_1 + \mathbf{u}_2) \right) \cdot \mathbf{n} \mathbf{n} & \text{on } \Gamma. \end{cases}$$

The whole construction of the asymptotics follows from a recurrence argument, see for instance [4] for a similar context. Assume the sequences  $(\mathbf{u}_n)$  and  $(\mathbf{p}_n)$  known until rank  $n = N - 1$ , then the Sturm-Liouville problem (29) uniquely defines  $\mathbf{p}_N$  whose trace on  $\Gamma$  is inserted into (30) as a data to determine the interior part  $\mathbf{u}_N$ .

### 5.3. Construction of equivalent conditions.

5.3.1. *Order 0.* Since the equations in (31) are independent of  $\varepsilon$ , the equivalent condition of order 0 is "exact" (i.e.  $\mathbf{u}_\varepsilon^0 = \mathbf{u}_0$ ). This condition is the stress free boundary condition

$$\mathbf{T}(\mathbf{u}_0) = 0 \quad \text{on } \Gamma.$$

5.3.2. *Order 1.* According to (31) and (32), the truncate expansion  $\mathbf{u}_{1,\varepsilon} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1$  solves the elastic equation in  $\Omega_s$  together with the equivalent condition

$$\mathbf{T}(\mathbf{u}_{1,\varepsilon}) = \varepsilon \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} \quad \text{on } \Gamma.$$

Writting  $\mathbf{u}_0 = \mathbf{u}_{1,\varepsilon} - \varepsilon \mathbf{u}_1$ , we infer

$$\mathbf{T}(\mathbf{u}_{1,\varepsilon}) - \varepsilon \omega^2 \rho_f \mathbf{u}_{1,\varepsilon} \cdot \mathbf{n} \mathbf{n} = -\varepsilon^2 \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} \quad \text{on } \Gamma.$$

Neglecting the term of order  $\varepsilon^2$  in the previous right-hand side, we obtain the asymptotic model of order 1: we define  $\mathbf{u}_\varepsilon^1$  such that it solves the elastic equation in  $\Omega_s$  together with the equivalent condition

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - \varepsilon \omega^2 \rho_f \mathbf{u}_\varepsilon^1 \cdot \mathbf{n} \mathbf{n} = 0 \quad \text{on } \Gamma.$$

5.3.3. *Order 2.* According to (31), (32), and (34), the truncate expansion  $\mathbf{u}_{2,\varepsilon} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2$  solves the elastic equation in  $\Omega_s$  with the condition set on  $\Gamma$

$$(37) \quad \mathbf{T}(\mathbf{u}_{2,\varepsilon}) = \varepsilon \rho_f \omega^2 (\mathbf{u}_0 + \varepsilon \mathbf{u}_1) \cdot \mathbf{n} \mathbf{n} - \varepsilon^2 \mathcal{H} \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} \quad \text{on } \Gamma.$$

Using  $\mathbf{u}_0 + \varepsilon \mathbf{u}_1 = \mathbf{u}_{2,\varepsilon} - \varepsilon^2 \mathbf{u}_2$  and  $\mathbf{u}_0 = \mathbf{u}_{2,\varepsilon} - \varepsilon \mathbf{u}_1 - \varepsilon^2 \mathbf{u}_2$ , we obtain on  $\Gamma$ :

$$\mathbf{T}(\mathbf{u}_{2,\varepsilon}) - \varepsilon \omega^2 \rho_f (1 - \varepsilon \mathcal{H}) \mathbf{u}_{2,\varepsilon} \cdot \mathbf{n} \mathbf{n} = -\varepsilon^3 \omega^2 \rho_f \mathbf{u}_2 \cdot \mathbf{n} \mathbf{n} + \varepsilon^3 \mathcal{H} \omega^2 \rho_f (\mathbf{u}_1 + \varepsilon \mathbf{u}_2) \cdot \mathbf{n} \mathbf{n}$$

Neglecting the terms of order  $\varepsilon^3$  in the right-hand sides of the previous condition, we obtain the equivalent condition of order 2

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - \varepsilon \omega^2 \rho_f (1 - \varepsilon \mathcal{H}) \mathbf{u}_\varepsilon^2 \cdot \mathbf{n} \mathbf{n} = 0 \quad \text{on } \Gamma.$$

5.3.4. *Order 3.* According to (31), (32), (34) and (36), the truncate expansion  $\mathbf{u}_{3,\varepsilon} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \varepsilon^3 \mathbf{u}_3$  solves the elastic equation in  $\Omega_s$  with the condition set on  $\Gamma$

$$(38) \quad \mathbf{T}(\mathbf{u}_{3,\varepsilon}) = \mathbf{T}(\mathbf{u}_{2,\varepsilon}) + \varepsilon^3 \rho_f \omega^2 \left[ \frac{1}{3} (4\mathcal{H}^2 - \mathcal{K} + \Delta_\Gamma + \kappa^2 \mathbb{I}) \mathbf{u}_0 + (-\mathcal{H} \mathbf{u}_1 + \mathbf{u}_2) \right] \cdot \mathbf{n} \mathbf{n}.$$

According to (37), there holds

$$\begin{aligned} \mathbf{T}(\mathbf{u}_{3,\varepsilon}) - \varepsilon \omega^2 \rho_f \left[ 1 - \varepsilon \mathcal{H} + \frac{\varepsilon^2}{3} (4\mathcal{H}^2 - \mathcal{K} + \Delta_\Gamma + \kappa^2 \mathbb{I}) \right] (\mathbf{u}_{3,\varepsilon} \cdot \mathbf{n}) \mathbf{n} = \\ - \varepsilon^4 \rho_f \omega^2 \left[ (\mathbf{u}_3 - \mathcal{H}(\mathbf{u}_2 + \varepsilon \mathbf{u}_3)) + \frac{1}{3} (4\mathcal{H}^2 - \mathcal{K} + \Delta_\Gamma + \kappa^2 \mathbb{I}) (\mathbf{u}_1 + \varepsilon \mathbf{u}_2 + \varepsilon^2 \mathbf{u}_3) \right] \cdot \mathbf{n} \mathbf{n}. \end{aligned}$$

Neglecting the terms of order  $\varepsilon^4$  in the right-hand sides of the previous condition, we obtain the equivalent condition of order 3

$$\mathbf{T}(\mathbf{u}_\varepsilon^3) - \varepsilon \omega^2 \rho_f \left[ 1 - \varepsilon \mathcal{H} + \frac{\varepsilon^2}{3} (4\mathcal{H}^2 - \mathcal{K} + \Delta_\Gamma + \kappa^2 \mathbb{I}) \right] (\mathbf{u}_\varepsilon^3 \cdot \mathbf{n}) \mathbf{n} = 0 \quad \text{on } \Gamma.$$

5.4. **Error estimates for the remainders.** The validation of the asymptotic expansion (25)-(26) consists in proving estimates for remainders  $(\mathbf{r}_\varepsilon^N, r_\varepsilon^N)$  defined in  $\Omega_s$  and  $\Omega_f^\varepsilon$  as

$$(39) \quad \mathbf{r}_\varepsilon^N = \mathbf{u}_\varepsilon - \sum_{n=0}^N \varepsilon^n \mathbf{u}_n \quad \text{in } \Omega_s, \quad \text{and} \quad r_\varepsilon^N(\mathbf{x}) = \mathbf{p}_\varepsilon(\mathbf{x}) - \sum_{n=0}^N \varepsilon^n \mathbf{p}_n(y_\alpha, \frac{y_3}{\varepsilon}) \quad \text{for all } \mathbf{x} \in \Omega_f^\varepsilon.$$

The convergence result is the following statement.

**Theorem 5.2.** *Under Assumptions 2.2-2.4-2.5-2.6 and for  $\varepsilon$  small enough, the solution  $(\mathbf{u}_\varepsilon, \mathbf{p}_\varepsilon)$  of problem (1) has a two-scale expansion which can be written in the form (25)-(26), with  $\mathbf{u}_j \in \mathbf{H}^1(\Omega_s)$  and  $\mathbf{p}_j \in \mathbf{H}^1(\Gamma \times (0, 1))$ . For each  $N \in \mathbb{N}$ , the remainders  $(\mathbf{r}_\varepsilon^N, r_\varepsilon^N)$  satisfy*

$$(40) \quad \|\mathbf{r}_\varepsilon^N\|_{1,\Omega_s} + \sqrt{\varepsilon} \|r_\varepsilon^N\|_{1,\Omega_f^\varepsilon} \leq C_N \varepsilon^{N+1}$$

with a constant  $C_N$  independent of  $\varepsilon$ .

*Proof.* This proof is rather classical, see for instance the proof of [4, Th. 2.1] where the authors consider an interface problem for the Laplacian operator set in a domain with a thin layer. The error estimate (40) is obtained through an evaluation of the right-hand sides when the elasto-acoustic operator is applied to  $(\mathbf{r}_\varepsilon^N, r_\varepsilon^N)$ . By construction, the remainder  $(\mathbf{r}_\varepsilon^N, r_\varepsilon^N)$  solves the problem

$$(41) \quad \begin{cases} \Delta r_\varepsilon^N + \kappa^2 r_\varepsilon^N = f_{N,\varepsilon} & \text{in } \Omega_f^\varepsilon \\ \nabla \cdot \underline{\sigma}(\mathbf{r}_\varepsilon^N) + \omega^2 \rho \mathbf{r}_\varepsilon^N = 0 & \text{in } \Omega_s \\ \partial_n r_\varepsilon^N = \rho_f \omega^2 \mathbf{r}_\varepsilon^N \cdot \mathbf{n} + g_{N,\varepsilon} & \text{on } \Gamma \\ \mathbf{T}(\mathbf{r}_\varepsilon^N) = -r_\varepsilon^N \mathbf{n} & \text{on } \Gamma \\ r_\varepsilon^N = 0 & \text{on } \Gamma^\varepsilon. \end{cases}$$

Here, the right-hand sides are explicit :

$$f_{N,\varepsilon} = \varepsilon^{N-1} [F_N - \sum_{l=1}^N \varepsilon^{l-2+N} R_\varepsilon^{N-l} \mathbf{p}_l] \quad \text{in } \Omega_f^\varepsilon,$$

and

$$g_{N,\varepsilon} = \rho_f \omega^2 \varepsilon^N \mathbf{u}^N \cdot \mathbf{n} \quad \text{on } \Gamma.$$

We have the following estimates for the residues  $f_{N,\varepsilon}$  and  $g_{N,\varepsilon}$

$$\|f_{N,\varepsilon}\|_{0,\Omega_f^\varepsilon} = \mathcal{O}(\varepsilon^{N-\frac{1}{2}}) \quad \text{and} \quad \|g_{N,\varepsilon}\|_{0,\Gamma} = \mathcal{O}(\varepsilon^N).$$

We can apply Theorem 6 to the couple  $(\mathbf{u}, \mathbf{p}) = (\mathbf{r}_\varepsilon^N, r_\varepsilon^N)$ , and we obtain

$$\|\mathbf{r}_\varepsilon^N\|_{1,\Omega_s} + \|r_\varepsilon^N\|_{1,\Omega_f} \leq C_N \varepsilon^{N-\frac{1}{2}}.$$

Writting  $\mathbf{r}_\varepsilon^N = \mathbf{r}_\varepsilon^{N+2} + \varepsilon^{N+2}\mathbf{u}_{N+2} + \varepsilon^{N+1}\mathbf{u}_{N+1}$  and  $r_\varepsilon^N = r_\varepsilon^{N+2} + \varepsilon^{N+2}\mathbf{p}_{N+2} + \varepsilon^{N+1}\mathbf{p}_{N+1}$ , we apply the previous estimate to the couple  $(\mathbf{r}_\varepsilon^{N+2}, r_\varepsilon^{N+2})$  and we use estimates  $\|\mathbf{u}_l\|_{1,\Omega_s} = \mathcal{O}(1)$  and  $\|\mathbf{p}_l\|_{1,\Omega_f} = \mathcal{O}(\varepsilon^{-\frac{1}{2}})$  to infer the optimal estimates (40).  $\square$

**5.5. Validation of equivalent conditions.** We consider the elastic problem (7) with an equivalent condition and at a fixed frequency  $\omega$  satisfying Assumption 2.4. The functional setting for the problem (7) is described by the Hilbert space

- $\mathbf{H}^1(\Omega_s)$  when  $k$  belongs to  $\{0, 1, 2\}$
- $\mathbf{V} = \{\mathbf{u} \in \mathbf{H}^1(\Omega_s) \mid \mathbf{u} \cdot \mathbf{n}|_\Gamma \in \mathbf{H}^1(\Gamma)\}$  for the Ventcel's problem (when  $k = 3$ ).

The space  $\mathbf{V}$  endowed with its natural norm is a Hilbert space. To unify the notations, we introduce a common variational framework

- $\mathbf{V}^k$  denotes the space  $\mathbf{H}^1(\Omega_s)$  when  $k \in \{0, 1, 2\}$
- $\mathbf{V}^3$  denotes the space  $\mathbf{V}$ .

The main result of this section is the following statement, that is for all  $k \in \{0, 1, 2, 3\}$  the problem (7) is well-posed, and its solution satisfies uniform  $\mathbf{H}^1$  estimates.

**Proposition 5.3.** *Under Assumptions 2.2-2.4-2.5, for all  $k \in \{0, 1, 2, 3\}$  there are constants  $\varepsilon_k, C_k > 0$  such that for all  $\varepsilon \in (0, \varepsilon_k)$ , the problem (7) with a data  $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$  has a unique solution  $\mathbf{u}_\varepsilon^k \in \mathbf{V}^k$  which satisfies the uniform estimates:*

$$(42a) \quad \|\mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C_k \|\mathbf{f}\|_{0,\Omega_s} \quad \text{for all } k \in \{0, 1, 2\},$$

$$(42b) \quad \|\mathbf{u}_\varepsilon^3\|_{1,\Omega_s} + \varepsilon^{\frac{3}{2}} \|\nabla_\Gamma(\mathbf{u}_\varepsilon^3 \cdot \mathbf{n})\|_{0,\Gamma} \leq C_3 \|\mathbf{f}\|_{0,\Omega_s}.$$

The key for the proof of Prop. 5.3 is the following Lemma.

**Lemma 5.4.** *Under Assumptions 2.2-2.4-2.5, for all  $k \in \{0, 1, 2, 3\}$  there exists constants  $\varepsilon_k, C_k > 0$  such that for all  $\varepsilon \in (0, \varepsilon_k)$ , any solution  $\mathbf{u}_\varepsilon^k \in \mathbf{V}^k$  of problem (7) with a data  $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$  satisfies the uniform estimate:*

$$(43) \quad \|\mathbf{u}_\varepsilon^k\|_{0,\Omega_s} \leq C_k \|\mathbf{f}\|_{0,\Omega_s}.$$

**Remark 5.5.** *For  $k = 0$ , the Proposition 5.3 and the Lemma 5.4 hold for all  $\varepsilon > 0$ .*

The Lemma 5.4 is proved in Sec. 6.1. As a consequence of this Lemma, each solution of the problem (7) satisfies uniform  $\mathbf{H}^1$ -estimates (42a)-(42b). Then, the proof of the Prop. 5.3 is a consequence of the Fredholm alternative since the problem (7) is of Fredholm type.

## 6. ANALYSIS OF EQUIVALENT CONDITIONS

In this section, we first prove the Lemma 5.4, i.e. uniform  $\mathbf{L}^2$ -estimates (43) for the solution of problem (7), Sec. 6.1. In Sec. 6.2, we prove that the solution  $\mathbf{u}_\varepsilon^k$  of problem (7) satisfies uniform  $\mathbf{H}^1$  error estimates (8) and we infer the Theorem 3.2.

We focus on the proof of Lemma 5.4 for  $k = 3$  since the proof for  $k \in \{0, 1, 2\}$  is more classical. Hence we consider the problem

$$(44) \quad \begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}_\varepsilon) + \omega^2 \rho \mathbf{u}_\varepsilon = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_\varepsilon) + \mathbf{B}_{3,\varepsilon}(\mathbf{u}_\varepsilon \cdot \mathbf{n})\mathbf{n} = 0 & \text{on } \Gamma. \end{cases}$$

To prepare for the proof, we recall the variational formulation for the elastic displacement. If  $\mathbf{u} \in \mathbf{V}^3$  is solution of (44), then it satisfies for all  $\mathbf{v} \in \mathbf{V}^3$ :

$$(45) \quad \int_{\Omega_s} (\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\varepsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \bar{\mathbf{v}}) \, d\mathbf{x} - \varepsilon \omega^2 \rho_f \int_{\Gamma} \mathcal{J}_\varepsilon \mathbf{u} \cdot \mathbf{n} \bar{\mathbf{v}} \cdot \mathbf{n} \, d\sigma \\ + \frac{\varepsilon^3}{3} \omega^2 \rho_f \int_{\Gamma} \nabla_{\Gamma}(\mathbf{u} \cdot \mathbf{n}) \nabla_{\Gamma}(\bar{\mathbf{v}} \cdot \mathbf{n}) \, d\sigma = - \int_{\Omega_s} \mathbf{f} \cdot \bar{\mathbf{v}} \, d\mathbf{x} ,$$

where  $\mathcal{J}_\varepsilon$  is a function defined on  $\Gamma$  as  $\mathcal{J}_\varepsilon = \left(1 - \varepsilon \mathcal{H} + \frac{\varepsilon^2}{3} (4\mathcal{H}^2 - \mathcal{K} - \kappa^2)\right)$  which tends to 1 when  $\varepsilon$  goes to 0.

**6.1. Proof of Lemma 5.4 : Uniform  $L^2$  estimate of the elastic displacement.** Reductio ad absurdum: We assume that there is a sequence  $(\mathbf{u}_m) \in L^2(\Omega_s)$ ,  $m \in \mathbb{N}$ , of solutions of the elastic equation (44) associated with a parameter  $\varepsilon_m$  and a right-hand side  $\mathbf{f}_m \in L^2(\Omega_s)$ :

$$(46a) \quad \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_m) + \omega^2 \rho \mathbf{u}_m = \mathbf{f}_m \quad \text{in } \Omega_s ,$$

$$(46b) \quad \mathbf{T}(\mathbf{u}_m) - \varepsilon_m \omega^2 \rho_f \mathcal{J}_{\varepsilon_m} \mathbf{u}_m \cdot \mathbf{n} \mathbf{n} - \frac{\varepsilon_m^3}{3} \omega^2 \rho_f \Delta_{\Gamma}(\mathbf{u}_m \cdot \mathbf{n}) \mathbf{n} = 0 \quad \text{on } \Gamma ,$$

satisfying the following conditions

$$(47a) \quad \varepsilon_m \rightarrow 0 \quad \text{as } m \rightarrow \infty ,$$

$$(47b) \quad \|\mathbf{u}_m\|_{0,\Omega_s} = 1 \quad \text{for all } m \in \mathbb{N} ,$$

$$(47c) \quad \|\mathbf{f}_m\|_{0,\Omega_s} \rightarrow 0 \quad \text{as } m \rightarrow \infty .$$

**6.1.1. Estimates of the sequence  $\{\mathbf{u}_m\}$ .** We first prove that the sequence  $\{\mathbf{u}_m\}$  is bounded in  $\mathbf{H}^1(\Omega_s)$  (but not in  $\mathbf{V}^3$ ). We particularize the elastic variational formulation (45) for the sequence  $\{\mathbf{u}_m\}$ : For all  $\mathbf{v} \in \mathbf{V}^3$ ,

$$(48) \quad \int_{\Omega_s} (\underline{\underline{\sigma}}(\mathbf{u}_m) : \underline{\underline{\varepsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u}_m \bar{\mathbf{v}}) \, d\mathbf{x} - \varepsilon_m \omega^2 \rho_f \int_{\Gamma} \mathcal{J}_{\varepsilon_m} \mathbf{u}_m \cdot \mathbf{n} \bar{\mathbf{v}} \cdot \mathbf{n} \, d\sigma \\ + \frac{\varepsilon_m^3}{3} \omega^2 \rho_f \int_{\Gamma} \nabla_{\Gamma}(\mathbf{u}_m \cdot \mathbf{n}) \nabla_{\Gamma}(\bar{\mathbf{v}} \cdot \mathbf{n}) \, d\sigma = - \int_{\Omega_s} \mathbf{f}_m \cdot \bar{\mathbf{v}} \, d\mathbf{x} .$$

Choosing  $\mathbf{v} = \mathbf{u}_m$  in (48), we obtain with the help of condition (47b) the following uniform bound

$$(49) \quad \int_{\Omega_s} \underline{\underline{C}} |\underline{\underline{\varepsilon}}(\mathbf{u}_m)|^2 \, d\mathbf{x} - \varepsilon_m \omega^2 \rho_f \int_{\Gamma} \mathcal{J}_{\varepsilon_m} |\mathbf{u}_m \cdot \mathbf{n}|^2 \, d\sigma + \frac{\varepsilon_m^3}{3} \omega^2 \rho_f \int_{\Gamma} |\nabla_{\Gamma}(\mathbf{u}_m \cdot \mathbf{n})|^2 \, d\sigma \\ \leq \omega^2 \rho + \|\mathbf{f}_m\|_{0,\Omega_s} .$$

**Remark 6.1.** The tensor  $\underline{\underline{\varepsilon}}(\mathbf{u})$  is symmetric, hence thanks to the assumptions 2.2 (i)-(iii) there holds

$$\forall \mathbf{u} \in \mathbf{H}^1(\Omega_s) , \quad \int_{\Omega_s} \underline{\underline{C}} \underline{\underline{\varepsilon}}(\mathbf{u}) : \underline{\underline{\varepsilon}}(\bar{\mathbf{u}}) \, d\mathbf{x} \geq \alpha \|\underline{\underline{\varepsilon}}(\mathbf{u})\|_{0,\Omega_s}^2 .$$

Therefore, using the Korn inequality (which is available since  $\Omega_s$  is a smooth domain), we infer : For all  $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$

$$\int_{\Omega_s} \underline{\underline{C}} \underline{\underline{\varepsilon}}(\mathbf{u}) : \underline{\underline{\varepsilon}}(\bar{\mathbf{u}}) \, d\mathbf{x} \geq \alpha C \|\mathbf{u}\|_{1,\Omega_s}^2 - \alpha c \|\mathbf{u}\|_{0,\Omega_s}^2 .$$

Combining the trace inequality

$$(50) \quad \forall \mathbf{u} \in \mathbf{H}^1(\Omega_s) , \quad \|\mathbf{u} \cdot \mathbf{n}\|_{0,\Gamma} \leq C_1 \|\mathbf{u}\|_{1,\Omega_s}$$

and the Korn inequality, we infer for  $m$  large enough

$$\alpha C \|\mathbf{u}_m\|_{1,\Omega_s}^2 - \varepsilon_m \beta \omega^2 \rho_f \|\mathbf{u}_m \cdot \mathbf{n}\|_{0,\Gamma}^2 + \frac{\varepsilon_m^3}{3} \omega^2 \rho_f \|\nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n})\|_{0,\Gamma}^2 \leq C_2 + \|\mathbf{f}_m\|_{0,\Omega_s},$$

with a constant  $\beta > 0$  such that  $\mathcal{J}_\varepsilon \leq \beta$  for  $\varepsilon$  small enough (remind  $\mathcal{J}_\varepsilon \rightarrow 1$  as  $\varepsilon \rightarrow 0$ ), and  $C_2 = \alpha c + \omega^2 \rho$ . Using again the trace inequality (50), we obtain

$$(\alpha C - \varepsilon_m \beta \omega^2 \rho_f C_1^2) \|\mathbf{u}_m\|_{1,\Omega_s}^2 + \frac{\varepsilon_m^3}{3} \omega^2 \rho_f \|\nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n})\|_{0,\Gamma}^2 \leq C_2 + \|\mathbf{f}_m\|_{0,\Omega_s}.$$

Then, using (47a), there holds:  $\alpha C - \varepsilon_m \beta \omega^2 \rho_f C_1^2 > 0$ , for  $m$  large enough. According to (47c) we infer that the sequence  $\{\mathbf{u}_m\}$ , resp.  $\{(\varepsilon_m)^{\frac{3}{2}} \nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n})\}$ , is bounded in  $\mathbf{H}^1(\Omega_s)$ , resp. in  $\mathbf{L}^2(\Gamma)$ :

$$(51a) \quad \|\mathbf{u}_m\|_{1,\Omega_s} \leq C,$$

$$(51b) \quad (\varepsilon_m)^{\frac{3}{2}} \|\nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n})\|_{0,\Gamma} \leq C.$$

Another consequence of (51a) is that the sequence  $\{\mathbf{u}_m \cdot \mathbf{n}\}$  is bounded in  $H^{\frac{1}{2}}(\Gamma)$ .

$$(52) \quad \|\mathbf{u}_m \cdot \mathbf{n}\|_{\frac{1}{2},\Gamma} \leq C.$$

**6.1.2. Limit of the sequence and conclusion.** The domain  $\Omega_s$  being bounded, the embedding of  $\mathbf{H}^1(\Omega_s)$  in  $\mathbf{L}^2(\Omega_s)$  is compact. Hence as a consequence of (51a), using the Rellich Lemma we can extract a subsequence of  $\{\mathbf{u}_m\}$  (still denoted by  $\{\mathbf{u}_m\}$ ) which is converging in  $\mathbf{L}^2(\Omega_s)$ , and we can assume that the sequence  $\{\underline{\nabla} \mathbf{u}_m\}$  is strongly converging in  $\mathbf{L}^2(\Omega_s)$ . As a consequence of (52), up to the extraction of a subsequence, we can assume that the sequence  $\{\mathbf{u}_m \cdot \mathbf{n}\}$  is weakly converging in  $L^2(\Gamma)$ : We deduce that there is  $\mathbf{u} \in \mathbf{L}^2(\Omega_s)$  such that

$$(53) \quad \begin{cases} \underline{\nabla} \mathbf{u}_m \rightharpoonup \underline{\nabla} \mathbf{u} & \text{in } \mathbf{L}^2(\Omega_s) \\ \mathbf{u}_m \rightarrow \mathbf{u} & \text{in } \mathbf{L}^2(\Omega_s) \\ \mathbf{u}_m \cdot \mathbf{n} \rightarrow \mathbf{u} \cdot \mathbf{n} & \text{in } L^2(\Gamma). \end{cases}$$

A consequence of the strong convergence in  $\mathbf{L}^2(\Omega_s)$  and (47b) is that  $\|\mathbf{u}\|_{0,\Omega_s} = 1$ . As a consequence of (51b), we can extract a subsequence of  $\{(\varepsilon_m)^{\frac{3}{2}} \nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n})\}$  (still denoted by  $\{(\varepsilon_m)^{\frac{3}{2}} \nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n})\}$ ) which is weakly converging to a function  $\mathbf{t} \in \mathbf{L}^2(\Gamma)$

$$(54) \quad (\varepsilon_m)^{\frac{3}{2}} \nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n}) \rightharpoonup \mathbf{t} \quad \text{in } \mathbf{L}^2(\Gamma).$$

Using Assumption 2.4, we are going to prove that  $\mathbf{u} = 0$ , which will contradict  $\|\mathbf{u}\|_{0,\Omega_s} = 1$ , and finally prove estimate (43). Let  $\mathbf{v} \in \mathbf{V}^3$  be a test function in (48)

$$(55) \quad \int_{\Omega_s} (\underline{\sigma}(\mathbf{u}_m) : \underline{\varepsilon}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u}_m \bar{\mathbf{v}}) \, d\mathbf{x} - \varepsilon_m \omega^2 \rho_f \int_\Gamma \mathcal{J}_{\varepsilon_m} \mathbf{u}_m \cdot \mathbf{n} \bar{\mathbf{v}} \cdot \mathbf{n} \, d\sigma \\ + \frac{\varepsilon_m^3}{3} \omega^2 \rho_f \int_\Gamma \nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n}) \nabla_\Gamma(\bar{\mathbf{v}} \cdot \mathbf{n}) \, d\sigma = - \int_{\Omega_s} \mathbf{f}_m \cdot \bar{\mathbf{v}} \, d\mathbf{x}.$$

According to (53), (54), and (47a), taking limits as  $m \rightarrow +\infty$ , there holds

$$\varepsilon_m \int_\Gamma \mathcal{J}_{\varepsilon_m} \mathbf{u}_m \cdot \mathbf{n} \bar{\mathbf{v}} \cdot \mathbf{n} \, d\sigma \rightarrow 0 \quad \text{and} \quad \frac{\varepsilon_m^3}{3} \int_\Gamma \nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n}) \nabla_\Gamma(\bar{\mathbf{v}} \cdot \mathbf{n}) \, d\sigma \rightarrow 0.$$

Hence, according to (53), (47a) and (47c), taking limits as  $m \rightarrow +\infty$ , we deduce from the previous equalities (55)  $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$  satisfies for all  $\mathbf{v} \in \mathbf{H}^1(\Omega_s)$ :

$$\int_{\Omega_s} (\underline{\sigma}(\mathbf{u}) : \underline{\varepsilon}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \bar{\mathbf{v}}) \, d\mathbf{x} = 0.$$

Integrating by parts (see (24)) we find that  $\mathbf{u}$  satisfies the problem

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}) = 0 & \text{on } \Gamma . \end{cases}$$

By Assumption 2.4, we deduce

$$\mathbf{u} = 0 \quad \text{in } \Omega_s ,$$

which contradicts  $\|\mathbf{u}\|_{0,\Omega_s} = 1$  and ends the proof of Lemma 5.4.

**6.2. Proof of error estimates.** In this section we prove the Theorem 3.2. Since the problem (7) is of Fredholm type, it is sufficient to show that any solution  $\mathbf{u}_\varepsilon^k$  of (7) satisfies the error estimate (8)

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C\varepsilon^{k+1} .$$

We prove hereafter the estimate (8) in two steps, Sec. 6.2.1 and Sec. 6.2.2.

**6.2.1. Step (i).** The first step consists to derive an expansion of  $\mathbf{u}_\varepsilon^k$  and to show that the truncated expansions of  $\mathbf{u}_\varepsilon^k$  and  $\mathbf{u}_\varepsilon$  coincide up to the order  $\varepsilon^k$ :

$$(56) \quad \mathbf{u}_\varepsilon = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \mathbf{r}_\varepsilon^k ,$$

$$(57) \quad \mathbf{u}_\varepsilon^k = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \tilde{\mathbf{r}}_\varepsilon^k .$$

Hereafter, we justify the expansion (57). By construction,  $\mathbf{u}_\varepsilon^k$  admits an expansion

$$\mathbf{u}_\varepsilon^k = \mathbf{v}_0 + \varepsilon \mathbf{v}_1 + \varepsilon^2 \mathbf{v}_2 + \cdots + \varepsilon^k \mathbf{v}_k + \tilde{\mathbf{r}}_\varepsilon^k$$

where each term  $\mathbf{v}_n$ , for  $0 \leq n \leq k$ , satisfies the problem (30) as well as the term  $\mathbf{u}_n$ . Using the spectral Assumption 2.4, we infer that for all  $0 \leq l \leq k$ ,  $\mathbf{v}_n = \mathbf{u}_n$  in  $\Omega_s$ , and the expansion (57) holds.

Hence,

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_s} = \|\mathbf{r}_\varepsilon^k - \tilde{\mathbf{r}}_\varepsilon^k\|_{1,\Omega_s} .$$

The estimate of the remainder  $\mathbf{r}_\varepsilon^k$  is already proved in Thm 5.2 (Sec. 5.4) :  $\|\mathbf{r}_\varepsilon^k\|_{1,\Omega_s} \leq C\varepsilon^{k+1}$ . In the next step, we prove estimates for the remainder  $\tilde{\mathbf{r}}_\varepsilon^k$ .

**6.2.2. Step (ii).** According to (57), the remainder  $\tilde{\mathbf{r}}_\varepsilon^k$  satisfies the elastic equation in  $\Omega_s$ . Applying the impedance operator  $\mathbf{T} + \mathbf{B}_{k,\varepsilon}$  (where  $\mathbf{B}_{k,\varepsilon}(\mathbf{u}) := \mathbf{B}_{k,\varepsilon}(\mathbf{u} \cdot \mathbf{n})\mathbf{n}$ ) to the remainder  $\tilde{\mathbf{r}}_\varepsilon^k$ , we prove hereafter that

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^k) + \mathbf{B}_{k,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^k \cdot \mathbf{n})\mathbf{n} = \mathcal{O}(\varepsilon^{k+1}) \quad \text{on } \Gamma .$$

Since  $\mathbf{u}_\varepsilon^0 = \mathbf{u}_0$ , then  $\tilde{\mathbf{r}}_\varepsilon^0 = 0$  (i.e. the expansion (56) is exact at the order 0). Relying on the construction of equivalent conditions detailed in section 5.3, there holds

$$\begin{aligned} \mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^1) + \mathbf{B}_{1,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^1 \cdot \mathbf{n})\mathbf{n} &= \varepsilon^2 \omega^2 \rho_f \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} \\ \mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^2) + \mathbf{B}_{2,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^2 \cdot \mathbf{n})\mathbf{n} &= \varepsilon^3 \omega^2 \rho_f ((1 - \varepsilon \mathcal{H}) \mathbf{u}_2 + \mathcal{H} \mathbf{u}_1) \cdot \mathbf{n} \mathbf{n} \\ \mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^3) + \mathbf{B}_{3,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^3 \cdot \mathbf{n})\mathbf{n} &= \\ \varepsilon^4 \omega^2 \rho_f \left[ (\mathbf{u}_3 - \mathcal{H}(\mathbf{u}_2 + \varepsilon \mathbf{u}_3)) + \frac{1}{3} (4\mathcal{H}^2 - \mathcal{K} + \Delta_\Gamma + \kappa^2 \mathbb{I}) (\mathbf{u}_1 + \varepsilon \mathbf{u}_2 + \varepsilon^2 \mathbf{u}_3) \right] \cdot \mathbf{n} \mathbf{n} \end{aligned}$$

on  $\Gamma$ . According to the estimate (43), we infer the uniform estimate

$$\|\tilde{\mathbf{r}}_\varepsilon^k\|_{1,\Omega_s} \leq C\varepsilon^{k+1} ,$$

which ends the proof of Theorem 3.2.

## APPENDIX A. EQUIVALENT CONDITIONS FOR A FOURIER-ROBIN EXTERNAL BOUNDARY CONDITION

In this section, we consider the elasto-acoustic transmission problem complemented with a Fourier-Robin external boundary condition

$$(58) \quad \begin{cases} \Delta p_\varepsilon + \kappa^2 p_\varepsilon = 0 & \text{in } \Omega_f^\varepsilon \\ \nabla \cdot \underline{\sigma}(\mathbf{u}_\varepsilon) + \omega^2 \rho \mathbf{u}_\varepsilon = \mathbf{f} & \text{in } \Omega_s \\ \partial_n p_\varepsilon = \rho_f \omega^2 \mathbf{u}_\varepsilon \cdot \mathbf{n} & \text{on } \Gamma \\ \mathbf{T}(\mathbf{u}_\varepsilon) = -p_\varepsilon \mathbf{n} & \text{on } \Gamma \\ \partial_n p_\varepsilon - i\kappa p_\varepsilon = 0 & \text{on } \Gamma^\varepsilon. \end{cases}$$

We remind that the construction of equivalent conditions (ECs) set on the surface  $\Gamma$  consists to exhibit surfacic differential operators  $F_\varepsilon$

$$F_\varepsilon : \mathcal{C}^\infty(\Gamma) \rightarrow \mathcal{C}^\infty(\Gamma),$$

together with  $\tilde{\mathbf{u}}_\varepsilon$  solution of the problem

$$(59a) \quad \nabla \cdot \underline{\sigma}(\tilde{\mathbf{u}}_\varepsilon) + \omega^2 \rho \tilde{\mathbf{u}}_\varepsilon = \mathbf{f} \quad \text{in } \Omega_s,$$

$$(59b) \quad \mathbf{T}(\tilde{\mathbf{u}}_\varepsilon) + F_\varepsilon(\tilde{\mathbf{u}}_\varepsilon \cdot \mathbf{n}) \mathbf{n} = 0 \quad \text{on } \Gamma,$$

see Sec. 2.3. The equivalent condition is of order  $k$  when a uniform estimate

$$\|\mathbf{u}_\varepsilon - \tilde{\mathbf{u}}_\varepsilon\|_{1, \Omega_s} \leq C \varepsilon^{k+1}$$

holds, and then we define  $\mathbf{u}_\varepsilon^k = \tilde{\mathbf{u}}_\varepsilon$ .

**A.1. Statement of Equivalent Conditions.** In this section, we present Equivalent Conditions (ECs) up to the first order associated with the exact problem (58) and satisfied by  $\mathbf{u}_\varepsilon^k$ . Hence,  $\mathbf{u}_\varepsilon^k$  solves the elastic equation (60a) set in  $\Omega_s$  together with an EC (60b) set on  $\Gamma$ :

$$(60a) \quad \nabla \cdot \underline{\sigma}(\mathbf{u}_\varepsilon^k) + \omega^2 \rho \mathbf{u}_\varepsilon^k = \mathbf{f} \quad \text{in } \Omega_s,$$

$$(60b) \quad \mathbf{T}(\mathbf{u}_\varepsilon^k) + F_{k,\varepsilon}(\mathbf{u}_\varepsilon^k \cdot \mathbf{n}) \mathbf{n} = 0 \quad \text{on } \Gamma.$$

Here  $F_{k,\varepsilon}$  is a surfacic differential operator acting on functions defined on  $\Gamma$  for  $k = 0, 1$  as

$$F_{0,\varepsilon}(= F_0) = -i\omega c \rho_f \quad \text{on } \Gamma,$$

$$F_{1,\varepsilon} = -i\omega c \rho_f (1 + \varepsilon (-2\mathcal{H} + i\kappa^{-1} \Delta_\Gamma)) \quad \text{on } \Gamma.$$

Hence, Equivalent Conditions of order  $k = 0, 1$  write

*Order 0.*

$$(61) \quad \mathbf{T}(\mathbf{u}_0) - i\omega c \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0 \quad \text{on } \Gamma \quad (\mathbf{u}_\varepsilon^0 = \mathbf{u}_0),$$

*Order 1.*

$$(62) \quad \mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c \rho_f (1 + \varepsilon (-2\mathcal{H} + i\kappa^{-1} \Delta_\Gamma)) (\mathbf{u}_\varepsilon^1 \cdot \mathbf{n}) \mathbf{n} = 0 \quad \text{on } \Gamma.$$

The well-posedness of the problem (60a)-(61) is proved in Sections A.3 and A.4. The EC (62) is a degenerate Ventcel condition with no coerciveness since the Laplacian-Beltrami operator  $\Delta_\Gamma$  appears in (62) with a non-negative sign

$$\operatorname{Re} F_{1,\varepsilon} = \varepsilon c^2 \rho_f \Delta_\Gamma.$$

However, since a small and real parameter  $\varepsilon$  appears in front of the operator  $\Delta_\Gamma$  in the EC (62), we propose hereafter to modify this condition by using a " $\mathbf{T}(\mathbf{u}) \cdot \mathbf{n}$ -to- $\mathbf{u} \cdot \mathbf{n}$ " formulation, (63). Then, the well-posedness of the problem (60a)-(63) is proved in Sections A.3 and A.5. We derive in Section A.6 the differential operators  $F_{k,\varepsilon}$  for  $k \in \{0, 1\}$  which appear in (60b). Formal calculus to derive these ECs are presented in Sec. A.6.

**A.2. A Modified Equivalent Condition.** We introduce the operator  $N_\varepsilon := (F_\varepsilon)^{-1}$ . Then the Equivalent Condition (59b) for  $\mathbf{u}_\varepsilon$  can be rewritten as :

$$N_\varepsilon (\mathbf{T}(\mathbf{u}_\varepsilon) \cdot \mathbf{n}) + \mathbf{u}_\varepsilon \cdot \mathbf{n} = 0 \quad \text{on } \Gamma .$$

In this framework, a “ $\mathbf{T}(\mathbf{u}) \cdot \mathbf{n}$ -to- $\mathbf{u} \cdot \mathbf{n}$ ” EC of order 1 writes:

$$N_{1,\varepsilon} (\mathbf{T}(\mathbf{u}_\varepsilon^1) \cdot \mathbf{n}) + \mathbf{u}_\varepsilon^1 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma ,$$

where  $N_{1,\varepsilon}$  denotes a local approximation of  $N_\varepsilon$ . The expression of  $N_{1,\varepsilon}$  can be obtained by a formal Taylor expansion of  $(F_{1,\varepsilon})^{-1}$ :

$$(F_{1,\varepsilon})^{-1} = (-i\omega c\rho_f)^{-1} (1 - \varepsilon(-2\mathcal{H} + i\kappa^{-1}\Delta_\Gamma)) + \mathcal{O}(\varepsilon^2) .$$

Hence, the expression of  $N_{1,\varepsilon}$  is given by

$$N_{1,\varepsilon} = i(\omega c\rho_f)^{-1} (1 - \varepsilon(-2\mathcal{H} + i\kappa^{-1}\Delta_\Gamma)) .$$

**Remark A.1.** This approximation restores coerciveness property since

$$\text{Re } N_{1,\varepsilon} = \varepsilon\omega^{-2}\rho_f^{-1}\Delta_\Gamma$$

is a non-positive local operator, see (82).

The modified EC of order 1 writes :

$$(63) \quad i(\omega c\rho_f)^{-1} (1 - \varepsilon(-2\mathcal{H} + i\kappa^{-1}\Delta_\Gamma)) (\mathbf{T}(\mathbf{u}_\varepsilon^1) \cdot \mathbf{n}) + \mathbf{u}_\varepsilon^1 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma .$$

The main point is that the results of existence and uniqueness holds for the problem (60a)-(63), see Prop. A.5 for  $k = 1$ . The EC (63) is used in practice by introducing  $\varphi_\varepsilon = \mathbf{T}(\mathbf{u}_\varepsilon^1) \cdot \mathbf{n}|_\Gamma$  as a new unknown. Hence the equivalent model with a modified EC of order 1 writes : Find  $\mathbf{u}_\varepsilon^1 \in \mathbf{H}^1(\Omega_s)$  such that

$$(64) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon^1) + \omega^2 \rho \mathbf{u}_\varepsilon^1 = \mathbf{f} & \text{in } \Omega_s \\ N_{1,\varepsilon} (\mathbf{T}(\mathbf{u}_\varepsilon^1) \cdot \mathbf{n}) + \mathbf{u}_\varepsilon^1 \cdot \mathbf{n} = 0 & \text{on } \Gamma \end{cases}$$

**A.3. Analysis of Equivalent Conditions.** The elastic equation (60a) set with the equivalent condition of order 0 (61) or with the modified equivalent condition of order 1 (63) has a unique solution  $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$  provided  $\omega$  is not an eigenfrequency of the problem

$$(65) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}) = 0 \quad \text{and} \quad \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma . \end{cases}$$

Hence, we work under the following spectral assumption

**Assumption A.2.** The angular frequency  $\omega$  is not an eigenfrequency of the problem (65).

**Remark A.3.** For axisymmetric bodies  $\Omega_s$  (balls, ellipsoids, ...) and for a discrete set of frequencies  $\omega$ , there exist non trivial solutions  $\mathbf{u}$  to (65), [16, 21] . Such a solution  $\mathbf{u}$ , resp. frequency  $\omega$ , is called a Jones mode, resp. a Jones frequency. However, Jones eigenmodes do not exist for generic domains, [14].

**Remark A.4.** Under the spectral assumption A.2, it is possible to adapt the proof of Th. 6 and to prove uniform estimates (6) for the elasto-acoustic field  $(\mathbf{u}_\varepsilon, p_\varepsilon)$  solution of (58) as  $\varepsilon \rightarrow 0$ .

The main result of this section is the following statement, that is the problem (60a)-(61) for  $k = 0$  (resp. the problem (60a) for  $k = 1$  with the EC (63)) is well-posed, and its solution satisfies uniform  $\mathbf{H}^1$  estimates.

**Proposition A.5.** Under Assumptions 2.2-A.2-2.5, for  $k = 0, 1$  there exists constants  $\varepsilon_k, C_k > 0$  such that for all  $\varepsilon \in (0, \varepsilon_k)$ , the problem (60a)-(61) when  $k = 0$ , resp. (60a)-(63) when  $k = 1$ , with a data  $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$  has a unique solution  $\mathbf{u}_\varepsilon^k \in \mathbf{H}^1(\Omega_s)$  which satisfies the uniform estimates:

$$(66) \quad \|\mathbf{u}_\varepsilon^k\|_{1,\Omega_s} \leq C_k \|\mathbf{f}\|_{0,\Omega_s} .$$



**Remark A.6.** The result for  $k = 0$  stated in Prop. A.5 holds for all  $\varepsilon$ .

*Proof.* The proof of Prop. A.5 is based on the following Lemma A.7, resp. Lemma A.8, for  $k = 0$ , resp. for  $k = 1$ . The Lemma A.7, resp. Lemma A.8, is proved in Sec. A.4, resp. in Sec. A.5. We then conclude to the Prop. A.5 for  $k = 0, 1$  applying the alternative of Fredholm.  $\square$

**Lemma A.7.** Under Assumptions 2.2-A.2-2.5, there exists a constant  $C_0 > 0$  such that any solution  $\mathbf{u}_0 \in \mathbf{H}^1(\Omega_s)$  of the problem (60a)-(61) with a data  $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$  satisfies the estimate :

$$(67) \quad \|\mathbf{u}_0\|_{0,\Omega_s} \leq C_0 \|\mathbf{f}\|_{0,\Omega_s}.$$

To prepare for the proof of Lemma A.7, we introduce the variational formulation for the elastic displacement : if  $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$  is a solution of (60a)-(61), then it satisfies for all  $\mathbf{v} \in \mathbf{H}^1(\Omega_s)$ :

$$(68) \quad \int_{\Omega_s} (\underline{\sigma}(\mathbf{u}) : \underline{\varepsilon}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \bar{\mathbf{v}}) \, d\mathbf{x} - i\omega c \rho_f \int_{\Gamma} \mathbf{u} \cdot \mathbf{n} \bar{\mathbf{v}} \cdot \mathbf{n} \, d\sigma = - \int_{\Omega_s} \mathbf{f} \cdot \bar{\mathbf{v}} \, d\mathbf{x}.$$

**Lemma A.8.** Under Assumptions 2.2-A.2-2.5, there exists constants  $\varepsilon_1, C_1 > 0$  such that for all  $\varepsilon \in (0, \varepsilon_1)$  any solution  $\mathbf{u}_\varepsilon^1 \in \mathbf{H}^1(\Omega_s)$  of the problem (64) with a data  $\mathbf{f} \in \mathbf{L}^2(\Omega_s)$  satisfies the uniform estimates:

$$(69) \quad \|\mathbf{u}_\varepsilon^1\|_{0,\Omega_s} \leq C_1 \|\mathbf{f}\|_{0,\Omega_s}.$$

To prepare for the proof of Lemma A.8, we first note that the problem (64) is equivalent to find  $(\mathbf{u}_\varepsilon^1, \varphi_\varepsilon) \in \mathbf{H}^1(\Omega_s) \times H^1(\Gamma)$  such that

$$(70a) \quad \nabla \cdot \underline{\sigma}(\mathbf{u}_\varepsilon^1) + \omega^2 \rho \mathbf{u}_\varepsilon^1 = \mathbf{f} \quad \text{in } \Omega_s,$$

$$(70b) \quad \mathbf{T}(\mathbf{u}_\varepsilon^1) \cdot \mathbf{n} = \varphi_\varepsilon \quad \text{on } \Gamma,$$

$$(70c) \quad -\Delta_\Gamma \varphi_\varepsilon - i \frac{\kappa}{\varepsilon} \mathcal{I}_\varepsilon \varphi_\varepsilon = -\frac{\omega^2 \rho_f}{\varepsilon} \mathbf{u}_\varepsilon^1 \cdot \mathbf{n} \quad \text{on } \Gamma,$$

where  $\mathcal{I}_\varepsilon$  is a function defined on  $\Gamma$  as  $\mathcal{I}_\varepsilon = (1 + 2\varepsilon \mathcal{H})$  which tends to 1 when  $\varepsilon$  goes to 0. We then introduce the variational formulation for  $(\mathbf{u}, \varphi) = (\mathbf{u}_\varepsilon^1, \varphi_\varepsilon) \in \mathbf{H}^1(\Omega_s) \times H^1(\Gamma)$  solution of the problem (70a)-(70b)-(70c) :

Find  $(\mathbf{u}, \varphi) = (\mathbf{u}_\varepsilon^1, \varphi_\varepsilon) \in \mathbf{H}^1(\Omega_s) \times H^1(\Gamma)$  such that for all  $(\mathbf{v}, \psi) \in \mathbf{H}^1(\Omega_s) \times H^1(\Gamma)$

$$(71a) \quad \int_{\Omega_s} (\underline{\sigma}(\mathbf{u}) : \underline{\varepsilon}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \bar{\mathbf{v}}) \, d\mathbf{x} - \int_{\Gamma} \varphi \bar{\mathbf{v}} \cdot \mathbf{n} \, d\sigma = - \int_{\Omega_s} \mathbf{f} \cdot \bar{\mathbf{v}} \, d\mathbf{x}$$

$$(71b) \quad \int_{\Gamma} (\nabla_\Gamma \varphi \nabla_\Gamma \bar{\psi} - i \frac{\kappa}{\varepsilon} \mathcal{I}_\varepsilon \varphi \bar{\psi}) \, d\sigma = -\frac{\omega^2 \rho_f}{\varepsilon} \int_{\Gamma} \mathbf{u} \cdot \mathbf{n} \bar{\psi} \, d\sigma.$$

**A.4. Proof of Lemma A.7 : Uniform  $\mathbf{L}^2$  estimate of the elastic displacement.** Reductio ad absurdum: We assume that there is a sequence  $\{\mathbf{u}_m\} \in \mathbf{H}^1(\Omega_s)$ ,  $m \in \mathbb{N}$ , of solutions of the problem (60a)-(61) associated with a right-hand side  $\mathbf{f}_m \in \mathbf{L}^2(\Omega_s)$ :

$$(72a) \quad \nabla \cdot \underline{\sigma}(\mathbf{u}_m) + \omega^2 \rho \mathbf{u}_m = \mathbf{f}_m \quad \text{in } \Omega_s,$$

$$(72b) \quad \mathbf{T}(\mathbf{u}_m) - i\omega c \rho_f \mathbf{u}_m \cdot \mathbf{n} = 0 \quad \text{on } \Gamma,$$

satisfying the following conditions

$$(73a) \quad \|\mathbf{u}_m\|_{0,\Omega_s} = 1 \quad \text{for all } m \in \mathbb{N},$$

$$(73b) \quad \|\mathbf{f}_m\|_{0,\Omega_s} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

A.4.1. *Estimates of the sequence  $\{\mathbf{u}_m\}$ .* We first prove that the sequence  $\{\mathbf{u}_m\}$  is bounded in  $\mathbf{H}^1(\Omega_s)$ . We particularize the elastic variational formulation (68) for the sequence  $\{\mathbf{u}_m\}$ : For all  $\mathbf{v} \in \mathbf{H}^1(\Omega_s)$ :

$$(74) \quad \int_{\Omega_s} (\underline{\sigma}(\mathbf{u}_m) : \underline{\varepsilon}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u}_m \bar{\mathbf{v}}) \, d\mathbf{x} - i\omega c \rho_f \int_{\Gamma} \mathbf{u}_m \cdot \mathbf{n} \bar{\mathbf{v}} \cdot \mathbf{n} \, d\sigma = - \int_{\Omega_s} \mathbf{f}_m \cdot \bar{\mathbf{v}} \, d\mathbf{x}.$$

We first take the imaginary part of the previous equality (74) when  $\mathbf{v} = \mathbf{u}_m$ . We obtain with the help of the condition (73b) the following uniform bound

$$(75) \quad \|\mathbf{u}_m \cdot \mathbf{n}\|_{0,\Gamma} \leq C.$$

Then, taking the real part of the equality (74) when  $\mathbf{v} = \mathbf{u}_m$ , we obtain with the help of conditions (73a)-(73b) a uniform bound in  $\mathbf{H}^1(\Omega_s)$  for the sequence  $\{\mathbf{u}_m\}$ :

$$(76) \quad \|\mathbf{u}_m\|_{1,\Omega_s} \leq C.$$

A.4.2. *Limit of the sequence and conclusion.* The domain  $\Omega_s$  being bounded, the embedding of  $\mathbf{H}^1(\Omega_s)$  in  $\mathbf{L}^2(\Omega_s)$  is compact. Hence as a consequence of (76), using the Rellich Lemma we can extract a subsequence of  $\{\mathbf{u}_m\}$  (still denoted by  $\{\mathbf{u}_m\}$ ) which is converging in  $\mathbf{L}^2(\Omega_s)$ , and we can assume that the sequence  $\{\underline{\nabla} \mathbf{u}_m\}$  is weakly converging in  $\mathbf{L}^2(\Omega_s)$ : We deduce that there is  $\mathbf{u} \in \mathbf{L}^2(\Omega_s)$  such that

$$(77) \quad \begin{cases} \underline{\nabla} \mathbf{u}_m \rightharpoonup \underline{\nabla} \mathbf{u} & \text{in } \mathbf{L}^2(\Omega_s) \\ \mathbf{u}_m \rightarrow \mathbf{u} & \text{in } \mathbf{L}^2(\Omega_s). \end{cases}$$

A consequence of the strong convergence in  $\mathbf{L}^2(\Omega_s)$  and (73a) is that  $\|\mathbf{u}\|_{0,\Omega_s} = 1$ .

Another consequence of (76) is that the sequence  $\{\mathbf{u}_m \cdot \mathbf{n}\}$  is bounded in  $H^{\frac{1}{2}}(\Gamma)$ . Therefore, we can assume that the sequence  $\{\mathbf{u}_m \cdot \mathbf{n}\}$  is strongly converging to a limit  $\mathbf{u} \cdot \mathbf{n}$  in  $L^2(\Gamma)$ . Using Assumption A.2, we are going to prove that  $\mathbf{u} = 0$ , which will contradict  $\|\mathbf{u}\|_{0,\Omega_s} = 1$ , and finally prove estimate (67).

Let  $\mathbf{v} \in \mathbf{H}^1(\Omega_s)$  be a test function in (74)

$$(78) \quad \int_{\Omega_s} (\underline{\sigma}(\mathbf{u}_m) : \underline{\varepsilon}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u}_m \bar{\mathbf{v}}) \, d\mathbf{x} - i\omega c \rho_f \int_{\Gamma} \mathbf{u}_m \cdot \mathbf{n} \bar{\mathbf{v}} \cdot \mathbf{n} \, d\sigma = - \int_{\Omega_s} \mathbf{f}_m \cdot \bar{\mathbf{v}} \, d\mathbf{x}.$$

Hence, according to (77) and (73b), taking limits as  $m \rightarrow +\infty$ , we deduce from the previous equalities (78):  $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$  satisfies for all  $\mathbf{v} \in \mathbf{H}^1(\Omega_s)$ :

$$\int_{\Omega_s} (\underline{\sigma}(\mathbf{u}) : \underline{\varepsilon}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \bar{\mathbf{v}}) \, d\mathbf{x} - i\omega c \rho_f \int_{\Gamma} \mathbf{u} \cdot \mathbf{n} \bar{\mathbf{v}} \cdot \mathbf{n} \, d\sigma = 0.$$

Taking  $\mathbf{v} = \mathbf{u}$  as test function in the previous equality, and then taking the imaginary part, we infer

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma.$$

Finally, integrating by parts (see (24)) we find that  $\mathbf{u}$  satisfies the problem (65)

$$\begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}) = 0 \quad \text{and} \quad \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma. \end{cases}$$

By Assumption A.2, we deduce

$$\mathbf{u} = 0 \quad \text{in } \Omega_s,$$

which contradicts  $\|\mathbf{u}\|_{0,\Omega_s} = 1$  and ends the proof of Lemma A.7.

**A.5. Proof of Lemma A.8 : Uniform  $L^2$  estimate of the elastic displacement.** Reductio ad absurdum: We assume that there is a sequence  $(\mathbf{u}_m, \varphi_m) \in \mathbf{H}^1(\Omega_s) \times H^1(\Gamma)$ ,  $m \in \mathbb{N}$ , of solutions of (70a)-(70b)-(70c) associated with a parameter  $\varepsilon_m$  and a right-hand side  $\mathbf{f}_m \in L^2(\Omega_s)$ :

$$(79a) \quad \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_m) + \omega^2 \rho \mathbf{u}_m = \mathbf{f}_m \quad \text{in } \Omega_s,$$

$$(79b) \quad \mathbf{T}(\mathbf{u}_m) \cdot \mathbf{n} = \varphi_m \quad \text{on } \Gamma,$$

$$(79c) \quad -\Delta_\Gamma \varphi_m - i \frac{\kappa}{\varepsilon_m} \mathcal{I}_{\varepsilon_m} \varphi_m = -\frac{\omega^2 \rho_f}{\varepsilon_m} \mathbf{u}_m \cdot \mathbf{n} \quad \text{on } \Gamma.$$

satisfying the following conditions

$$(80a) \quad \varepsilon_m \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

$$(80b) \quad \|\mathbf{u}_m\|_{0,\Omega_s} = 1 \quad \text{for all } m \in \mathbb{N},$$

$$(80c) \quad \|\mathbf{f}_m\|_{0,\Omega_s} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

**A.5.1. Estimates of the sequence  $\{\mathbf{u}_m\}$ .** We first prove that the sequence  $\{\mathbf{u}_m\}$  is bounded in  $\mathbf{H}^1(\Omega_s)$ . We particularize the variational formulation (71a)-(71b) for  $\mathbf{v} = \mathbf{u}_m$  and  $\psi = \varphi_m$ :

$$(81a) \quad \int_{\Omega_s} (\underline{\underline{C}}|\underline{\underline{\varepsilon}}(\mathbf{u}_m)|^2 - \omega^2 \rho |\mathbf{u}_m|^2) \, d\mathbf{x} - \int_\Gamma \varphi_m \bar{\mathbf{u}}_m \cdot \mathbf{n} \, d\sigma = - \int_{\Omega_s} \mathbf{f}_m \cdot \bar{\mathbf{u}}_m \, d\mathbf{x}$$

$$(81b) \quad \int_\Gamma (|\nabla_\Gamma \varphi_m|^2 - i \frac{\kappa}{\varepsilon_m} \mathcal{I}_{\varepsilon_m} |\varphi_m|^2) \, d\sigma = -\frac{\omega^2 \rho_f}{\varepsilon_m} \int_\Gamma \mathbf{u}_m \cdot \mathbf{n} \bar{\varphi}_m \, d\sigma.$$

It leads to

$$(82) \quad \int_{\Omega_s} (\underline{\underline{C}}|\underline{\underline{\varepsilon}}(\mathbf{u}_m)|^2 - \omega^2 \rho |\mathbf{u}_m|^2) \, d\mathbf{x} + \frac{\varepsilon_m}{\omega^2 \rho_f} \int_\Gamma |\nabla_\Gamma \varphi_m|^2 \, d\sigma + i \frac{1}{\omega c \rho_f} \int_\Gamma \mathcal{I}_{\varepsilon_m} |\varphi_m|^2 \, d\sigma = - \int_{\Omega_s} \mathbf{f}_m \cdot \bar{\mathbf{u}}_m \, d\mathbf{x}.$$

We first take the imaginary part of the previous equality (82). We obtain with the help of conditions (80a)-(80c) the following uniform bound

$$(83) \quad \|\varphi_m\|_{0,\Gamma} \leq C,$$

since  $\mathcal{I}_{\varepsilon_m}$  tends to 1 when  $m$  goes to  $\infty$ . Then, taking the real part of the equality (82), we obtain with the help of conditions (80a)-(80b)-(80c)-(83) the following uniform bound

$$(84) \quad \int_{\Omega_s} \underline{\underline{C}}|\underline{\underline{\varepsilon}}(\mathbf{u}_m)|^2 \, d\mathbf{x} + \frac{\varepsilon_m}{\omega^2 \rho_f} \int_\Gamma |\nabla_\Gamma \varphi_m|^2 \, d\sigma \leq C.$$

Hence, as a consequence of the Korn inequality, the sequence  $\{\mathbf{u}_m\}$  is bounded in  $\mathbf{H}^1(\Omega_s)$ :

$$(85) \quad \|\mathbf{u}_m\|_{1,\Omega_s} \leq C,$$

and the sequence  $\{(\varepsilon_m)^{\frac{1}{2}} \nabla_\Gamma \varphi_m\}$  is bounded in  $\mathbf{L}^2(\Gamma)$ :

$$(86) \quad (\varepsilon_m)^{\frac{1}{2}} \|\nabla_\Gamma \varphi_m\|_{0,\Gamma} \leq C.$$

**A.5.2. Limit of the sequence and conclusion.** The domain  $\Omega_s$  being bounded, the embedding of  $\mathbf{H}^1(\Omega_s)$  in  $\mathbf{L}^2(\Omega_s)$  is compact. Hence as a consequence of (85), using the Rellich Lemma we can extract a subsequence of  $\{\mathbf{u}_m\}$  (still denoted by  $\{\mathbf{u}_m\}$ ) which is converging in  $\mathbf{L}^2(\Omega_s)$ , and we can assume that the sequence  $\{\underline{\underline{\nabla}}\mathbf{u}_m\}$  is weakly converging in  $\mathbf{L}^2(\Omega_s)$ : We deduce that there exists  $\mathbf{u} \in \mathbf{L}^2(\Omega_s)$  such that

$$(87) \quad \begin{cases} \underline{\underline{\nabla}}\mathbf{u}_m \rightharpoonup \underline{\underline{\nabla}}\mathbf{u} & \text{in } \mathbf{L}^2(\Omega_s) \\ \mathbf{u}_m \rightarrow \mathbf{u} & \text{in } \mathbf{L}^2(\Omega_s). \end{cases}$$

A consequence of the strong convergence in  $\mathbf{L}^2(\Omega_s)$  and (80b) is that  $\|\mathbf{u}\|_{0,\Omega_s} = 1$ . As a consequence of (85), the sequence  $\{\mathbf{u}_m \cdot \mathbf{n}\}$  is bounded in  $H^{\frac{1}{2}}(\Gamma)$ ; hence, up to the extraction of a subsequence, we can assume that the sequence  $\{\mathbf{u}_m \cdot \mathbf{n}\}$  is strongly converging to a limit  $\mathbf{u} \cdot \mathbf{n}$  in  $L^2(\Gamma)$ :

$$(88) \quad \mathbf{u}_m \cdot \mathbf{n} \rightarrow \mathbf{u} \cdot \mathbf{n} \quad \text{in } L^2(\Gamma) .$$

Using Assumption A.2, we are going to prove that  $\mathbf{u} = 0$ , which will contradict  $\|\mathbf{u}\|_{0,\Omega_s} = 1$ , and finally prove estimate (69).

Let  $\psi$  a test function in the formulation (71b) multiplied by  $\varepsilon$  and then written for  $\varepsilon = \varepsilon_m$  and  $\varphi = \varphi_m$ . It leads to

$$(89) \quad \int_{\Gamma} (\varepsilon_m \nabla_{\Gamma} \varphi_m \nabla_{\Gamma} \bar{\psi} - i\kappa \mathcal{I}_{\varepsilon_m} \varphi_m \bar{\psi}) d\sigma = -\omega^2 \rho_f \int_{\Gamma} \mathbf{u}_m \cdot \mathbf{n} \bar{\psi} d\sigma .$$

As a consequence of (86), we can extract a subsequence still denoted by  $\{(\varepsilon_m)^{\frac{1}{2}} \nabla_{\Gamma} \varphi_m\}$  which is weakly converging to a function  $\mathbf{t} \in \mathbf{L}^2(\Gamma)$

$$(90) \quad (\varepsilon_m)^{\frac{1}{2}} \nabla_{\Gamma} \varphi_m \rightharpoonup \mathbf{t} \quad \text{in } \mathbf{L}^2(\Gamma) .$$

As a consequence of (83), and using the Banach-Alaoglu theorem, we can assume that the sequence  $\{\varphi_m\}$  is weakly converging to a limit  $\varphi$  in  $L^2(\Gamma)$ . Hence, taking limits as  $m \rightarrow +\infty$ , there holds

$$(91) \quad \int_{\Gamma} \mathcal{I}_{\varepsilon_m} \varphi_m \bar{\psi} d\sigma \rightarrow \int_{\Gamma} \varphi \bar{\psi} d\sigma .$$

According to (91), (90), (88) and (80a), taking limits as  $m \rightarrow +\infty$ , we deduce from the previous equalities (89)

$$-i\kappa \int_{\Gamma} \varphi \bar{\psi} d\sigma = -\omega^2 \rho_f \int_{\Gamma} \mathbf{u} \cdot \mathbf{n} \bar{\psi} d\sigma .$$

Since the previous equality holds for all  $\psi \in L^2(\Gamma)$ , we infer

$$(92) \quad \varphi = -i\omega c \rho_f \mathbf{u} \cdot \mathbf{n} \quad \text{on } \Gamma .$$

Let  $\mathbf{v} \in \mathbf{H}^1(\Omega_s)$  be a test function in (71a) particularized for  $\mathbf{u} = \mathbf{u}_m$  and  $\varphi = \varphi_m$

$$(93) \quad \int_{\Omega_s} (\underline{\sigma}(\mathbf{u}_m) : \underline{\varepsilon}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u}_m \bar{\mathbf{v}}) d\mathbf{x} - \int_{\Gamma} \varphi_m \bar{\mathbf{v}} \cdot \mathbf{n} d\sigma = - \int_{\Omega_s} \mathbf{f}_m \cdot \bar{\mathbf{u}}_m d\mathbf{x} .$$

Hence, according to (90), (80a), (80b) and (80c), taking limits as  $m \rightarrow +\infty$ , we deduce from the previous equalities (92) and (93) :  $\mathbf{u} \in \mathbf{H}^1(\Omega_s)$  satisfies for all  $\mathbf{v} \in \mathbf{H}^1(\Omega_s)$ :

$$\int_{\Omega_s} (\underline{\sigma}(\mathbf{u}) : \underline{\varepsilon}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \bar{\mathbf{v}}) d\mathbf{x} + i\omega c \rho_f \int_{\Gamma} \mathbf{u} \cdot \mathbf{n} \bar{\mathbf{v}} \cdot \mathbf{n} d\sigma = 0 .$$

Finally, integrating by parts (see (24)) we find that  $\mathbf{u}$  satisfies the problem (65)

$$\begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}) = 0 \quad \text{on } \mathbf{u} \cdot \mathbf{n} = 0 & \text{on } \Gamma . \end{cases}$$

By Assumption A.2, we deduce

$$\mathbf{u} = 0 \quad \text{in } \Omega_s ,$$

which contradicts  $\|\mathbf{u}\|_{0,\Omega_s} = 1$  and ends the proof of Lemma A.8.

**A.6. Elements for the proof of Equivalent Conditions.** After the scaling  $h \rightarrow Y_3 = \varepsilon^{-1}h$  in the thin layer  $\Omega_f^\varepsilon$ , the transmission problem with a Fourier-Robin boundary condition writes :

$$(94) \quad \begin{cases} \varepsilon^{-2}[\partial_3^2 \mathbf{p}_\varepsilon + \sum_{n \geq 1} \varepsilon^n L^n \mathbf{p}_\varepsilon] = 0 & \text{in } \Gamma \times (0, 1) \\ \varepsilon^{-1} \partial_3 \mathbf{p}_\varepsilon = \rho_f \omega^2 \mathbf{u}_\varepsilon \cdot \mathbf{n} & \text{on } \Gamma \times \{0\} \\ \varepsilon^{-1} \partial_3 \mathbf{p}_\varepsilon - i\kappa \mathbf{p}_\varepsilon = 0 & \text{on } \Gamma \times \{1\} \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon) + \omega^2 \rho \mathbf{u}_\varepsilon = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_\varepsilon) = -\mathbf{p}_\varepsilon \mathbf{n} & \text{on } \Gamma . \end{cases}$$

We insert the following Ansatz in equations (94)

$$\begin{aligned} \mathbf{u}_\varepsilon(\mathbf{x}) &= \mathbf{u}_0(\mathbf{x}) + \varepsilon \mathbf{u}_1(\mathbf{x}) + \varepsilon^2 \mathbf{u}_2(\mathbf{x}) + \dots , \\ \mathbf{p}_\varepsilon(\mathbf{x}) &= \mathbf{p}_0(\mathbf{x}; \varepsilon) + \varepsilon \mathbf{p}_1(\mathbf{x}; \varepsilon) + \varepsilon^2 \mathbf{p}_2(\mathbf{x}; \varepsilon) + \dots , \quad \mathbf{p}_j(\mathbf{x}; \varepsilon) = \mathbf{p}_j(y_\alpha, \frac{h}{\varepsilon}) . \end{aligned}$$

Then we get the following two families of problems, coupled by their boundary conditions on  $\Gamma$  (i.e. when  $Y_3 = 0$ ):

$$(95) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_n) + \omega^2 \rho \mathbf{u}_n = \mathbf{f} \delta_0^n & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_n) = -\mathbf{p}_n \mathbf{n} & \text{on } \Gamma \end{cases}$$

$$(96) \quad \begin{cases} \partial_3^2 \mathbf{p}_n = - \sum_{l+p=n, l \geq 1} L^l \mathbf{p}_p & \text{for } Y_3 \in (0, 1) \\ \partial_3 \mathbf{p}_n = \rho_f \omega^2 \mathbf{u}_{n-1} \cdot \mathbf{n} & \text{for } Y_3 = 0 \\ \partial_3 \mathbf{p}_n - i\kappa \mathbf{p}_{n-1} = 0 & \text{for } Y_3 = 1 . \end{cases}$$

We explicit in §A.6.1 the first terms  $(\mathbf{u}_0, \mathbf{p}_0)$  and  $(\mathbf{u}_1, \mathbf{p}_1)$ . We give several details for the calculus in §A.6.2. Then we derive equivalent conditions in §A.6.3.

**A.6.1. First terms.** In the case  $n = 0$ ,  $\mathbf{u}_0$  solves the problem

$$(97) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_0) - i\omega c \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0 & \text{on } \Gamma , \end{cases}$$

and then we obtain

$$(98) \quad \mathbf{p}_0(= \mathbf{p}_0(y_\alpha)) = (i\kappa)^{-1} \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}|_\Gamma .$$

At the step  $n = 1$ , we find that  $\mathbf{u}_1$  solves the boundary value problem

$$(99) \quad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_1) - i\omega c \rho_f \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} = -(i\kappa)^{-1} \rho_f \omega^2 (-2\mathcal{H} - (i\kappa)^{-1} \Delta_\Gamma) (\mathbf{u}_0 \cdot \mathbf{n})|_\Gamma \mathbf{n} & \text{on } \Gamma . \end{cases}$$

and

$$(100) \quad \mathbf{p}_1(y_\alpha, Y_3) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}|_\Gamma Y_3 + b_1(y_\alpha) ,$$

where

$$b_1(y_\alpha) = (i\kappa)^{-1} \rho_f \omega^2 (-2\mathcal{H} - (i\kappa)^{-1} \Delta_\Gamma) (\mathbf{u}_0 \cdot \mathbf{n})|_\Gamma + (i\kappa)^{-1} \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}|_\Gamma .$$

A.6.2. *Formal calculus.* In the case  $n = 0$ , we obtain from (96)

$$\mathbf{p}_0 = \mathbf{p}_0(y_\alpha),$$

where  $\mathbf{p}_0(y_\alpha)$  has to be determined. Then (95) yields  $\mathbf{u}_0$  solves the problem

$$(101) \quad \begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_0) = -\mathbf{p}_0(y_\alpha) \mathbf{n} & \text{on } \Gamma. \end{cases}$$

At step  $n = 1$ , according to (96) since  $\mathbf{p}_0 = \mathbf{p}_0(y_\alpha)$  there holds

$$(102) \quad \begin{cases} \partial_3^2 \mathbf{p}_1 = 0 & \text{for } Y_3 \in (0, 1) \\ \partial_3 \mathbf{p}_1 = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} & \text{for } Y_3 = 0 \\ \partial_3 \mathbf{p}_1 = i\kappa \mathbf{p}_0 & \text{for } Y_3 = 1. \end{cases}$$

Since the right-hand side of the first equation in (102) is zero, the function  $\partial_3 \mathbf{p}_1 (= \partial_3 \mathbf{p}_1(y_\alpha)) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}$  is independent of the variable  $Y_3$ . Hence, there exists two functions  $a_1, b_1$  defined on  $\Gamma$  such that  $\mathbf{p}_1(y_\alpha, Y_3) = a_1(y_\alpha) Y_3 + b_1(y_\alpha)$ . Using the last two equations in the system (102),  $a_1$  satisfies

$$\begin{cases} a_1(y_\alpha) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} & \text{on } \Gamma \\ a_1(y_\alpha) = i\kappa \mathbf{p}_0(y_\alpha). \end{cases}$$

We infer that  $\mathbf{p}_0$  satisfies (98) and  $\mathbf{p}_1$  writes as follow

$$\mathbf{p}_1(y_\alpha, Y_3) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}|_\Gamma Y_3 + b_1(y_\alpha),$$

where  $b_1(y_\alpha) = \mathbf{p}_1(y_\alpha, 0)$  has to be determined. Using the above expression of  $\mathbf{p}_0$  we can explicit the boundary condition in (101)

$$(103) \quad \mathbf{T}(\mathbf{u}_0) - i\omega c \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0 \quad \text{on } \Gamma.$$

Hence  $\mathbf{u}_0$  solves the problem (97). According to (95),  $\mathbf{u}_1$  solves

$$(104) \quad \begin{cases} \nabla \cdot \underline{\sigma}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in } \Omega_s \\ \mathbf{T}(\mathbf{u}_1) = -\mathbf{p}_1(y_\alpha, 0) \mathbf{n} & \text{on } \Gamma. \end{cases}$$

In the problem (104), the right-hand side term  $\mathbf{p}_1(y_\alpha, 0) = b_1(y_\alpha)$  has to be explicit. At step  $n = 2$ , according to (96) we find

$$(105) \quad \begin{cases} \partial_3^2 \mathbf{p}_2 = -2\mathcal{H} \partial_3 \mathbf{p}_1(y_\alpha) - \Delta_\Gamma \mathbf{p}_0(y_\alpha) - \kappa^2 \mathbf{p}_0(y_\alpha) & \text{for } Y_3 \in (0, 1) \\ \partial_3 \mathbf{p}_2 = \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n} & \text{for } Y_3 = 0 \\ \partial_3 \mathbf{p}_2 = i\kappa \mathbf{p}_1(y_\alpha, 1) & \text{for } Y_3 = 1. \end{cases}$$

Since the right-hand side of the first equation in (105) does not depend on  $Y_3$ , there exists functions  $a_2, b_2$  and  $c_2$  such that  $\mathbf{p}_2(y_\alpha, Y_3) = a_2(y_\alpha) Y_3^2 + b_2(y_\alpha) Y_3 + c_2(y_\alpha)$ , where the couple  $(a_2, b_2)$  solves the system

$$(106) \quad \begin{cases} 2a_2(y_\alpha) = -2\mathcal{H} \partial_3 \mathbf{p}_1(y_\alpha) - \Delta_\Gamma \mathbf{p}_0(y_\alpha) - \kappa^2 \mathbf{p}_0(y_\alpha) & \text{for } Y_3 \in (0, 1) \\ b_2(y_\alpha) = \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n} & \text{for } Y_3 = 0 \\ 2a_2(y_\alpha) + b_2(y_\alpha) = i\kappa (\partial_3 \mathbf{p}_1(y_\alpha) + b_1(y_\alpha)) & \text{for } Y_3 = 1. \end{cases}$$

We eliminate  $a_2$  and  $b_2$  in the system (106) to explicit the function  $b_1$ :

$$b_1(y_\alpha) = (i\kappa)^{-1} (-2\mathcal{H} \partial_3 \mathbf{p}_1(y_\alpha) - \Delta_\Gamma \mathbf{p}_0(y_\alpha) - \kappa^2 \mathbf{p}_0(y_\alpha) + \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}) - \partial_3 \mathbf{p}_1(y_\alpha),$$

Since  $\partial_3 \mathbf{p}_1(y_\alpha) = \rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n}|_\Gamma$  and  $\mathbf{p}_0(y_\alpha)$  is given by (98), we deduce

$$b_1(y_\alpha) = (i\kappa)^{-1} \rho_f \omega^2 (-2\mathcal{H} - (i\kappa)^{-1} \Delta_\Gamma) (\mathbf{u}_0 \cdot \mathbf{n})|_\Gamma + (i\kappa)^{-1} \rho_f \omega^2 \mathbf{u}_1 \cdot \mathbf{n}|_\Gamma,$$

and  $\mathbf{p}_1$  satisfies (100). Now we can explicit the boundary condition in equation (104):

$$(107) \quad \mathbf{T}(\mathbf{u}_1) - i\omega c \rho_f \mathbf{u}_1 \cdot \mathbf{n} \mathbf{n} = -(i\kappa)^{-1} \rho_f \omega^2 (-2\mathcal{H} - (i\kappa)^{-1} \Delta_\Gamma) (\mathbf{u}_0 \cdot \mathbf{n})|_\Gamma \mathbf{n} \quad \text{on } \Gamma.$$

and  $\mathbf{u}_1$  solves the problem (99).

A.6.3. *Construction of equivalent conditions.* The equivalent condition of order 0 is exact (in the following sense :  $\mathbf{u}_\varepsilon^0 = \mathbf{u}_0$  (101)) and it writes

$$\mathbf{T}(\mathbf{u}_0) - i\omega c \rho_f \mathbf{u}_0 \cdot \mathbf{n} \mathbf{n} = 0 \quad \text{on } \Gamma.$$

We introduce the truncated expansion  $\mathbf{u}_{1,\varepsilon} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1$ . Adding equations (103) and  $\varepsilon$  times (107) we obtain

$$\mathbf{T}(\mathbf{u}_{1,\varepsilon}) - i\omega c \rho_f \mathbf{u}_{1,\varepsilon} \cdot \mathbf{n} \mathbf{n} = -\varepsilon (i\kappa)^{-1} \rho_f \omega^2 (-2\mathcal{H} - (i\kappa)^{-1} \Delta_\Gamma) (\mathbf{u}_0 \cdot \mathbf{n})|_\Gamma \mathbf{n} \quad \text{on } \Gamma.$$

We infer

$$\mathbf{T}(\mathbf{u}_{1,\varepsilon}) - i\omega c \rho_f \mathbf{u}_{1,\varepsilon} \cdot \mathbf{n} \mathbf{n} - i\omega c \rho_f (-2\mathcal{H} - (i\kappa)^{-1} \Delta_\Gamma) ((\varepsilon \mathbf{u}_{1,\varepsilon} - \varepsilon^2 \mathbf{u}_2) \cdot \mathbf{n}) \mathbf{n} = 0 \quad \text{on } \Gamma.$$

Neglecting the term of order  $\varepsilon^2$  in the left-hand side, we obtain the equivalent condition of order 1:

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c \rho_f (1 + 2\varepsilon \mathcal{H}) \mathbf{u}_\varepsilon^1 \cdot \mathbf{n} \mathbf{n} + \varepsilon c^2 \rho_f \Delta_\Gamma (\mathbf{u}_\varepsilon^1 \cdot \mathbf{n}) \mathbf{n} = 0 \quad \text{on } \Gamma.$$

#### APPENDIX B. EXPANSION OF THE HELMHOLTZ OPERATOR IN NORMAL COORDINATES

The aim of this appendix is to expand the Helmholtz operator set in the domain  $\Omega_f^\varepsilon$  in power series of  $\varepsilon$  and to explicit the first terms of the expansion, see (112) and Prop. B.3. We first write the Helmholtz operator in normal coordinates, Prop. B.1.

We set  $(y_\alpha, y_3)$  a *normal coordinate system* [8, 5] to the surface  $\Gamma$  on the manifold  $\Omega_f^\varepsilon$  :  $y_\alpha$  ( $\alpha \in \{1, 2\}$ ) is a tangential coordinate on  $\Gamma$  and  $y_3 = h$  is the distance to the surface  $\Gamma$ . We set  $g_{ij}$  the metric tensor associated with this coordinate system:

$$(108) \quad g_{ij} = \langle \mathbf{y}_i, \mathbf{y}_j \rangle_{\mathbb{R}^3} \quad \text{where} \quad \mathbf{y}_i = \frac{\partial}{\partial y_i} \quad \text{are the associated coordinate vector fields.}$$

We denote by  $\partial_\alpha f$ ,  $\alpha = 1, 2$ , the partial derivative with respect to the tangential coordinate  $y_\alpha$  on  $\Gamma$ , and  $\partial_3^h$  is the partial derivative with respect to the normal coordinate  $y_3 = h$ . We set  $\nabla$  the standard connexion (covariant derivative) on the manifold  $\Omega_f^\varepsilon$  associated with the Euclidean scalar product on  $\mathbb{R}^3$  [12]. We denote by  $D^h$  the connexion on  $\Gamma_h$  induced by  $\nabla$ : Here  $D_\alpha^h$  is the covariant derivative on the manifold  $\Gamma_h$ , which is the surface contained in  $\Omega_f^\varepsilon$  at a distance  $h$  of the surface  $\Gamma$  (see Figure 1).

In the framework above, it is possible to expand the Helmholtz operator in power series of  $\varepsilon$ . We first expand the connexion and then the Laplace operator. Let  $a_{\alpha\beta}(h)$  be the metric tensor of the manifold  $\Gamma_h$ . The metric tensor in such a coordinate system writes [8, 5]

$$(109) \quad a_{\alpha\beta}(h) = a_{\alpha\beta} - 2b_{\alpha\beta} h + b_\alpha^\gamma b_{\gamma\beta} h^2.$$

Here,  $b_{\alpha\beta}$  is the *curvature tensor* in  $\Gamma$ ,  $b_\alpha^\gamma = a^{\gamma\beta} b_{\alpha\beta}$  (We use the summation convention of repeated two dimensional indices represented by greek letters). The inverse of the metric tensor  $a^{\alpha\beta}(h)$  expands in power series of  $h$

$$a^{\alpha\beta}(h) = a^{\alpha\beta} + 2b^{\alpha\beta} h + \mathcal{O}(h^2).$$

The following proposition gives a normal parameterization of the Laplace operator  $\Delta$  applied to a function  $f$ .

**Proposition B.1.** *For a given function  $f$  defined in the domain  $\Omega_f^\varepsilon$ , there holds for all  $h \in (0, \varepsilon)$*

$$(110) \quad \Delta f = (\partial_3^h)^2 f - b_\alpha^\alpha(h) \partial_3^h f + a^{\alpha\beta}(h) D_\alpha^h \partial_\beta f \quad \text{on } \Gamma_h.$$

*Proof.* The intrinsic Laplace operator is the intrinsic divergence of the gradient operator:

$$\Delta f = \nabla \cdot \mathbf{T} \quad \text{where} \quad \mathbf{T} = \nabla f.$$

Here  $f$  is a function and  $\mathbf{T} = (T_\alpha, T_3)$  is a 1-form field in  $\Omega_f^\varepsilon$ .

We can explicit the Laplace operator in normal coordinates by means of a *normal parameterization* of the intrinsic divergence operator denoted by  $\nabla \cdot$ : there holds for all  $h \in (0, \varepsilon)$  [24, Prop. 3.19] :

$$(111) \quad \nabla \cdot \mathbf{T} = \partial_3^h T_3 + a^{\alpha\beta}(h) D_\alpha^h T_\beta - b_\alpha^\alpha(h) T_3 \quad \text{on } \Gamma_h.$$

Since  $f$  is a function, the components of the 1-form field  $\mathbf{T}$  writes in normal coordinates as

$$T_\alpha = \partial_\alpha f \quad \text{and} \quad T_3 = \partial_3^h f \quad \text{on} \quad \Gamma_h.$$

We infer for all  $h \in (0, \varepsilon)$

$$\Delta f = (\partial_3^h)^2 f - b_\alpha^\alpha(h) \partial_3^h f + a^{\alpha\beta}(h) D_\alpha^h \partial_\beta f \quad \text{on} \quad \Gamma_h.$$

□

**Remark B.2.** There holds  $D_\alpha^h \partial_\beta f = \partial_\alpha \partial_\beta f - \Gamma_{\alpha\beta}^\gamma(h) \partial_\gamma f$  on  $\Gamma_h$ , where  $\Gamma_{jk}^l(h)$  denotes a Christoffel symbol in the normal coordinates [12, eq. (3.5)].

We denote by  $L = L(y_\alpha, h; D_\alpha^h, \partial_3^h)$  the operator  $\Delta + \kappa^2 \mathbb{I}$  written in the coordinates  $(y_\alpha, h)$ . The operator  $L$  expands in power series of  $h$  with intrinsic coefficients with respect to  $\Gamma_0 = \Gamma$ , see [5, Appendix A.1] for the three dimensional Maxwell operator set on a boundary layer. Then, we make the scaling  $Y_3 = \varepsilon^{-1} h$  to work on a manifold independent of  $\varepsilon$ . Thus, the problem is stated on the manifold  $\Omega_f = \Gamma \times (0, 1)$ . The three-dimensional harmonic Helmholtz operator is then written  $L(\varepsilon)$  in  $\Omega_f$ . Using expansions of the covariant derivative and the metric, the operator  $L(\varepsilon)$  expand in power series of  $\varepsilon$  [5, 12].

The following proposition provides the expansion of the first terms  $L^k$ ,  $k = 0, 1, 2$ , appearing in the expansion

$$(112) \quad L(\varepsilon) = \varepsilon^{-2} \sum_{k=0}^{\infty} \varepsilon^k L^k,$$

the coefficients  $L^k$  are intrinsic operators.

**Proposition B.3.** For  $k = 0, 1, 2$ , the operators  $L^k = L^k(y_\alpha, Y_3; D_\alpha, \partial_3)$  in the expansion (112) write

$$(113) \quad L^0 = \partial_3^2, \quad L^1 = 2\mathcal{H}\partial_3, \quad \text{and} \quad L^2 = a^{\alpha\beta} D_\alpha \partial_\beta + \kappa^2 \mathbb{I} - c_\alpha^\alpha Y_3 \partial_3.$$

Here,  $\partial_3$  is the partial derivative with respect to  $Y_3$ ,  $\mathcal{H} = -\frac{1}{2} b_\alpha^\alpha$  is the mean curvature of the surface  $\Gamma$ , and  $D_\alpha$  is the covariant derivative on  $\Gamma$ .

**Remark B.4.** The sign of  $\mathcal{H}$  depends on the orientation of the surface  $\Gamma$ . As a convention, the unit normal vector  $\mathbf{n}$  on the surface  $\Gamma$  is outwardly oriented to  $\Omega_s$ , see Figure 1.

*Proof.* There holds

$$b_\alpha^\alpha(h) = b_\alpha^\alpha + h c_\alpha^\alpha + \mathcal{O}(h^2),$$

[24, §4.3.1], where  $c_{\alpha\beta} = b_\alpha^\gamma b_{\gamma\beta}$ . The Christoffel symbols  $\Gamma_{jk}^l(h)$  defined on the manifold  $\Gamma_h$  expand as  $\Gamma_{jk}^l(h) = \Gamma_{jk}^l + \mathcal{O}(h)$  [12], where  $\Gamma_{jk}^l$  are the Christoffel symbols defined on  $\Gamma$ . We infer the following expansion  $D_\alpha^h \partial_\beta f = D_\alpha \partial_\beta f + \mathcal{O}(h)$ , where

$$D_\alpha \partial_\beta f = \partial_\alpha \partial_\beta f - \Gamma_{\alpha\beta}^\gamma \partial_\gamma f.$$

We make the scaling  $h = \varepsilon Y_3$  in (110) and there holds

$$(114) \quad \Delta = \varepsilon^{-2} \partial_3^2 - \varepsilon^{-1} (b_\alpha^\alpha + \varepsilon Y_3 c_\alpha^\alpha + \mathcal{O}(\varepsilon^2)) \partial_3 + (a^{\alpha\beta} + \mathcal{O}(\varepsilon)) (D_\alpha \partial_\beta + \mathcal{O}(\varepsilon)) \quad \text{in} \quad \Omega_f.$$

We perform the identification of terms with the power  $-2$ ,  $-1$ , and  $0$  in  $\varepsilon$ , and we infer the expression of operators  $L^0$ ,  $L^1$ , and  $L^2$  (113). □

**Remark B.5.** We identify the operator  $a^{\alpha\beta} D_\alpha \partial_\beta$  with the Laplacian-Beltrami operator  $\Delta_\Gamma$  defined on  $\Gamma$ . Since  $c_\alpha^\alpha = 4\mathcal{H}^2 - 2\mathcal{K}$ , there holds  $L^2 = \Delta_\Gamma + \kappa^2 \mathbb{I} - 2(2\mathcal{H}^2 - \mathcal{K}) Y_3 \partial_3$ . We use this expression of the operator  $L^2$  to derive an Equivalent Condition of order 1 for an Absorbing Boundary Condition set on  $\Gamma^\varepsilon$ , see Sec. A.



## REFERENCES

- [1] V. Andreev and A. Samarski. *Méthode aux différences pour les équations elliptiques*. Edition de Moscou, Moscou, 1978.
- [2] X. Antoine, H. Barucq, and L. Vernhet. High-frequency asymptotic analysis of a dissipative transmission problem resulting in generalized impedance boundary conditions. *Asymptot. Anal.*, 26(3-4):257–283, 2001.
- [3] A. Bendali and K. Lemrabet. The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation. *SIAM J. Appl. Math.*, 56(6):1664–1693, 1996.
- [4] G. Caloz, M. Costabel, M. Dauge, and G. Vial. Asymptotic expansion of the solution of an interface problem in a polygonal domain with thin layer. *Asymptot. Anal.*, 50(1-2):121–173, 2006.
- [5] G. Caloz, M. Dauge, E. Faou, and V. Péron. On the influence of the geometry on skin effect in electromagnetism. *Computer Methods in Applied Mechanics and Engineering*, 200(9-12):1053–1068, 2011.
- [6] G. Caloz, M. Dauge, and V. Péron. Uniform estimates for transmission problems with high contrast in heat conduction and electromagnetism. *Journal of Mathematical Analysis and Applications*, 370(2):555–572, 2010.
- [7] M. Costabel, M. Dauge, and S. Nicaise. Corner Singularities and Analytic Regularity for Linear Elliptic Systems. Part I: Smooth domains. 211 pages v2: Improvement of layout.
- [8] M. Dauge, E. Faou, and V. Péron. Comportement asymptotique à haute conductivité de l'épaisseur de peau en électromagnétisme. *C. R. Math. Acad. Sci. Paris*, 348(7-8):385–390, 2010.
- [9] M. Durán and J.-C. Nédélec. Un problème spectral issu d'un couplage élasto-acoustique. *M2AN Math. Model. Numer. Anal.*, 34(4):835–857, 2000.
- [10] M. Duruflé, V. Péron, and C. Poignard. Time-harmonic Maxwell equations in biological cells—the differential form formalism to treat the thin layer. *Confluentes Math.*, 3(2):325–357, 2011.
- [11] B. Engquist and J.C. Nédélec. Effective boundary condition for acoustic and electromagnetic scattering in thin layers. *Technical Report of CMAP*, 278, 1993.
- [12] E. Faou. Elasticity on a thin shell: formal series solution. *Asymptot. Anal.*, 31(3-4):317–361, 2002.
- [13] H. Haddar, P. Joly, and H.-M. Nguyen. Generalized impedance boundary conditions for scattering problems from strongly absorbing obstacles: the case of Maxwell's equations. *Math. Models Methods Appl. Sci.*, 18(10):1787–1827, 2008.
- [14] T. Hargé. Valeurs propres d'un corps élastique. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(13):857–859, 1990.
- [15] T. Huttunen, J. P. Kaipio, and P. Monk. An ultra-weak method for acoustic fluid-solid interaction. *J. Comput. Appl. Math.*, 213(1):166–185, 2008.
- [16] D. S. Jones. Low-frequency scattering by a body in lubricated contact. *Quart. J. Mech. Appl. Math.*, 36(1):111–138, 1983.
- [17] O. D. Lafitte. Diffraction in the high frequency regime by a thin layer of dielectric material. I. The equivalent impedance boundary condition. *SIAM J. Appl. Math.*, 59(3):1028–1052 (electronic), 1999.
- [18] O. D. Lafitte and G. Lebeau. Équations de Maxwell et opérateur d'impédance sur le bord d'un obstacle convexe absorbant. *C. R. Acad. Sci. Paris Sér. I Math.*, 316(11):1177–1182, 1993.
- [19] K. Lemrabet. Le problème de Ventcel pour le système de l'élasticité dans un domaine de  $\mathbf{R}^3$ . *C. R. Acad. Sci. Paris Sér. I Math.*, 304(6):151–154, 1987.
- [20] M. A. Leontovich. Approximate boundary conditions for the electromagnetic field on the surface of a good conductor. In *Investigations on radiowave propagation*, volume 2, pages 5–12. Printing House of the USSR Academy of Sciences, Moscow, 1948.
- [21] C. J. Luke and P. A. Martin. Fluid-solid interaction: acoustic scattering by a smooth elastic obstacle. *SIAM J. Appl. Math.*, 55(4):904–922, 1995.
- [22] P. Monk and V. Selgas. An inverse fluid-solid interaction problem. *Inverse Probl. Imaging*, 3(2):173–198, 2009.
- [23] D. Natroshvili, A.-M. Sändig, and W. L. Wendland. Fluid-structure interaction problems. In *Mathematical aspects of boundary element methods (Palaiseau, 1998)*, volume 414 of *Chapman & Hall/CRC Res. Notes Math.*, pages 252–262. Chapman & Hall/CRC, Boca Raton, FL, 2000.
- [24] V. Péron. *Modélisation mathématique de phénomènes électromagnétiques dans des matériaux à fort contraste*. PhD thesis, Université Rennes 1, 2009.
- [25] S. M. Rytov. Calcul du skin effect par la méthode des perturbations. *Journal of Physics*, 11(3):233–242, 1940.
- [26] K. Schmidt and S. Tordeux. High order transmission conditions for thin conductive sheets in magneto-quasistatics. *ESAIM: M2AN*, 45(6):1115–1140, Nov 2011.
- [27] T.B.A. Senior, J.L. Volakis, and Institution of Electrical Engineers. *Approximate Boundary Conditions in Electromagnetics*. IEE Electromagnetic Waves Series. Inst of Engineering & Technology, 1995.

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