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Complexity of greedy edge-colouring

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Abstract: The *Grundy index* of a graph $G = (V, E)$ is the greatest number of colours that the greedy edge-colouring algorithm can use on G . We prove that the problem of determining the Grundy index of a graph $G = (V, E)$ is NP-hard for general graphs. We also show that this problem is polynomial-time solvable for caterpillars. More specifically, we prove that the Grundy index of a caterpillar is $\Delta(G)$ or $\Delta(G) + 1$ and present a polynomial-time algorithm to determine it exactly.

Key-words: Edge colouring, greedy colouring, greedy algorithm, line graphs, caterpillars, NP-complete.

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Complexité de coloration gloutonne d'arêtes

Résumé : L'*indice Grundy* d'un graphe G est le plus grand nombre de couleurs que l'algorithme glouton de coloration d'arêtes peut utiliser pour G . Nous prouvons que le problème de déterminer l'indice Grundy d'un graphe est NP-dur en général. Nous montrons également que ce problème peut être résolu en temps polynômial pour les chenilles. Plus spécifiquement, nous prouvons que l'indice Grundy d'une chenille est $\Delta(G)$ ou $\Delta(G)+1$ et nous présentons un algorithme polynômial pour le déterminer exactement.

Mots-clés : Coloration d'arêtes, coloration gloutonne, algorithme glouton, graphe des lignes, chenilles, NP-complet.

1 Introduction

All the graphs considered in this paper are loopless.

A k -colouring of a graph $G = (V, E)$ is a surjective mapping $c : V \rightarrow \{1, 2, \dots, k\}$ such that $c(u) \neq c(v)$ for any edge $uv \in E$. The *chromatic number* is $\chi(G) = \min\{k \mid G \text{ admits a } k\text{-colouring}\}$. On the algorithmic point of view, finding the chromatic number of a graph is a hard problem. For all $k \geq 3$ it is NP-complete to decide if a graph admits a k -colouring (see [2]). Furthermore, it is NP-hard to approximate the chromatic number within $|V(G)|^{\varepsilon_0}$ for some positive constant ε_0 as shown by Lund and Yannakakis [5].

Hence lots of heuristics have been developed to colour a graph. The most basic and widespread because it works on-line is the greedy algorithm. Given a vertex ordering $\sigma = v_1 < \dots < v_n$ of $V(G)$, this algorithm colours the vertices in the order v_1, \dots, v_n , assigning to v_i the smallest positive integer not used on its lower-indexed neighbours. A colouring resulting of the greedy algorithm is called a *greedy colouring*. The *Grundy number* $\Gamma(G)$ is the largest k such that G has a greedy k -colouring. Easily, $\chi(G) \leq \Gamma(G) \leq \Delta(G) + 1$.

Zaker [6] showed that for any fixed k , one can decide in polynomial time if a given graph has Grundy number at most k . However determining the Grundy number of a graph is NP-hard [6], and given a graph G , it is even NP-complete to decide if $\Gamma(G) = \Delta(G) + 1$ as shown by Havet and Sampaio [3]. In addition, Asté et al. [1] showed that for any constant $c \geq 1$, it is NP-complete to decide if $\Gamma(G) \leq c \cdot \chi(G)$.

Graph colouring of many graph classes has also been studied. One of the classes is the one of line graphs. The *line graph* of a graph G , denoted $L(G)$, is the graph whose vertices are the edges of G , with $ef \in E(L(G))$ whenever e and f share an endvertex. Colouring line graphs corresponds to edge-colouring. A k -edge-colouring of a graph G is a surjective mapping $\phi : E(G) \rightarrow \{1, \dots, k\}$ such that if two edges e and f are adjacent (i.e share an endvertex), then $\phi(e) \neq \phi(f)$. A k -edge colouring may also be seen as a partition of the edge set of G into k disjoint *matchings* $M_i = \{e \mid \phi(e) = i\}$, $1 \leq i \leq k$. By edge-colouring we mean either the mapping ϕ or the partition.

The *chromatic index* $\chi'(G)$ of a graph G is the least k such that G admits a k -edge-colouring. It is easy to see that $\chi'(G) = \chi(L(G))$. Obviously, $\Delta(G) \leq \chi'(G)$ and Shannon's and Vizing's theorems state that $\chi'(G) \leq \max\{\frac{3}{2}\Delta(G); \Delta(G) + \mu(G)\}$, where $\mu(G)$ is the maximum number of edges between two vertices of G . Holyer [4] showed that for any $k \geq 3$, it is NP-complete to decide if a k -regular graph has chromatic index k .

One can apply the greedy algorithm to colour a line graph. It corresponds to the following greedy algorithm for edge-colouring. Given a graph $G = (V, E)$ and an edge ordering $\theta = e_1 < \dots < e_n$, assign to e_i the least positive integer that was not already assigned to lower-indexed edges adjacent to it. An edge-colouring obtained by this process is called a *greedy edge-colouring* and it has the following property:

For every $j < i$, every edge e in M_i is adjacent to an edge in M_j . (P)

Note that an edge-colouring satisfying (P) is a greedy edge-colouring relative to any edge ordering in which the edges of M_i precede those of M_j when $i < j$.

The *Grundy index* $\Gamma'(G)$ of a graph G is the largest number of colours of a greedy edge-colouring of G . Notice that $\Gamma'(G) = \Gamma(L(G))$. By definition, $\chi'(G) \leq \Gamma'(G)$. Furthermore, as an edge is incident to at most $2\Delta(G) - 2$ other edges ($\Delta(G) - 1$ at each endvertex), colouring the edges greedily uses at most $2\Delta(G) - 1$ colours. So $\Delta(G) \leq \Gamma'(G) \leq 2\Delta(G) - 1$. There are graphs for which the Grundy index equals the maximum degree: stars for example. On the opposite, for any Δ there is a tree with maximum degree Δ and Grundy index $2\Delta(G) - 1$.

In this paper, we study the complexity of finding the Grundy index of a graph. We prove that it is NP-hard by showing that the following problem is co-NP-complete.

MINIMUM GREEDY EDGE-COLOURINGInstance: A graph G .Question: $\Gamma'(G) = \Delta(G)$?

The proof is a reduction from 3-EDGE-COLOURABILITY OF CUBIC GRAPHS which was proved to be NP-complete by Holyer [4]. We recall that a *cubic graph* is a 3-regular graph. The reduction also proves that it is co-NP-complete to decide if $\Gamma'(G) = \chi'(G)$.

3-EDGE-COLOURABILITY OF CUBIC GRAPHSInstance: A cubic graph G .Question: Is G 3-edge colourable?

We then extend the result to a more general problem.

 f -GREEDY EDGE-COLOURINGInstance: A graph G .Question: $\Gamma'(G) \leq f(\Delta(G))$?

We show that for any function f such that $k \leq f(k) \leq 2k - 2$, the problem f -GREEDY EDGE-COLOURING is co-NP-Complete.

Since determining the Grundy index is NP-hard, a natural question to ask is for which class of graphs it can be done in polynomial time. In Section 3, we consider *caterpillars* which are trees such that the deletion of all leaves results in a path, called *backbone*. We show that if T is a caterpillar then $\Gamma'(T) \leq \Delta(T) + 1$ and then give a linear-time algorithm to compute the Grundy index of a caterpillar.

2 Co-NP-completeness results

The aim of this section is to prove that f -GREEDY EDGE-COLOURING is co-NP-complete for every function f such that $k \leq f(k) \leq 2k - 2$ for all k .

For sake of clarity, we first show that MINIMUM GREEDY EDGE-COLOURING is co-NP-Complete.

Theorem 1. MINIMUM GREEDY EDGE-COLOURING is co-NP-Complete.

MINIMUM GREEDY EDGE-COLOURING is clearly in co-NP, because a greedy edge-colouring of a graph G with at least $\Delta(G) + 1$ colours is a certificate that $\Gamma'(G) > \Delta(G)$.

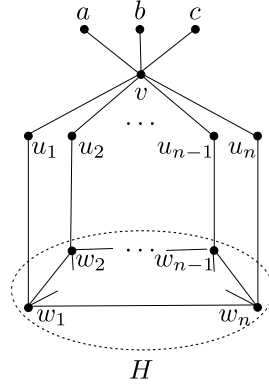
We now prove the co-NP-completeness by reduction from 3-EDGE-COLOURABILITY OF CUBIC GRAPHS.

Let H be a cubic graph on n vertices w_1, \dots, w_n . Let G be the graph defined by $V(G) = V(H) \cup \{u_1, \dots, u_n\} \cup \{v, a, b, c\}$ and $E(G) = E(H) \cup \{u_i w_i \mid 1 \leq i \leq n\} \cup \{v u_i \mid 1 \leq i \leq n\} \cup \{av, bv, cv\}$. See Figure 1.

In G , $d(v) = n + 3$, while the degree of all other vertices is at most 4. Thus, $\Delta(G) = d(v) = n + 3$ because $n \geq 4$ has H is cubic. Moreover, every edge of G is adjacent to at most $n + 3$ edges so $\Gamma'(G) \leq n + 4 = \Delta(G) + 1$. Hence the Grundy index of G is either $\Delta(G)$ or $\Delta(G) + 1$. The co-NP-completeness of MINIMUM GREEDY EDGE-COLOURING follows directly from the following claim.

Claim 1. $\chi'(H) = 3$ if and only if $\Gamma'(G) = \Delta(G) + 1$.

Proof. (\Rightarrow) Suppose that there exists a 3-edge-colouring ϕ of H . Let us extend ϕ into a greedy edge-colouring of G with $\Delta(G) + 1 = n + 4$ colours. Set $\phi(av) = 1$, $\phi(bv) = 2$, $\phi(cv) = 3$, and for all $1 \leq i \leq n$, $\phi(u_i w_i) = 4$ and $\phi(u_i v) = i + 4$. Notice that every vertex w_i is incident to an

Figure 1: Graph G obtained from a cubic graph H .

edge of H of each colour in $\{1, 2, 3\}$ since H is cubic. Then it is straightforward to check that ϕ is a greedy $(n + 4)$ -edge-colouring of G .

(\Leftarrow) Suppose that there is a greedy $(n + 4)$ -edge-colouring of G . Some edge is coloured $n + 4$. But such an edge has to be adjacent to at least $n + 3$ edges and thus to be one of the vu_i , say vu_n . The edge vu_n is adjacent to exactly $n + 3$ edges. So by Property (P), all edges adjacent to vu_n receive distinct colours in $\{1, \dots, n + 3\}$.

Let us first prove by induction on $1 \leq j \leq n$ that the edge e_j incident to vu_n labelled $n + 5 - j$ is one of the vu_i , the result holding for $j = 1$. Suppose now that $j \geq 2$. The edge e_j must have degree at least $n + 5 - j$ since it is adjacent to vu_n and one edge of each colour in $\{1, \dots, n + 4 - j\}$ by Property (P). Hence e_j must be incident to v since $u_n w_n$ is adjacent to four edges. Then e_j must have degree at least $n + 3$ since it is adjacent to the $j - 1$ edges e_l for $1 \leq l < j$ and one edge of each colour in $\{1, \dots, n + 4 - j\}$. Hence e_j is one of the vu_i .

Hence, without loss of generality, we may assume that $\phi(vu_i) = i + 4$ for all $1 \leq i \leq n$. The edge vu_i is adjacent to an edge coloured 4. This edge must be $u_i w_i$ since the edges av , bv and cv are adjacent to at most 2 edges coloured in $\{1, 2, 3\}$. Thus $\phi(u_i w_i) = 4$ for all $1 \leq i \leq n$.

Now every edge $u_i w_i$ is adjacent to three edges, one of each colour in $\{1, 2, 3\}$. Since $\phi(vu_i) \geq 5$, these three edges must be the three edges incident to w_i in H . Thus all the edges of H are coloured in $\{1, 2, 3\}$. Hence the restriction to ϕ to H is a 3-edge-colouring. \square

Remark 1. Observe that the graph G has chromatic index $\Delta(G)$. Indeed colour the edges adjacent to v with the colours $1, \dots, \Delta(G)$ and then extend greedily this colouring to the other edges. Since all the remaining edges are adjacent to at most 6 edges they will all get a colour less or equal to 7. Since $\Delta(G) \geq 7$, we obtain a $\Delta(G)$ -edge colouring. Hence the above reduction shows that it is co-NP-complete to decide if $\Gamma'(G) = \chi'(G)$.

Theorem 1 may be generalized as follows.

Theorem 2. *Let f be a function such that $k \leq f(k) \leq 2k - 2$ for all $k \in \mathbb{N}$. f -GREEDY EDGE-COLOURING is co-NP-Complete.*

Proof. f -GREEDY EDGE-COLOURING is clearly in co-NP, because a greedy edge-colouring of a graph G with more than $f(\Delta(G))$ colours is a certificate that $\Gamma'(G) > f(\Delta(G))$.

We now prove the co-NP-completeness by reduction from 3-EDGE-COLOURABILITY OF CUBIC GRAPHS.

Let H be a cubic graph on n vertices w_1, \dots, w_n and let G be the graph defined as in the proof of Theorem 1. Set $p = f(n + 3) - (n + 3)$. Then $0 \leq p \leq n + 1$. For $1 \leq i \leq p$, let T_i be the tree with vertex set $\{a_i, b_i, c_i, t_i\} \cup \{a_{i,j}, b_{i,j}, c_{i,j}, s_{i,j}, t_{i,j} \mid 1 \leq j \leq n - 1\}$ and edge set $\{a_i t_i, b_i t_i, c_i t_i\} \cup \bigcup_{j=1}^{n-1} \{a_{i,j} t_{i,j}, b_{i,j} t_{i,j}, c_{i,j} t_{i,j}, t_{i,j} s_{i,j}, s_{i,j} t_i\}$. Let G' be a graph obtained from the disjoint union of G and the T_i by adding the edge $u_n t_i$ for all $1 \leq i \leq p$. See Figure 2.

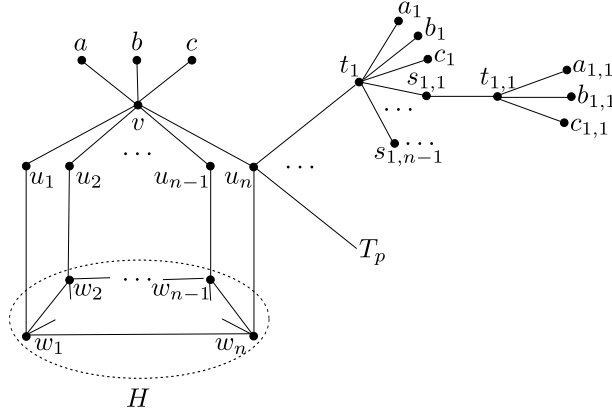


Figure 2: The graph G' obtained from a cubic graph H .

Observe that $\Delta(G') = n + 3$ and the vertices of degree $n + 3$ are v, t_1, \dots, t_p and u_n when $p = n + 1$. Moreover every edge is adjacent to at most $n + 3 + p$, so $\Gamma'(G) \leq n + 3 + p + 1 = f(\Delta(G')) + 1$. The co-NP-completeness of f -GREEDY EDGE-COLOURING follows directly from the following claim.

Claim 2. $\chi'(H) = 3$ if and only if $\Gamma'(G') = f(\Delta(G')) + 1$.

(\Rightarrow) Suppose that there exists a 3-edge-colouring ϕ of H . Let us extend ϕ into a greedy edge-colouring of G' with $f(\Delta(G')) + 1 = n + p + 4$ colours. We first extend it into a greedy $(n + 4)$ -colouring of G as we did in the proof of Theorem 1. In particular, we have $\phi(u_n w_n) = 4$ and $\phi(u_n v) = n + 4$. For all $1 \leq i \leq p$ and all $1 \leq j \leq n - 1$, we set $\phi(t_i a_i) = 1$, $\phi(t_i b_i) = 2$, $\phi(t_i c_i) = 3$, $\phi(t_{i,j} a_{i,j}) = 1$, $\phi(t_{i,j} b_{i,j}) = 2$, $\phi(t_{i,j} c_{i,j}) = 3$, $\phi(t_{i,j} s_{i,j}) = j + 3$, and $\phi(t_i u_n) = n + 4 + i$. Then it is straightforward to check that ϕ is a greedy $(n + p + 4)$ -edge-colouring of G' .

(\Leftarrow) Suppose that G' admits a greedy $(n + p + 4)$ -edge-colouring ϕ . For all $1 \leq i \leq p$, there is an edge e_i coloured $n + 4 + i$. This edge must have to be adjacent to at least $n + 3 + i$ edges by Property (P). So all the e_i must be in $F = \{v u_n\} \cup \{u_n t_i \mid 1 \leq i \leq p\}$. Now the edge e_p is adjacent to an edge e_0 coloured $n + 4$. This edge is adjacent to at least $n + 4$ -edges, one of each colour in $\{1, \dots, n + 3\}$ and e_p . Hence it also has to be in F . Since $|F|$, all the edges in F are coloured with distinct labels in $\{n + 4, \dots, n + p + 4\}$.

Now applying the same reasoning as in the proof of Theorem 1, we derive that the restriction to ϕ to H is a 3-edge-colouring. \square

3 Greedy edge-colouring of caterpillars

In this section, we show a polynomial-time algorithm solving GREEDY EDGE-COLOURING for caterpillars. Recall that caterpillars are trees such that the deletion of all leaves results in a path, called backbone.

We first show some properties of a greedy edge-colouring of a caterpillar where we can see that the Grundy index of a caterpillar T is at most $\Delta(T) + 1$. We then give a polynomial-time algorithm that computes its Grundy index.

3.1 Grundy index of a caterpillar

Lemma 3. *Let T be a caterpillar and v a vertex in its backbone. In every greedy edge-colouring of T , the colours $1, \dots, d(v) - 2$ appear on the edges incident to v .*

Proof. By the contrapositive. Let c be a greedy edge-colouring of T . Suppose that a colour $\alpha \in \{1, \dots, d(v) - 2\}$ is not assigned to any edge incident to v . Then, since all the edges incident to v have different colours, at least three colours strictly greater than $d(v) - 2$ appear on three edges incident to v . One of these colours, say β must appear on an edge e incident with a leaf. But e is uniquely adjacent to edges incident to v . So e is adjacent to no edge coloured α . Since $\alpha \leq d(v) - 2 < \beta$, the edge-colouring c is not greedy. \square

Lemma 4. *Let c be a greedy edge-colouring of a caterpillar T and v a vertex in the backbone of T . If two edges e_1 and e_2 incident to v receive colours greater than $d(v) - 1$, then e_1 and e_2 are two edges of the backbone and the edges incident to v and leaves are coloured $1, \dots, d(v) - 2$.*

Proof. Suppose by way of contradiction that one of these two edges, say e_1 , is incident to a leaf. Then e_1 is adjacent to $d(v) - 1$ other edges, and one of them, namely e_2 , is assigned a colour greater than $d(v) - 1$. Thus e_1 is adjacent to at most $d(v) - 2$ edges whose colour is less or equal to $d(v) - 1$. So, there is a colour α in $\{1, \dots, d(v) - 1\}$ such that no edge incident to e_1 is coloured α . This contradicts the fact that c is greedy. Hence e_1 and e_2 are edges of the backbone.

Now by Lemma 3, there must be edges incident to v of each colour in $\{1, \dots, d(v) - 2\}$. So the $d(v) - 2$ edges distinct from e_1 and e_2 , which are the edges linking v and leaves are coloured in $\{1, \dots, d(v) - 2\}$. \square

We use Lemma 4 to prove the following theorem.

Theorem 5. *If T is a caterpillar, then $\Gamma'(T) \leq \Delta(T) + 1$.*

Proof. Set $\Delta(T) = \Delta$. Suppose by way of contradiction that it is possible to greedily colour T with $\Delta + 2$ colours. Let e be an edge coloured $\Delta + 2$. It must be adjacent to at least $\Delta + 1$ edges, one of each colour $1, \dots, \Delta + 1$. Thus, the edge e is in the backbone. According to Lemma 4, the edges e_1 and e_2 adjacent to e with colours Δ and $\Delta + 1$ are in the backbone. Furthermore all the edges adjacent to e which are neither e_1 nor e_2 are coloured in $\{1, \dots, \Delta - 2\}$. Hence e is adjacent to no edge coloured $\Delta - 1$, a contradiction. \square

Theorem 5 is tight. For instance, consider the caterpillar T_k with backbone (t, u, v, w) for which the vertex t has degree $k - 1$, u has degree 2 and v and w degree k . An edge-colouring in which the $k - 2$ edges incident to t and a leaf are coloured with $1, \dots, k - 2$, the $k - 1$ edges incident to w and a leaf with $1, \dots, k - 1$, the $k - 2$ edges incident to v and a leaf with $1, \dots, k - 2$, the edge tu with $k - 1$, the edge vw with k and the edge uv with $k + 1$ is greedy.

3.2 Finding the Grundy index of a caterpillar

Let us now examine in more details when a caterpillar T has Grundy index $\Delta(T) + 1$.

Let T be a caterpillar with backbone $P = (v_1, v_2, \dots, v_n)$. The first edge of P is v_1v_2 . For any edge $e = v_iv_{i+1} \in P$, removing e from T gives two caterpillars T_e^- and T_e^+ , the first one containing v_i and the second one containing v_{i+1} . For convenience, the backbone of T_e^- is

$P_e^- = (v_i, v_{i-1}, \dots, v_1)$ and the backbone of T_e^+ is $P_e^+ = (v_{i+1}, \dots, v_n)$. Hence the first edge of T_e^- is (v_i, v_{i-1}) and the first edge of T_e^+ is (v_{i+1}, v_{i+2}) .

Lemma 6. *Let T be a caterpillar of maximum degree Δ with backbone $P = (v_1, \dots, v_n)$. Then $\Gamma'(T) = \Delta + 1$ if and only if there is an edge $e \in E(P) \setminus \{v_1v_2, v_{n-1}v_n\}$ such that*

1. *one endvertex of e has degree Δ , and*
2. *one of the two caterpillars T_e^- and T_e^+ has a greedy edge-colouring such that the first edge of its backbone is coloured Δ and the other has a greedy edge-colouring such that its first edge of its backbone is coloured $\Delta - 1$.*

Proof. Assume that T has a greedy $(\Delta + 1)$ -edge-colouring. Let e be an edge coloured $\Delta + 1$. By Lemma 4, e is in the backbone and incident to a vertex of degree Δ , proving (i). Moreover, the edge e is adjacent to an edge coloured Δ and another one labelled $\Delta - 1$. Again by Lemma 4 these two edges must also be in the backbone. In particular, e is not v_1v_2 nor $v_{n-1}v_n$ because these two edges are adjacent to a unique edge of the backbone. Moreover the greedy edge-colourings induced on T_e^- and T_e^+ clearly satisfy (ii).

Conversely, assume that there is an edge $e \in E(P) \setminus \{v_1v_2, v_{n-1}v_n\}$ satisfying (i) and (ii). Let ϕ^- and ϕ^+ be the greedy edge-colourings of T_e^- and T_e^+ respectively as in (ii). Let ϕ be the edge-colouring of T defined by $\phi(e) = \Delta + 1$, $\phi(f) = \phi^-(f)$ for all $f \in T_e^-$ and $\phi(f) = \phi^+(f)$ for all $f \in T_e^+$. We claim that ϕ is a greedy edge-colouring. Clearly, since ϕ^- and ϕ^+ are greedy, it suffices to prove that e is adjacent to an edge of every colour i in $\{1, \dots, \Delta\}$. Since ϕ^+ and ϕ^- satisfy (ii), then e is adjacent to an edge labelled Δ and an edge labelled $\Delta - 1$. Now, e is incident to a vertex v of degree Δ . This vertex is incident to e and an edge f in the backbone. The edge f is the first edge of a tree T_f in $\{T_e^+, T_e^-\}$. In the greedy edge-colouring of T_f , the edge f has a colour greater than $\Delta - 2$, so the $\Delta - 2$ edges incident to v which are not e nor f have all one colour in $1, \dots, \Delta - 2$. Hence e is adjacent to an edge of every colour in $\{1, \dots, \Delta\}$. \square

In view of this lemma, it is useful to decide when a caterpillar has a greedy edge-colouring with a prescribed colour on the first edge of its backbone.

Lemma 7. *Let T be a caterpillar with backbone P with first edge is $e = uv$. Then T has a greedy edge-colouring such that e is coloured k if and only if one of the following holds:*

1. $d(u) \geq k$ or $d(v) \geq k$;
2. $d(u) = k - 1$ and T_e^+ admits a greedy edge-colouring such that the first edge of P_e^+ is coloured $k - 1$.

Proof. Let $e = uv$ with u the first vertex of P . Assume first that T has a greedy edge-colouring such that e is coloured k and that e is incident to no vertex of degree k . Then the edges incident to u must be coloured by $1, \dots, d(u) - 1$ and the edges incident to v and a leaf are coloured by $1, \dots, d(v) - 2$. Hence the edge incident to e and coloured $k - 1$ must be the first edge of P_e^+ is coloured $k - 1$ by Property (P). So the edge incident to e and coloured $k - 2$ must be incident to u , and thus $d(u) - 1 \geq k - 2$, that is $d(u) \geq k - 1$.

Assume now that (i) holds. Let x be a vertex in $\{u, v\}$ with degree at least k . One can colour all the edges incident to x with $1, \dots, d(x)$ such that e is coloured k and then extend this edge-colouring greedily to obtain the desired greedy edge-colouring of T .

Finally assume that (ii) holds. Let ϕ be a greedy-edge colouring of T_e^+ such that the first edge of P_e^+ is coloured $k - 1$. One can extend it by assigning k to e , $1, \dots, k - 2$ to the $k - 2$ edges incident to u and leaves. It is routine to check that this is a greedy-edge colouring of T . \square

Lemma 7 translates easily in a linear-time recursive algorithm. Using this, and Lemma 6, one derive a linear-time algorithm for computing the Grundy index of a given caterpillar.

Theorem 8. *Determining the Grundy index of a caterpillar T can be done in $O(|V(T)|)$.*

Proof. Theorem 5 and Lemma 6 imply that Algorithm 1 return the Grundy index of T provided that we have a subroutine $\text{FirstEdge}(T, P, k)$ that returns 'yes' if a caterpillar T with backbone P admits a greedy-edge colouring such that the first edge of P is coloured k .

Algorithm 1: GrundyIndex(T)

Input: A caterpillar T .

Output: $\Gamma'(T)$.

- (1) Let $P = (v_1, v_2, \dots, v_n)$ be the backbone of T . Compute $d(v_i)$ for all $i \leq v_n$ and compute $\Delta = \Delta(T)$.
- (2) **for** $i = 2$ to $n - 2$ **do**
 - (3) $e := v_i v_{i+1}$;
 - (4) **if** $d(v_i) = \Delta$ or $d(v_{i+1}) = \Delta$ **then**
 - (5) **if** $\text{FirstEdge}(T_e^+, P_e^+, \Delta) = \text{TRUE}$ and $\text{FirstEdge}(T_e^-, P_e^-, \Delta - 1) = \text{TRUE}$ **then**
 - (6) **return** $\Delta + 1$;
 - (7) **if** $\text{FirstEdge}(T_e^+, P_e^+, \Delta - 1) = \text{TRUE}$ and $\text{FirstEdge}(T_e^-, P_e^-, \Delta) = \text{TRUE}$ **then**
 - (8) **return** $\Delta + 1$;
- (9) **Return** Δ ;

Such a subroutine FirstEdge may be obtained by Algorithm 2 according to Lemma 7.

Algorithm 2: FirstEdge

Input: A caterpillar T with backbone P and an integer k .

Output: *TRUE* if there is a greedy k -edge-colouring of T with first edge of P coloured k , and *FALSE* otherwise.

- (1) Let u be the first vertex of P and v its second. (So uv is the first edge.)
- (2) **If** $d(u) \geq k$ or $d(v) \geq k$ **then**
 - (3) **return** *TRUE*;
- (4) **If** $d(u) \geq k - 1$ **then**
 - (5) **return** $\text{FirstEdge}(T - u, P - u, k - 1)$;
- (6) **return** *FALSE*;

Let us now examine the complexity of Algorithm 1. Let us first observe that $\text{FirstEdge}(T, P, k)$ makes a constant number of operations before calling $\text{FirstEdge}(T - u, P - u, k - 1)$. Hence an easy induction show that it makes $O(k)$ operations in total.

Algorithm 1 first compute (line 1) the degrees of all the v_i , which can be done in time $O(|V(T)|)$ and then takes the maximum of all this values which can also be done in time $O(|V(T)|)$.

In a second phase (line 2 to 8), for each edge $e \in P$ which is incident to a vertex of degree Δ , Algorithm 1 makes at most four calls of FirstEdge with last parameter $\Delta - 1$ or Δ . Hence for each $e \in P$ it makes $O(\Delta)$ operations. Let S be the set of vertices of degree Δ , The number of edges of P incident to a vertex of degree Δ is at most $2|S|$. But every vertex in S is adjacent to at least $\Delta - 2$ leaves. Hence $|V(T)| \geq |S| + (\Delta - 2)|S|$, so $|S| \leq |V(T)|/(\Delta - 1)$. Hence, in this second phase, the algorithm makes at most $O\left(2 \times \frac{|V(T)|}{\Delta - 1} \Delta\right) = O(|V(T)|)$ operations.

Thus, in total, Algorithm 1 makes $O(|V(T)|)$ operations. □

References

- [1] M. Asté, F. Havet and C. Linhares-Sales. Grundy number and products of graphs. *Discrete Mathematics* 310(9):1482–1490, 2010.
- [2] M. R. Garey and D. S. Johnson, *Computers and intractability. A guide to the theory of NP-completeness*. A Series of Books in the Mathematical Sciences. W. H. Freeman and Co., San Francisco, Calif., 1979.
- [3] F. Havet and L. Sampaio. On the Grundy number of a graph. In Proceedings of the International Symposium on Parameterized and Exact Computation(IPEC), *Lecture Notes on Computer science* 6478, December 2010.
- [4] I. Holyer. The NP-completeness of edge-coloring. *SIAM J. Computing* 2:225–231, 1981.
- [5] C. Lund and M. Yannakakis. On the hardness of approximating minimization problems. In *Proc. 25th ACM Symposium on Theory of Computing*, pages 286-293, 1993.
- [6] M. Zaker. Results on the Grundy chromatic number of graphs. *Discrete Math.* 306(23):3166–3173, 2006.



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