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## Homological reconstruction and simplification in $\mathbb{R}^3$

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**Abstract:** We consider the problem of deciding whether the persistent homology group of a simplicial pair  $(K, L)$  can be realized as the homology  $H_*(X)$  of some complex  $X$  with  $L \subset X \subset K$ . We show that this problem is NP-complete even if  $K$  is embedded in  $\mathbb{R}^3$ .

As a consequence, we show that it is NP-hard to simplify level and sublevel sets of scalar functions on  $\mathbb{S}^3$  within a given tolerance constraint. This problem has relevance to the visualization of medical images by isosurfaces. We also show an implication to the theory of well groups of scalar functions: not every well group can be realized by some level set, and deciding whether a well group can be realized is NP-complete.

**Key-words:** Topological persistence, homological simplification, isosurface simplification, NP-completeness

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## Reconstruction et simplification homologique dans $\mathbb{R}^3$

**Résumé :** Nous considérons le problème de décider si le groupe d'homologie persistant de la paire simpliciale  $(K, L)$  peut être réalisée comme l'homologie  $H_*(X)$  d'un complexe  $X$  vérifiant  $L \subset X \subset K$ . Nous montrons que ce problème est NP-complet, même si  $K$  est plongé dans  $\mathbb{R}^3$ .

Nous en déduisons qu'il est NP-dur de simplifier les niveaux de fonctions scalaires sur  $\mathbb{S}^3$  avec une tolérance fixée. Ce problème est pertinent pour la visualisation des isosurfaces dans les images médicales. Nous montrons également une conséquence pour la théorie des "well groups" de fonctions scalaires: il n'est pas toujours possible de réaliser un *well group* comme un ensemble de niveau, et décider si une telle réalisation est possible est NP-dur.

**Mots-clés :** Persistance topologique, simplification homologique, simplification d'iso-surfaces, NP-complétude

## 1 Introduction

In this paper, we establish NP-completeness of a variety of related problems that ask for an object in  $\mathbb{R}^3$  with a prescribed homology.

In the most basic setting, we have a point cloud in  $\mathbb{R}^d$  that samples a shape and want to retrieve information on the sampled shape. There exists a whole spectrum of possibilities regarding the type of sought information. At the coarsest level, we can content ourselves with the homology groups which record the “holes” of a given dimension, hereafter referred to as *homological features* (connected components, cycles, cavities and so on). At a finer level, we may be interested in building an approximation of the shape, reflecting as accurately as possible both its geometry and topology. The standard way is to construct a *simplicial complex* using the data points as vertices, such as for instance the  $\alpha$ -complex, the Rips complex or the Čech complex [11, 10]. All three constructions have in common to depend upon a scale parameter  $\alpha$  and to get bigger as  $\alpha$  increases. In the ideal case, we expect the complex to have the right homology for some suitable value of  $\alpha$  [18, 7, 5, 1]. Unfortunately, depending on the sampling, it may happen that such a value of  $\alpha$  does not exist. Nonetheless, we might still be able to infer the true homology of the shape hidden in the noisy data using persistent homology [13, 8, 6]. Given two scale parameters  $\alpha_1$  and  $\alpha_2$ , the *persistent homology groups* record the homological features that persist from  $\alpha_1$  to  $\alpha_2$ . Under very weak hypotheses, we know that the persistent homology is precisely that of the sampled shape. The persistent homology can be computed efficiently (i.e., in polynomial time).

A natural question is then to ask for a complex that carries the persistent homology: given a complex  $K$  and a subcomplex  $L$ , can we find a subcomplex of  $K$  that contains  $L$  and whose homological features are precisely those common to  $L$  and  $K$ ? Our answer is that sometimes we cannot, and deciding whether we can is NP-complete. This answer was first given in the general case by Attali and Lieutier [2], who posed the restriction to complexes embedded in  $\mathbb{R}^3$  as an open problem. We resolve this problem by proving NP-completeness even for complexes embedded in  $\mathbb{R}^3$ . Note that in  $\mathbb{R}^2$  a solution always exists and can be computed in polynomial time. The above problem concentrates on building a complex whose homology matches perfectly the persistent homology of  $L$  into  $K$ : all the homological noise has been removed. We call such an object a *homological reconstruction*. However, when it does not exist, it is still relevant to look for a complex nested between  $L$  and  $K$  and whose homology is as close as possible to the persistent homology of  $L$  into  $K$ : as much noise as possible has been removed. We call such a complex a *homological simplification* and prove that finding one is also an NP-hard problem.

In the field of visualization and image analysis, another common setting consists in describing a shape through a continuous function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  instead of a point cloud in  $\mathbb{R}^d$ . For instance, a medical image may be a collection of density measurements over a grid of 3D points and is best modeled as a continuous map over a certain domain of  $\mathbb{R}^3$ . In the ideal case, the shape is a sublevel set of the function,  $f^{-1}(-\infty, t]$ . Unfortunately, noise can plague the data. As the parameter  $t$  increases, sublevel sets inflate and we can track the evolution of their homology. Features that appear and disappear quickly are considered topological noise, and we can consider the common features of two sublevel sets as those of a denoised sublevel set. The question now becomes: can we find another cleaner function, close enough to the original one, whose sublevel set has the denoised homology, i.e., a *sublevel set reconstruction*? Again, in  $\mathbb{R}^2$  such a denoised function always exists and can be found in polynomial time [3], while in  $\mathbb{R}^3$  the problem does not always have a solution. The corresponding optimization problem asks for a *sublevel set simplification*, i.e., a function close to the original one that minimizes the number of homological features of the sublevel set. Often, one is also interested in the homology of a level set,  $f^{-1}(t)$ . We show how it can be related to the (persistent) homology of sublevel sets, and consider the corresponding *level set reconstruction/simplification* problems.

Further in this direction, Edelsbrunner et al. introduced the well group [14, 4] as a denoised version of the homology group of a level set. Again, we can ask whether one can find a *realization* of the well group, i.e., a cleaner function whose level set has the same homology as the well group?

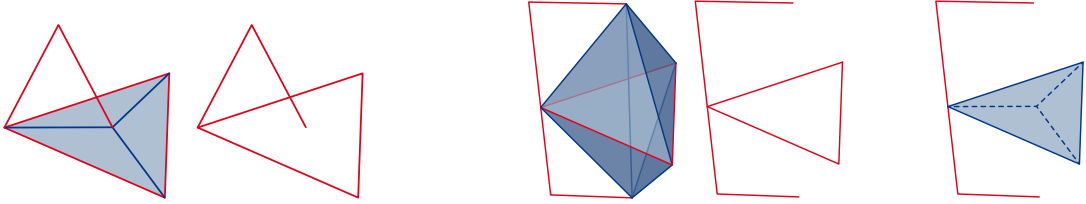


Figure 1: Left: Example of a simplicial pair  $(K, L)$  having no homological reconstruction. Right: Example of a simplicial pair having a homological reconstruction as a subspace, but not as a subcomplex.

We shall see in this paper that all of these related problems are NP-hard, as a consequence of the NP-completeness of the homological reconstruction problem.

## 1.1 Background and notations

We are only concerned with topological spaces that are triangulable by a finite simplicial complex, so simplicial and singular homology are isomorphic and we make no distinction between the two. In particular, we use the simplicial versions of the Excision and Mayer-Vietoris sequence theorems, which have less restrictive assumptions than their singular counterparts. If  $K$  is an abstract simplicial complex, we denote by  $|K|$  its geometric realization. Throughout this article, we consider homology with coefficients in an arbitrary field  $\mathbb{F}$ , so the homology groups are finite-dimensional  $\mathbb{F}$ -vector spaces and there is no torsion. Note that for simplicial complexes  $K$  embedded in  $\mathbb{R}^3$ , this is in fact not a restriction, since due to the absence of torsion in  $\mathbb{R}^3$  the Betti numbers are independent of the choice of coefficients [15, §3.3].

Given a topological space  $\mathcal{K}$ , we write  $H_*(\mathcal{K}) = \bigoplus_i H_i(\mathcal{K})$  for the direct sum of homology groups in all dimensions, and  $\beta(\mathcal{K}) = \sum_{i \geq 0} \beta_i(\mathcal{K})$  for the *total Betti number*. If  $(\mathcal{K}, \mathcal{L})$  is a pair of topological spaces  $\mathcal{L} \subset \mathcal{K}$ , the inclusion  $\mathcal{L} \hookrightarrow \mathcal{K}$  induces a homomorphism  $H_*(\mathcal{L}) \rightarrow H_*(\mathcal{K})$ , which is denoted by  $H_*(\mathcal{L} \hookrightarrow \mathcal{K})$ . The rank of this map is the *persistent Betti number* of the inclusion  $\mathcal{L} \hookrightarrow \mathcal{K}$  and is denoted by  $\beta(\mathcal{L} \hookrightarrow \mathcal{K}) = \text{rank } H_*(\mathcal{L} \hookrightarrow \mathcal{K})$ ; the image  $\text{im } H_*(\mathcal{L} \hookrightarrow \mathcal{K})$  is a *persistent homology group*.

A *piecewise linear function* on a topological space  $\mathcal{K}$  is a continuous function  $f : \mathcal{K} \rightarrow \mathbb{R}$  such that there exists a finite triangulation of  $\mathcal{K}$  on which  $f$  is simplexwise linear.

## 2 Homological reconstruction of simplicial pairs

**Lemma 1.** *Consider a triple of topological spaces  $\mathcal{L} \subset \mathcal{X} \subset \mathcal{K}$  with finite Betti numbers. Then*

$$\beta(\mathcal{X}) \geq \beta(\mathcal{L} \hookrightarrow \mathcal{K}).$$

This property suggests the following definition:

**Definition 1** (Homological reconstruction). *Consider a triple of topological spaces  $\mathcal{L} \subset \mathcal{X} \subset \mathcal{K}$  with finite Betti numbers. Then  $\mathcal{X}$  is called a homological reconstruction of the pair  $(\mathcal{K}, \mathcal{L})$  if  $\beta(\mathcal{X}) = \beta(\mathcal{L} \hookrightarrow \mathcal{K})$ .*

We will often omit “homological” since there is no ambiguity in this paper. An equivalent condition for  $\mathcal{X}$  being a reconstruction is that  $H_*(\mathcal{L} \hookrightarrow \mathcal{X})$  is surjective and  $H_*(\mathcal{X} \hookrightarrow \mathcal{K})$  is injective, as defined in [2]. Not every pair  $(\mathcal{K}, \mathcal{L})$  admits a reconstruction; a simple counterexample is shown in Fig. 1. The use of topological spaces in the definition (as opposed to simplicial complexes) is motivated by the following observation. Let  $(K, L)$  be a simplicial pair. Then there might be a reconstruction of  $(|K|, |L|)$ , but not as a subcomplex of  $K$ . An example is shown in Fig. 1.

## 2.1 NP-completeness of homological reconstruction

We now focus our attention on spaces that are geometric realizations of finite simplicial complexes embedded in  $\mathbb{R}^3$ .

**Theorem 1.** *The following problem is NP-complete: Given as input a simplicial pair  $(K, L)$  embedded in  $\mathbb{R}^3$ , decide whether there exists a reconstruction  $\mathcal{X}$  of  $(|K|, |L|)$ . The problem is also NP-complete if  $\mathcal{X}$  is required to be a subcomplex of  $K$ .*

*Proof.* We prove this by reduction from 3-SAT. Recall that a Boolean formula  $\Phi$  is in 3CNF if it is a conjunction of several clauses, each of which is a disjunction of three literals, a literal being either a variable or its negation. Given a 3CNF formula  $\Phi$ , we construct a simplicial pair  $(K_\Phi, L_\Phi)$  embedded in  $\mathbb{R}^3$  and prove that  $(|K_\Phi|, |L_\Phi|)$  has a reconstruction (as a subcomplex of  $K_\Phi$ ) if and only if  $\Phi$  has a satisfying assignment (see Lemmas 2 and 3 below).  $\square$

For this, we associate to the 3CNF formula  $\Phi$  a simplicial pair  $(K_\Phi, L_\Phi)$  in such a way that any reconstruction  $\mathcal{X}$  of  $(K_\Phi, L_\Phi)$  has a trivial homology, i.e.,

$$\beta_d(\mathcal{X}) = \beta_d(|L_\Phi| \hookrightarrow |K_\Phi|) = \begin{cases} 1 & \text{if } d = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This means that  $\mathcal{X}$  has a single connected component, no loops, and no cavities.  $\mathcal{X}$  has to fill all loops or cavities in  $L_\Phi$  and has to connect the different connected components of  $L_\Phi$  by adding to  $L_\Phi$  portions of  $K_\Phi$  without creating any new loops or cavities. In the figures, red represents the subcomplex  $L_\Phi$ .

**The variable gadget.** The variable gadget is a simplicial pair  $(V_i, W_i)$  as depicted in Fig. 2, left. The simplicial complex  $V_i$  contains 4 edges forming a cycle. The two red edges belong to  $W_i$ , but not the two blue ones. One of the blue edges will be called  $\text{True}_i$  and the other one will be called  $\text{False}_i$ . The key property of this construction is that any reconstruction of the pair  $(|V_i|, |W_i|)$  cannot contain both edges  $\text{True}_i$  and  $\text{False}_i$ , for otherwise they would create a 1-cycle together with the red ones. This property will allow us to match the presence of the edge  $\text{True}_i$  to a *true* assignment of the variable  $v_i$ .

**The clause gadget.** The clause gadget is a simplicial pair  $(C_j, D_j)$  as depicted in Fig. 2, right. The simplicial complex  $D_j$  contains a cycle ABCDE. The cycle is closed with two surfaces in  $C_j$  (thereafter

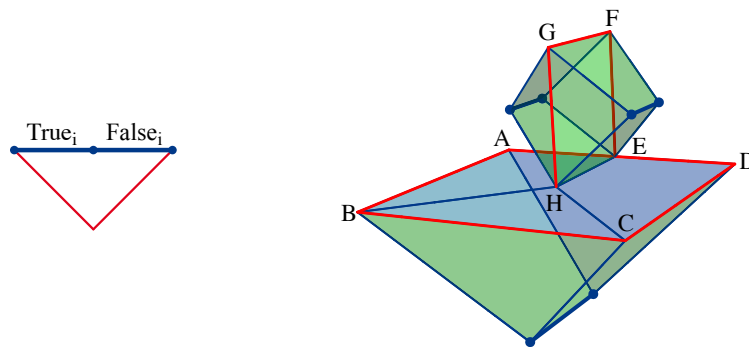


Figure 2: Variable (left) and clause (right) gadgets for the reduction of homological reconstruction to 3-SAT.



referred to as the lower hemisphere and the disk) thereby creating a cavity. Furthermore, the complex  $D_j$  contains an arc that ends inside the disk. Whenever we fill the cycle ABCDE with the disk, this connects the two endpoints of the arc, thus creating a new cycle, which we close twice in  $C_j$  by a left hemisphere and a right hemisphere. Consider one bold edge in the interior of each hemisphere, which is where the clause gadget will connect to the variable gadgets.

The key property of this clause gadget is that at least one of the 3 bold edges must be present in any reconstruction  $X$  of the pair  $(|C_j|, |D_j|)$ . Indeed, the cycle ABCDE in  $X$  must be filled up. If it is filled by the lower hemisphere, we are done. If it is filled by the disk, we have a new cycle EFGH in  $X$  which in turn must be killed either by the left or by the right hemisphere. In any case,  $X$  contains at least one of the hemispheres and thus one of the three bold edges.

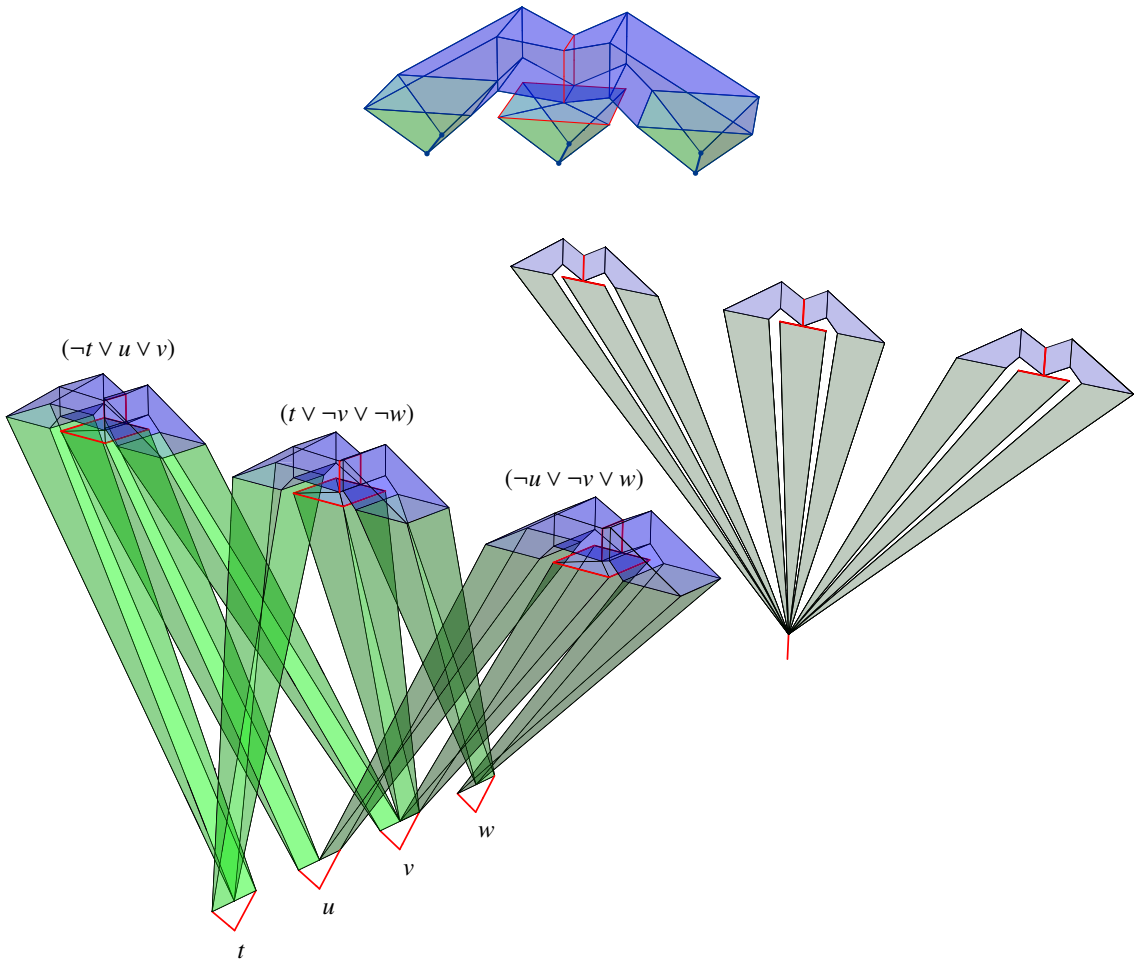


Figure 3: Embedding of the clause gadget with aligned hemispheres (top), and the simplicial pair  $(K_\Phi, L_\Phi)$  generated in the reduction from the 3SAT instance  $(\neg t \wedge u \wedge v) \vee (t \wedge \neg v \wedge \neg w) \vee (\neg u \wedge \neg v \wedge w)$  (bottom left), with projection orthogonal to one of the coordinate axes (bottom right).

**Correspondence with a formula.** Given a 3CNF formula  $\Phi$  with  $n$  clauses  $c_1, \dots, c_n$  and  $m$  variables  $v_1, \dots, v_m$ , we construct a 2-dimensional pair  $(K_\Phi, L_\Phi)$  as follows. For each variable  $v_i$  we take a copy  $(V_i, W_i)$  of the variable gadget. For each clause  $c_j$ , we take a copy  $(C_j, D_j)$  of the clause gadget; for each literal  $ev_i$  of  $c_j$ , we identify one of the bold edges of  $C_j$  to  $\text{False}_i$  if  $e$  is a negation and  $\text{True}_i$  otherwise. See Fig. 3 for an example.

First notice that  $\beta_2(L_\Phi) = 0$  (i.e.,  $L_\Phi$  has no cavities). Second, we can assume that  $\beta_0(K_\Phi) = 1$  (i.e.,  $K_\Phi$  is connected). Indeed, if  $K_\Phi$  is disconnected, it means that the 3SAT problem (and the reconstruction problem) can be decomposed into 2 independent subproblems with disjoint sets of variables, which can be solved separately. Last,  $\beta_1(L_\Phi \hookrightarrow K_\Phi) = 0$  (i.e., the cycles in  $L_\Phi$  are boundaries in  $K_\Phi$ ). Indeed, the only 1-cycles in  $L_\Phi$  are the 1-cycles  $ABCDE$  in each  $D_j$ , and they are filled in  $K_\Phi$ . This means that we are looking for a reconstruction with trivial homology.

**From a reconstruction to a satisfying assignment.** Let  $\mathcal{X}$  be a homological reconstruction of the pair  $(|K_\Phi|, |L_\Phi|)$ . We do not assume that  $\mathcal{X}$  is the geometric realization of some subcomplex of  $K_\Phi$ . Assign to each variable  $v_i$  the value *true* if the edge  $\text{True}_i$  is contained in  $\mathcal{X}$ , and *false* otherwise. For each clause gadget  $(C_j, D_j)$ , at least one bold edge is contained in  $\mathcal{X}$ . If this edge corresponds to a positive literal  $v_i$ , this means that  $\text{True}_i$  is in  $\mathcal{X}$ ,  $v_i$  is *true* and the clause is satisfied. If the edge corresponds to a negative literal  $\neg v_i$ , this implies that  $\text{False}_i$  is in  $\mathcal{X}$ .  $\text{True}_i$  is thus not in  $\mathcal{X}$ , so  $v_i$  was assigned *false* and the clause is satisfied. We have thus shown that the assignment of the variables makes the formula evaluate to *true*:

**Lemma 2.** *If  $(|K_\Phi|, |L_\Phi|)$  has a homological reconstruction, then  $\Phi$  has a satisfying assignment.*

**From a satisfying assignment to a reconstruction.** Given a satisfying assignment for the formula  $\Phi$ , we construct a subcomplex  $X \subset K_\Phi$  such that  $|X|$  is a reconstruction of  $(|K_\Phi|, |L_\Phi|)$ . We start with  $X = L_\Phi$  and add to  $X$  a selected set of simplices from  $K_\Phi$ . For each clause  $c_j$ , we pick one literal that evaluates to *true* and close the cycle in the clause gadget complex  $D_j$  correspondingly. If the literal corresponds to the bold edge of the lower hemisphere, we add this hemisphere. Otherwise, we add the disk and the hemisphere that contains the bold edge corresponding to the selected literal.

The only 2-cycles in  $K_\Phi$  are in the clause gadgets. As we did not create any 2-cycle in  $X$ , it follows that  $\beta_2(X) = 0$ . By construction, filling the clause gadgets never introduced both  $\text{True}_i$  and  $\text{False}_i$  in  $X$ . Indeed, it could only introduce  $\text{True}_i$  if  $v_i$  was assigned the value *true* and  $\text{False}_i$  if it was assigned the value *false*. Therefore, the 1-cycle in the variable gadgets do not appear in  $X$ . Also, for each clause gadget, we filled the  $ABCDE$  1-cycle, and whenever we created an extra  $EFGH$  1-cycle by adding the disk, we immediately filled it with the left or right hemisphere. Now we only need to check that the construction did not create any “non-local” 1-cycles. Since for each clause we have only used one of the literals which evaluate to *true*, the only contact a clause gadget in  $X$  has with the rest of  $X$  is through a single bold edge, and the clause gadget can be collapsed to that edge. After collapsing all clause gadgets, all that remains are disconnected variable gadgets with at most 3 edges each, and so  $\beta_1(X) = 0$ . We finally add to  $X$  just enough edges from  $K_\Phi$  so that it becomes connected, without creating any extra cycles in the process. This is possible since we assumed that  $K_\Phi$  is connected. Thus we have  $\beta_0(X) = 1$ . We conclude:

**Lemma 3.** *If  $\Phi$  has a satisfying assignment, then there is a subcomplex  $X$  of  $K_\Phi$  such that  $|X|$  is a homological reconstruction of  $(|K_\Phi|, |L_\Phi|)$ .*

**Embedding.** Later, we have to consider not only an embedding of  $K_\Phi$ , but also a triangulation of its complement. The following fact will be useful:

**Lemma 4.** *There is a triangulation of  $\mathbb{S}^3$  with size polynomial in the size of  $K_\Phi$  and having  $K_\Phi$  as a subcomplex.*

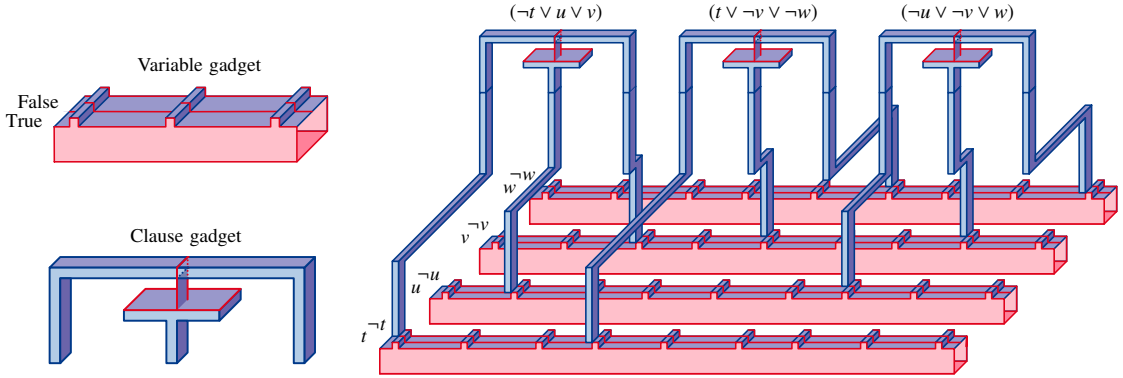


Figure 4: Example of 3-SAT reduction using a 3D grid embedding.

*Proof.* First, referring to Fig. 3, it is clear that  $K_\Phi$  can be embedded in  $\mathbb{R}^3$ . Indeed, we can align the clause gadgets and the variable gadgets along two lines parallel to the coordinate axes and make each clause gadget look like a small body with three long tentacles that connect to the variable gadgets.

We can subdivide the space by first projecting  $K_\Phi$  onto a plane orthogonal to the line carrying the variable gadgets. We get a polygonal region whose complement can easily be triangulated inside a bounding box without adding any new vertex or edge. Extending each triangle in the direction of the projection, we get a collection of tubes whose interiors do not intersect  $K_\Phi$ . Thus, the tubes can easily be triangulated while respecting  $K_\Phi$  to obtain a polynomial size triangulation of a bounding box of the construction, which can trivially be extended to a polynomial size triangulation of  $\mathbb{S}^3$ .  $\square$

We want to remark that a similar construction can be realized even if we restrict edges and faces of  $L$  and  $K$  to be edges and faces of a 3D grid (see Fig. 4). This means that a variant of Theorem 1 can also be shown for cubical complexes arising from 3D image data.

**Corollary 1.** *The homological simplification problem is NP-hard: Given as input a simplicial pair  $(K, L)$  embedded in  $\mathbb{R}^3$ , find a complex  $X$  minimizing  $\beta(X)$  subject to  $L \subset X \subset K$ .*

*Proof.* To determine if a reconstruction exists, we can first find a complex  $X$  minimizing  $\beta(X)$  subject to  $L \subset X \subset K$ . We then only need to check if its Betti number matches the lower bound  $\beta(L \hookrightarrow K)$ .  $\square$

### 3 Reconstruction and simplification of level and sublevel sets

Given a real-valued function  $f$ , we write  $F_t$  for the  $t$ -level set  $f^{-1}(t)$ ,  $F_{\leq t}$  for the (closed)  $t$ -sublevel set  $f^{-1}((-\infty, t])$ , and  $F_{< t}$  for the open  $t$ -sublevel set  $f^{-1}((-\infty, t))$ . In this paper we shall only consider real-valued piecewise linear functions. Note that level and sublevel sets of a simplexwise linear function on a simplicial complex  $K$  are not necessarily subcomplexes of  $K$ , but subcomplexes of an appropriate subdivision of  $K$ . Moreover, we have the following property:

**Proposition 1** ([16, 17]). *Let  $f$  be a simplexwise linear function on a simplicial complex  $K$ . Let  $K(t)$  be the induced subcomplex of  $K$  on  $\{v \in \text{vert } K : f(v) \leq t\}$ . Then  $K(t)$  is homotopy equivalent to the sublevel set  $F_{\leq t}$ . If  $t \neq f(v)$  for all  $v \in \text{vert } K$ , then  $K(t)$  is also homotopy equivalent to the open sublevel set  $F_{< t}$ .*

**Definition 2** (Sublevel set reconstruction). *Let  $f, g$  be piecewise linear functions and consider real parameters  $t$  and  $\delta$ . The function  $g$  is called a sublevel set  $(t, \delta)$ -reconstruction of  $f$  if  $\|g - f\|_\infty \leq \delta$  and  $G_{\leq t}$  is a reconstruction of the pair  $(F_{\leq t+\delta}, F_{\leq t-\delta})$ , i.e.,  $\beta(G_{\leq t}) = \beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta})$ .*

Note that  $F_{\leq t-\delta} \subseteq G_{\leq t} \subseteq F_{\leq t+\delta}$ , so that  $\beta(G_{\leq t}) \geq \beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta})$ . A sublevel set  $(t, \delta)$ -reconstruction is thus also a minimizer of  $\beta(G_{\leq t})$ .

### 3.1 NP-completeness of sublevel set reconstruction

**Theorem 2.** *The sublevel set reconstruction problem is NP-complete: Given as input a piecewise linear function  $f$  on a simplicial complex embedded in  $\mathbb{R}^3$  and parameters  $t$  and  $\delta$ , decide whether there exists a sublevel set  $(t, \delta)$ -reconstruction  $g$  of  $f$ . The problem is also NP-complete if  $g$  is required to be simplexwise linear.*

*Proof.* We proceed by reduction from the previous section. Let  $(K, L) = (K_\Phi, L_\Phi)$  be a simplicial pair defined by a 3-SAT instance  $\Phi$ . We construct an instance of the level set simplification problem by defining a simplexwise linear function  $f : |\text{sd } K| \rightarrow \mathbb{R}$  on the barycentric subdivision of  $K$  via its values on the vertices of  $\text{sd } K$  (recall that a vertex  $\sigma$  of  $\text{sd } K$  is a simplex of  $K$ ). Let

$$f : \sigma \mapsto \begin{cases} -2 & \text{if } \sigma \in L, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Note that for every function  $g$  with  $\|g - f\|_\infty \leq 1$ , the 0-sublevel set  $G_{\leq 0}$  contains  $L$  and is contained in  $K$ . We show that the following propositions are equivalent:

- (a)  $(|K|, |L|)$  has a reconstruction;
- (b) there exists a subcomplex  $X \subset K$  such that  $|X|$  is a reconstruction of  $(|K|, |L|)$ ;
- (c)  $f$  has a simplexwise linear sublevel set  $(0, 1)$ -reconstruction  $g$ .
- (d)  $f$  has a sublevel set  $(0, 1)$ -reconstruction  $g$ .

First, (a)  $\implies$  (b) follows from Lemmas 2 and 3. To show (b)  $\implies$  (c), we define a simplexwise linear function  $g$  on  $|\text{sd } K|$  by its values on the vertices of  $\text{sd } K$  (the simplices of  $K$ ):

$$g : \sigma \mapsto \begin{cases} -2 & \text{if } \sigma \in L, \\ -1 & \text{if } \sigma \in X \setminus L, \\ 1 & \text{if } \sigma \in K \setminus X. \end{cases} \quad (2)$$

We have  $\|g - f\|_\infty = 1$ . By Proposition 1, the sublevel set  $G_{\leq 0}$  is homotopy equivalent to  $|X|$  and hence is a reconstruction of the pair  $(|K|, |L|) \simeq (F_{\leq 1}, F_{\leq -1})$ . Finally, (c)  $\implies$  (d) is trivial and (d)  $\implies$  (a) follows directly with  $G_{\leq 0}$  as a reconstruction of  $(F_{\leq 1}, F_{\leq -1}) \simeq (|K|, |L|)$ .  $\square$

**Corollary 2.** *The sublevel set simplification problem is NP-hard: Given as input a simplexwise linear function  $f$  on a simplicial complex embedded in  $\mathbb{R}^3$  and parameters  $t$  and  $\delta$ , find a simplexwise linear function  $g$  minimizing  $\beta(G_{\leq t})$  subject to  $\|g - f\|_\infty \leq \delta$ .*

*Proof.* To determine if  $f$  has a sublevel set  $(t, \delta)$ -reconstruction, we can first find a simplexwise linear minimizer of  $\beta(G_{\leq t})$ . We then only need to check if  $\beta(G_{\leq t})$  matches the lower bound  $\beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta})$ , which can be done in time polynomial in the size of  $K$ .  $\square$

### 3.2 Betti numbers of level sets and sublevel sets

The Betti numbers of level and sublevel sets are related by the following formula:

**Lemma 5.** *Let  $f$  be a piecewise linear function on  $\mathbb{S}^n$ ,  $n > 1$ , and let  $t \in \text{int}(\text{im } f)$ . Then*

$$\beta_d(F_t) = \beta_d(F_{\leq t}) + \beta_{n-d-1}(F_{< t}).$$

*Proof.* By exactness of the Mayer-Vietoris sequence [19, §4.6] for  $\mathbb{S}^n$ ,  $F_{\leq t}$ , and  $F_{\geq t}$ , we have [12]

$$\beta_d(F_t) = \beta_d(F_{\leq t}) + \beta_d(F_{\geq t}) + \begin{cases} -1 & \text{if } d = 0, \\ 1 & \text{if } d = n - 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

By Alexander duality [15, §3.3] and the duality of homology and cohomology with field coefficients [15, §3.1], we have

$$\widetilde{H}_d(F_{\geq t}) \cong \widetilde{H}^{n-d-1}(F_{< t}) \cong \widetilde{H}_{n-d-1}(F_{< t}),$$

where  $\widetilde{H}_d$  denotes the  $d$ th reduced homology group. Recall that

$$\beta_d(X) = \text{rank}(\widetilde{H}_d(X)) + \begin{cases} 1 & \text{if } d = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We thus have

$$\beta_d(F_{\geq t}) = \beta_{n-d-1}(F_{< t}) + \begin{cases} 1 & \text{if } d = 0, \\ -1 & \text{if } d = n - 1, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

By combining Eqs. (3) and (4), we obtain the stated equality.  $\square$

For all  $f, g$  with  $\|g - f\|_\infty \leq \delta$  and  $t \pm \delta \in \text{int}(\text{im } f)$  we have  $t \in \text{int}(\text{im } g)$  and thus by Lemmas 1 and 5  $\beta(G_t) \geq \beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta}) + \beta(F_{< t-\delta} \hookrightarrow F_{< t+\delta})$ . This motivates the following definition:

**Definition 3** (Level set reconstruction). *Let  $f, g$  be piecewise linear functions on  $\mathbb{S}^n$  and consider real parameters  $t$  and  $\delta$  with  $t \pm \delta \in \text{int}(\text{im } f)$ . The function  $g$  is called a level set  $(t, \delta)$ -reconstruction of  $f$  if  $\|g - f\|_\infty \leq \delta$  and  $\beta(G_t) = \beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta}) + \beta(F_{< t-\delta} \hookrightarrow F_{< t+\delta})$ .*

A level set  $(t, \delta)$ -reconstruction is thus also a minimizer of  $\beta(G_{\leq t})$ .

### 3.3 NP-completeness of level set reconstruction

**Definition 4.** *Let  $f$  be a piecewise linear function. A homological regular value of  $f$  is a number  $t \in \mathbb{R}$  such that  $H_*(F_{< t} \hookrightarrow F_{\leq t})$  is an isomorphism.*

We remark that for piecewise linear functions, this definition is equivalent to the one given in [8].

**Lemma 6.** *Let  $f$  be a piecewise linear function on  $\mathbb{S}^n$ ,  $n > 1$ . If  $t \pm \delta \in \text{int}(\text{im } f)$  are regular values of  $f$  and  $g$  is a level set  $(t, \delta)$ -reconstruction of  $f$ , then  $t$  is a regular value of  $g$ .*

*Proof.* By hypothesis  $t \pm \delta$  are regular values of  $f$  and hence  $\beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta}) = \beta(F_{< t-\delta} \hookrightarrow F_{< t+\delta})$ . Since  $g$  is a level set reconstruction, by Lemma 5 we have

$$\beta(G_t) = \beta(G_{\leq t}) + \beta(G_{< t}) = \beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta}) + \beta(F_{< t-\delta} \hookrightarrow F_{< t+\delta}).$$

It follows that  $\beta(G_{< t}) = \beta(G_{\leq t}) = \beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta})$ . Since

$$\beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta}) \leq \beta(G_{< t} \hookrightarrow G_{\leq t}) \leq \beta(G_{\leq t}),$$

we conclude that  $H_*(G_{< t} \hookrightarrow G_{\leq t})$  is an isomorphism.  $\square$

**Lemma 7.** *Let  $f$  and  $g$  be piecewise linear functions on  $\mathbb{S}^n$ ,  $n > 1$ . Assume that  $t \pm \delta \in \text{int}(\text{im } f)$  are regular values of  $f$  and  $t \in \text{int}(\text{im } g)$  is a regular value of  $g$ . Then  $g$  is a sublevel set  $(t, \delta)$ -reconstruction of  $f$  if and only if  $g$  is a level set  $(t, \delta)$ -reconstruction of  $f$ .*

*Proof.* By hypothesis  $t \pm \delta$  are regular values of  $f$  and  $t$  is a regular value of  $g$ . Thus:

$$\beta(G_t) = 2\beta(G_{\leq t}),$$

$$\beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta}) + \beta(F_{< t-\delta} \hookrightarrow F_{< t+\delta}) = 2\beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta}).$$

It follows that  $g$  is a sublevel set  $(t, \delta)$ -reconstruction if and only if it is a level set  $(t, \delta)$ -reconstruction.  $\square$

**Theorem 3.** *The sublevel set simplification problem is NP-complete: Given as input a simplexwise linear function on a triangulation of  $\mathbb{S}^3$  and parameters  $t$  and  $\delta$ , decide whether there exists a level set  $(t, \delta)$ -reconstruction  $g$  of  $f$ . The problem is also NP-complete if  $g$  is required to be simplexwise linear.*

*Proof.* We reuse the same reduction as in Theorem 2. Since we need functions defined on the sphere, we triangulate the complement of  $K$  to obtain a triangulation  $S$  of the sphere with size polynomial in the size of  $K$  and  $K \subset S$  as in Lemma 4. We extend  $f$  from Eq. (1) to a simplexwise linear function  $\tilde{f}$  on  $|\text{sd } S|$ :

$$\tilde{f} : \sigma \mapsto \begin{cases} f(\sigma) & \text{if } \sigma \in K, \\ 2 & \text{otherwise.} \end{cases}$$

We then prove that propositions (a)–(d) in the proof of Theorem 2 and (e), (f) below are equivalent.

- (e)  $\tilde{f}$  has a simplexwise linear level set  $(0, 1)$ -reconstruction  $\tilde{g}$ .
- (f)  $\tilde{f}$  has a level set  $(0, 1)$ -reconstruction  $\tilde{g}$ .

We trivially have (e)  $\implies$  (f). Now we prove that (f)  $\implies$  (d). Proposition 1 implies that the values  $\pm 1$  are regular values of  $\tilde{f}$ . By Lemma 6, the value 0 is a regular value of  $\tilde{g}$ . Lemma 7 then proves that  $\tilde{g}$  is a sublevel set reconstruction of  $\tilde{f}$ . Now let  $g$  be the restriction of  $\tilde{g}$  to  $K$ . Since the sublevel sets  $F_{\leq t}$  and  $\tilde{F}_{\leq t}$  are homotopy equivalent for  $t \leq 1$ , and the sublevel sets  $G_{\leq t}$  and  $\tilde{G}_{\leq t}$  are homotopy equivalent for  $t \leq 0$ , it follows that  $g$  is a sublevel set reconstruction of  $f$ .

Next we prove that (b)  $\implies$  (e). We extend  $g$  from Eq. (2) to  $\tilde{g} : |\text{sd } S| \rightarrow \mathbb{R}$  as above for  $\tilde{f}$  and notice that 0 is a regular value of  $\tilde{g}$ , which by Lemma 7 implies that  $\tilde{g}$  is a level set reconstruction.  $\square$

**Corollary 3.** *The level set simplification problem is NP-hard: Given a piecewise linear function  $f$  on  $\mathbb{S}^3$  and parameters  $t$  and  $\delta$ , find a simplexwise linear function  $g$  minimizing  $\beta(G_t)$  subject to  $\|g - f\|_\infty \leq \delta$ .*

*Proof.* To determine if  $f$  has a level set  $(t, \delta)$ -reconstruction, we can first find a minimizer of  $\beta(G_t)$ . We then only need to check if  $\beta(G_t)$  matches the lower bound  $\beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta}) + \beta(F_{< t-\delta} \hookrightarrow F_{< t+\delta})$ .  $\square$

## 4 Realizations of well groups

We now discuss how the previous results relate to the concept of well groups, which were defined in [14] as a robust version of the homology group of a level set.

Let  $f : \mathcal{K} \rightarrow \mathbb{R}$  be a piecewise linear function. For  $\delta \geq 0$ , the  $\delta$ -well group of  $t \in \mathbb{R}$  is defined as

$$W_*(f, t, \delta) = \bigcap_{g: \|g-f\|_\infty \leq \delta} \text{im } H_*(G_t \hookrightarrow F_{[t-\delta, t+\delta]}),$$

where  $F_{[a,b]} = f^{-1}([a, b])$ . Intuitively, it captures the homology common to all perturbed level sets.

Clearly, the rank of the well group provides a lower bound on the Betti number of the  $t$ -level set of any  $g$  with  $\|g - f\|_\infty \leq \delta$ . We say that the well group is *realized* by such a function  $g$  if  $\text{im } H_*(G_t \hookrightarrow F_{[t-\delta, t+\delta]}) = W_*(f, t, \delta)$ , or equivalently, if  $\beta(G_t) = \text{rank } W_*(f, t, \delta)$ . As it turns out, the lower bound cannot always be achieved, and hence not every well group is realizable.

More generally, for an interval  $[a, b]$  we have

$$W_*(f, [a, b], \delta) = \bigcap_{g: \|g-f\|_\infty \leq \delta} \text{im } H_*(G_{[a,b]} \hookrightarrow F_{[a-\delta, b+\delta]})$$

In fact [4], the well group is already given by the intersection of just two persistent homology groups:

$$W_*(f, [a, b], \delta) = \text{im } H_*(F_{[a-\delta, b-\delta]} \hookrightarrow F_{[a-\delta, b+\delta]}) \cap \text{im } H_*(F_{[a+\delta, b+\delta]} \hookrightarrow F_{[a-\delta, b+\delta]}).$$

The following formula expresses the rank of the well group in terms of persistent Betti numbers.

**Theorem 4** ([4]). *Let  $f : \mathcal{K} \rightarrow \mathbb{R}$  be a piecewise linear function and let  $a \leq b$  and  $\delta \in \mathbb{R}$  be such that  $a \pm \delta, b \pm \delta$  are regular values of  $f$ . Then*

$$\begin{aligned} \text{rank } W_*(f, [a, b], \delta) &= \beta(F_{\leq b-\delta} \hookrightarrow F_{\leq b+\delta}) \\ &\quad - \beta((F_{\leq b-\delta}, \emptyset) \hookrightarrow (\mathcal{K}, F_{\geq a+\delta})) \\ &\quad + \beta((\mathcal{K}, F_{\geq a+\delta}) \hookrightarrow (\mathcal{K}, F_{\geq a-\delta})) \\ &\quad - \beta((F_{\leq b+\delta}, \emptyset) \hookrightarrow (\mathcal{K}, F_{\geq a-\delta})). \end{aligned}$$

#### 4.1 The well group of a level set

We now establish a close relation between a well group and the level set reconstruction problem:

**Theorem 5.** *Let  $f$  be a piecewise linear function on  $\mathbb{S}^n$  with  $t \pm \delta \in \text{int}(\text{im } f)$ . A piecewise linear function  $g$  realizes the well group  $W_*(f, t, \delta)$  if and only if it is a level set  $(t, \delta)$ -reconstruction of  $f$ .*

*Proof.* For every  $s \in \mathbb{R}$ , there is  $\epsilon > 0$  such that all values in  $[s - \epsilon, s)$  and in  $(s, s + \epsilon]$  are regular, and hence  $H_*(F_{\leq s-\epsilon}) \cong H_*(F_{< s})$  and  $H_*(F_{\leq s+\epsilon}) \cong H_*(F_{\leq s})$ . Moreover,  $W_*(f, s, \delta) \cong W_*(f, [s - \epsilon, s + \epsilon], \delta)$ . Choose  $\epsilon$  such that the above holds for  $s = t \pm \delta$ . Let  $a = t - \epsilon$  and  $b = t + \epsilon$ . Now  $a \pm \delta, b \pm \delta$  are regular values and we can apply Theorem 4.

The second and forth terms in the formula of Theorem 4 vanish. To see this, note that  $t \pm \delta \in \text{int}(\text{im } f)$  implies  $F_{\leq b \pm \delta} = F_{\leq t + \epsilon \pm \delta} \subsetneq \mathbb{S}^n$  for  $\epsilon$  small enough, and thus  $\beta_n(F_{\leq b \pm \delta}) = 0$ . Similarly,  $F_{\geq a \pm \delta} = F_{\geq t - \epsilon \pm \delta} \neq \emptyset$  and thus  $\beta_0(\mathbb{S}^n, F_{\geq a \pm \delta}) = 0$ . Moreover,  $\beta_d(\mathbb{S}^n) = 0$  for  $d \notin \{0, n\}$ . Since the induced homomorphism  $H_*((F_{\leq b \pm \delta}, \emptyset) \hookrightarrow (\mathbb{S}^n, F_{\geq a \pm \delta}))$  factors as  $H_*(F_{\leq b \pm \delta}) \rightarrow H_*(\mathbb{S}^n) \rightarrow H_*(\mathbb{S}^n, F_{\geq a \pm \delta})$ , we have

$$\beta((F_{\leq b \pm \delta}, \emptyset) \hookrightarrow (\mathbb{S}^n, F_{\geq a \pm \delta})) = 0.$$

Finally, by the duality theorem of extended persistence [9], we can rewrite the third term as

$$\beta_d((\mathbb{S}^n, F_{\geq a+\delta}) \hookrightarrow (\mathbb{S}^n, F_{\geq a-\delta})) = \beta_{n-d}(F_{\leq a-\delta} \hookrightarrow F_{\leq a+\delta}).$$

This yields

$$\text{rank } W_*(f, t, \delta) = \beta(F_{\leq t-\delta} \hookrightarrow F_{\leq t+\delta}) + \beta(F_{< t-\delta} \hookrightarrow F_{< t+\delta}). \quad \square$$

Together with Theorem 3, we have:

**Corollary 4.** *The well group realization problem is NP-complete: Given a piecewise linear function  $f : |K| \subseteq \mathbb{S}^3 \rightarrow \mathbb{R}$  and parameters  $t$  and  $\delta$ , decide whether the well group  $W_*(f, t, \delta)$  can be realized.*

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