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Observer Design for Lur'e Systems With Multivalued Mappings: A Passivity Approach

Bernard Brogliato and W. P. M. H. (Maurice) Heemels

***Abstract*—This paper considers the design of state observers for Lur'e systems with multivalued mappings in the feedback path. In particular, we focus on maximal monotone mappings that that do not require any compactness and local boundedness properties and include various models for relays, friction characteristics and complementarity conditions. We propose two types of observers that are constructed by rendering a suitable operator passive in an appropriate sense. The well-posedness properties of the observer dynamics are carefully analyzed and the global asymptotic stability of the observation error is formally proven.**

***Index Terms*—Differential inclusion, Lur'e system, maximal monotone mapping, multivalued system, normal cone, observer, passivity, positive realness.**

I. Introduction

Differential inclusions are ubiquitous in many engineering fields such as mechanics (mechanical systems with unilateral constraints and/or friction), electrical engineering (switched circuits), hybrid systems (relay control systems, discontinuous dynamical systems adopting generalized solutions concepts), economics (projected dynamical systems describing oligopolistic markets or traffic networks) and so on, see, e.g., [5], [11], and [20]. In this paper, we focus on the class of Lur'e systems with multivalued mappings in the feedback path. Such systems consist of the interconnection of a linear system given by

$$\begin{cases} \dot{x}(t) = Ax(t) - Gw(t) + Bu(t) \\ z(t) = Hx(t) \\ y(t) = Cx(t) \end{cases} \tag{1a}$$

with a multivalued nonlinearity of the form

$$w(t) \in \rho\left(z(t)\right) \tag{1b}$$

where for each $z \in \mathbb{R}^l$ $\rho(z)$ is a subset of $\mathbb{R}^l$ containing possibly an infinite number of elements, $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$ denote the state and the measured output, respectively, $z(t) \in \mathbb{R}^l$ and $w(t) \in \mathbb{R}^l$ are the variables going into and coming out of the nonlinearity, respectively, and $u(t) \in \mathbb{R}^m$ is the control input at time $t \in \mathbb{R}$. Systems of the considered type may arise as a natural consequence of modelling (e.g. friction models in mechanical systems, descriptions of ideal diodes in electrical circuits, etc.) or as a consequence of using generalized solution concepts for discontinuous dynamical systems (e.g., Filippov solutions [11]). Examples include various important classes of nonsmooth dynamical systems such as certain piecewise linear systems, linear relay systems, mechanical systems with friction, linear complementarity systems [5], [14], and electric circuits with switching elements and MOS transistors [1], [21]. In this paper we consider the problem of observer design for such Lur'e systems with multivalued

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mappings in the feedback path. In particular, we will focus on the case where $\rho(\cdot)$ is maximal monotone [15]. The requirement that the mapping is monotone is an extension of the usually considered concept of continuous, sector bounded nonlinearity.

Only few works on observability and observer design methodologies for differential inclusions are available in the literature. The observability properties of a class of differential inclusions, different from those studied in this paper, are examined in [3], while in [8] observability for a class of linear complementarity systems is studied, however, without considering the problem of observer design. The only exception in the literature that really considers observer design for multivalued systems is formed by the work in [18]. However, the differential inclusions considered in [18] are different from those considered in this paper, as compactness and local boundedness properties of the multivalued maps are assumed. These conditions are typically not satisfied by the multivalued right-hand sides that we work with.

II. Preliminaries

A. Notation and Definitions

For a symmetric matrix A, we denote its smallest eigenvalue by $\lambda_{\min}(A)$ and its largest eigenvalue by $\lambda_{\max}(A)$. A (not necessarily symmetric) square matrix $P \in \mathbb{R}^{n\times n}$ is called positive definite (denoted by $P \succ 0$), if $x^\top P x > 0$ for all $x \in \mathbb{R}^n$, $x \neq 0$. Similarly, it is called nonnegative definite (denoted by $P \succeq 0$), if $x^\top P x \geq 0$ for all $x \in \mathbb{R}^n$. For a nonnegative definite and symmetric matrix P, we denote its square root by $R = P^{1/2}$, which satisfies $R = R^\top \succeq 0$ and $R^2 = P$. The matrix triple (A, B, C), where B has full column rank, is called *strictly passive*, if there exist a $P = P^\top \succ 0$ and a $Q = Q^\top \succ 0$ such that

$$PA + A^\top P = -Q \tag{2a}$$
$$B^\top P = C. \tag{2b}$$

The material that follows is taken from [15] and [17]. A mapping $\rho : \mathbb{R}^l \hookrightarrow \mathbb{R}^l$ is said to be *multivalued* if it assigns to each element $x \in \mathbb{R}^l$ a subset $\rho(x) \subset \mathbb{R}^l$ (which may be empty). The domain of the mapping $\rho(\cdot)$, $\mathrm{dom}(\rho)$ is defined as $\mathrm{dom}(\rho) = \{x \in \mathbb{R}^l | \rho(x) \neq \emptyset\}$. We define the graph of the mapping $\rho(\cdot)$ as $\mathrm{Graph}\ \rho = \{(x, x^*) | x^* \in \rho(x)\}$. A multivalued mapping $\rho(\cdot)$ is said to be *monotone*, if for all $x_1, x_2 \in \mathrm{dom}(\rho)$ and all $x_1^* \in \rho(x_1)$ and $x_2^* \in \rho(x_2)$ it holds that $\langle x_1^* - x_2^*, x_1 - x_2\rangle \geq 0$, where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^l$. In addition, $\|\cdot\|$ denotes the corresponding Euclidean norm. A multivalued mapping $\rho(\cdot)$ is said to be *maximal monotone*, if $\rho(\cdot)$ is monotone and no enlargement of its graph is possible without destroying monotonicity, or more precisely, if for every pair $(x_1, x_1^*) \notin \mathrm{Graph}\ \rho$ there exists a point $(x_2, x_2^*) \in \mathrm{Graph}\ \rho$ with $\langle x_1^* - x_2^*, x_1 - x_2\rangle < 0$. For two (multivalued) functions $F : \mathbb{R}^k \hookrightarrow \mathbb{R}^l$ and $G : \mathbb{R}^l \hookrightarrow \mathbb{R}^m$, $G \circ F$ denotes their composition, i.e. for $x \in \mathbb{R}^k$ we define $G \circ F(x) = G(F(x)) := \bigcup_{y\in F(x)} G(y)$.

For an interval $I \subseteq \mathbb{R}$, we denote by $\mathcal{L}^1(I, \mathbb{R}^n)$ and $\mathcal{L}^1_{loc}(I, \mathbb{R}^n)$ the Lebesgue space of integrable and locally integrable functions, respectively, from I to $\mathbb{R}^n$. A function $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is separately measurable if $f(\cdot, x) : I \to \mathbb{R}^n$ is measurable for all $x \in \mathbb{R}^n$. An absolutely continuous (AC) function $f : [a, b] \to \mathbb{R}^n$ is a function that can be written as $f(x) - f(a) = \int_a^x \dot{f}(t)dt$ for any $x \geq a$ for a function $\dot{f} \in \mathcal{L}^1([a, b], \mathbb{R}^n)$, which is considered as its derivative. An absolutely continuous function is almost everywhere differentiable. A function $f : I \to \mathbb{R}^n$ is locally AC if it is AC for any bounded interval $[a, b] \subset I$.

The normal cone to a convex set $S \subset \mathbb{R}^n$ at a point $x \in S$ is $N(S; x) = \{z \in \mathbb{R}^n \mid \langle z, y - x\rangle \leq 0, \text{ for all } y \in S\}$. A convention is that $N(S; x) = \emptyset$ if $x \notin S$. Next we present a technical lemma that will be of importance in the remainder of the paper and can be derived from [19, Theor. 12.43].

Lemma 2.1: Let $\mathcal{H} : \mathbb{R}^n \to \mathbb{R}^l$ be the affine mapping given by $\mathcal{H}(x) = Hx + h$ for all $x \in \mathbb{R}^n$ corresponding to the matrix $H \in \mathbb{R}^{l\times n}$ and the vector $h \in \mathbb{R}^l$ and suppose that H has full row rank l. Then the following statements hold. If $\rho : \mathbb{R}^l \hookrightarrow \mathbb{R}^l$ is a maximal monotone mapping with $\mathrm{dom}\ \rho \neq \emptyset$, then the mapping $x \mapsto H^\top \rho(Hx + h)$ is also maximal monotone. □

B. Monotone Differential Inclusions

Let us consider the differential inclusion (DI)

$$\dot{x}(t) \in -F(x(t)) + f(t), \quad x(t_0) = x_0 \in \mathrm{dom}(F). \tag{3}$$

Definition 2.2: A locally AC trajectory $x : [t_0, \infty) \to \mathbb{R}^n$ is called a solution to the DI and the initial condition in (3), if $x(t_0) = x_0$, for all $t \in [t_0, \infty)$ $x(t) \in \mathrm{dom}(F)$, and for almost all $t \in [t_0, \infty)$ $\dot{x}(t) \in -F(x(t)) + f(t)$. □

The following result is a generalization of the Hille-Yosida Theorem [12, Theor. 3.7.1].

Theorem 2.3: Let $F : \mathbb{R}^n \hookrightarrow \mathbb{R}^n$ be a maximal monotone operator and $f : \mathbb{R} \to \mathbb{R}^n$ be locally AC. Then the DI and the initial condition in (3) with $x_0 \in \mathrm{dom}(F)$ possess a unique locally AC solution $x(\cdot)$ on $[t_0, +\infty)$ in the sense of Definition 2.2. □

C. Perturbed Moreau's Sweeping Process

Let us now deal with another class of DIs consisting of so-called *perturbed Moreau's sweeping processes*, see [10], in which the multivalued mapping $\rho(\cdot)$ can possibly be time-varying. In particular, we consider $\rho(t, x) = N(S(t); x)$ the normal cone to the set $S(t)$ at x. The resulting DIs are then given by

$$\begin{cases} -\dot{x}(t) \in N(S(t); x(t)) + f(t, x(t)) & \text{a.e. } t \in [t_0, +\infty), \\ x(t_0) = x_0 \in S(t_0) \end{cases} \tag{4}$$

in case one is interested in locally AC solutions. The difference between (3) and (4) is reflected by the fact that in (3) the multivalued mapping $F(\cdot)$ depends on x only and $f(\cdot)$ on the time only, while in (4) both the multivalued mapping and $f(\cdot,\cdot)$ depend both on the time t and the state x. In addition, the solution trajectories of (4) might have to be considered in a larger function class consisting of functions of local bounded variation when the set $S(\cdot)$ is not varying in a locally AC manner (see (A2) below). In that case, (4) has to be interpreted as a measure differential inclusion, a term coined by J.J. Moreau (see e.g. [10], [16] and [1] for an introduction to Moreau's sweeping process) in which the solutions are of bounded variation and can allow for discontinuities in the state trajectory $x(\cdot)$. Some of the well-posedness and observer design techniques presented below can be extended to this case as well, see [7] for more details.

It is noteworthy that when $S(t)$ is closed, convex and nonempty for each t, then $N(S(t); \cdot)$ defines a maximal monotone mapping for each fixed t as proven in [19, Coroll. 12.18]

Let us now present existence and uniqueness results for the inclusions of the form (4).

Theorem 2.4: [10, Theor. 1] Let $S(\cdot)$ satisfy the following assumptions.

(A1) For each $t \geq t_0$, $S(t)$ is a nonempty, closed and convex subset of $\mathbb{R}^n$.

(A2) $S(\cdot)$ varies in an AC way, i.e. there exists an AC function v such that for any $y \in \mathbb{R}^n$ and $s, t \geq t_0$

$$|d(y, S(t)) - d(y, S(s))| \leq \|v(t) - v(s)\|,$$

where $d(y, S) = \inf\{\|y - x\| \mid x \in S\}$.

Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be a separately measurable map on $I = [t_0, t_1]$ with $t_1 < +\infty$ such that:

- For every $\eta > 0$ there exists a nonnegative function $k_\eta(\cdot) \in \mathcal{L}^1(I, \mathbb{R})$ such that for all $t \in I$ and for any $(x, y) \in B(0, \eta) \times B(0, \eta)$ one has $\|f(t, x) - f(t, y)\| \leq k_\eta(t)\|x - y\|$;
- there exists a nonnegative function $\beta(\cdot) \in \mathcal{L}^1(I, \mathbb{R})$ such that, for all $t \in I$ and for all $x \in \bigcup_{s \in I} S(s)$, $\|f(t, x)\| \leq \beta(t)(1 + \|x\|)$.

Then for any $x_0 \in S(t_0)$ the inclusion (4) has a unique AC solution $x(\cdot)$ on I (in a sense similar to Definition 2.2). □

The first condition is a kind of local Lipschitz continuity property in the second variable of $f(\cdot, \cdot)$ and the second condition is a natural growth condition. In case $t_1 = \infty$ then the theorem provides a result on the existence and uniqueness of a *locally* AC solution in a straightforward manner (in which $k_\eta(\cdot)$ and $\beta(\cdot)$ become $\mathcal{L}^1_{loc}$-functions and $S(\cdot)$ varies in a locally AC manner).

III. Problem Statement

Consider the system (1), where a time-independent multivalued map $\rho(\cdot)$ is in the feedback loop, which yields the following DI:

$$\begin{cases} \dot{x}(t) = Ax(t) - Gw(t) + Bu(t) \\ w(t) \in \rho\left(Hx(t)\right) \\ y(t) = Cx(t), \quad x(0) = x_0 \in \mathrm{dom}(\rho \circ H), \end{cases} \tag{5}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $G \in \mathbb{R}^{n \times l}$, $H \in \mathbb{R}^{l \times n}$ and $C \in \mathbb{R}^{p \times n}$. Moreover, we assume that G has full column rank, the mapping $\rho : \mathbb{R}^l \hookrightarrow \mathbb{R}^l$ is maximal monotone and $u(\cdot)$ is locally AC. To make the observer design meaningful, it is natural to assume that the system (5) whose state $x(\cdot)$ is to be estimated by an observer, allows solutions for all relevant initial states and input functions.

Assumption 3.1: The system in (5) possesses a locally AC solution $x(\cdot)$ on $[0, +\infty)$ for any $x(0) \in \mathrm{dom}(\rho \circ H)$ and locally AC input functions $u(\cdot)$. □

Such conditions can be verified by results in the mathematical literature on DIs [9], [11] (in case of local boundedness of $\rho(\cdot)$) or the results in Section II above, which also apply when $\rho(z)$ is an unbounded set for some $z \in \mathbb{R}^l$.

The first proposed observer ("basic" observer scheme) for the system (5) has the following form:

$$\begin{cases} \dot{\hat{x}}(t) = (A - LC)\hat{x}(t) - G\hat{w}(t) + Ly(t) + Bu(t) \\ \hat{w}(t) \in \rho\left(H\hat{x}(t)\right) \end{cases} \tag{6}$$

where $L \in \mathbb{R}^{n \times p}$ is the observer gain and $H\hat{x}(0) \in \mathrm{dom}(\rho)$. The second proposed observer ("extended" observer scheme) has the following form:

$$\begin{cases} \dot{\hat{x}}(t) = (A - LC)\hat{x}(t) - G\hat{w}(t) + Ly(t) + Bu(t) \\ \hat{w}(t) \in \rho\left((H - KC)\hat{x}(t) + Ky(t)\right) \end{cases} \tag{7}$$

where $L \in \mathbb{R}^{n \times p}$ and $K \in \mathbb{R}^{l \times p}$ are the observer gains and $\hat{x}(0)$ is such that $(H - KC)\hat{x}(0) + Ky(0) \in \mathrm{dom}(\rho)$. The basic observer is a special case of the extended observer with $K = 0$. The idea of the extended observer was also used for Lipschitz continuous (single-valued) systems in [2]. Although the basic observer is much more natural and closer to the observer structures for linear systems, there exist multivalued systems as in (1) for which no basic observer can be constructed by the design procedure given below, but one can still find an extended observer as will be demonstrated in Example 4.1 below.

Problem 3.2: The problem of observer design consists in finding the gain L for the basic observer or the gains L and K for the extended observer, such that:

- **Observer well-posedness:** for each solution $x(\cdot)$ to the observed plant (5) there exists a unique solution $\hat{x}(\cdot)$ to the observer dynamics on $[0, \infty)$.
- **Asymptotic state recovery:** $\hat{x}(\cdot)$ asymptotically recovers $x(\cdot)$, i.e. $\lim_{t \to \infty}[\hat{x}(t) - x(t)] = 0$. □

Remark 3.3: As we only assume the existence of solutions in Assumption 3.1, it might be the case that the observed plant (5) allows for multiple solutions given an initial condition $x(0) = x_0$. However, as the solution trajectory $x(\cdot)$ enters the observers (6) and (7) through the measured output $y(\cdot)$, the observer dynamics has different terms in the right-hand side corresponding to the different solutions. Indeed, let $x^a(\cdot)$ and $x^b(\cdot)$ be two solutions of the observed plant for $x(0) = x_0$ and let $y^a = Cx^a$ and $y^b = Cx^b$ be the corresponding outputs. When the observer dynamics produces unique solutions given an initial condition and given the driving inputs u and y, as imposed in Problem 3.2, the observer has for each individual solution trajectory of the observed plant a unique response. To be more precise, for initial condition $\hat{x}(t_0) = \hat{x}_0$ and external inputs $u(\cdot)$ and $y(\cdot)$, where the latter can be $y = y^a = Cx^a$ or $y = y^b = Cx^b$, the observer has two *different* solutions, say $\hat{x}^a(\cdot)$ and $\hat{x}^b(\cdot)$, respectively. The problem definition above should be interpreted in the sense that it requires in this case that both $\lim_{t \to \infty}[\hat{x}^a(t) - x^a(t)] = 0$ and $\lim_{t \to \infty}[\hat{x}^b(t) - x^b(t)] = 0$. In essence, this is not different than for observer design for linear or smooth nonlinear systems with unique trajectories given an initial condition and exogenous inputs: the observer just recovers asymptotically the state trajectory that corresponds to the input and output trajectories that are actually fed to it. From a practical point of view, the given interpretation is meaningful, because the actual physical plant (for which (5) is only a model) typically behaves according to only one of the possible solutions as allowed by the model, for instance, due to the presence of small disturbances. □

IV. Main Results

In this section we will prove that if the gains L and K are chosen such that the triple $(A - LC, G, H)$ (respectively $(A - LC, G, H - KC)$) is strictly passive, then the obtained observer (6) [(7), respectively] will satisfy the requirements mentioned in Problem 3.2. To compute the gains L and K such that $(A - LC, G, H - KC)$ is strictly passive, one can solve the matrix (in)equalities:

$$\begin{aligned} & -(A - LC)^\top P - P(A - LC) \succ 0 \\ & P = P^\top \succ 0 \\ & G^\top P = H - KC \end{aligned} \tag{8}$$

where Q in (2) can then be taken as $Q := -(A - LC)^\top P - P(A - LC)$. Condition (8) is a linear matrix inequality (LMI) in P, K, $L^\top P$, which can be solved efficiently. For necessary and sufficient conditions for the existence of solutions to (8), see for instance [2].

To motivate the introduction of the extended observer next to the basic observer, consider the following example.

Example 4.1: Consider (5) with the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ C = (1\ 0);\ G = (1\ 0)^\top;\ H = (0\ 1) \tag{9}$$

and B arbitrary. As A is unstable, the triple (A, G, H) is not (strictly) passive by itself. If we try to find a matrix L such that $(A - LC, G, H)$ is strictly passive, we have to satisfy

$$-(A - LC)^\top P - P(A - LC) \succ 0,\ P = P^\top \succ 0 \text{ and } G^\top P = H$$

for some matrix P. Since the condition $G^\top P = H$ requires $(1\ 0)P = (0\ 1)$, we can conclude that the first row of P must be equal to $(0\ 1)$,

which obstructs the positive definiteness of P. Hence, one cannot make $(A-LC, G, H)$ strictly passive by suitable choice of L. Consequently, we will not be able to find a basic observer using the results below as they require $(A-LC, G, H)$ to be strictly passive. However, with $L = (2\ 0)^\top$ and $K = -1$, we get $A - LC = -I_2$ and $H - KC = (1\ 1)$. Since the matrix $P = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ is positive definite and symmetric and satisfies $Q := -(A - LC)^\top P - P(A - LC) \succ 0$ and $G^\top P = H - KC$, $(A - LC, G, H - KC)$ is strictly passive and an extended observer can be designed based on the theory presented next. □

A. Well-Posedness of the Observer

To prove that strict passivity of $(A - LC, G, H - KC)$ guarantees the proper behavior of the observer, we start with two lemmas on well-posedness. We will start with the case $K = 0$ (the basic observer)

Lemma 4.2: **[Time-independent $\rho(\cdot)$, basic observer, AC solutions]** Consider the system (5) and the basic observer (6). We assume that the triple $(A - LC, G, H)$ is strictly passive, G has full column rank, $\rho(\cdot)$ is maximal monotone, and Assumption 3.1 holds. Let $u(\cdot)$ be a locally AC input function and $x(\cdot)$ a corresponding locally AC solution to (5) with output trajectory $y(\cdot)$ for some $x(0) \in \mathrm{dom}(\rho \circ H)$. Then the corresponding observer dynamics (6) has a unique locally AC solution on $[0, \infty)$ for any initial state $\hat{x}(0) \in \mathrm{dom}(\rho \circ H)$. □

Proof: Since the triple $(A - LC, G, H)$ is strictly passive and G has full column rank there exist positive definite and symmetric matrices P and Q that satisfy (2) with $K = 0$. Applying the change of variables introduced in [6]

$$\xi = R\hat{x}, \tag{10}$$

where $R = P^{1/2}$, transforms (6) into

$$\begin{cases} \dot{\xi}(t) = R(A - LC)R^{-1}\xi(t) - RG\hat{w}(t) + RBu(t) + RLy(t) \\ \hat{w}(t) \in \rho\left(HR^{-1}\xi(t)\right). \end{cases} \tag{11}$$

Since $H\hat{x}(0) \in \mathrm{dom}(\rho)$, we have $HR^{-1}\xi(0) \in \mathrm{dom}(\rho)$. Define the mapping $\beta : \mathbb{R}^n \hookrightarrow \mathbb{R}^n$ as $\beta(\xi) = R^{-1}H^\top\rho(HR^{-1}\xi)$. Using the strict passivity condition that yields $G^\top P = H$, (11) can be rewritten as

$$\dot{\xi}(t) \in R(A - LC)R^{-1}\xi(t) - \beta(\xi(t)) + RBu(t) + RLy(t) \tag{12}$$

where $\xi(0) \in \mathrm{dom}(\beta)$. From the strict passivity condition (2) (with $K = 0$) and the full-column rank of G, it follows that $H = G^\top P$ and $HR^{-1} = G^\top R$ have full row rank. Together with the fact that $\rho(\cdot)$ is maximal monotone we have that $\beta(\cdot)$ is maximal monotone as well due to Lemma 2.1. From the strict passivity condition (8) it follows that $R^{-1}(A - LC)^\top R + R(A - LC)R^{-1}$ is negative definite, which means that the mapping $\xi \mapsto -R(A - LC)R^{-1}\xi$ is monotone by definition. Maximality of the mapping $\xi \mapsto -R(A - LC)R^{-1}\xi$ follows from linearity, see [4, Prop. 2.3]. Hence, the mapping $\xi \mapsto -R(A - LC)R^{-1}\xi + \beta(\xi)$ is maximal monotone as the sum of maximal monotone mappings is maximal monotone again [19, Coroll. 12.44]. Since the signal $u(\cdot)$ is locally AC, and $y(\cdot)$ is locally AC due to Assumption 3.1, existence and uniqueness of locally AC solutions to (12) and (6) follow now from Theorem 2.3. □

In the following lemma we address the question of well-posedness of the extended observer scheme. Since in this case the multivalued mapping in (7) is *time-dependent*, we will consider a particular class of mappings $\rho(\cdot)$ that corresponds to Section II-C. Actually, in this case it will turn out that the second and third condition in (8), i.e. the existence of a symmetric positive definite matrix P such that $G^\top P = H - KC$ suffices to prove well-posedness.

Lemma 4.3: **[Time-independent $\rho(\cdot) = N(S; \cdot)$, extended observer, AC solutions]** Consider the system (5) and the extended observer (7) with $\rho(\cdot) = N(S; \cdot)$, where the set $S \subset \mathbb{R}^l$ is assumed to be nonempty, closed and convex. Suppose that Assumption 3.1 holds and assume that there exists a symmetric positive definite matrix P such that $G^\top P = H - KC$ and G has full column rank. Let the signal $u(\cdot)$ be locally AC and let $x(\cdot)$ be a corresponding locally AC solution to (5) with output trajectory $y(\cdot)$ for some $x(0)$ with $Hx(0) \in S$. Then the corresponding observer dynamics (7) has a unique locally AC solution on $[0, \infty)$ for each $\hat{x}(0)$ with $(H - KC)\hat{x}(0) + Ky(0) \in S = \mathrm{dom}(\rho)$. □

Proof: Let us introduce the change of variable (10) for (7), where as before, $R = P^{1/2}$. In the same way as in the proof of Lemma 4.2, (7) is transformed into

$$\begin{aligned} \dot{\xi}(t) \in\, & R(A - LC)R^{-1}\xi(t) - R^{-1}(H - KC)^\top\rho \\ & \times \left((H - KC)R^{-1}\xi(t) + Ky(t)\right) + RBu(t) + RLy(t). \end{aligned} \tag{13}$$

Let $S'(t) = \{\xi \in \mathbb{R}^n \mid (H - KC)R^{-1}\xi + Ky(t) \in S\} \subset \mathbb{R}^n$. Since G has full column rank l, $(H - KC)R^{-1} = G^\top R$ has full row rank l. As S is nonempty by the hypothesis, this implies that $S'(t)$ is nonempty for each t. Indeed, S is a subset of $\mathbb{R}^l$ and the full row rank of $(H - KC)R^{-1}$ implies that for any l-dimensional vector z of S. one can find at least one ξ such that $(H - KC)R^{-1}\xi = z - Ky(t)$. By [15, Prop. 1.2.4], the closedness and convexity of S carry over to $S'(t)$ for each t. Hence, condition (A1) of Theorem 2.4 is satisfied. In addition, the fact that $y(\cdot)$ is locally AC implies that $S'(\cdot)$ varies in a locally AC manner as in (A2). Consider now $N(S'(t); x)$. By applying Lemma 2.1 (for fixed t) we obtain that $N(S'(t); \xi) = R^{-1}(H - KC)^\top N(S, (H - KC)R^{-1}\xi + Ky(t))$. Therefore, one can rewrite (13) as

$$\begin{aligned} -\dot{\xi}(t) + R(A - LC)R^{-1}\xi(t) - RBu(t) \\ -RLy(t) \in N\left(S'(t); \xi(t)\right) \end{aligned} \tag{14}$$

where $\xi(0) = R\hat{x}(0) \in S'(0)$. The description (14) fits within (4) with $S'(\cdot)$ satisfying the conditions of Theorem 2.4. Since $u(\cdot)$ and $y(\cdot)$ are locally AC, the result follows now from Theorem 2.4. □

B. Asymptotic Recovery of the State

The following theorem states the main result of the paper.

Theorem 4.4: Consider the observed system (5) and either the basic observer (6) or the extended observer (7), where $(A - LC, G, H)$ or $(A - LC, G, H - KC)$, respectively, is strictly passive with corresponding positive definite and symmetric matrices P and Q satisfying (2). Assume also that the additional conditions of Lemma 4.2 or Lemma 4.3, respectively, are satisfied. Let $x(\cdot)$ be a locally AC solution to (5) for $x(0) \in \mathrm{dom}(\rho \circ H)$ and locally AC input $u : [0, +\infty) \to \mathbb{R}^m$. Then the observer (6) or (7), respectively, has for each $\hat{x}(0)$ with $H\hat{x}(0) \in \mathrm{dom}(\rho)$ or $(H - KC)\hat{x}(0) + Ky(0) \in \mathrm{dom}(\rho) = S$, respectively, a unique locally AC solution $\hat{x}(\cdot)$, which exponentially recovers the state x in the sense that the observation error $e(t) := x(t) - \hat{x}(t)$ satisfies the exponential decay bound

$$\|e(t)\| \le \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}\,\|e(0)\|\, e^{\left(-\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}t\right)} \tag{15}$$

for $t \in \mathbb{R}_+$. □

Proof: Using Lemma 4.2 or Lemma 4.3 for the basic and extended observers, respectively, it follows that for each locally AC solution to the observer plant (5), the observer also has a locally AC solution $\hat{x}(\cdot)$ provided $H\hat{x}(0) \in \mathrm{dom}(\rho)$ or $(H - KC)\hat{x}(0) + Ky(0) \in \mathrm{dom}(\rho) = S$, respectively. Hence, the observation error $e(\cdot) = x(\cdot) - \hat{x}(\cdot)$ is also locally AC and satisfies for the extended observer (7) almost everywhere the error dynamics, obtained by subtracting (5) and (7)

$$\dot{e}(t) = (A - LC)e(t) - G\left(w(t) - \hat{w}(t)\right) \tag{16a}$$

$$w(t) \in \rho\left(Hx(t)\right) \tag{16b}$$
$$\hat{w}(t) \in \rho\left(Hx(t) - (H - KC)e(t)\right). \tag{16c}$$

Note that the error dynamics for the basic observer is obtained as a special case of (16) by taking $K = 0$. We consider now the candidate Lyapunov function $V(e) = (1/2)e^\top Pe$. Since e is locally AC, $t \mapsto V(e(t))$ is also locally AC, and the derivative $\dot{V}(e(t))$ exists for almost all t. Hence, $\dot{V}(e(t))$ satisfies for almost all t

$$\begin{aligned}\dot{V}\left(e(t)\right) &= e^\top(t)P\dot{e}(t)\\ &= e^\top(t)P\left((A - LC)e(t) - G\left(w(t) - \hat{w}(t)\right)\right)\\ &= -\frac{1}{2}e^\top(t)Qe(t)\\ &\quad - e^\top(t)(H - KC)^\top\left(w(t) - \hat{w}(t)\right)\end{aligned} \tag{17}$$

for some $w(t)$, $\hat{w}(t)$ satisfying (16b) and (16c). Since

$$\begin{aligned}&e^\top(t)(H - KC)^\top\left(w(t) - \hat{w}(t)\right)\\ &\quad = \left\langle Hx(t) - \left\{(H - KC)\hat{x}(t) + Ky(t)\right\}, w(t) - \hat{w}(t)\right\rangle\end{aligned}$$

with $w(t) \in \rho(Hx(t))$ and $\hat{w}(t) \in \rho(H\hat{x}(t) + K(y(t) - \hat{y}(t)))$, it follows from the monotonicity of $\rho(\cdot)$ that $e^\top(t)(H - KC)^\top(w(t) - \hat{w}(t)) \geq 0$. Note that in the case of the extended observer and thus under the conditions of Lemma 4.3, that $\rho(\cdot) = N(S;\cdot)$ is also monotone. Therefore,

$$\dot{V}\left(e(t)\right) \leq -\frac{1}{2}e^\top(t)Qe(t). \tag{18}$$

As $e^\top Qe \geq \lambda_{\min}(Q)e^\top e \geq (\lambda_{\min}(Q)/\lambda_{\max}(P))e^\top Pe = (2\lambda_{\min}(Q)/\lambda_{\max}(P))V(e)$ for all $e \in \mathbb{R}^n$, we have that

$$\dot{V}\left(e(t)\right) \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}V\left(e(t)\right). \tag{19}$$

Clearly, this implies that

$$\begin{aligned}\lambda_{\min}(P)\left\|e(t)\right\|^2 &\leq V\left(e(t)\right) \leq e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}t}V\left(e(0)\right)\\ &\leq \lambda_{\max}(P)e^{-\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}t}\left\|e(0)\right\|^2.\end{aligned}$$

This proves the exponential recovery of the state. The condition (15) is obtained by taking the square root of the inequality above. □

Two remarks on the above results are in order.

Remark 4.5: The above derived methodologies were successfully applied to the observer design problem for an experimental setup of deep sea oil drilling equipment in the preliminary version [13] of this paper. The multivalued mapping was caused by the presence of dry friction with Stribeck effect that could be well described by a multivalued friction characteristic. The underlying linear model was not passive itself. Only the extended observer turned out to be applicable in this case to exponentially recover the state of the drilling equipment. See [13] for more details.

Remark 4.6: Possible extensions of the observer design methodologies to other classes of differential inclusions are possible. For instance, extensions of Theorem 4.4 to more general systems of the form

$$\begin{cases}\dot{x}(t) = Ax(t) - Gw(t) + Bu(t)\\ w(t) \in \rho\left(Hx(t) - Dw(t) + Eu(t)\right)\\ y(t) = Cx(t)\end{cases} \tag{20}$$

may be obtained provided one can prove the well-posedness property of the corresponding observer. The stability proof can be carried over *mutatis mutandis*, but the hard part of the proof would be obtaining existence and uniqueness of solutions $\hat{x}(\cdot)$ for the observer dynamics. Another direction for extension of Lemma 4.3, for which partial results are reported in [7], is the consideration of perturbed Moreau's sweeping processes in which the set S becomes time-varying as well (possibly not in a locally AC manner). This results in state trajectories that possess discontinuities and as such, the observer design problem has to be tackled for systems with state jumps, which is known to be a highly nontrivial problem. For polyhedral sets, one can find observer design methods in the spirit of this paper in the accompanying report [7].

V. Conclusions

In this paper, we presented novel methodologies for the observer design for a class of multivalued systems consisting of Lur'e systems with maximal monotone multivalued mappings in the feedback path. As the considered class of systems is nonsmooth and multivalued, the existing literature on nonlinear observer design does not apply to this setting. We proposed two observer structures (the basic and the extended observer) and carefully examined the existence and uniqueness of solutions given an initial condition and input function (well-posedness), as this is not *a priori* guaranteed. The observer design is constructive in nature as it is based on rendering the linear part of the observation error dynamics strictly passive by choosing appropriate observer gains. Under the natural assumption that the observed system has a solution, when the control input belongs to a certain admissible class, it is shown that there exists a unique solution for the estimated state, and that the observer recovers the state of the original system exponentially.

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