# Trajectory Tracking Control of Multiconstraint Complementarity Lagrangian Systems

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***Abstract*—In this study, one considers the tracking control problem of a class of nonsmooth fully actuated Lagrangian systems subject to frictionless unilateral constraints. The task under consideration contains both free-motion and constraint-motion phases. A switching controller that guarantees an approximate tracking is designed. Particular attention is paid to transition (impacting and detachment) between different phases of motion. The exogenous signals that assure the stabilization on (take-off from) some constaints are explicitly defined. This paper extends previous works on the topic as it considers more than one constraint for $n$-degree-of-freedom systems. Numerical examples illustrate the main results.**

***Index Terms*—Complementarity problem, impacts, Lagrangian systems, nonsmooth systems, stability, tracking control.**

## I. Introduction

THE control of mechanical systems subject to unilateral constraints has been the object of many studies in the past fifteen years. Such systems, which consist of three main ingredients (see (1) below) are highly nonlinear nonsmooth dynamical systems. Theoretical aspects of their Lyapunov stability and the related stabilization issues have been studied in [9], [17], [19], [31]. The specific yet important task of the stabilization of impacting transition phases was analyzed and experimentally tested in [16], [28], [29], [32]–[34]. From the point of view of tracking control of complementarity Lagrangian systems along general constrained/unconstrained paths, such studies focus on a module of the overall control problem. The problem of robust impact detection with only position measurement received attention in [5]. One of the first works formulating the control of complete robotic tasks via unilateral constraints and complementarity conditions was presented in [15]. In that work the impacts were considered inelastic and the control problem was solved using a time optimal problem. The tracking control problem under consideration, involving systems that undergo transitions from free to constrained motions, and vice-versa, along an infinity of cycles, was formulated and studied in [8] for the 1-dof (degree-of-freedom) case and in [4] for the $n$-dof case. Both of these works consider systems with only one unilateral frictionless constraint. In this paper we not only consider the multiconstraint case but the results in Section VII relax some very hard to verify conditions imposed in [4] to assure the stability. Moreover the accurate design of the control law that guarantees the detachment from the constraints is formulated and incorporated in the stability analysis for the first time. Considering multiple constraints may be quite important in applications like virtual reality and haptic systems, where typical tasks involve manipulating objects modelled as rigid bodies [11] in complex environments with many unilateral constraints. We note that in the case of a single nonsmooth impact the exponential stability and bounded-input bounded state (BIBS) stability was studied in [24] using a state feedback control law. A study for a multiple degree-of-freedom linear systems subject to nonsmooth impacts can be found in [25]. That approach proposes a proportional-derivative control law in order to study BIBS stability via Lyapunov techniques. Other approaches for the tracking control of nonsmooth mechanical systems can be found in [12], [18], [23], and [27]. The analysis and control of systems subject to unilateral constraints also received attention in [3].

This paper focuses on the problem of tracking control of complementarity Lagrangian systems [26] subject to frictionless unilateral constraints whose dynamics may be expressed as

$$\begin{cases} M(X)\ddot{X} + C(X,\dot{X})\dot{X} + G(X) = U + \nabla F(X)\lambda_X \\ 0 \le \lambda_X \perp F(X) \ge 0, \\ \text{Collision rule} \end{cases} \tag{1}$$

where $X(t) \in \mathbb{R}_n$ is the vector of generalized coordinates, $M(X) = M^T(X) \in \mathbb{R}^{n\times n}$ is the positive definite inertia matrix, $F(X) \in \mathbb{R}^m$ represents the distance to the constraints, $C(X,\dot{X})$ is the matrix containing Coriolis and centripetal forces, $G(X)$ contains conservative forces, $\lambda_X \in \mathbb{R}^m$ is the vector of the Lagrangian multipliers associated to the constraints and $U \in \mathbb{R}^n$ is the vector of generalized torque inputs. For the sake of completeness we precise that $\nabla$ denotes the Euclidean gradient $\nabla F(X) = (\nabla F_1(X), \ldots, \nabla F_m(X)) \in \mathbb{R}^{n\times m}$ where $\nabla F_i(X) \in \mathbb{R}^n$ represents the vector of partial derivatives of $F_i(\cdot)$ w.r.t. the components of $X$. We assume that the functions $F_i(\cdot)$ are continuously differentiable and that $\nabla F_i(X) \neq 0$ for all $X$ with $F_i(X) = 0$. It is worth to precise here that for a given function $f(\cdot)$ its derivative w.r.t. the time $t$ will be denoted by $\dot{f}(\cdot)$. For any function $f(\cdot)$ the limit to the right at the instant $t$ will be denoted by $f(t^+)$ and the limit to the left will be denoted by $f(t^-)$. A simple jump of the function $f(\cdot)$ at the moment $t = t_\ell$ is denoted $\sigma_f(t_\ell) = f(t_\ell^+) - f(t_\ell^-)$. The Dirac measure at time $t$ is $\delta_t$.

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*Definition 1:* A Linear Complementarity Problem (LCP) is a system given by

$$\begin{cases} \lambda \geq 0 \\ A\lambda + b \geq 0 \\ \lambda^T(A\lambda + b) = 0 \end{cases} \tag{2}$$

which is compactly re-written as

$$0 \leq \lambda \perp A\lambda + b \geq 0. \tag{3}$$

Such an LCP has a unique solution for all $b$ if and only if $A$ is a P-matrix, i.e., all its principal minors are positive [10].

The admissible domain associated to the system (1) is the closed set $\Phi$ where the system can evolve and it is described as follows:

$$\Phi = \{X \in \mathbb{R}^n \mid F(X) \geq 0\} = \bigcap_{1 \leq i \leq m} \Phi_i$$

where $\Phi_i = \{X \in \mathbb{R}^n \mid F_i(X) \geq 0\}$ considering that a vector is non-negative if and only if all its components are non-negative. In order to have a well-posed problem with a physical meaning we consider that $\Phi$ contains at least a closed ball of positive radius.

*Definition 2:* A singularity of the boundary $\partial\Phi$ of $\Phi$ is the intersection of two or more codimension one surfaces $\Sigma_i = \{X \in \mathbb{R}^n \mid F_i(X) = 0\}$.

The presence of $\partial\Phi$ may induce some impacts that must be included in the dynamics of the system. It is obvious that $m > 1$ allows both simple impacts (when one constraint is involved) and multiple impacts (when singularities or surfaces of codimension larger than 1 are involved). Let us introduce the following notion of $p_\epsilon$-impact.

*Definition 3:* Let $\epsilon \geq 0$ be a fixed real number. We say that a $p_\epsilon$-impact occurs at the instant $t$ if

$$\|F_I(X(t))\| \leq \epsilon, \quad \prod_{i \in I} F_i(X(t)) = 0$$

where $I = \{i_1, i_2, \ldots, i_p\} \subset 1, \ldots, m$, $F_I(X) = (F_{i_1}(X), F_{i_2}(X), \ldots, F_{i_p}(X))^T$.

If $\epsilon = 0$ the $p$ surfaces $\Sigma_i$, $i \in I$ are stroked simultaneously and a $p-$ impact occurs. When $\epsilon > 0$ the system collides $\partial\Phi$ in a neighborhood of the intersection $\bigcap_{i \in I} \Sigma_i$.

*Definition 4:* [20], [26] The tangent cone to $\Phi = \{X \in \mathbb{R}^n \mid F_i(X) \geq 0,\ \forall i = 1, \ldots, m\}$ at $X \in \mathbb{R}^n$ is defined as:

$$T_\Phi(X) = \{z \in \mathbb{R}^n \mid z^T \nabla F_i(X) \geq 0,\ \forall i = J(X)\}$$

where $J(X) \triangleq \{i \in \{1, \ldots, m\} \mid F_i(X) \leq 0\}$ is the index set of active constraints. When $X \in \Phi \setminus \partial\Phi$ one has $J(X) = \emptyset$ and $T_\Phi(X) = \mathbb{R}^n$.

The normal cone to $\Phi$ at $X$ is defined as the polar cone to $T_\Phi(X)$

$$N_\Phi(X) = \{y \in \mathbb{R}^n \mid \forall z \in T_\Phi(X), y^T z \leq 0\}.$$

The collision (or restitution) rule in (1), is a relation between the post-impact velocity and the pre-impact velocity. Among the various models of collision rules, Moreau's rule is an extension of Newton's law which is energetically consistent [14], [21] and is numerically tractable [1]. For these reasons throughout this paper the collision rule will be defined by Moreau's relation [26]

$$\dot{X}(t_\ell^+) = -e\dot{X}(t_\ell^-) + \operatorname*{arg\,min}_{z \in T_\Phi(X(t_\ell))} \frac{(1+e)}{2} \times [z - \dot{X}(t_\ell^-)]^T M(X(t_\ell))[z - \dot{X}(t_\ell^-)] \tag{4}$$

where $\dot{X}(t_\ell^+)$ is the post-impact velocity, $\dot{X}(t_\ell^-)$ is the pre-impact velocity and $e \in [0, 1]$ is the restitution coefficient. Denoting by $T$ the kinetic energy of the system, we can compute the kinetic energy loss at the impact time $t_\ell$ as [21]

$$T_L(t_\ell) = -\frac{1-e}{2(1+e)} \Big[ [\dot{X}(t_\ell^+) - \dot{X}(t_\ell^-)]^T M(X(t_\ell)) \cdot [\dot{X}(t_\ell^+) - \dot{X}(t_\ell^-)] \Big] \leq 0. \tag{5}$$

The collision rule can be rewritten considering the vector of generalized velocities as an element of the tangent space to the configuration space of the system, equipped with the kinetic energy metric. Doing so (see [6, section 6.2]), the discontinuous velocity components $\dot{X}_{norm}$ and the continuous ones $\dot{X}_{tang}$ are identified. Precisely, $\begin{pmatrix} \dot{X}_{norm} \\ \dot{X}_{tang} \end{pmatrix} = \mathcal{M}\dot{X}$, $\mathcal{M} = \begin{pmatrix} \mathbf{n} \\ \mathbf{t} \end{pmatrix} M(X)$ where $\mathbf{n}$ represents the unitary normal vectors $\mathbf{n}_i = M^{-1}(X(t_\ell))\nabla F_i(X(t_\ell)) / \sqrt{\nabla F_i(X(t_\ell))^T M^{-1}(X(t_\ell)) \nabla F_i(X(t_\ell))}$ $i \in J(X(t_\ell))$ (see Definition 4) and $\mathbf{t}$ represents mutually independent unitary vectors $\mathbf{t}_i$ such that $\mathbf{t}_i^T M(X(t_\ell))\mathbf{n}_j = 0$, $\forall i, j$. In this case the collision rule (4) at the impact time $t_\ell$ becomes the generalized Newton's rule $\begin{pmatrix} \dot{X}_{norm}(t_\ell^+) \\ \dot{X}_{tang}(t_\ell^+) \end{pmatrix} = -\eta \begin{pmatrix} \dot{X}_{norm}(t_\ell^-) \\ \dot{X}_{tang}(t_\ell^-) \end{pmatrix}$, $\eta = diag(e_1, \ldots, e_m, 0, \ldots, 0)$ where $e_i$ is the restitution coefficient w.r.t. the surface $\Sigma_i$. For the sake of simplicity we consider in this paper that all the restitution coefficients are equal, i.e., $e_1 = \cdots = e_m \triangleq e$.

*Remark 1:*

1) If $X \in \Sigma_1 \bigcap \Sigma_2$ and the angle $\angle(\Sigma_1, \Sigma_2) \leq \pi$ then in the neighborhood of $X$ one has $\Phi \simeq T_\Phi(X)$.
2) Let $m = 1$. The case $e = 0$ is called a plastic impact and the case $e = 1$ is called an elastic impact. In the first case the normal component of the velocity becomes zero and in the second case the normal component of the velocity changes only its direction and preserves its magnitude. As we can easily see from (5) in the second case there is no loss of kinetic energy at the impact moment.
3) One recalls that we deal with frictionless unilateral constraints. Some frictional contact laws that fit within the nonsmooth mechanic framework (1) can be found in [17].

The structure of the paper is as follows: in Section II one presents some basic concepts and prerequisites necessary for the further developments. Section III is devoted to the controller design. In Section IV one defines the desired (or "exogenous") trajectories entering the dynamics. The desired contact-force that must occur on the phases where the motion is constrained, is explicitly defined in Section V. Section VI focuses on the strategy for take-off at the end of the constraint phases. The main results related to the closed-loop stability analysis are presented in Section VII. One example and concluding remarks end the paper.

The following standard notations will be adopted: $||\cdot||$ is the Euclidean norm, $b_p \in \mathbb{R}^p$ and $b_{n-p} \in \mathbb{R}^{n-p}$ are the vectors formed with the first $p$ and the last $n-p$ components of $b \in \mathbb{R}^n$, respectively. $N_\Phi(X_p = 0)$ is the normal cone $N_\Phi(X)$ to $\Phi$ at $X$ [20], [26] when $X$ satisfies $X_p = 0$, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ represent the smallest and the largest eigenvalues of a symmetric matrix, respectively.

## II. Basic Concepts

### A. Typical Task

Following [8] the time axis can be split into intervals $\Omega_k$ and $I_k$ corresponding to specific phases of motion. Due to the singularities of $\partial\Phi$ that must be taken into account, the constrained-motion phases need to be decomposed in sub-phases where some specific constraints are active. Between two such sub-phases a transition phase occurs only when the number of active constraints increases. This means that a typical task can be represented in the time domain as

$$t \in \mathbb{R}^+ = \Omega_0^\emptyset \cup \left[ \bigcup_{k\geq 1} \left( I_k \cup \left( \bigcup_{i=1}^{m_k} \Omega_k^{J_{k,i}} \right) \right) \right]$$
$$J_{k,m_k} \subset J_{k+1,1}, J_{k,m_k} \subset J_{k,m_k-1} \subset \cdots \subset J_{k,1} \qquad (6)$$

where the superscript $J_{k,i} = \{j \in \{1,\ldots,m\} \mid F_j(X) = 0\}$ represents the set of active constraints during the corresponding motion phase, and $I_k$ denotes the transient between two $\Omega_k$ phases when the number of active constraints increases. Without loss of generality we suppose that the system is initialized in the interior of $\Phi$ at a free-motion phase. The impacts during $I_k$ involve $p = |J_{k,1}|$ constraints ($p_\epsilon$-impacts). Furthermore we shall prove that the first impact of $I_k$ is a $p_\epsilon$-impact with $\epsilon$ bounded by a parameter chosen by the designer. When the number of active constraints decreases there is no impact, thus no other transition phases are needed. We note that $J_{k,i} = \emptyset$ corresponds to free-motion $(F(X) > 0)$.

Since the tracking control problem involves no difficulty during the $\Omega_k$ phases, *the central issue is the study of the passages between them (the design of transition phases $I_k$ and detachment conditions), and the stability of the trajectories evolving along* (6) (i.e., an infinity of cycles). It is noteworthy that the passage $\Omega_k^{J_{k,i}} \longrightarrow \Omega_k^{J_{k,i+1}}$ consists of detachments from some constraints. In Section VI we consider that $p$ constraints are active and we give the conditions to smoothly take-off from $r$ of them. It is clear that once we know how to do that, we can manage all the transitions mentioned above. Throughout the paper, the sequence $I_k \cup \left(\bigcup_{i=1}^{m_k} \Omega_k^{J_{k,i}}\right)$ will be referred to as the cycle $k$ of the system's evolution. For robustness reasons during transition phases $I_k$ we impose a closed-loop dynamics (containing impacts) that mimics somehow the bouncing-ball dynamics (see e.g., [6]).

### B. Exogenous Signals Entering the Dynamics

In this section we introduce the trajectories playing a role in the dynamics and the design of the controller. Some instants that will be used further are also defined.

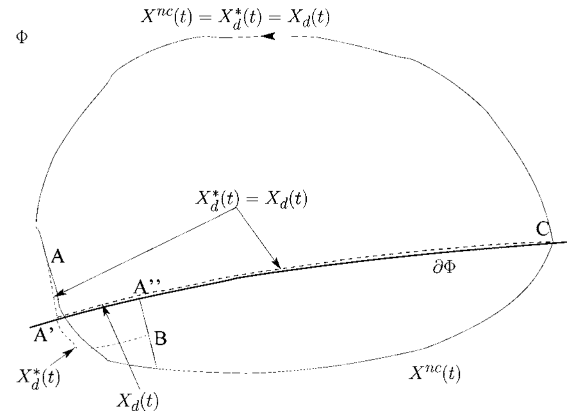


Fig. 1. Closed-loop desired trajectory and control signals.

- $X^{nc}(\cdot)$ denotes the desired trajectory of the unconstrained system (i.e., the trajectory that the system should track if there were no constraints). We suppose that $F(X^{nc}(t)) < 0$ for some $t$, otherwise the problem reduces to the tracking control of a system with no constraints.
- $X_d^*(\cdot)$ denotes the signal entering the control input and playing the role of the desired trajectory during some parts of the motion.
- $X_d(\cdot)$ represents the signal entering the Lyapunov function. This signal is set on the boundary $\partial\Phi$ after the first impact of each cycle.

The signals $X_d^*(\cdot)$ and $X_d(\cdot)$ coincide on the $\Omega_k$ phases while $X^{nc}(\cdot)$ is used to define everywhere $X_d^*(\cdot)$ and $X_d(\cdot)$. These three functions coincide only on the $\Omega_k^\emptyset$ phases.

Throughout the paper we consider $I_k = [\tau_0^k, t_f^k]$, where $\tau_0^k$ is chosen by the designer as the start of the transition phase $I_k$ and $t_f^k$ is the end of $I_k$. We note that all superscripts $(\cdot)^k$ will refer to the cycle $k$ of the system motion. We also use the following notations:

- $t_0^k$ is the first impact during the cycle $k$;
- $t_\infty^k$ is the accumulation point of the sequence $\{t_\ell^k\}_{\ell\geq 0}$ of the impact instants during the cycle $k$ ($t_f^k \geq t_\infty^k$);
- $\tau_1^k$ will be explicitly defined later and represents the instant when the signal $X_d^*(\cdot)$ reaches a given value chosen by the designer in order to impose a closed-loop dynamics with impacts during the transition phases;
- $t_d^{k,i}$ is the desired detachment instant at the end of the phase $\Omega_k^{J_{k,i}}$.

It is noteworthy that $t_0^k$, $t_\infty^k$ are state-dependent whereas $\tau_0^k$, $\tau_1^k$ and $t_d^{k,i}$ are exogenous and imposed by the designer. To better understand the definition of these specific instants, in Fig. 1 we represent the exogenous signals $X^{nc}(\cdot)$, $X_d(\cdot)$, $X_d^*(\cdot)$ during a sequence $\Omega_{k-1}^{J_{k-1}} \cup I_k \cup \Omega_k^{J_{k,1}} \cup \Omega_k^{J_{k,2}}$ when the motion is simplified as follows:

- during the transition phase we take into account only the constraints that must be activated $J_{k,1} \setminus J_{k-1,m_{k-1}}$.
- at the end of the phase $\Omega_k^{J_{k,1}}$ we take into account only the constraints that must be deactivated $J_{k,1} \setminus J_{k,2}$.

The points $A$, $A'$, $A''$ and $C$ in Fig. 1 correspond to the moments $\tau_0^k$, $t_0^k$, $t_f^k$ and $t_d^{k,1}$ respectively. We have seen that the choice of $\tau_0^k$ plays an important role in the stability criterion given by Proposition 1. On the other hand in Fig. 1 we see that starting from $A$ the desired trajectory $X_d(\cdot) = X_d^*(\cdot)$ is deformed compared to $X^{nc}(\cdot)$. In order to reduce this deformation, the time $\tau_0^k$ and implicitly the point $A$ must be close to $\partial\Phi$ (see also Fig. 4). Further details on the choice of $\tau_0^k$ will be given later. Taking into account just the constraints $J_{k,1} \setminus J_{k,2}$ we can identify $t_d^{k,1}$ with the moment when $X_d(\cdot)$ and $X^{nc}(\cdot)$ rejoin at $C$. See also Fig. 4 for an illustration on an example.

### *C. Stability Analysis Criteria*

The system (1) is a complex nonsmooth and nonlinear dynamical system which involves continuous and discrete time phases. A stability framework for this type of systems has been proposed in [8] and extended in [4]. This is an extension of the Lyapunov second method adapted to closed-loop mechanical systems with unilateral constraints. Since we use this criterion in the following tracking control strategy it is worth to clarify the framework and to introduce some definitions.

Let us define $\Omega$ as the complement in $\mathbb{R}^+$ of $I = \bigcup_{k\geq 1} I_k$ and assume that the Lebesgue measure of $\Omega$, denoted $\lambda[\Omega]$, equals infinity. Consider $x(\cdot)$ the state of the closed-loop system in (1) with some feedback controller $U(X,\ \dot{X},\ X_d^*,\ \dot{X}_d^*,\ \ddot{X}_d^*)$.

*Definition 5 (Weakly Stable System [4]):* The closed loop system is called weakly stable if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $\|x(0)\| \leq \delta(\epsilon) \Rightarrow \|x(t)\| \leq \epsilon$ for all $t \geq 0$, $t \in \Omega$. The system is asymptotically weakly stable if it is weakly stable and $\lim_{t\in\Omega, t\longrightarrow\infty} x(t) = 0$. Finally, the practical weak stability holds if there exists $0 < R < +\infty$ and $t^* < +\infty$ such that $\|x(t)\| < R$ for all $t > t^*$, $t \in \Omega$.

Weak stability is therefore Lyapunov stability without looking at the transition phases. Consider $V(\cdot)$ such that there exists strictly increasing functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying the conditions: $\alpha(0) = 0$, $\beta(0) = 0$ and $\alpha(\|x\|) \leq V(x,t) \leq \beta(\|x\|)$.

*Definition 6:* A transition phase $I_k$ is called finite if it involves a sequence of impact times $(t_\ell^k)_{0\leq\ell\leq N}$, $N \leq \infty$ with the accumulation point $t_N^k < \infty$ (for the sake of simplicity we shall denote the accumulation point by $t_\infty^k$ even if $N < \infty$).

In the sequel all the transition phases are supposed finite, which implies that $e < 1$ (in [2] it is shown that $e = 1$ implies that $t_\infty^k = +\infty$). The following criterion is inspired from [4], and will be used to study the stability of the system (1).

*Proposition 1 (Weak Stability):* Assume that the task admits the representation (6) and that

a) $\lambda[I_k] < +\infty,\ \forall k \in \mathbb{N}$;
b) outside the impact accumulation phases $[t_0^k,\ t_\infty^k]$ one has $\dot{V}(x(t),t) \leq -\gamma V(x(t),t)$ for some constant $\gamma > 0$;
c) $\sum_{\ell\geq 0}\left[V(t_{\ell+1}^{k-}) - V(t_\ell^{k+})\right] \leq K_1 V^{p_1}(\tau_0^k),\ \forall k \in \mathbb{N}$ for some $p_1 \geq 0, K_1 \geq 0$;
d) the system is initialized on $\Omega_0$ such that $V(\tau_0^1) \leq 1$;
e) $\sum_{\ell\geq 0}\sigma_V(t_\ell^k) \leq K_2 V^{p_2}(\tau_0^k) + \xi,\ \forall k \in \mathbb{N}$ for some $p_2 \geq 0, K_2 \geq 0$ and $\xi \geq 0$.

If $p = \min\{p_1,p_2\} < 1$ then $V(\tau_0^k) \leq \delta(\gamma,\xi),\ \forall k \geq 2$, where $\delta(\gamma,\xi)$ is a function that can be made arbitrarily small by increasing the value of $\gamma$. The system is practically weakly stable with $R = \alpha^{-1}(\delta(\gamma,\xi))$.

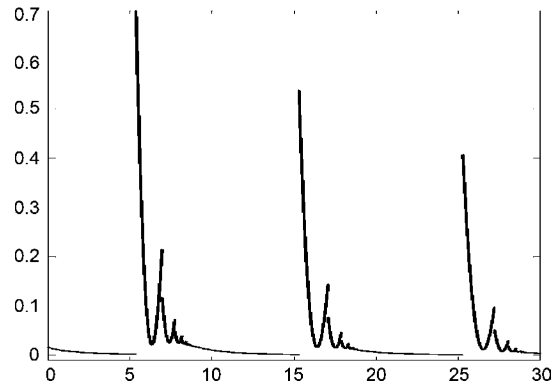


Fig. 2. Typical evolution of the Lyapunov function of weakly stable systems.

*Proof:* From assumption (b) one has

$$V(t_f^k) \leq V(t_\infty^k)e^{-\gamma(t_f^k - t_\infty^k)}.$$

It is clear that condition (c) combined with (e) leads to

$$V(t_\infty^k) \leq V(\tau_0^k) + K_1 V^{p_1}(\tau_0^k) + K_2 V^{p_2}(\tau_0^k) + \xi.$$

Considering $p < 1$, the assumption (d) guarantees that $\max\{V(\tau_0^k), V^{p_1}(\tau_0^k), V^{p_2}(\tau_0^k)\} \leq V^p(\tau_0^k) \leq 1$ and we get

$$V(t_f^k) \leq e^{-\gamma(t_f^k - t_\infty^k)}\left[1 + K_1 + K_2 + \xi\right] \triangleq \delta(\gamma,\xi).$$

From assumption (b) one has $V(\tau_0^{k+1}) \leq V(t_f^k)$ and thus $V(\tau_0^k) \leq \delta(\gamma,\xi),\ \forall k \geq 2$. The term $\delta(\gamma,\xi)$ can be made as small as desired increasing either $\gamma$ or the length of the interval $[t_\infty^k,\ t_f^k]$. The proof is completed by the relation $\alpha(\|x\|) \leq V(x,t),\ \forall x, t$. ■

*Remark 2:* Since the Lyapunov function is exponentially decreasing on the $\Omega_k$ phases, assumption (d) in Proposition 1 means that the system is initialized on $\Omega_0$ sufficiently far from the moment when the trajectory $X^{nc}(\cdot)$ leaves the admissible domain.

Precisely, the weak stability is characterized by an "almost decreasing" Lyapunov function $V(x(\cdot),\cdot)$ as illustrated in Fig. 2.

*Remark 3:* It is worth to point out the local character of the stability criterion proposed by Proposition 1. This character is firstly given by condition (d) of the statement and secondly by the synchronization constraints of the control law and the motion phase of the system (see (6) and (9) below).

The practical stability is very useful because attaining asymptotic stability is not an easy task for the unilaterally constrained systems described by (1) especially when $n \geq 2$ and $M(q)$ is not a diagonal matrix (i.e., there are inertial couplings, which is the general case).

### *D. Dissipativity and Tracking Versus Stabilization*

Let us make a parenthesis to highlight the major discrepancy between the trajectory tracking problem and the stabilization problem. To this aim let us first recall that the dynamics in (1) and (4) can be equivalently rewritten as the *measure differential inclusion* [1], [6], [21], [26]

$$\begin{cases} \Big\{ -M(q(t))dv - [C(X(t),v(t^+))v(t^+) - G(X(t)) \\ \qquad +U(t)]dt \Big\} \in N_{T_\Phi(X(t))}(w(t)) \\ w(t) = \frac{v(t^+)+ev(t^-)}{1+e} \end{cases} \tag{7}$$

where $dv$ is the differential measure associated with the velocity $v(\cdot)$ that is a right-continuous function of local bounded variation, $v(\cdot)$ is equal almost everywhere to $\dot{X}(\cdot)$, $X(\cdot)$ is absolutely continuous and $X(t) - X(0) = \int_{[0,t]} v(s)ds$. The right-hand-side is the normal cone to the tangent cone, where the cones are as in Definition 4. As shown in [9] and [7, section 3.9.4, 6.8.2, 7.2.4], a crucial property for stabilization is that the *cone complementarity problem*

$$N_{T_\Phi(X(t))}(w(t)) \ni \xi \perp w(t) \in T_\Phi(X(t)) \tag{8}$$

defines a maximal monotone mapping $\xi \mapsto w$, because the two cones $T_\Phi(\cdot)$ and $N_\Phi(\cdot)$ are polar cones [20], and $N_{T_\Phi(X(t))}(\cdot) \subseteq N_\Phi(\cdot)$. This maximal monotonicity property allows one to use dissipativity arguments in an absolute stability framework to derive a Lyapunov function. Let us consider now the tracking control problem. The new (closed-loop) state vector is $(\tilde{X}, \dot{\tilde{X}})$. Therefore the right-hand-side of the closed-loop measure differential inclusion becomes the normal cone $N_{T_\Phi(\tilde{X}(t)+X_d(t))}(\tilde{w}(t) + w_d(t))$, with $w_d(t) = v_d(t^+) + ev_d(t^-)/1 + e$. The sets $T_{\Phi_t}(\cdot) \triangleq T_\Phi(\cdot + X_d(t))$ and $N_{T^t_{\Phi_t}}(\cdot) \triangleq N_{T_{\Phi_t}}(\cdot + w_d(t))$ are now time-varying, and the monotonicity property is generally lost. This explains why the trajectory tracking problem is much more intricate than its stabilization counterpart.

## III. Controller Design

In order to overcome some difficulties that can appear in the controller definition, the dynamical (1) will be expressed in the generalized coordinates introduced by McClamroch & Wang [22], which allow one to split the generalized coordinates into a "normal" and a "tangential" parts, with a suitable diffeomorphic transformation $q = Q(X)$. We suppose that the generalized coordinates transformation holds globally in $\Phi$, which may obviously not be the case in general. However, the study of the singularities that might be generated by the coordinates transformation is out of the scope of this paper. Let us consider $D = [I \vdots O] \in \mathbb{R}^{m\times n}$, $I \in \mathbb{R}^{m\times m}$ the identity matrix. The new coordinates will be $q = Q(X) \in \mathbb{R}^n$, with $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, $q_1 = \begin{bmatrix} q_1^1 \\ \vdots \\ q_1^m \end{bmatrix}$ such that $\Phi = \{q \mid Dq \geq 0\}$[1]. The tangent cone $T_\Phi(q_1 = 0) = \{v \mid Dv \geq 0\}$ is the space of admissible velocities on the boundary of $\Phi$.

[1]In particular it is implicitly assumed that the functions $F_i(\cdot)$ in (1) are linearly independent.

The controller used here consists of different low-level control laws for each phase of the system. More precisely, the switching controller can be expressed as

$$T(q)U = \begin{cases} U_{nc} & \text{for } t \in \Omega_k^\emptyset \\ U_t^J & \text{for } t \in I_k \\ U_c^J & \text{for } t \in \Omega_k^J \end{cases} \tag{9}$$

where $T(q) = \begin{pmatrix} T_1(q) \\ T_2(q) \end{pmatrix} \in \mathbb{R}^{n\times n}$ is full-rank under some basic assumptions (see [22]). The dynamics becomes

$$\begin{cases} M_{11}(q)\ddot{q}_1 + M_{12}(q)\ddot{q}_2 + C_1(q,\dot{q})\dot{q} + g_1(q) = T_1(q)U + \lambda \\ M_{21}(q)\ddot{q}_1 + M_{22}(q)\ddot{q}_2 + C_2(q,\dot{q})\dot{q} + g_2(q) = T_2(q)U \\ q_1^i \geq 0, q_1^i\lambda_i = 0,\ \lambda_i \geq 0,\ 1 \leq i \leq m \\ \text{Collision rule} \end{cases} \tag{10}$$

where the set of complementary relations can be written more compactly as $0 \leq \lambda \perp Dq \geq 0$.

In the sequel $U_{nc}$ coincides with the fixed-parameter controller proposed in [30] and the closed-loop stability analysis of the system is based on Proposition 1. First, let us introduce some notations: $\tilde{q} = q - q_d$, $\bar{q} = q - q_d^*$, $s = \dot{\tilde{q}} + \gamma_2\tilde{q}$, $\bar{s} = \dot{\bar{q}} + \gamma_2\bar{q}$, $\dot{q}_e = \dot{q}_d - \gamma_2\tilde{q}$ where $\gamma_2 > 0$ is a scalar gain and $q_d(\cdot)$, $q_d^*(\cdot)$ represent the desired trajectories defined in the previous section. Using the above notations the controller is given by $T(q)U \triangleq$

$$\begin{cases} U_{nc} = M(q)\ddot{q}_e + C(q,\dot{q})\dot{q}_e + G(q) - \gamma_1 s \\ U_t^J = U_{nc}, t \leq t_0^k \\ U_t^J = M(q)\ddot{q}_e + C(q,\dot{q})\dot{q}_e + G(q) - \gamma_1\bar{s},\ t > t_0^k \\ U_c^J = U_{nc} - P_d + K_f(P_q - P_d) \end{cases} \tag{11}$$

where $\gamma_1 > 0$ is a scalar gain, $K_f > 0$, $P_q = D^T\lambda$ and $P_d = D^T\lambda_d$ is the desired contact force during persistently constrained motion. It is clear that during $\Omega_k^J$ not all the constraints are active and, therefore, some components of $\lambda$ and $\lambda_d$ are zero.

In order to prove the stability of the closed-loop system (9) –(11) we will use the following positive definite function:

$$V(t,s,\tilde{q}) = \frac{1}{2}s^T M(q)s + \gamma_1\gamma_2\tilde{q}^T\tilde{q}. \tag{12}$$

## IV. Tracking Control Framework

### A. Design of the Desired Trajectories

In this paper we treat the tracking control problem for the closed-loop dynamical system (9)–(11) with the complete desired path a priori taking into account the complementarity conditions and the impacts. In order to define the desired trajectory let us consider the motion of a virtual and unconstrained particle perfectly following a trajectory (represented by $X^{nc}(\cdot)$ on Fig. 1) with an orbit that leaves the admissible domain for a given period. Therefore, the orbit of the virtual particle can be split into two parts, one of them belonging to the admissible domain (inner part) and the other one outside the admissible domain (outer part). In the sequel we deal with the tracking control strategy when the desired trajectory is constructed such that:

(i) when no activated constraints, it coincides with the trajectory of the virtual particle (the desired path and velocity are defined by the path and velocity of the virtual particle, respectively);

(ii) when $p \leq m$ constraints are active, its orbit coincides with the projection of the outer part of the virtual particle's orbit on the surface of codimension $p$ defined by the activated constraints ($X_d$ between $A''$ and $C$ in Fig. 1);

(iii) the desired detachment moment and the moment when the virtual particle re-enters the admissible domain (with respect to $p \leq m$ constraints) are synchronized.

Therefore we have not only to track a desired path but also to impose a desired velocity allowing the motion synchronization on the admissible domain. The main difficulties here consist of:

- stabilizing the system on $\partial\Phi$ during the transition phases $I_k$ and incorporating the velocity jumps in the overall stability analysis;
- deactivating some constraints at the moment when the unconstrained trajectory re-enters the admissible domain with respect to them;
- maintaining a persistently constrained motion between the moment when the system was stabilized on $\partial\Phi$ and the detachment moment.

*Remark 4:* The problem can be relaxed considering that we want to track only a desired path like $X^{nc}(\cdot)$ (without imposing a desired velocity on the inner part of the desired trajectory and/or a given period to complete a cycle). In this way the synchronization problem (iii) disappears and we can assume there exists a twice differentiable desired trajectory outside $[t_0^k,\ t_f^k]$ that assures the detachment when the force control is dropped. In other words, in this case we have to design the desired trajectory only during $I_k$ phases.

### B. Design of $q_d^*(\cdot)$ and $q_d(\cdot)$ on the Phases $i_k$

During the transition phases the system must be stabilized on $\partial\Phi$. Obviously, this does not mean that all the constraints have to be activated (i.e., $q_1^i(t) = 0,\ \forall i = 1, \ldots, m$). Let us consider that only the first $p$ constraints (eventually reordering the coordinates) define the border of $\Phi$ where the system must be stabilized. The following methodology will be used to define $q_d^*(\cdot)$:

1) During a small period $\delta > 0$ chosen by the designer the desired velocity becomes zero preserving the twice differentiability of $q_d^*(\cdot)$. For instance we can use the following definition:

$$q_d^*(t) = q^{nc}\left(\tau_0^k + \frac{(t-\tau_0^k-\delta)^2(t-\tau_0^k)}{\delta^2}\right),\ t \in [\tau_0^k, \tau_0^k+\delta]$$

which means $q_d^*(\tau_0^k + \delta) = q_d^*(\tau_0^k) = q^{nc}(\tau_0^k)$, $\dot{q}_d^*(\tau_0^k + \delta) = 0$ and $\dot{q}_d^*(\tau_0^k) = \dot{q}^{nc}(\tau_0^k)$.

2) The last $n-p$ components of $q_d^*(\cdot)$ are frozen

$$(q_d^*)_{n-p}(t) = q_{n-p}^{nc}(\tau_0^k), \quad t \in (\tau_0^k + \delta, t_f^k]. \tag{13}$$

3) For a fixed $\varphi > 0$ the moment $\tau_1^k$ is chosen by the designer as the instant when the limit conditions $\left(q_d^i\right)^*(\tau_1^k) = -\nu V^{1/3}(\tau_0^k), \left(\dot{q}_d^i\right)^*(\tau_1^k) = 0,\ \forall i = 1, \ldots, p$, hold. On $[\tau_0^k + \delta, \tau_1^k)$ we define $q_d^*(\cdot)$ as a twice differentiable decreasing signal. Precisely, denoting $t' = t-(\tau_0^k+\delta)/\tau_1^k-(\tau_0^k+\delta)$, the components $\left(q_d^i\right)^*(\cdot), i = 1, \ldots, p$ of $(q_d^*)_p(\cdot)$ are defined as $\left(q_d^i\right)^*(t) =$

$$\begin{cases} a_3^i(t')^3 + a_2^i(t')^2 + a_0^i, & t \in [\tau_0^k + \delta, \min\{\tau_1^k; t_0^k\}] \\ -\varphi V^{1/3}(\tau_0^k), & t \in (\min\{\tau_1^k; t_0^k\}, t_f^k] \end{cases} \tag{14}$$

where $V(\cdot)$ is defined in (12) and the coefficients are

$$\begin{aligned} a_3^i &= 2[\left(q^i\right)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k)] \\ a_2^i &= -3[\left(q^i\right)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k)] \\ a_0^i &= \left(q^i\right)^{nc}(\tau_0^k). \end{aligned} \tag{15}$$

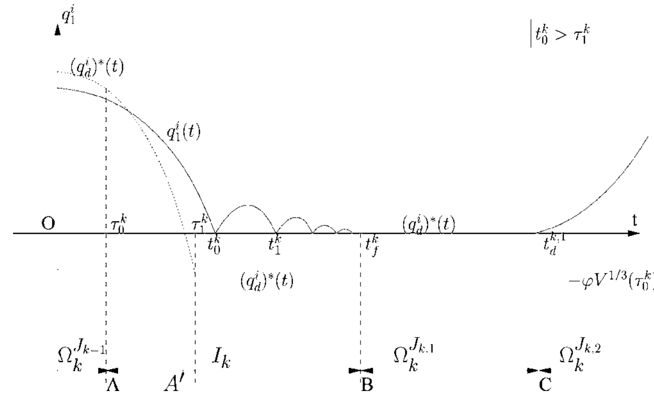


Fig. 3. Design of $q_{1d}^*$ on the transition phases $I_k$.

The rationale behind the choice of $q_d^*(\cdot)$ is on one hand to assure a robust stabilization on $\partial\Phi$, mimicking the bouncing-ball dynamics; on the other hand to enable one to compute suitable upper-bounds that will help using Proposition 1 (hence $V^{1/3}(\cdot)$ terms in (14) with $V(\cdot)$ in (12)).

*Remark 5:* Two different situations are possible. The first one is given by $t_0^k > \tau_1^k$ (see Fig. 3) and we shall prove that in this situation all the jumps of the Lyapunov function in (12) are negative. The second situation was pointed out in [4] and is given by $t_0^k < \tau_1^k$. In this situation the first jump at $t_0^k$ in the Lyapunov function may be positive. It is noteworthy that $q_d^*(\cdot)$ will then have a jump at the time $t_0^k$ since $(q_d^i)^*(t_0^{k+}) = -\varphi V^{1/3}(\tau_0^k), \forall i = 1, \ldots, p$ (see (14)).

In order to limit the deformation of the desired trajectory $q_d^*(\cdot)$ w.r.t. the unconstrained trajectory $q^{nc}(\cdot)$ during the $I_k$ phases (see Figs. 1 and 3), we impose in the sequel

$$\|q_p^{nc}(\tau_0^k)\| \leq \psi \tag{16}$$

where $\psi > 0$ is chosen by the designer. It is obvious that a smaller $\psi$ leads to smaller deformation of the desired trajectory and to smaller deformation of the real trajectory as we shall see in Section VIII. Nevertheless, due to the tracking error, $\psi$ cannot be chosen zero. We also note that $\|q_p^{nc}(\tau_0^k)\| \leq \psi$ is a practical way to choose $\tau_0^k$.

During the transition phases $I_k$ we define $(q_d)_{n-p}(t) = (q_d^*)_{n-p}(t)$. Assuming a finite accumulation period, the impact process can be considered in some way equivalent to a plastic impact. Therefore, $(q_d)_p(\cdot)$ and $(\dot{q}_d)_p(\cdot)$ are set to zero on the right of $t_0^k$.

## V. Design of the Desired Contact Force During Constraint Phases

For the sake of simplicity we consider the case of the constraint phase $\Omega_k^J, J \neq \emptyset$ with $J = \{1, \ldots, p\}$. Obviously a sufficiently large desired contact force $P_d$ assures a constrained movement on $\Omega_k^J$. Nevertheless at the end of the $\Omega_k^J$ phases a detachment from some surfaces $\Sigma_i$ has to take place. It is clear that a take-off implies not only a well-defined desired trajectory but also some small values of the corresponding contact force components. On the other hand, if the components of the desired contact force decrease too much a detachment can take place before the end of the $\Omega_k^J$ phases which can generate other impacts. Therefore we need a lower bound of the desired force which assures the contact during the $\Omega_k^J$ phases.

Dropping the time argument, the dynamics of the system on $\Omega_k^J$ can be written as

$$\begin{cases} M(q)\ddot{q} + F(q,\dot{q}) = U_c + D_p^T \lambda_p \\ 0 \leq q_p \perp \lambda_p \geq 0 \end{cases} \tag{17}$$

where $F(q,\dot{q}) = C(q,\dot{q})\dot{q} + G(q)$ and $D_p = [I_p \vdots O] \in \mathbb{R}^{p\times n}$. On $\Omega_k^J$ the system is permanently constrained which implies $q_p(\cdot) = 0$ and $\dot{q}_p(\cdot) = 0$. In order to assure these conditions it is sufficient to have $\lambda_p > 0$.

In the following let us denote $M^{-1}(q) = \begin{pmatrix} [M^{-1}(q)]_{p,p} & [M^{-1}(q)]_{p,n-p} \\ [M^{-1}(q)]_{n-p,p} & [M^{-1}(q)]_{n-p,n-p} \end{pmatrix}$ and $C(q,\dot{q}) = \begin{pmatrix} C(q,\dot{q})_{p,p} & C(q,\dot{q})_{p,n-p} \\ C(q,\dot{q})_{n-p,p} & C(q,\dot{q})_{n-p,n-p} \end{pmatrix}$ where the meaning of each component is obvious.

*Proposition 2:* On $\Omega_k^J$ the constraint motion of the closed-loop system (17), (9), (11) is assured if the desired contact force is defined by

$$\begin{aligned} \Big(\lambda_d\Big)_p \triangleq \beta - \frac{\bar{M}_{p,p}(q)}{1+K_f}\Big([M^{-1}(q)]_{p,p}C_{p,n-p}(q,\dot{q}) \\ + [M^{-1}(q)]_{p,n-p}C_{n-p,n-p}(q,\dot{q}) + \gamma_1[M^{-1}(q)]_{p,n-p}t\Big)s_{n-p} \end{aligned} \tag{18}$$

where $\bar{M}_{p,p}(q) = \big([M^{-1}(q)]_{p,p}\big)^{-1} = \big(D_p M^{-1}(q) D_p^T\big)^{-1}$ is the inverse of the Delassus' matrix (see [1], [6] for the definition) and $\beta \in \mathbb{R}^p,\ \beta > 0$.

*Proof:* First, we notice that the second relation in (17) implies on $\Omega_k^J$ (see [13])

$$0 \leq \ddot{q}_p \perp \lambda_p \geq 0 \Leftrightarrow 0 \leq D_p\ddot{q} \perp \lambda_p \geq 0. \tag{19}$$

From (17) and (11) one easily gets

$$\ddot{q} = M^{-1}(q)\big[-F(q,\dot{q}) + U_{nc} + (1+K_f)D_p^T(\lambda - \lambda_d)_p\big].$$

Combining the last two equations we obtain the following LCP with unknown $\lambda_p$:

$$\begin{aligned} 0 \leq D_p M^{-1}(q)\big[-F(q,\dot{q}) + U_{nc} - (1+K_f)D_p^T(\lambda_d)_p\big] \\ +(1+K_f)D_p M^{-1}(q)D_p^T\lambda_p \perp \lambda_p \geq 0. \end{aligned} \tag{20}$$

Since $(1+K_f)D_pM^{-1}(q)D_p^T > 0$ and hence is a P-matrix, the LCP (20) has a unique solution and one deduces that $\lambda_p > 0$ if and only if

$$\begin{aligned} &\frac{\bar{M}_{p,p}(q)}{1+K_f}D_pM^{-1}(q)\big[U_{nc} - F(q,\dot{q}) - (1+K_f)D_p^T(\lambda_d)_p\big] < 0 \\ &\Leftrightarrow (\lambda_d)_p > \frac{\bar{M}_{p,p}(q)}{1+K_f}D_pM^{-1}(q)\big[U_{nc} - F(q,\dot{q})\big] \\ &\Leftrightarrow (\lambda_d)_p = \beta + \frac{\bar{M}_{p,p}(q)}{1+K_f}D_pM^{-1}(q)\big[U_{nc} - F(q,\dot{q})\big] \end{aligned} \tag{21}$$

with $\beta \in \mathbb{R}^p,\ \beta > 0$. Since $U_{nc} - F(q,\dot{q}) = M(q)\ddot{q}_e - C(q,\dot{q})s - \gamma_1 s, (\ddot{q}_e)_p = 0$ and $s_p = 0$, (21) rewrites as (18) and the proof is finished. It is noteworthy that

$$\begin{aligned} \lambda_p &= -\frac{\bar{M}_{p,p}(q)}{1+K_f}D_pM^{-1}(q)\big[U_{nc} - F(q,\dot{q}) \\ &\qquad - (1+K_f)D_p^T(\lambda_d)_p\big] \\ &= (\lambda_d)_p - \frac{\bar{M}_{p,p}(q)}{1+K_f}D_pM^{-1}(q)\big[U_{nc} - F(q,\dot{q})\big] = \beta \end{aligned}$$

■

*Remark 6:* The control law used in this paper with the design of $\lambda_d$ described above leads to the following closed-loop dynamics on $\Omega_k^J$

$$\begin{cases} M_{p,n-p}(q)\dot{s}_{n-p} + C_{p,n-p}(q,\dot{q})s_{n-p} = (1+K_f)(\lambda - \lambda_d)_p \\ M_{n-p,n-p}(q)\dot{s}_{n-p} + C_{n-p,n-p}(q,\dot{q})s_{n-p} + \gamma_1 s_{n-p} = 0 \\ q_p = 0, \quad \lambda_p = \beta. \end{cases}$$

It is noteworthy that the closed-loop dynamics is nonlinear and therefore, we do not use the feedback stabilization proposed in [22].

## VI. Strategy for Take-off at the End of Constraint Phases $\Omega_k^j$

We have discussed in the previous sections the necessity of a trajectory with impacts in order to assure the robust stabilization on $\partial\Phi$ in finite time and, the design of the desired trajectory to stabilize the system on $\partial\Phi$. Now, we are interested in finding the conditions on the control signal $U_c^J$ that assure the take-off at the end of the constrained phases $\Omega_k^J$. We consider the phase $\Omega_k^J$ expressed as the time interval $[t_f^k, t_d^k)$. The dynamics on $[t_f^k, t_d^k)$ is given by (17) and the system is permanently constrained, which implies $q_p(\cdot) = 0$ and $\dot{q}_p(\cdot) = 0$. Let us also consider that the first $r$ constraints ($r < p$) have to be deactivated. Thus, the detachment takes place at $t_d^k$ if $\ddot{q}_r(t_d^{k+}) > 0$ which requires $\lambda_r(t_d^{k-}) = 0$. The last $p - r$ constraints remain active which means $\lambda_{p-r}(t_d^{k-}) > 0$.

To simplify the notation we drop the arguments $t$ and $q$ in many equations of this section. We decompose the LCP matrix (which is the Delassus' matrix $D_p M^{-1}(q) D_p^\top$ multiplied by $(1+K_f)$) as

$$(1+K_f)D_pM^{-1}(q)D_p^\top = \begin{pmatrix} A_1(q) & A_2(q) \\ A_2(q)^T & A_3(q) \end{pmatrix}$$

with $A_1 \in \mathbb{R}^{r\times r}$, $A_2 \in \mathbb{R}^{r\times(p-r)}$ and $A_3 \in \mathbb{R}^{(p-r)\times(p-r)}$.

*Proposition 3:* For the closed-loop system (17), (9), (11) the decrease of the active constraints number from $p$ to $p-r$ (with $r<p$), is possible if

$$\begin{pmatrix} (\lambda_d)_r(t_d^k) \\ (\lambda_d)_{p-r}(t_d^k) \end{pmatrix} = \begin{pmatrix} \left(A_1 - A_2A_3^{-1}A_2^T\right)^{-1}\left(b_r - A_2A_3^{-1}b_{p-r}\right) - C_1 \\ C_2 + A_3^{-1}\left(b_{p-r} - A_2^T(\lambda_d)_r\right) \end{pmatrix} \quad (22)$$

where

$$b_p \triangleq b(q,\dot q, U_{nc}) \triangleq D_pM^{-1}(q)[U_{nc} - F(q,\dot q)] \geq 0$$

and $C_1 \in \mathbb{R}^r, C_2 \in \mathbb{R}^{p-r}$ such that $C_1 \geq 0, C_2 > 0$.

*Proof:* From (11) and (17) one gets

$$\ddot q_p(t) = b_p + (1+K_f)D_pM^{-1}(q)D_p^T(\lambda - \lambda_d).$$

Therefore the LCP (19) rewrites as

$$0 \leq \begin{pmatrix} \lambda_r \\ \lambda_{p-r} \end{pmatrix} \perp \begin{pmatrix} b_r + A_1(\lambda-\lambda_d)_r + A_2(\lambda-\lambda_d)_{p-r} \\ b_{p-r} + A_2^T(\lambda-\lambda_d)_r + A_3(\lambda-\lambda_d)_{p-r} \end{pmatrix} \geq 0. \quad (23)$$

Under the conditions $\lambda_r = 0$ and $\lambda_{p-r} > 0$ one has

$$0 \leq \lambda_{p-r} \perp b_{p-r} - A_2^T(\lambda_d)_r + A_3(\lambda-\lambda_d)_{p-r} \geq 0$$

with the solution

$$\lambda_{p-r} = -A_3^{-1}\left(b_{p-r} - A_2^T(\lambda_d)_r - A_3(\lambda_d)_{p-r}\right). \quad (24)$$

Thus $\lambda_{p-r} > 0$ is equivalent to

$$(\lambda_d)_{p-r} > A_3^{-1}\left(b_{p-r} - A_2^T(\lambda_d)_r\right)$$

which leads to the second part of definition (22). Furthermore, replacing $(\lambda_d)_{p-r}$ in (24) we get $\lambda_{p-r} = C_2$ and $b_r + A_1(\lambda - \lambda_d)_r + A_2(\lambda-\lambda_d)_{p-r} \geq 0$ yields the first part of definition (22). To conclude, the solution of the LCP (23) is $\lambda_p = \begin{pmatrix} 0 \\ C_2 \end{pmatrix} \in \mathbb{R}^p$ and $(\lambda_d)_p$ is defined by (22). ■

*Proposition 4:* The closed-loop system (17), (9), (11) is permanently constrained on $[t_f^k, t_d^k)$ and a smooth detachment is guaranteed on $[t_d^k, t_d^k+\bar\epsilon)$ ($\bar\epsilon$ is a small positive real number chosen by the designer) if

(i) $(\lambda_d)_p(\cdot)$ is defined on $[t_f^k, t_d^k)$ by (22) where $C_1$ is replaced by $C_1(t-t_d^k)$.

(ii) On $[t_d^k, t_d^k+\bar\epsilon)$

$$q_d^*(t) = q_d(t) = \begin{pmatrix} q_r^*(t) \\ q_{n-r}^{nc}(t) \end{pmatrix}$$

where $q_r^*(\cdot)$ is a twice differentiable function such that

$$\begin{array}{ll} q_r^*(t_d^k) = 0, & q_r^*(t_d^k+\bar\epsilon) = q_r^{nc}(t_d^k+\bar\epsilon) \\ \dot q_r^*(t_d^k) = 0, & \dot q_r^*(t_d^k+\bar\epsilon) = \dot q_r^{nc}(t_d^k+\bar\epsilon) \end{array} \quad (25)$$

and $\ddot q_r^*(t_d^{k+}) = a > \max\left(0, -A_1(q)(\lambda_d)_r(t_d^{k-})\right)$.

*Proof:*

(i) The uniqueness of solution of the LCP (19) guarantees that (18) and (22) agree if $C_1 < 0$. In other words, replacing $C_1$ by $C_1(t-t_d^k)$ in (22) we assure a constrained motion on $[t_f^k, t_d^k)$ and the necessary conditions for detachment on $[t_d^k, t_d^k+\bar\epsilon)$.

(ii) Obviously (25) is imposed in order to assure the twice differentiability of the desired trajectory. Finally, straightforward computations show that

$$\sigma_{\ddot q_r(t_d^k)} = \ddot q_r^*(t_d^{k+}) + A_1(q)(\lambda_d)_r(t_d^{k-})$$

which means that the detachment is guaranteed and no other impacts occur when the desired acceleration satisfies $\ddot q_r^*(t_d^{k+}) > \max\left(0, -A_1(q)(\lambda_d)_r(t_d^{k-})\right)$. ■

## VII. Closed-Loop Stability Analysis

In the case $\Phi = \mathbb{R}^n$, the function $V(t,s,\tilde q)$ in (12) can be used in order to prove the closed-loop stability of the system (10), (11) (see for instance [7]). In the case studied here ($\Phi \subset \mathbb{R}^n$) the analysis becomes more complex.

To simplify the notation $V(t,s(t),\tilde q(t))$ is denoted as $V(t)$. In order to introduce the main result of this paper we make the next assumption, which is verified in practice for dissipative systems.

*Assumption 1:* The controller $U_t$ in (11) assures that all the transition phases are finite (see Definition 6) and the accumulation point $t_\infty^k$ is smaller than $t_d^{k,1}$ for all $k \in \mathbb{N}$.

Since outside $[t_0^k,\ t_f^k]$ we will show that the Lyapunov function exponentially decreases, we may presume that all the impacts take place during $I_k$.

*Lemma 1:* Consider the closed-loop system (9)–(11) with $(q_d^*)_p(\cdot)$ defined on the interval $[\tau_0^k,\ t_0^k]$ as in (14)–(13). Let us also suppose that condition (b) of Proposition 1 is satisfied. The following inequalities hold:

$$\begin{array}{l} \|\tilde q(t_0^{k-})\| \leq \sqrt{\dfrac{V(\tau_0^k)}{\gamma_1\gamma_2}}, \|s(t_0^{k-})\| \leq \sqrt{\dfrac{2V(\tau_0^k)}{\lambda_{\min}(M(q))}} \\ \|\dot{\tilde q}(t_0^{k-})\| \leq \left(\sqrt{\dfrac{2}{\lambda_{\min}(M(q))}} + \sqrt{\dfrac{\gamma_2}{\gamma_1}}\right) V^{1/2}(\tau_0^k). \end{array} \quad (26)$$

Furthermore, if $t_0^k \leq \tau_1^k$ and $t_0^k$ is a $p_{\epsilon_k}$-impact one has

$$\begin{array}{l} \|(q_d)_p(t_0^{k-})\| \leq \epsilon_k + \sqrt{\dfrac{V(\tau_0^k)}{\gamma_1\gamma_2}} \\ \|(\dot q_d)_p(t_0^{k-})\| \leq K + K'V^{1/3}(\tau_0^k) \end{array} \quad (27)$$

where $\epsilon_k \leq \max\{\psi, \sqrt{\bar{p}}\varphi V^{1/3}(\tau_0^k)\} + \sqrt{V(\tau_0^k)/\gamma_1\gamma_2}\}$, and $K, K' > 0$ are some constant real numbers that will be defined in the proof

*Proof:* See Appendix A. ■

The main result of this paper can be stated as follows.

*Theorem 1:* Let Assumption 1 hold, $e \in [0, 1)$ and $(q_d^*)_p(\cdot)$ defined as in (13)–(14). The closed-loop system (9)–(11) initialized on $\Omega_0$ such that $V(\tau_0^0) \leq 1$, satisfies the requirements of Proposition 1 and is therefore practically weakly stable with the closed-loop state $x(\cdot) = [s(\cdot), \tilde{q}(\cdot)]$ and $R = \sqrt{e^{-\gamma(t_f^k - t_\infty^k)}(1 + K_1 + K_2 + \xi)/\rho}$ where $\rho = \min\{\lambda_{\min}(M(q))/2; \gamma_1\gamma_2\}$ and $K_1, K_2$ are defined in the proof.

*Proof:* See Appendix B. ■

*Remark 7:* Since the closed-loop system (9)–(11) satisfies the requirements of Proposition 1 one also deduces $V(\tau_0^k) \leq \delta(\gamma, \xi)$, so $\epsilon_k \leq \max\{\psi, \sqrt{\bar{p}}\varphi\delta(\gamma,\xi)^{1/3}\} + \sqrt{\delta(\gamma,\xi)/\gamma_1\gamma_2}\}$, $\forall k \geq 1$. In other words the sequence $\{\epsilon_k\}_k$ is uniformly upperbounded and the upperbound can be decreased by adjusting the parameters $\psi$ and $\gamma$.

## VIII. Illustrative Example

### A. Planar Two-Link Rigid-Joint Manipulator With One Constraint

The main issues of the control scheme proposed in this paper are first emphasized simulating the behavior of a planar two-link rigid-joint manipulator in presence of one unilateral constraint. The lengths $l_1$, $l_2$ of the manipulator's links are set to 0.5 m, and their masses $m_1$, $m_2$ are set to 1 kg, $g$ is the gravity acceleration. Denoting by $\theta_i$ the joint angle of the link $i$ and $I_i$ the moment of inertia of link $i$ about the axis that passes through the center of mass and is parallel to the $OZ$ axis, the dynamics of the two-link manipulator is given by (1) with $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$, $G = \begin{bmatrix} G_1 \\ G_1 \end{bmatrix}$

$$\begin{cases} M_{11} = \frac{m_1 l_1^2}{4} + m_2\left(l_1^2 + \frac{l_2^2}{4} l_1 l_2 \cos\theta_2\right) + I_1 + I_2 \\ M_{12} = M_{21} = \frac{m_2 l_2^2}{4} + \frac{m_2 l_1 l_2}{2}\cos\theta_2 + I_2 \\ M_{22} = \frac{m_2 l_2^2}{4} + I_2 \end{cases}$$

$$\begin{cases} C_{11} = -m_2 l_1 l_2 \dot{\theta}_2 \sin\theta_2 \\ C_{12} = -\frac{m_2 l_1 l_2}{2}\dot{\theta}_2 \sin\theta_2 \\ C_{21} = \frac{m_2 l_1 l_2}{2}\dot{\theta}_1 \sin\theta_2 \\ C_{22} = 0 \end{cases}$$

$$\begin{cases} G_1 = \frac{g}{2}[l_1(2m_1 + m_2)\cos\theta_1 + m_2 l_2 \cos(\theta_1 + \theta_2)] \\ G_2 = \frac{m_2 g l_2}{2}\cos(\theta_1 + \theta_2) \end{cases}.$$

The dynamics can be rewritten in the cartesian coordinates using the change of variables

$$q = \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) \end{pmatrix} = Q(X). \quad (28)$$

The admissible domain is the upper half plane $y \geq 0$ (here $m = 1$ and $q_1 = y$) and the unconstrained desired trajectory $q^{nc}(\cdot)$ is given by a circle that violates the constraint. Precisely, the end effector must follow a half-circle, stabilize on the constraint $(y = 0)$ and move on the constraint until the point where the circle $q^{nc}(\cdot)$ re-enters the admissible domain. Thus (6) writes as $\mathbb{R}^+ = \Omega_0^{\emptyset} \cup I_1 \cup \Omega_1^{\{1\}} \cup \Omega_1^{\emptyset} \cup I_2 \cup \Omega_2^{\{1\}} \cup \Omega_2^{\emptyset} \cup \cdots$, with $m_k = 2$ for all $k \geq 0$, $J_{k,1} = \{1\}$, $J_{k,2} = \emptyset$. Using the $q-$ coordinates $x_d^*$ is frozen during the transition phases $I_k$ while $y_d^*$ is defined by (14)–(15). Furthermore, the controller $T(q)U$ is computed by (11) where $K_f$ is set to 0.5 and $(\lambda_d)_y$ (i.e., the desired contact force corresponding to the constraint $y = 0$) is given by (18) where $\beta$ has a decreasing profile like in item (i) of Proposition 4. The impacts are imposed using the parameter $\varphi = 100$ in (14)–(15). The numerical simulations are done with the Moreau's time-stepping algorithm of the SICONOS software platform (http://siconos.gforge.inria.fr). The choice of a time-stepping algorithm was mainly dictated by the presence of accumulations of impacts which render the use of event-driven methods difficult [1]. A further reason to choose the SICONOS software platform for the simulation of the complementarity systems is its capability to solve LCPs[2]. Let us set $e = 0.7$, $\gamma_1 = 8$, $\gamma_2 = 7$, 10 s the period of each cycle and 30 s the final simulation time. First, let us point out [Fig. 4 (left)] the influence of $\psi$ (i.e., the choice of $\tau_0^k$) on the deformation of the real trajectory w.r.t. the desired unconstrained one. As we have pointed out in Section IV the deformation gets smaller when $\psi > 0$ decreases. It is noteworthy that the tangential approach

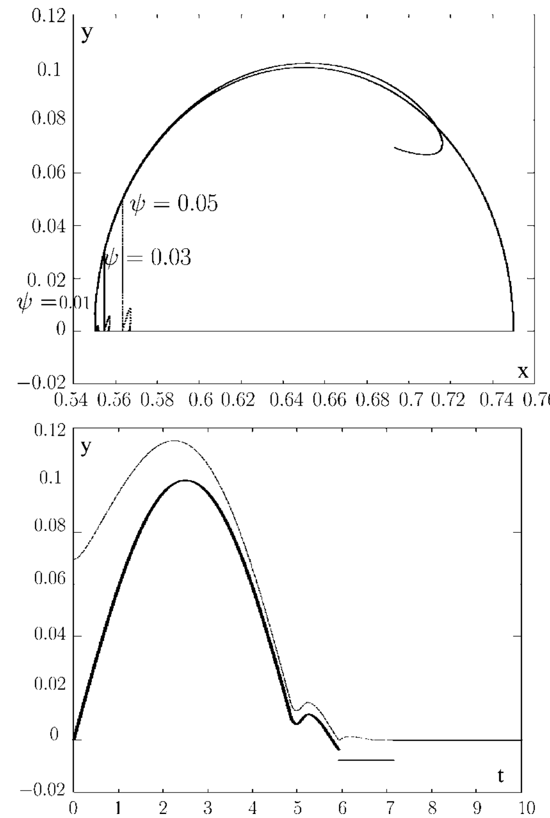


Fig. 4. Up: The influence of $\psi$ on the real trajectory's deformation for controller's gains set to $\gamma_1 = 8$, $\gamma_2 = 7$. Down: $y(t) = q^1(t)$ (dashed) and $y_d^*(t) = (q_d^1)^*(t)$ (solid) during the first cycle ($\psi = 0.01$).

[2]The control scheme proposed in this paper may require to solve an LCP of dimension $\bar{p} \approx 10$ (reasonable in some control applications). But this requires a specific solver since the usual "hybrid" methods must treat $2^{\bar{p}}$ cases and quickly become inefficient [1].

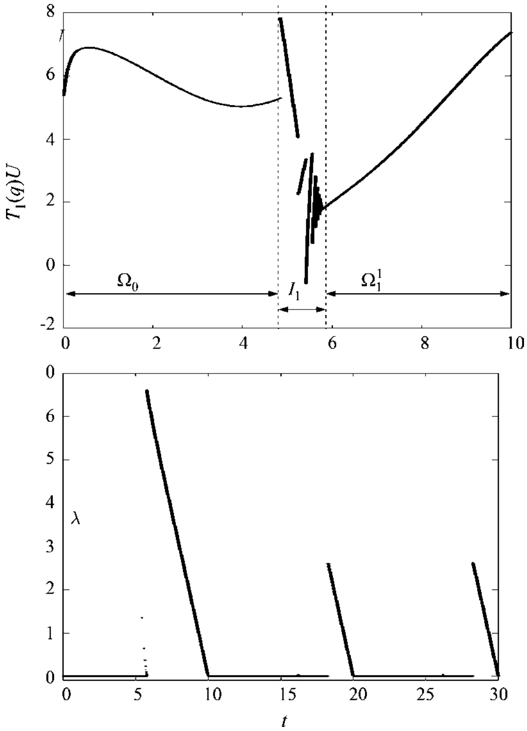


Fig. 5. Up: The switching controller during the first 10 s; Down: Variation of the contact force $\lambda$.

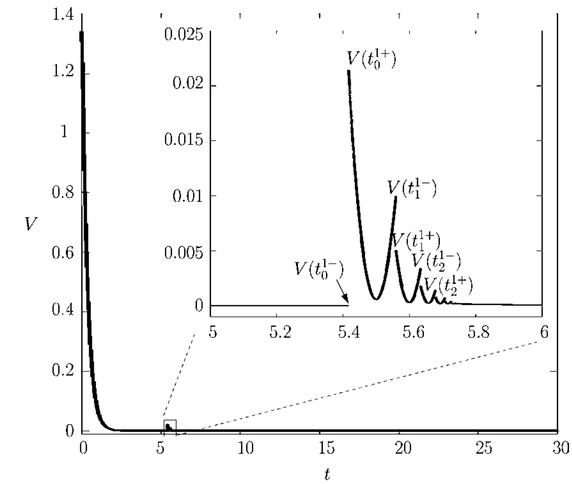


Fig. 6. Variation of the Lyapunov function for $\gamma_1 = 8,\ \gamma_2 = 7$; Zoom: Variation of the Lyapunov function during the phase $I_0$.

corresponding to $\psi = 0$ lacks of robustness and is unreliable due to the nonzero initial tracking errors.

On Fig. 4 one sees that since $q_d(\cdot) = q_d^*(\cdot)$ before the first impact $t_0^k$ of each cycle and $t_0^k < \tau_1^k$, there exists a jump at the moment $t_0^k \approx 6$ s in $q_d(\cdot), q_d^*(\cdot)$, respectively, and both signals are set to zero at $t \approx 7.2$ s. The jump of $q_d(\cdot)$ induces a positive jump in the variation of $V(\cdot)$ (details are in Appendix B) The switches of the controller during the first 10 s are depicted in Fig. 5. Clearly since the velocity jumps, the controller jumps as well.

The Fig. 5 presents the variation of the contact force $\lambda$. One sees that $\lambda$ remains 0 during the free motion phases. The contact force $\lambda$ is designed as a decreasing linear function during constrained motion phases $\Omega_k^{\{1\}}$ in order to allow a smooth detachment at the end of these phases. It is worth to mention that the magnitude of $\lambda$ depends indirectly on $V(\tau_0^k)$. Precisely, when $V(\tau_0^k)$ approaches zero the system tends to a tangential stabilization on $\partial\Phi$, which implies larger values of $t_0^k$ and consequently smaller length of $[t_f^k,\ t_d^{k,1}]$ and smaller magnitude of the contact force measured by $\lambda$ (see Proposition 4).

Fig. 6 shows that the tracking error described by the Lyapunov function rapidly decreases and remains close to 0. In other words the practical weak stability is guaranteed. On the zoom made in Fig. 6 one can also observe the behavior of $V(\cdot)$ during the stabilization on $\partial\Phi$, that is an almost decreasing function.

### *B. Planar Two-Link Rigid-Joint Manipulator With Two Constraints*

In the sequel we introduce another constraint into the previous dynamics. Precisely we impose an admissible domain $\Phi = \{(x, y) \mid y \geq 0, 0.7 - x \geq 0\}$. Let us also consider an unconstrained desired trajectory given by the circle $\{(x, y) \mid (x - 0.7)^2 + y^2 = 0.5\}$ that violates both constraints. In other words, the two-link planar manipulator must track a quarter-circle; stabilize on and then follow the line $\Sigma_1 = \{(x, y) \mid y = 0\}$; stabilize on the intersection of $\Sigma_1$ and $\Sigma_2 = \{(x, y) \mid x = 0.7\}$; detach from $\Sigma_1$ and follow $\Sigma_2$ until the unconstrained circle re-enters $\Phi$ and finally take-off from $\Sigma_2$ in order to repeat the previous steps. Therefore, we have: $\mathbb{R}^+ = \Omega_0^{\emptyset} \cup I_1 \cup \Omega_1^{J_{1,1}} \cup I_2 \cup \Omega_2^{J_{2,1}} \cup \Omega_2^{J_{2,2}} \cup \Omega_2^{J_{2,3}} \cup I_3 \cup \Omega_2^{J_{3,1}} \cup I_4 \cup \cdots$ with $J_{1,1} = \{1\}$, $J_{2,1} = \{1, 2\}$, $J_{2,2} = \{2\}$, $J_{2,3} = \emptyset$, etc. We note that during $I_{2k+1}$ the system is stabilized on $\Sigma_1$ (1-impacts) while during $I_{2k}$ the system is stabilized on $\Sigma_1 \cap \Sigma_2$ ($2_{\epsilon_k}$ -impacts).

The numerical values used for the dynamical model are again $l_1 = l_2 = 0.5$ m, $I_1 = I_2 = 1$ kg.m$^2$, $m_1 = m_2 = 1$ kg and the restitution coefficient $e = 0.7$. The impacts are imposed by $\varphi = 100$ in (14) and (15) the beginning of transition phases are defined using $\psi = 0.05$ in (16). We impose a period of 10 s for two consecutive cycles and we simulate the dynamics during 60 s. Setting the controller gains $\gamma_1 = 15,\ \gamma_2 = 15$ we see in Fig. 7 (left) that the desired trajectory is accurately followed. The jumps in the variation of the Lyapunov function are pointed out in Fig. 7.

In this case we have imposed a constant contact-force $\lambda_1$ during the motion on the surface $\Sigma_1$ (see Fig. 8 (left)) and a decreasing contact-force, that allows a smooth detachment, during the motion on $\Sigma_2$ (see Fig. 8 (right)). On Fig. 9 the values of the multipliers $\lambda_1$ and $\lambda_2$ during the transition phase $I_2$ (stabilization in the corner) are depicted.

## IX. Conclusion

In this paper, we have proposed a methodology to study the tracking control of fully actuated Lagrangian systems subject to multiple frictionless unilateral constraints and multiple impacts. The main contribution of the work is twofold: first, it formulates a general control framework and second, it provides a stability analysis for the class of systems under consideration. It is noteworthy that even in the simplest case of only one frictionless unilateral constraint the paper already presents some notable

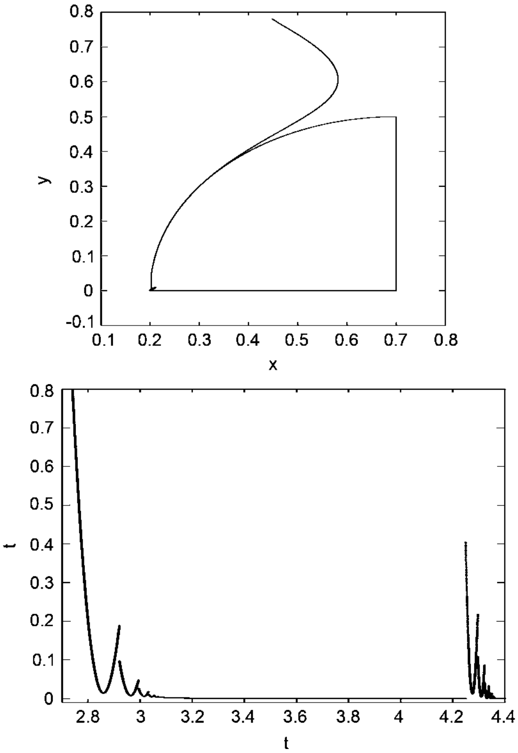


Fig. 7. Up: The trajectory of the system during six cycles; Down: Zoom on the variation of the Lyapunov function on the first two transition phases.

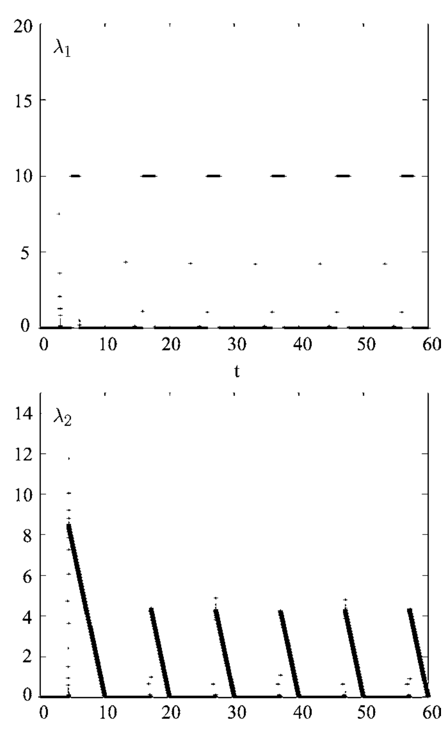


Fig. 8. Up: Variation of the contact force during the motion on $\Sigma_1$; Down: Variation of the contact force during the motion on $\Sigma_2$.

improvements with respect to the existing works. Precisely, the stability analysis result is significantly more general than those presented in [4] and [8] and, each element entering the dynamics

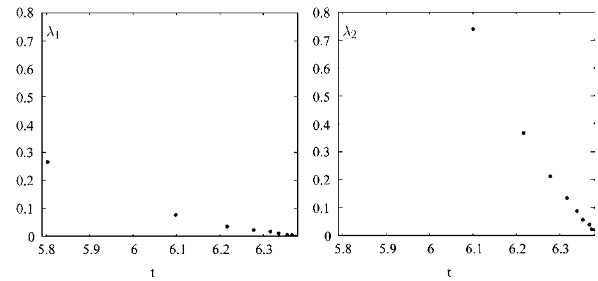


Fig. 9. Zoom on the transition phase $I_2$ with $2_\epsilon$-impacts dots are the impulsive force magnitude at impacts).

(desired trajectory, contact force) is explicitly defined. Numerical simulations are done with the SICONOS software platform [1] in order to illustrate the results.

## APPENDIX

*Proof of Lemma 1:* From (12) we can deduce on one hand that

$$V(t_0^{k-}) \geq \gamma_1\gamma_2 \|\tilde{q}(t_0^{k-})\|^2$$

and on the other hand

$$V(t_0^{k-}) \geq \frac{1}{2} s(t_0^{k-})^T M(q(t_0^{k-})) s(t_0^{k-}).$$

Since condition (b) of Proposition 1 is satisfied one has $V(\tau_0^k) \geq V(t_0^{k-})$ and the first two inequalities in (26) become trivial. Let us recall that $s(t) = \dot{\tilde{q}}(t) + \gamma_2 \tilde{q}(t)$ which implies $\|\dot{\tilde{q}}(t_0^{k-})\| \leq \|s(t_0^{k-})\| + \gamma_2 \|\tilde{q}(t_0^{k-})\|$. Combining this with the first two inequalities in (26) we derive the third inequality in (26).

For the rest of the proof we assume that $t_0^k \leq \tau_1^k$. Therefore $(q_d)_p(t_0^{k-}) = (q_d^*)_p(t_0^k)$. Since $(q_d)_p(\cdot)$ is a continuous function with all the components $q_d^i(\cdot)$ defined as decreasing functions on $[\tau_0^k + \delta, \tau_1^k]$, it is obvious that $\|(q_d)_p(t_0^{k-})\| \leq \max \{\|(q_d)_p(\tau_0^k + \delta)\|, \|(q_d)_p(\tau_1^k)\|\} = \max\{\psi, \sqrt{p}\varphi V^{1/3}(\tau_0^k)\}$. Furthermore

$$\begin{aligned}\|q_p(t_0^k)\| &\leq \|\tilde{q}_p(t_0^{k-})\| + \|(q_d)_p(t_0^{k-})\| \\ &\leq \sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}} + \max\{\psi, \sqrt{p}\varphi V^{1/3}(\tau_0^k)\}.\end{aligned}$$

Thus $t_0^k$ is a $p_{\epsilon_k}$-impact with $\epsilon_k \leq \max\{\psi, \sqrt{p}\varphi V^{1/3}(\tau_0^k)\} + \sqrt{V(\tau_0^k)/\gamma_1\gamma_2}\}$. From Definition 3 one has $\|q_p(t_0^k)\| \leq \epsilon_k$ and using

$$\|(q_d)_p(t_0^{k-})\| \leq \|\tilde{q}_p(t_0^{k-})\| + \|q_p(t_0^k)\|$$

one obtains the first inequality (27).

Let us denote $t'_k = t_0^k - \tau_0^k - \delta/\tau_1^k - \tau_0^k - \delta \in [0, 1]$. We recall here that $\tau_0^k$ was chosen such that $\|q_p^{nc}(\tau_0^k)\| \leq \psi$. From (14), (15) and the first inequality in (27), for $i = 1, \ldots, p$ one has

$$\begin{aligned}q_d^i(t_0^{k-}) = &\left[(q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k)\right] \\ &\times \left(2(t'_k)^3 - 3(t'_k)^2\right) + (q^i)^{nc}(\tau_0^k) \\ \leq &\, \epsilon_k + \sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}}.\end{aligned}$$

It follows that:

$$3(t_k')^2 - 2(t_k')^3 \geq \frac{(q^i)^{nc}(\tau_0^k) - \epsilon_k - \sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}}}{(q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k)}.$$

For $t > 0$ one has $2t - t^2 \geq 3t^2 - 2t^3$, therefore

$$2t_k' - (t_k')^2 \geq \frac{(q^i)^{nc}(\tau_0^k) - \epsilon_k - \sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}}}{(q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k)}$$

which means that

$$(1 - t_k')^2 \leq \frac{\sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}} + \varphi V^{1/3}(\tau_0^k) + \epsilon_k}{(q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k)}.$$

Straightforward computations lead to

$$|\dot{q}_d^i(t_0^{k-})| = \frac{6((q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k))}{\tau_1^k - \tau_0^k - \delta}\left(t_k' - (t_k')^2\right).$$

Since $t_k' - (t_k')^2 \leq 1 - t_k'$ and from (16) one has $(q^i)^{nc}(\tau_0^k) \leq \psi$, one arrives at

$$\begin{aligned}|\dot{q}_d^i(t_0^{k-})| &\leq \frac{6((q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k))}{\tau_1^k - \tau_0^k - \delta}(1 - t_k') \\ &\leq \frac{6\sqrt{(\psi + \varphi V^{1/3}(\tau_0^k))\left(\sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}} + \varphi V^{1/3}(\tau_0^k) + \epsilon_k\right)}}{\tau_1^k - \tau_0^k - \delta} \\ &= \frac{6}{\tau_1^k - \tau_0^k - \delta}\sqrt{\left\{\psi\epsilon_k + (\psi\varphi + \epsilon_k\varphi)V^{1/3}(\tau_0^k) + \varphi^2 V^{2/3}(\tau_0^k) + \frac{\varphi V^{5/6}(\tau_0^k) + \psi V^{1/2}(\tau_0^k)}{\sqrt{\gamma_1\gamma_2}}\right\}}.\end{aligned}$$

Since $V(\tau_0^k) < 1$ (thus $V^{p_1}(\tau_0^k) > V^{p_2}(\tau_0^k)$ for $p_1 < p_2$) one obtains

$$|\dot{q}_d^i(t_0^{k-})| \leq \frac{6\sqrt{\psi\epsilon_k + \left[\left(\frac{1}{\sqrt{\gamma_1\gamma_2}} + \varphi\right)(\varphi + \psi) + \epsilon_k\varphi\right]V^{1/3}(\tau_0^k)}}{\tau_1^k - \tau_0^k - \delta}.$$

Furthermore $\epsilon_k \leq \psi + \sqrt{p}\varphi V^{1/3}(\tau_0^k) + \sqrt{V(\tau_0^k)/\gamma_1\gamma_2}$ and

$$|\dot{q}_d^i(t_0^{k-})| \leq \frac{6\psi}{\tau_1^k - \tau_0^k - \delta} + \frac{6\sqrt{\left(\frac{2}{\sqrt{\gamma_1\gamma_2}} + (1+\sqrt{p})\varphi\right)(\varphi + \psi) + \psi\varphi}}{\tau_1^k - \tau_0^k - \delta}V^{1/3}(\tau_0^k).$$

Therefore, the second inequality in (27) holds with $K = 6p\psi/\tau_1^k - \tau_0^k - \delta$

$$K' = \frac{6\sqrt{p}}{\tau_1^k - \tau_0^k - \delta} \times \sqrt{\left(\frac{2}{\sqrt{\gamma_1\gamma_2}} + (1+\sqrt{p})\varphi\right)(\varphi + \psi) + \psi\varphi}.$$

*Proof of Theorem 1:* First we observe that conditions (a) and (d) of Proposition 1 hold when the hypothesis of the Theorem are verified. Thus in order to prove Theorem 1 it is sufficient to verify the conditions (b), (c) and (e) of Proposition 1. To this aim we shall also use the function $V_1(t,s) = 1/2s(t)^T M(q)s(t)$.

(b) Using that $\dot{M}(q) - 2C(q,\dot{q})$ is a skew-symmetric matrix, straightforward computations show that on $\mathbb{R}_+ \setminus \bigcup_{k\geq 1}[t_0^k, t_f^k]$ the time derivative of the Lyapunov function is given by

$$\dot{V}(t) = -\gamma_1 s^T s + 2\gamma_1\gamma_2\tilde{q}^T\dot{\tilde{q}} = -\gamma_1\|\dot{\tilde{q}}\|^2 - \gamma_1\gamma_2^2\|\tilde{q}\|^2.$$

On the other hand

$$\begin{aligned}V(t) &\leq \frac{\lambda_{\max}(M(q))}{2}\|s\|^2 + \gamma_1\gamma_2\|\tilde{q}\|^2 \\ &\leq \gamma^{-1}[\gamma_1\|\dot{\tilde{q}}\|^2 + \gamma_1\gamma_2^2\|\tilde{q}\|^2]\end{aligned}$$

where

$$\gamma^{-1} = \max\left\{\lambda_{\max}(M(q))\frac{1+2\gamma_2}{2\gamma_1}; \frac{\lambda_{\max}(M(q))(\gamma_2+2)+2\gamma_1}{2\gamma_1\gamma_2}\right\} > 0.$$

Therefore $\dot{V}(t) \leq -\gamma^{-1}V(t)$ on $\mathbb{R}_+ \setminus \bigcup_{k\geq 1}[t_0^k, t_f^k]$.

(c) By definition

$$\begin{aligned}V(t_{\ell+1}^{k-}) - V(t_\ell^{k+}) &= V_1(t_{\ell+1}^{k-}) - V_1(t_\ell^{k+}) \\ &+\gamma_1\gamma_2[(\tilde{q}^T(t_{\ell+1}^{k-}))\tilde{q}(t_{\ell+1}^{k-}) - (\tilde{q}^T(t_\ell^{k+}))\tilde{q}(t_\ell^{k+})]. \quad (29)\end{aligned}$$

On the other hand, straightforward computations show that

$$\begin{aligned}V_1(t_{\ell+1}^{k-}) - V_1(t_\ell^{k+}) &= \int_{(t_\ell^k, t_{\ell+1}^k)}\dot{V}_1(t)\mathrm{d}t \\ &= \gamma_1\gamma_2\int_{(t_\ell^k, t_{\ell+1}^k)} s_p^T(t)(q_d^*)_p(t)\mathrm{d}t \\ &- \gamma_1\int_{(t_\ell^k, t_{\ell+1}^k)} s(t)^T s(t)\mathrm{d}t. \quad (30)\end{aligned}$$

Furthermore

$$\int_{(t_\ell^k, t_{\ell+1}^k)} s(t)^T s(t)\mathrm{d}t = \int_{(t_\ell^k, t_{\ell+1}^k)} \|\dot{\tilde{q}}(t)\|^2 + \gamma_2^2\|\tilde{q}(t)\|^2\mathrm{d}t + \gamma_2[(\tilde{q}^T(t_{\ell+1}^{k-}))\tilde{q}(t_{\ell+1}^{k-}) - (\tilde{q}^T(t_\ell^{k+}))\tilde{q}(t_\ell^{k+})]. \quad (31)$$

Therefore, inserting successively (31) in (30) and (30) in (29) one arrives at

$$V(t_{\ell+1}^{k-}) - V(t_\ell^{k+}) \leq \gamma_1\gamma_2\int_{(t_\ell^k, t_{\ell+1}^k)} s_p^T(t)(q_d^*)_p(t)\mathrm{d}t. \quad (32)$$

In the sequel let us denote by $S(v)$ the sum of all the components of a vector $v$. Taking into account the definition (14) and the fact that $(q_d)_p$ and $(\dot{q}_d)_p$ are set to zero at $t_0^{k+}$ one obtains

$$\int_{(t_\ell^k, t_{\ell+1}^k)} s_p^T(t)(q_d^*)_p(t)\mathrm{d}t = -\varphi V^{1/3}(\tau_0^k) \cdot \left(\int_{(t_\ell^k, t_{\ell+1}^k)} S(\dot{q}_p(t))\mathrm{d}t + \gamma_2\int_{(t_\ell^k, t_{\ell+1}^k)} S(q_p(t))\mathrm{d}t\right).$$

Since $\varphi\gamma_2 V^{1/3}(\tau_0^k) \geq 0$ and $S(q_p(t)) \geq 0$ it follows that:

$$\int_{t_\ell^k}^{t_{\ell+1}^k} s_p^T(t)(q_d^*)_p(t)\mathrm{d}t \leq \varphi V^{1/3}(\tau_0^k)[S(q_p(t_\ell^k)) - S(q_p(t_{\ell+1}^k))].$$

Thus

$$\sum_{\ell \geq 0} \left[V(t_{\ell+1}^{k-}) - V(t_\ell^{k+})\right] \leq \gamma_1\gamma_2\varphi V^{1/3}(\tau_0^k) S(q_p(t_0^k)) \leq \gamma_1\gamma_2\varphi V^{1/3}(\tau_0^k)\sqrt{3}\|q_p(t_0^k)\|.$$

Since $t_0^k$ is a $p_{\epsilon_k}$-impact and $\epsilon_k \leq \psi + \sqrt{p}\varphi V^{1/3}(\tau_0^k) + \sqrt{V(\tau_0^k)/\gamma_1\gamma_2}$ one gets

$$\sum_{\ell \geq 0} \left[V(t_{\ell+1}^{k-}) - V(t_\ell^{k+})\right] \leq K_1 V^{p_1}(\tau_0^k)$$

where $K_1 = \sqrt{3}\gamma_1\gamma_2\varphi(\psi + \sqrt{p}\varphi + 1/\sqrt{\gamma_1\gamma 2}) > 0$ and $p_1 = 2/3$.

(e) First, let us compute the Lyapunov function's jumps at the instants $t_\ell^k, \ell \geq 1$. Using the continuity of the position $q(\cdot)$ and the definition of the desired trajectory $q_d(\cdot)$ on the $I_k$ phases (i.e., $q_d(t_\ell^{k+}) = q_d(t_\ell^{k-}), \dot{q}_d(t_\ell^{k+}) = 0 = \dot{q}_d(t_\ell^{k-})$) one gets

$$\begin{aligned}\sigma_V(t_\ell^k) &= V(t_\ell^{k+}) - V(t_\ell^{k-}) = \gamma_1\gamma_2\sigma_{\|\tilde{q}\|^2}(t_\ell^k) \\ &\quad + \frac{s^T(t_\ell^{k+})M_\ell^k s(t_\ell^{k+}) - s^T(t_\ell^{k-})M_\ell^k s(t_\ell^{k-})}{2} \\ &= T_L(t_\ell^k) + \gamma_2\tilde{q}(t_\ell^k)^T M_\ell^k \sigma_{\dot{q}}(t_\ell^k)\end{aligned} \quad (33)$$

where $M_\ell^k$ denotes the inertia matrix $M(q(t_\ell^k))$ and $T_L$ is the kinetic energy loss at the impact time $t_\ell^k$.

From (5) one has $T_L(t_\ell^k) \leq 0$ and (33) becomes $\sigma_V(t_\ell^k) \leq \gamma_2\tilde{q}(t_\ell^k)^T M_\ell^k \sigma_{\dot{q}}(t_\ell^k)$. Let us recall that $M_\ell^k \sigma_{\dot{q}}(t_\ell^k)$ is the percussion vector $\mathcal{P}$ (see [6, Chapter 1]). In the generalized coordinates introduced in Section III one obtains $M_\ell^k \sigma_{\dot{q}}(t_\ell^k) = D^T\mathcal{P}$ with $\lambda = \mathcal{P}\delta_{t_\ell^k}$. In other words the generalized coordinates introduced in Section III coincide with the so called quasi-coordinates [6] and the vector $\dot{q}_{tang}$ is equal to $\dot{q}_{n-m}$ (i.e., $\sigma_{\dot{q}}(t_\ell^k) = \begin{pmatrix}\sigma_{\dot{q}_m}(t_\ell^k) \\ \mathbf{0}_{n-m}\end{pmatrix}$ where $\mathbf{0}_{n-m}$ denotes the $n-m$ vector with all its components equal zero). Therefore

$$\sigma_V(t_\ell^k) \leq \gamma_2\tilde{q}(t_\ell^k)^T M_\ell^k \sigma_{\dot{q}}(t_\ell^k) = \gamma_2 q_p(t_\ell^k)^T\mathcal{P} = 0 \quad (34)$$

where we have used $(q_d)_p(t_\ell^{k+}) = 0 = (q_d)_p(t_\ell^{k-})$ and the last equality is stated using the complementarity relation entering the dynamics, which impose that $\mathcal{P}$ is orthogonal to $\partial\Phi$.

The Lyapunov function's jump corresponding to the first impact of each cycle can be computed as

$$\begin{aligned}\sigma_V(t_0^k) &= V(t_0^{k+}) - V(t_0^{k-}) \\ &= \gamma_1\gamma_2\sigma_{\|\tilde{q}\|^2}(t_0^k) \\ &\quad + \frac{s^T(t_0^{k+})M_0 s(t_0^{k+}) - s^T(t_0^{k-})M_0 s(t_0^{k-})}{2}.\end{aligned} \quad (35)$$

- It is clear that $t_0^k > \tau_1^k$ implies $q_d(t_0^{k+}) = q_d(t_0^{k-})$ and $\dot{q}_d(t_0^{k+}) = 0 = \dot{q}_d(t_0^{k-})$. Thus, the computations for $t_\ell^k, \ell \geq 1$ hold also for $t_0^k$.
- If $t_0^k \leq \tau_1^k$ one has $(q_d)_p(t_0^{k-}) \neq (q_d)_p(t_0^{k+}) = 0$ and $(\dot{q}_d)_p(t_0^{k-}) \neq (\dot{q}_d)_p(t_0^{k+}) = 0$. Then the initial jump of each cycle is given by

$$\begin{aligned}\sigma_V(t_0^k) &= T_L(t_0^k) + \dot{q}_d(t_0^{k-})^T M_0 \dot{q}(t_0^{k-}) \\ &+ \frac{\gamma_2^2}{2}\left(\tilde{q}(t_0^{k+})^T M_0 \tilde{q}(t_0^{k+}) - \tilde{q}(t_0^{k-})^T M_0 \tilde{q}(t_0^{k-})\right) \\ &+ \gamma_2\left(\dot{q}(t_0^{k+})^T M_0 \tilde{q}(t_0^{k+}) - \dot{\tilde{q}}(t_0^{k-})^T M_0 \tilde{q}(t_0^{k-})\right) \\ &- \frac{1}{2}\dot{q}_d(t_0^{k-})^T M_0 \dot{q}_d(t_0^{k-}).\end{aligned} \quad (36)$$

Since $T_L(t_0^k) \leq 0$ the (36) rewrites as

$$\begin{aligned}\sigma_V(t_0^k) \leq\ & \lambda_{\max}(M(q))\Big[\gamma_2\big(\|(\dot{q}_d)_p(t_0^{k-})\| \cdot \|\tilde{q}(t_0^{k-})\| + \\ & + \|\dot{q}(t_0^{k-})\| \cdot \|(q_d)_p(t_0^{k-})\|\big) + \frac{1}{2}\|(\dot{q}_d)_p(t_0^{k-})\|^2 \\ & + \frac{\gamma_2^2}{2}\big(\|q_p(t_0^k)\|^2 + \|\tilde{q}_p(t_0^{k-})\|^2 \\ & + 2\|(q_d)_p(t_0^{k-})\| \cdot \|\tilde{q}_{n-p}(t_0^{k-})\|\big) + \|(\dot{q}_d)_p(t_0^{k-})\| \cdot \|\dot{q}(t_0^{k-})\|\Big].\end{aligned} \quad (37)$$

Obviously $\|\dot{q}(t_0^{k-})\| = \|\dot{\tilde{q}}(t_0^{k-}) + (\dot{q}_d)_p(t_0^{k-})\|$ and Lemma 1 combined with $V(\tau_0^k) < 1$ yields

$$\|\dot{q}(t_0^{k-})\| \leq K + \left(\sqrt{\frac{2}{\lambda_{\min}(M)}} + \sqrt{\frac{\gamma_2}{\gamma_1}} + K'\right)V^{1/3}(\tau_0^k).$$

Therefore

$$\sigma_V(t_0^k) \leq K_2 V^{p_2}(\tau_0^k) + \xi$$

where $p_2 = 1/3$, $\xi = 3/2K^2 + \gamma_2\psi K + \gamma_2^2\psi^2/2$ and $K_2 = \lambda_{\max}(M(q))\Big[3KK' + 3/2(K')^2 + \gamma_2\psi K + \sqrt{2\gamma_2/\lambda_{\min}(M(q))\gamma_1} + (K' + K)\big(\gamma_2\sqrt{p}\varphi + 3\sqrt{\gamma_2/\gamma_1} + \sqrt{2/\lambda_{\min}(M(q))}\big) + \gamma_2\big(\sqrt{2/\lambda_{\min}(M(q))} + 3\sqrt{\gamma_2/\gamma_1}\big)(\psi + \sqrt{p}\varphi) + \gamma_2^2\psi\varphi\sqrt{p} + \gamma_2^2\varphi^2 p/2 + 4\gamma_2/\gamma_1 + \psi\gamma_2\left(2\sqrt{\gamma_2/\gamma_1} + \sqrt{2/\lambda_{\min}(M(q))} + K'\right)\Big]$.

Defining $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+, \alpha(\omega) = \rho\omega^2$ we get $\alpha(0) = 0$ and $\alpha(\|[s(t), \tilde{q}(t)]\|) \leq V(t, s, \tilde{q})$. Thus, Proposition 1 also yields

$$\begin{aligned}R &= \alpha^{-1}(e^{-\gamma(t_f^k - t_\infty^k)}(1 + K_1 + K_2 + \xi)) \\ &= \sqrt{\frac{e^{-\gamma(t_f^k - t_\infty^k)}(1 + K_1 + K_2 + \xi)}{\rho}}\end{aligned}$$

which ends the proof.

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