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# GLOBAL SOLVABILITY OF A NETWORKED INTEGRATE-AND-FIRE MODEL OF MCKEAN-VLASOV TYPE

F. DELARUE, J. INGLIS, S. RUBENTHALER, E. TANRÉ

ABSTRACT. We here investigate the well-posedness of a networked integrate-and-fire model describing an infinite population of neurons which interact with one another through their common statistical distribution. The interaction is of the self-excitatory type as, at any time, the potential of a neuron increases when some of the others fire: precisely, the kick it receives is proportional to the instantaneous proportion of firing neurons at the same time. From a mathematical point of view, the coefficient of proportionality, denoted by  $\alpha$ , is of great importance as the resulting system is known to blow-up for large values of  $\alpha$ . In the current paper, we focus on the complementary regime and prove that existence and uniqueness hold for all time when  $\alpha$  is small enough.

KEYWORDS: McKean nonlinear diffusion process; renewal process; first hitting time density estimates; integrate-and-fire network; nonhomogeneous diffusion process; neuroscience.

## 1. INTRODUCTION

The stochastic integrate-and-fire model of the membrane potential  $V$  across a neuron in the brain has received a huge amount of attention since its introduction (see [15] for a comprehensive review). The central idea is to model  $V$  by threshold dynamics, in which the potential is described by a simple linear (stochastic) differential equation up until it reaches a fixed threshold value  $V_F$ , at which point the neuron emits a ‘spike’. Experimentally, at this point an action potential is observed, whereby the potential increases very rapidly to a peak (hyperpolarization phase) before decreasing quickly to a reset value (depolarization phase). It then relatively slowly increases once more to the resting potential (refractory period).

Since spikes are stereotyped events, they are fully characterized by the times at which they occur. The integrate-and-fire model is part of a family of spiking neuron models which take advantage of this by modeling only the spiking times and disregarding the nature of the spike itself. Specifically, in the integrate-and-fire model we observe jumps in the action potential as the voltage is immediately reset to a value  $V_R$  whenever it reaches the threshold  $V_F$ , which is motivated by the fact that the time period during which the action potential is observed is very small. Despite its simplicity, versions of the integrate-and-fire model have been able to predict the spiking times of a neuron with a reasonable degree of accuracy [8, 9].

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Many extensions of the basic integrate-and-fire model have been studied in the neuroscientific literature, including ones in which attempts are made to include noise and to describe the situation when many integrate-and-fire neurons are placed in a network and interact with each other. Indeed, in [11] and [12] the following equation describing how the potential  $V_i$  of the  $i$ -th neuron in a network of  $N$  behaves in time is proposed:

$$\frac{d}{dt}V_i(t) = -\lambda V_i(t) + \frac{\alpha}{N} \sum_j \sum_{n=-\infty}^{\infty} \delta(t - t_n^j) + \frac{\beta}{N} \sum_{j \neq i} V_j(t) + I_i^{ext}(t) + \sigma \eta_i(t) \quad (1.1)$$

for  $V_i(t) < V_F$  and where  $V_i(t)$  is immediately reset to  $V_R$  when it reaches  $V_F$ . Here  $I_i^{ext}(t)$  represents the external input current to the neuron,  $\eta_i(t)$  is the noise (a white noise) which is importantly supposed to be independent from neuron to neuron, and the constants  $\lambda, \beta, \alpha$  and  $\sigma$  are chosen according to experimental data. Moreover, the interaction term is described in terms of  $t_n^j$ , which is the time of the  $n$ -th spike of neuron  $j$ , and the Dirac function  $\delta$ . Precisely, it says that whenever one of the other neurons in the network spikes, the potential across neuron  $i$  receives a ‘kick’ of size  $\alpha/N$ . The Dirac mass interactions give rise to the same kind of instantaneous behavior as the integrate-and-fire model. Although it is a simplification of reality, it produces some interesting phenomena from a biological perspective, as observed in [12].

In the case of a large network i.e. when  $N$  is large, many authors approximate the interaction term in the above finite system by an instantaneous rate  $\nu(t)$ , the so-called mean-firing rate (see for example [1, 2, 12, 14]). However, in the neuroscience literature little attention is paid to how this convergence is achieved. Mathematically the mean-field limit as  $N \rightarrow \infty$  must be taken, and it is not at all clear that one observes a propagation of chaos phenomenon. This is the subject of a forthcoming manuscript [5].

However, the question we consider here is whether or not solutions exist for the resulting nonlinear equation when the mean-field limit is taken, and if so what type of solutions are admissible. Again this is not treated rigorously in the neuroscience literature. At first glance such a question may seem classical, given the volume of results available that guarantee the existence of a solution to distribution dependent SDEs. However, as quickly became apparent in our analysis, in the excitatory case ( $\alpha > 0$ ) the problem is in fact a delicate one, for which, to our knowledge, there are no existing results available. This difficulty is due to the nature of the interactions, which introduce the strong possibility of a solution that ‘blows up’ in finite time. The validity of the study of this question, and its non-trivial nature, is further justified by the fact that several authors have recently been interested in exactly the same problem from a PDE perspective ([3, 4]). Despite some serious effort and very interesting related results on their part, we understand that they were not able to prove the existence of global solutions to the mean-field equation, which is the main result of the present paper.

**1.1. Precisions:** We now make precise the nonlinear equation of interest. Firstly, since the mathematical difficulties lie within the jump interaction term, we in fact

isolate it by making two simplifications. Indeed, we suppose that there is no external input current ( $I_i^{ext}(t) \equiv 0$ ), and that the interaction term is composed solely of the jump or reset part ( $\beta = 0$ ). Although this is a non-trivial simplification from a neuroscience perspective, it still captures all the mathematical complexity of the resulting mean-field equation.

Without loss of generality, we also take the firing threshold  $V_F = 1$  and the reset value  $V_R = 0$  for notational simplicity. The precise nonlinear stochastic mean-field equation under study here then is

$$X_t = X_0 + \int_0^t b(X_s)ds + \alpha \mathbb{E}(M_t) + \sigma W_t - M_t, \quad t \geq 0, \quad (1.2)$$

where  $X_0 < 1$  almost surely,  $\alpha \in \mathbb{R}$ ,  $\sigma > 0$ ,  $(W_t)_{t \geq 0}$  is a standard Brownian motion in  $\mathbb{R}$  and  $b: \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz continuous. The jumps, or resets, of the system are described by

$$M_t = \sum_{k \geq 1} \mathbf{1}_{[0,t]}(\tau_k)$$

where  $(\tau_k)_{k \geq 1}$  stands for the sequence of hitting times of 1 by the process  $(X_t)_{t \geq 0}$ . That is  $(M_t)_{t \geq 0}$  counts the number of times  $X_t$  hits the threshold before time  $t$ , so that  $\mathbb{E}(M_t)$  denotes the expected number of times the threshold is reached before  $t$ . This equation corresponds to what we would expect when the limit  $N \rightarrow \infty$  is taken in (1.1) when the above simplifications are made, where the resets are now included in the equation by the presence of the counting process  $(M_t)_{t \geq 0}$ . It is indeed the case that the associated finite system of  $N$  neurons converges to this nonlinear limit equation (again this is shown in our forthcoming manuscript [5]).

**1.2. PDE viewpoint and ‘blow-up’ phenomenon:** In fact, as mentioned above, equation (1.2) has been rigorously studied from the PDE viewpoint before. Indeed, when  $\sigma \equiv 1$ , the associated Fokker-Planck equation for the density  $p(t, y)dy = \mathbb{P}(X_t \in dy)$  is given by

$$\partial_t p(t, y) + \partial_y [(b(y) + \alpha e'(t)) p(t, y)] - \frac{1}{2} \partial_{yy}^2 p(t, y) = \delta_0(y) e'(t), \quad y < 1,$$

where  $e(t) = \mathbb{E}(M_t)$ , subject to  $p(t, 1) = 0$ ,  $p(t, -\infty) = 0$ ,  $p(0, y)dy = \mathbb{P}(X_0 \in dy)$  and where  $\delta_0$  is the Dirac measure at 0. Moreover, the condition that  $p(t, y)$  must remain a probability density translates into the fact that

$$e'(t) = \frac{d}{dt} \mathbb{E}(M_t) = -\frac{1}{2} \partial_y p(1, t), \quad \forall t > 0,$$

which describes the nonlinearity of the problem. In the case when  $b(x) = -x$ , this nonlinear Fokker-Planck equation is exactly the one studied in [3] and [4]. Therein, the authors look for solutions to this equation, and conclude that for some choices of parameters no global-in-time solutions exist. The term ‘blow-up’ is then used to describe the situation where the solution (defined in a weak sense) ceases to exist after some finite time. With our formulation of the problem, since the quantity  $e'(t)$  corresponds to the mean-firing rate of the infinite network, it is very natural to define a ‘blow-up’ time as a time when  $e'(t)$  becomes infinite. Intuitively, this can

be understood as a point in time at which a large proportion of the neurons in the network all spike at exactly the same time i.e. the network *synchronizes*.

In [3] and [4] it is shown that, in the cases  $\alpha = 0$  and  $\alpha < 0$  (the latter one being referred to as ‘self-inhibitory’ in neuroscience), the nonlinear Fokker-Planck equation has a unique solution that does not blow-up in finite time (we will say that the equation is *globally solvable*). However, in the so-called ‘self-excitatory’ framework, i.e. for  $\alpha > 0$ , existence of a solution for all time is left open. Instead, a negative result is established [3, Theorem 2.2], stating that, for any  $\alpha > 0$ , it is possible to find an initial probability distribution  $\mathbb{P}(X_0 \in dy)$  such that any solution must blow-up in finite time, i.e. such that  $e'(t) = \infty$  for some  $t > 0$ . As a remarkable point, it says that solvability in the long run may fail for possibly small values of  $\alpha$ . On the contrary, it does not come as a surprise that solutions may explode in a finite time when  $\alpha \geq 1$  ( $= V_F - V_R$ ).

**1.3. Present contribution.** In this paper we thus investigate the case  $\alpha \in (0, 1)$ . Our main contribution is to show that, given a starting point  $X_0 = x_0$ , we can find an explicit  $\alpha$  small enough so that there does indeed exist a unique global-in-time solution to (1.2) (and hence to the associated Fokker-Planck equation) which does not blow-up (see Theorem 2.4). In view of the above discussions, our result complements and goes further than those found in [3] and [4], and the surprising difficulty of the problem is reflected in the rather involved nature of our proofs.

Our result is for a general Lipschitz function  $b$ , but there are two important specific cases that we keep in mind: the Brownian case when  $b \equiv 0$  and the Ornstein-Uhlenbeck case when  $b(x) = -\lambda x$ ,  $\lambda \geq 0$ . The Ornstein-Uhlenbeck case is most relevant to neuroscience since it is often used in the literature, but surprising difficulties remain in the purely Brownian case. In both these cases we are able to give an explicit  $\alpha_0$  depending on the starting point  $x_0$  such that (1.2) has a global solution for all  $\alpha < \alpha_0$ . However, it is interesting to note that our explicit values do not appear to be optimal, in the sense that from our numerical simulations there seems to be a real ‘gap’ between our positive result and the negative result of [3]. Precisely, our simulations suggest that for a given  $x_0$  there exist global-in-time solutions for  $\alpha$  bigger than our explicit  $\alpha_0$ , while there exist solutions that blow-up that do not satisfy the conditions of [3]. Thus a very interesting, and perhaps very difficult question is to determine for a given initial starting point (or even for an initial distribution) the true critical value  $\alpha_c$  such that for  $\alpha < \alpha_c$  (1.2) has a unique global solution. A more in depth discussion of this is given in Section 3.

As already said, Equation (1.2) can be thought as of McKean-Vlasov-type, since the process  $X_t$  depends on the distribution of the solution itself. However, it is highly non-standard, since it actually depends on the distribution of the *hitting times* of the threshold by the solution. This renders the traditional approaches to McKean-Vlasov equations and propagation of chaos, such as those presented by Sznitman in [16], inapplicable, because we have no *a priori* smoothness on the law of the hitting times. Thus our results are also new in this context.

The general structure of the proof is quite typical of the methods used to investigate well-posedness of nonlinear systems, and especially well-posedness of Markovian

stochastic differential equations (or equivalently of partial differential equations of second order) involving some non-trivial nonlinearity, though the details are far from classical. Precisely, the first point is to tackle unique solvability in small time: when the parameter  $\alpha$  is (strictly) less than 1 and the mass of the initial condition decays linearly at the threshold, it is proven that the system induces a natural contraction in a well-chosen space provided the time duration is small enough. In this framework, the specific notion of a solution plays a crucial role as it defines the right space for the contraction. Below, solutions are sought in such a way that the mapping  $t \mapsto \mathbb{E}(M_t)$  is continuously differentiable. The proof is at the intersection between probability and PDEs. The second stage is then to extend existence and uniqueness from short to long times. Exactly as for PDEs of quasilinear type, the point is to prove that some key quantity is preserved as time goes by. Here, we prove that the system cannot accumulate too much mass in the right vicinity of 1. Equivalently, this amounts to showing that the Lipschitz constant of the mapping  $t \mapsto \mathbb{E}(M_t)$  cannot blow-up in a finite time. This is where the condition  $\alpha$  small enough comes in: when  $\alpha$  is small enough, we manage to give some estimates for the density of  $X_t$  in the neighbourhood of 1, the critical value of  $\alpha$  explicitly depending upon the available bound of the density. Generally speaking, we make use of standard Gaussian estimates of Aronson type of the density; in the cases  $b \equiv 0$  and  $b(x) = -\lambda x$ ,  $x \in \mathbb{R}$ , for some  $\lambda > 0$ , the bounds are explicit enough to derive a numerical  $\alpha_0$  up to which unique solvability holds (Section 8). Unfortunately, the estimates we use are rather poor as they mostly neglect the right behavior of the density of  $X_t$  at the boundary, thus yielding a non-optimal value. Anyhow, they serve as a starting point for proving a refined estimate of the gradient of the density at the boundary: this is the required ingredient for proving that, at any time  $t$ , the mass of  $X_t$  decays linearly in the neighbourhood of 1, uniformly in  $t$  in compact sets, and thus to apply the existence and uniqueness argument in small time. Such a refined estimate is proven on any interval where a solution does exist: below, it is thus referred to as an *a priori* estimate. Coupled with the result of existence and uniqueness in small time, it permits to prove by induction that existence and uniqueness hold on any finite interval and thus on the whole of  $[0, \infty)$ .

The layout of the paper is as follows. In the first section below we present the main results of our paper. Then, instead of proceeding directly to the proof, which is the mathematical core of our paper, we prefer to continue with a discussion of the afore mentioned criticality that is present in the system. Precisely we illustrate the blow-up phenomenon when  $\alpha$  is too large in the Brownian case. We hope this discussion will emphasise the surprising nature of the system, and convince the reader of the novelty of the problem. We then proceed to the proof of the main results. Section 5 is devoted to proving the existence and uniqueness of a solution to (1.2) in small time, while Section 6 proves Theorem 2.3 which shows that the system cannot blow-up in finite time.

## 2. MAIN RESULTS

As stated in the introduction, we are interested in solutions to the nonlinear McKean-Vlasov-type SDE

$$X_t = X_0 + \int_0^t b(X_s) ds + \alpha \mathbb{E}(M_t) + W_t - M_t, \quad t \geq 0, \quad (2.1)$$

where  $X_0 < 1$  almost surely,  $\alpha \in (0, 1)$  and  $(W_t)_{t \geq 0}$  is a standard Brownian motion with respect to a filtration  $(\mathcal{F}_t)_{t \geq 0}$  satisfying the usual conditions. The jumps, or resets, of the system are described by

$$M_t = \sum_{k \geq 1} \mathbf{1}_{[0, t]}(\tau_k), \quad (2.2)$$

where the sequence of stopping times  $(\tau_k)_{k \geq 0}$  are defined by  $\tau_0 = 0$  and

$$\tau_k = \inf\{t > \tau_{k-1} : X_{t-} \geq 1\}, \quad k \geq 1. \quad (2.3)$$

We assume that  $b : (-\infty, 1] \rightarrow \mathbb{R}$  is Lipschitz continuous such that

$$|b(x)| \leq \Lambda(|x| + 1), \quad |b(x) - b(y)| \leq K|x - y|, \quad \forall x, y \in (-\infty, 1].$$

**Remark 2.1.** *Although we take the intensity of the noise in (2.1) to be 1 rather than a general  $\sigma > 0$  as stated in the introduction, this results in no loss of generality. Indeed, by setting  $u = t/\sigma^2$  we can rewrite (2.1) as*

$$X_{\sigma^2 u} = X_0 + \int_0^{\sigma^2 u} b(X_s) ds + \alpha \mathbb{E}(M_{\sigma^2 u}^X) + W_{\sigma^2 u} - M_{\sigma^2 u}^X, \quad u \geq 0,$$

for an arbitrary  $\sigma > 0$ , where we have emphasized the dependence of the process  $M$  on  $X$  by writing  $M = M^X$ . Now, by Brownian scaling we have that  $W_{\sigma^2 u} = \sigma \widetilde{W}_u$  where  $\widetilde{W}$  is another Brownian motion starting from 0. Set  $\widetilde{X}_u = X_{\sigma^2 u}$ . Then, since the number of times the process  $X$  hits the threshold before time  $\sigma^2 u$  is equal to the number of time the process  $\widetilde{X}$  hits the threshold before time  $u$ , we have that  $M_{\sigma^2 u}^X = M_u^{\widetilde{X}}$ . Hence it follows that

$$\widetilde{X}_u = X_0 + \int_0^{\sigma^2 u} b(X_s) ds + \alpha \mathbb{E}(M_u^{\widetilde{X}}) + \sigma \widetilde{W}_u - M_u^{\widetilde{X}}, \quad u \geq 0.$$

By a change of variable we then have

$$\widetilde{X}_u = X_0 + \int_0^u \sigma^2 b(\widetilde{X}_s) ds + \alpha \mathbb{E}(M_u^{\widetilde{X}}) + \sigma \widetilde{W}_u - M_u^{\widetilde{X}}, \quad u \geq 0.$$

Thus, if we modify the function  $b$  accordingly,  $\widetilde{X}$  satisfies exactly the equation of interest, but now with an arbitrary noise intensity.

As discussed in the introduction, the key point is to look for a solution for which  $t \mapsto \mathbb{E}(M_t)$  is continuously differentiable, which would correspond to a solution that does not exhibit a finite time blow-up. This leads to the following definition of a solution to (2.1), where as usual  $\mathcal{C}^1[0, T]$  denotes the space of continuously differentiable functions on  $[0, T]$ .

**Definition 2.2** (Solution to (2.1)). *The coupled process  $(X_t, M_t)_{0 \leq t \leq T}$  will be said to be a solution to (2.1) up until time  $T$  if  $(M_t)_{0 \leq t \leq T}$  satisfies (2.2) subject to (2.3), the map  $([0, T] \ni t \mapsto \mathbb{E}(M_t)) \in \mathcal{C}^1[0, T]$  and  $(X_t)_{0 \leq t \leq T}$  is a strong solution of (2.1) up until time  $T$ .*

Our main result is given by the following two theorems. The first guarantees that, when  $\alpha$  is small enough, if there exists a solution to (2.1) on some finite time interval, then the solution does not blow-up on this interval.

**Theorem 2.3.** *For a given  $\epsilon \in (0, 1)$ , there exists a positive constant  $\alpha_0 \in (0, 1]$ , depending only upon  $\epsilon$ ,  $K$  and  $\Lambda$ , such that, for any  $\alpha \in (0, \alpha_0)$  and any positive time  $T > 0$ , there exists a constant  $\mathcal{M}_T$ , only depending on  $T$ ,  $\epsilon$ ,  $K$  and  $\Lambda$ , such that, for any initial condition  $X_0 = x_0 \leq 1 - \epsilon$ , any solution to (2.1) according to Definition 2.2 satisfies*

$$\forall t \in [0, T], \quad \frac{d}{dt} \mathbb{E}(M_t) \leq \mathcal{M}_T.$$

The second theorem is the main global existence and uniqueness result.

**Theorem 2.4.** *For any  $T > 0$ , initial condition  $X_0 = x_0 < 1$ , and  $\alpha < \alpha_0$ , where  $\alpha_0 = \alpha_0(x_0)$  is as in Theorem 2.3 (taking  $\epsilon = 1 - x_0$ ), there exists a unique solution to the nonlinear equation (2.1) on  $[0, T]$  according to Definition 2.2.*

**Remark 2.5.** *The proofs of these two results are presented below (the second relying on the first), but since they are rather technical we highlight here an important key fact that is used. The size of the parameter  $\alpha_0$  in Theorem 2.3 is found explicitly in terms of  $\epsilon$ ,  $K$  and  $\Lambda$  (Proposition 6.3), but more precisely it derives from the fact that in the course of our proof we must first show that, a priori, any solution on  $[0, T]$  to the nonlinear equation (2.1) with  $X_0 = x_0 \leq 1 - \epsilon$  satisfies<sup>1</sup>*

$$\frac{1}{dx} \mathbb{P}(X_t \in dx) < \frac{1}{\alpha}, \quad t \in [0, T], \quad (2.4)$$

*in a neighborhood of the threshold 1 (see Lemma 6.2). It is this restriction that determines the  $\alpha_0$  in Theorem 2.3, so that it depends only on the best a priori estimates available for the density on the left-hand side of (2.4). The stated explicit choice for  $\alpha_0$  in Proposition 6.3 merely ensures that (2.4) holds for all  $\alpha < \alpha_0$  for any potential solution.*

*This of course raises the question of the optimality of the condition (2.4), which is a very interesting problem, but is outside the scope of this paper.*

*An example derivation of this a priori estimate of the density is carried out in the case of  $b \equiv 0$  in Section 8 below, which leads to an explicit value of  $\alpha_0$ .*

We finish this section with a remark about an interesting closed formula for the nonlinearity, which is motivated by the proof of finite time blow-up in [3, Theorem 2.2].

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<sup>1</sup>In the whole paper, we use the very convenient notation  $\frac{1}{dx} \mathbb{P}(X \in dx)$  to denote the density at point  $x$  of the random variable  $X$  (whenever it exists).

**Remark 2.6** (A closed formula for  $\mathbb{E}(M_t)$ ). *Thanks to the knowledge about the existence of a solution for  $\alpha$  sufficiently small provided by Theorem 2.4, we can extract some interesting formulae by modifying the methods in [3].*

*Indeed, fix  $x_0 < 1$  and suppose that we are in the Ornstein-Uhlenbeck case, so that  $b(x) = -\lambda x$  with  $\lambda \geq 0$ . Let  $\alpha_0 = \alpha_0(x_0)$  be as in Theorem 2.3. For  $\alpha < \alpha_0$  a solution to (2.1) with  $X_0 = x_0 < 1$  exists for all time in the sense of Definition 2.2. Then the map  $t \mapsto e(t) = \mathbb{E}(M_t)$  satisfies*

$$\exp(\mu x_0) = \frac{e^\mu - 1}{\alpha\mu} + \int_0^\infty \hat{\nu}'(s) \exp(-\alpha\mu\hat{e}(s)) ds, \quad (2.5)$$

for all  $\mu > 0$ , where

$$\hat{\nu}(s) = \frac{\varphi(\exp(\lambda s)) - \varphi(0)}{\alpha\mu} \exp\left(-\frac{\mu^2}{4\lambda}[\exp(2\lambda s) - 1]\right) \exp(-\lambda s),$$

$$\hat{e}(s) = \int_0^s \exp(\lambda u) e'(u) du,$$

with  $\varphi(x) = \exp(\mu x)$ , for  $x \in \mathbb{R}$ . The proof of this is given in Section 7. In particular, in the case when  $\lambda = 0$ , (2.5) reads

$$\frac{\mu^2}{2} \int_0^\infty \exp\left(-\alpha\mu e(s) - \frac{\mu^2}{2}s\right) ds = 1 - \frac{\alpha\mu \exp(\mu x_0)}{\exp(\mu) - 1}. \quad (2.6)$$

*It seems to be an interesting question to determine whether this characterizes  $e$  and more generally to determine whether the existence of a solution to (2.5) implies the existence of a global solution to (2.1).*

### 3. DISCUSSION OF THE CRITICALITY AND THE ‘BLOW-UP’

As mentioned in the introduction, instead of proceeding directly to the proofs, we choose first to discuss the criticality of the system in this section, so that the reader may get a ‘feel’ for the system. We hope this will help motivate our study and illustrate some of the surprising subtleties. However, the reader may skip ahead to the proofs which begin in the next section if he or she chooses.

Theorem 2.4 says that, for a given initial condition  $x_0 < 1$ , existence and uniqueness of a solution to (2.1) holds when the self-excitation rate  $\alpha$  lies in  $(0, \alpha_0(x_0))$ , for some  $\alpha_0(x_0) \in (0, 1]$ , the value of which is given in the statement of Theorem 2.3. This raises several questions:

- (1) Does the model indeed exhibit several regimes for global solvability according to the values of  $\alpha$ ?
- (2) If so, are there two phases exactly (using an analogy with statistical mechanics)? To put it differently, does there exist a critical threshold  $\alpha_c$  such that global solvability holds for  $\alpha < \alpha_c$  and fails for  $\alpha > \alpha_c$ ? Or are there any more complex cases when global solvability fails for some  $\alpha^+$  and then holds for some  $\alpha^- > \alpha^+$ ?
- (3) Conversely, are there any cases when global solvability holds for any  $\alpha < 1$ ?

Although Theorem 2.4 does not provide any answers to these questions, it does say that, for a given initial condition  $x_0 < 1$ , the critical rate

$$\alpha_c^-(x_0) = \sup\{\alpha^- \in (0, 1] : \mathcal{G}(\alpha, x_0) \text{ holds for any } \alpha \in (0, \alpha^-)\}$$

is (strictly) positive, since it is larger than  $\alpha_0(x_0)$ , where  $\mathcal{G}(\alpha, x_0)$  stands for the property of global unique solvability of equation (2.1) when driven by  $\alpha$  and initialized with  $x_0$ .

In this framework, the lower bound  $\alpha_c^-(x_0) \geq \alpha_0(x_0)$ , with  $\alpha_0(x_0)$  given by Theorem 2.3, is complementary to the previous result in [3]: there it is proven that the existence of a global solution fails for some values of  $\alpha < 1$  and some initial conditions, including Dirac masses located right below the barrier 1, in the case when  $b(x) + \lambda x \geq 0$ , with  $\lambda > 0$ . This shows that, in some cases,  $\alpha_c^-(x_0)$  is indeed strictly less than 1. Actually, the result in [3] says more: for any value of  $\lambda > 0$ , there exists a critical threshold  $x_c < 1$  such that, for any  $x_0 \in (x_c, 1)$ , there exists some  $\alpha'_0(x_0) < 1$  such that global existence fails (in the sense that  $e'$  blows up in finite time) for any  $\alpha \in (\alpha'_0(x_0), 1)$ . In other words, the critical rate

$$\alpha_c^+(x_0) = \inf\{\alpha^+ \in (0, 1] : \mathcal{G}(\alpha, x_0) \text{ fails for any } \alpha \in (\alpha^+, 1)\}$$

is strictly less than 1 for  $x_0$  close to 1, with the convention that  $\alpha_c^+(x_0) = 1$  if the set in the right-hand side is empty. Obviously,

$$\alpha_c^-(x_0) \leq \alpha_c^+(x_0),$$

but, as written above, the following questions remain open:

- (1) Are there any cases when  $\alpha_c^-(x_0) \neq \alpha_c^+(x_0)$ ?
- (2) Are there any cases when  $\alpha_c^-(x_0) = 1$ ?

More generally, finding the optimal values of  $\alpha_c^-(x_0)$  or/and  $\alpha_c^+(x_0)$  turns out to be a really challenging problem.

**3.1. Illustration: The Brownian case.** To further highlight the criticality of the system, we here use our new results and some numerical simulations to illustrate the blow-up phenomenon in the Brownian case. It is interesting to note that even in this simple case, the above questions are non-trivial.

Consider equation (2.1) with  $b \equiv 0$ , set  $e(t) = \mathbb{E}(M_t)$  and fix  $X_0 = x_0 < 1$ . Then the conditions of Theorem 2.4 are trivially satisfied, and so we know that there exists a global-in-time solution for all  $\alpha \in (0, \alpha_0(x_0))$ .

One may then ask if we ever observe a blow-up phenomenon in this case. The affirmative answer can be seen by adapting the strategy in [3] (note that the result in [3] is written for a non-zero  $\lambda$  but a similar argument applies when  $\lambda = 0$ ). Precisely it can be shown that existence fails for  $\alpha > \inf_{\mu > 0} [[\exp(\mu) - 1]/[\mu \exp(\mu x_0)]]$ . Thus, for values of  $x_0$  such that  $\inf_{\mu > 0} [[\exp(\mu) - 1]/[\mu \exp(\mu x_0)]] < 1$ , there is at least one critical value that lies in between  $\alpha_0(x_0)$  and  $\inf_{\mu > 0} [[\exp(\mu) - 1]/[\mu \exp(\mu x_0)]]$  at which the system's behavior changes radically.

We can observe this radical change numerically, which also allows us to compare our rigorous bounds with the numerically suggested true critical value. To do this, we use a particle method to solve the nonlinear equation with  $b \equiv 0$ : we refer to

the forthcoming paper [5] for a complete description and analysis of the numerical scheme. Then, by investigating numerically the graphs of  $e(t) = \mathbb{E}(M_t)$  for different values of  $\alpha$ , we can detect the emergence of some discontinuity.

Indeed, in Figure 1 (where  $x_0$  is fixed as 0.8), we numerically observe that the graph of  $e$  is regular for  $\alpha = 0.38$  but has a jump for  $\alpha = 0.39$ . We also observe that the associated particle system keeps stable for  $\alpha = 0.38$  but blows up for  $\alpha = 0.39$ . From the observations we have for other values of  $\alpha$ , it seems that global solvability fails for  $\alpha \geq 0.39$  and holds for  $\alpha \leq 0.38$ . We thus expect, in this framework, that there exists a single critical value that lies between 0.38 and 0.39.

As a final elucidation, we present in Figure 2 the various regions of the  $\alpha$ -parameter space  $(0, 1)$  for  $x_0 = 0.8$ . In this case, we find that  $\inf_{\mu > 0} [(\exp(\mu) - 1)/(\mu \exp(0.8\mu))] \approx 0.539$ , which defines region **D** in which we know that global solvability fails. By the numerical experiments it seems that global solvability also fails in region **C**, while by the same experiments it seems that global solutions do exist for  $\alpha$  in region **B**. In this article we prove that global solutions certainly exist for  $\alpha \in \mathbf{A}$ .

To complete the picture, we of course must find the explicit value of  $\alpha_0(0.8)$ . As mentioned in Remark 2.5 this is dependent on finding  $\alpha_0(0.8)$  small enough so that (2.4) is satisfied for all  $\alpha \in (0, \alpha_0(0.8))$ . This is done explicitly in Section 8, and it turns out that we may chose  $\alpha_0(0.8) = 0.104$ .

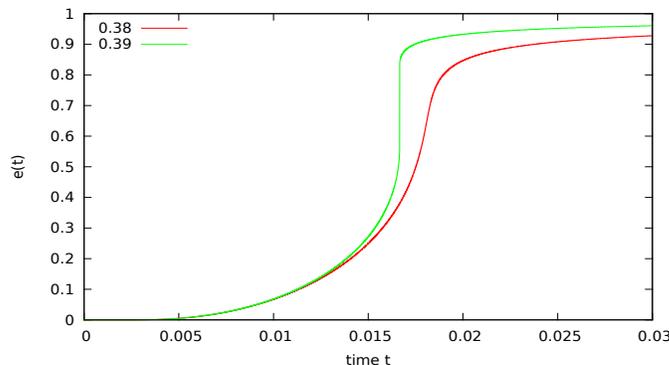


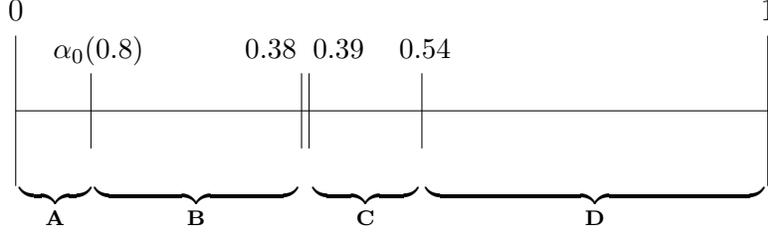
FIGURE 1. Plot of  $t \mapsto e(t)$  for  $x_0 = 0.8$ ,  $b(x) \equiv 0$ ,  $\alpha = 0.38$  (red) and  $\alpha = 0.39$  (green).

#### 4. SOLUTION AS A FIXED POINT

In this section we identify a solution to the nonlinear equation (2.1) as a fixed point of an appropriate map on an appropriate space. This will reduce the problem of finding a solution to identifying a fixed point of this map.

Let  $T > 0$ . For a general function  $e \in \mathcal{C}^1[0, T]$ , consider the linear SDE

$$X_t^e = X_0 + \int_0^t b(X_s^e) ds + \alpha e(t) + W_t - M_t^e, \quad t \in [0, T], \quad X_0 < 1 \text{ a.s.} \quad (4.1)$$

FIGURE 2. Critical regions of  $\alpha \in (0, 1)$ , for  $x_0 = 0.8$  and  $b(x) \equiv 0$ .

where  $(W_t)_{t \geq 0}$  is a standard Brownian motion,  $\alpha \in (0, 1)$ ,

$$M_t^e = \sum_{k \geq 1} \mathbf{1}_{[0, t]}(\tau_k^e) \quad (4.2)$$

and  $\tau_k^e = \inf\{t > \tau_{k-1}^e : X_{t-}^e \geq 1\}$  for  $k \geq 1$ ,  $\tau_0 = 0$ . The drift function  $b$  is assumed to be Lipschitz as above. Note that the solution to this SDE is well defined (by solving (4.1) iteratively from any  $\tau_k^e$  to the next  $\tau_{k+1}^e$  and by noticing that the jumping times  $(\tau_k)_{k \geq 0}$  cannot accumulate in finite time as the variations of  $(X_t^e)_{t \geq 0}$  on any  $[\tau_k^e, \tau_{k+1}^e)$ ,  $k \geq 0$ , are controlled in probability). We then define the map  $\Gamma$  by setting

$$\Gamma(e)(t) := \mathbb{E}(M_t^e). \quad (4.3)$$

We note that any fixed point of  $\Gamma$  that is continuously differentiable provides a solution to the nonlinear equation according to Definition 2.2 and vice versa. Thus, it is natural to look for a fixed point of  $\Gamma$  in a subspace of  $\mathcal{C}^1[0, T]$  where we are careful to uniformly control the size of the derivative. Moreover, since it is clear from the definition that  $\Gamma(e)(0) = 0$  and  $t \mapsto \Gamma(e)(t)$  is non-decreasing for any  $e \in \mathcal{C}^1[0, T]$ , we in fact restrict the domain of  $\Gamma$  to the closed subspace  $\mathcal{L}(T, A)$  of  $\mathcal{C}^1$  defined by

$$\mathcal{L}(T, A) := \left\{ e \in \mathcal{C}^1[0, T] : e(0) = 0, e(s) \leq e(t) \forall s \leq t, \sup_{0 \leq t \leq T} e'(t) \leq A \right\}$$

for some  $A \geq 0$ . The map  $\Gamma$  is thus defined as a map from  $\mathcal{L}(T, A)$  into the set of non-decreasing functions on  $[0, T]$ . It in fact depends on  $A$  as its domain of definition depends on  $A$ ; for this reason, it should be denoted by  $\Gamma^A$ . Anyhow, since the family  $(\Gamma^A)_{A \geq 0}$  is consistent in the sense that, for any  $A' \leq A$ , the restriction of  $\Gamma^A$  to  $\mathcal{L}(T, A')$  coincides with  $\Gamma^{A'}$ , we can get rid of the superscript  $A$  in  $\Gamma^A$  and then use the simpler notation  $\Gamma$ .

The following *a priori* stability result provides further information about where to look for fixed points, the proof of which we leave until the end of the section.

**Proposition 4.1.** *Given  $T > 0$ ,  $a \in (0, 1)$  and  $e \in \mathcal{L}(T, A)$  it holds that*

$$\left( (\forall t \in [0, T], e(t) \leq g_a(t)) \text{ and } (\mathbb{E}[(X_0)_+] \leq a) \right) \Rightarrow (\forall t \in [0, T], \Gamma(e)(t) \leq g_a(t)),$$

where  $(x)_+$  denotes the positive part of  $x \in \mathbb{R}$ , with

$$g_a(t) := \frac{a + (4 + \Lambda T^{1/2})t^{1/2}}{1 - \alpha} \exp\left(\frac{2\Lambda t}{1 - \alpha}\right). \quad (4.4)$$

Letting  $g(t) := g_1(t)$ ,  $t \geq 0$ , since  $X_0 < 1$  a.s., it thus makes sense to look for fixed points of  $\Gamma$  in the space

$$\mathcal{H}(T, A) := \{e \in \mathcal{L}(T, A) : e(t) \leq g(t)\}. \quad (4.5)$$

We equip  $\mathcal{H}(T, A)$  with the norm  $\|e\|_{\mathcal{H}(T, A)} = \|e\|_{\infty, T} + \|e'\|_{\infty, T}$  inherited from  $\mathcal{C}^1[0, T]$ . Here and throughout the paper,  $\|\cdot\|_{\infty, T}$  denotes the supremum norm on  $[0, T]$ .  $\mathcal{H}(T, A)$  is then a complete metric space, since it is a closed subspace of  $\mathcal{C}^1[0, T]$ .

For  $e \in \mathcal{H}(T, A)$  Proposition 4.1 implies that  $\Gamma(e)$  is finite, though it remains to show that  $\Gamma(e)$  is differentiable and that its derivative is bounded by  $A$  in order to check that  $\Gamma$  indeed maps  $\mathcal{H}(T, A)$  into itself, for a suitable value of  $A$  and  $T$ . The stability of  $\mathcal{H}(T, A)$  by  $\Gamma$  is discussed in Section 5.3.

#### 4.1. Proof of Proposition 4.1:

*Proof of Proposition 4.1:* Fix  $T > 0$ . We first note that we may write

$$\begin{aligned} M_t^e &= \sup_{s \leq t} [(Z_s^e)_+], \\ Z_t^e &= X_t^e + M_t^e = X_0 + \int_0^t b(X_s^e) ds + \alpha e(t) + W_t, \quad t \in [0, T], \end{aligned} \quad (4.6)$$

where  $\lfloor x \rfloor$  denotes the floor part of  $x \in \mathbb{R}$ . Indeed, one can see that for  $t \in [\tau_k^e, \tau_{k+1}^e)$ ,  $k \geq 0$ ,

$$\begin{aligned} \sup_{s \leq t} [(Z_s^e)_+] &= \max_{0 \leq j \leq k-1} \left( \sup_{s \in [\tau_j^e, \tau_{j+1}^e)} [(X_s^e + j)_+], \sup_{s \in [\tau_k^e, t)} [(X_s^e + k)_+] \right) \\ &= \max_{0 \leq j \leq k-1} (j + 1, k) = M_t^e, \end{aligned}$$

using the fact that  $X_t^e < 1$  for all  $t \geq 0$ .

Then, given  $t \in [0, T]$  such that  $Z_t^e \geq 0$ , let  $\rho^e := \sup\{s \in [0, t] : Z_s^e < 0\}$  ( $\sup \emptyset = 0$ ). Pay attention that  $\rho^e$  is not a stopping time and that it depends on  $t$ . Then, for  $s \in [\rho^e, t]$ ,

$$|b(X_s^e)| \leq \Lambda(1 + |X_s^e|) \leq \Lambda(1 + |Z_s^e| + M_s^e) = \Lambda(1 + (Z_s^e)_+ + M_s^e). \quad (4.7)$$

By (4.6), we know that  $M_s^e \leq \sup_{0 \leq r \leq s} (Z_r^e)_+$ . Therefore,

$$|b(X_s^e)| \leq \Lambda \left( 1 + 2 \sup_{0 \leq r \leq s} (Z_r^e)_+ \right).$$

By (4.1), we obtain:

$$Z_t^e \leq Z_{\rho^e}^e + \Lambda \int_{\rho^e}^t \left( 1 + 2 \sup_{0 \leq r \leq s} (Z_r^e)_+ \right) ds + \alpha e(t) + W_t - W_{\rho^e}. \quad (4.8)$$

If  $\rho^e > 0$ , then  $Z_{\rho^e}^e = 0$  as, obviously,  $(Z_s^e)_{0 \leq s \leq T}$  is a continuous process. If  $\rho^e = 0$ , then  $Z_{\rho^e}^e = X_0 \geq 0$ . Therefore,

$$(Z_t^e)_+ \leq (X_0)_+ + \Lambda \int_0^t \left( 1 + 2 \sup_{0 \leq r \leq s} (Z_r^e)_+ \right) ds + \alpha e(t) + 2 \sup_{0 \leq s \leq t} |W_s|. \quad (4.9)$$

Obviously, the above inequality still holds if  $Z_t^e \leq 0$ . Therefore, taking the supremum in the left-hand side, taking the expectation and applying Doob's maximal inequality for martingales, we see that

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} (Z_s^e)_+ \right] \leq \mathbb{E}[(X_0)_+] + \Lambda \int_0^t \left( 1 + 2\mathbb{E} \left[ \sup_{0 \leq r \leq s} (Z_r^e)_+ \right] \right) ds + \alpha e(t) + 4t^{1/2}, \quad (4.10)$$

for all  $t \in [0, T]$ , which shows that  $\mathbb{E}[\sup_{0 \leq t \leq T} (Z_t^e)_+]$  is finite. In particular, if  $\mathbb{E}[(X_0)_+] \leq a$  and

$$\forall t \in [0, T], \quad e(t) \leq g_a(t), \quad \text{with } g_a(t) := \frac{a + (4 + \Lambda T^{1/2})t^{1/2}}{1 - \alpha} \exp\left(\frac{2\Lambda t}{1 - \alpha}\right),$$

and  $R^e$  is the deterministic hitting time

$$R^e := \inf \left\{ t \in [0, T] : \mathbb{E} \left[ \sup_{0 \leq s \leq t} (Z_s^e)_+ \right] > g_a(t) \right\} \quad (\inf \emptyset = +\infty),$$

then, for any  $t \in [0, R^e \wedge T]$ ,

$$\begin{aligned} & \mathbb{E} \left[ \sup_{0 \leq s \leq t} (Z_s^e)_+ \right] \\ & \leq a + \Lambda \int_0^t (1 + 2g_a(s)) ds + \alpha g_a(t) + 4t^{1/2} \\ & < (a + (4 + \Lambda T^{1/2})t^{1/2}) \left[ 1 + \int_0^t \frac{2\Lambda}{1 - \alpha} \exp\left(\frac{2\Lambda s}{1 - \alpha}\right) ds \right] + \alpha g_a(t) \\ & = (1 - \alpha)g_a(t) + \alpha g_a(t) = g_a(t). \end{aligned}$$

Now, by the continuity of the paths of  $Z^e$  and by the finiteness of  $\mathbb{E}[\sup_{0 \leq t \leq T} (Z_t^e)_+]$ , we deduce that  $\mathbb{E}[\sup_{0 \leq s \leq t} (Z_s^e)_+]$  is continuous in  $t$ . Therefore, if  $R^e < T$ , then  $\mathbb{E}[\sup_{0 \leq s \leq R^e} (Z_s^e)_+]$  must be equal to  $g(R^e)$ , but, by the above inequalities, this sounds as a contradiction. By (4.6), this proves the announced bound.  $\square$

## 5. EXISTENCE AND UNIQUENESS IN SMALL TIME

The main result of this section is the following:

**Theorem 5.1.** *Suppose there exist  $\beta, \epsilon > 0$  such that  $\mathbb{P}(X_0 \in dx) \leq \beta(1 - x)dx$  for any  $x \in (1 - \epsilon, 1]$  and that the density of  $X_0$  on the interval  $(1 - \epsilon, 1]$  is differentiable at point 1. Then there exist constants  $A_1 \geq 0$  and  $T_1 \in (0, 1]$ , depending upon  $\beta, \epsilon, \alpha, \Lambda$  and  $K$  only, such that*

$$\Gamma(\mathcal{H}(A_1, T_1)) \subset \mathcal{H}(A_1, T_1).$$

Moreover, for all  $e_1, e_2 \in \mathcal{H}(A_1, T_1)$ ,

$$\|\Gamma(e_1) - \Gamma(e_2)\|_{\mathcal{H}(A_1, T_1)} \leq \frac{1}{2} \|e_1 - e_2\|_{\mathcal{H}(A_1, T_1)}.$$

Hence there exists a unique fixed point of the restriction of  $\Gamma$  to  $\mathcal{H}(A_1, T_1)$ , which provides a solution to (2.1) according to Definition 2.2 up until time  $T_1$  (such that  $[0, T_1] \ni t \mapsto \mathbb{E}(M_t)$  is in the space  $\mathcal{H}(A_1, T_1)$ ).

**Remark 5.2.** The assumption  $\mathbb{P}(X_0 \in dx) \leq \beta(1-x)dx$ , for  $x \in (1-\epsilon, 1]$ , says that the distribution of  $X_0$  has a density on the interval  $(1-\epsilon, 1]$ , and that the density is bounded by  $\beta(1-x)$ . This is a key assumption to prove Theorem 5.1 since it prevents the hitting time of 1 by  $X$  from having a Dirac mass at the initial time 0. In the proof, it permits to show that  $\Gamma(e)$  is Lipschitz continuous when  $e$  itself is Lipschitz continuous. By contrast, we feel that the proof could be written without requiring the density of  $X_0$  at point 1 to be differentiable. Here the differentiability of the density forces  $\Gamma(e)$  to be continuously differentiable at time 0 when  $e$  is continuously differentiable, as specified in the definition of  $\mathcal{L}(T, A)$ . This proves to be really convenient in the whole analysis, but does not affect the final result stated in Theorem 2.4.

**5.1. Representation of  $\Gamma$ .** Let  $T > 0$ . As a first step towards understanding the map  $\Gamma$  defined above, we note that, given  $e \in \mathcal{L}(T, A)$ , using the definitions we can write

$$\begin{aligned} \Gamma(e)(t) &= \mathbb{E}(M_t^e) = \mathbb{E}\left(\sum_{k \geq 1} \mathbf{1}_{[0, t]}(\tau_k^e)\right) \\ &= \sum_{k \geq 1} \int_0^t \mathbb{P}(\tau_{k+1}^e \in (s, t] | \tau_k^e = s) \mathbb{P}(\tau_k^e \in ds) + \mathbb{P}(\tau_1^e \leq t), \end{aligned}$$

where  $\mathbb{P}(\tau_k^e \in ds)$  is a convenient abuse of notation for denoting the law of  $\tau_k^e$  and  $\mathcal{B}(\mathbb{R}) \ni A \mapsto \mathbb{P}(\tau_{k+1}^e \in A | \tau_k^e = s)$  stands for the conditional law of  $\tau_{k+1}^e$  given  $\tau_k^e = s$ . Here  $\mathcal{B}(\mathbb{R})$  is the Borel  $\sigma$ -algebra on  $\mathbb{R}$ . Moreover, observing that the solution  $X^e$  to (4.1) is a Markov process (which restarts from 0 at time  $\tau_k^e$  when  $k \geq 1$ ), we may write

$$\Gamma(e)(t) = \mathbb{E}(M_t^e) = \sum_{k \geq 1} \int_0^t \mathbb{P}(\tau_1^{e^{\#s}} \leq t - s | X_0^{e^{\#s}} = 0) \mathbb{P}(\tau_k^e \in ds) + \mathbb{P}(\tau_1^e \leq t), \quad (5.1)$$

where  $e^{\#s}$  stands for the mapping  $([0, T-s] \ni t \mapsto e(t+s) - e(s)) \in \mathcal{L}(T-s, A)$ .

With this decomposition it is clear that in order to analyse  $\Gamma(e)$ , and more importantly the derivative of  $\Gamma(e)$  (recall we are looking for a fixed point in  $\mathcal{H}(T, A)$ ), we must analyse the densities of the first hitting times of a barrier by a *non-homogeneous* diffusion processes with a general Lipschitz drift term. Indeed, formally taking the derivative with respect to  $t$  in (5.1) introduces terms involving the density of  $\tau_1^e$ , where we recall that

$$\tau_1^e = \inf\{t > 0 : X_t^e \geq 1\} = \inf\left\{t > 0 : X_0 + \int_0^t b(X_s^e) ds + W_t \geq 1 - \alpha e(t)\right\}.$$

The analysis of such densities is well-known to be a difficult problem. These problems remain even in the case where  $b \equiv 0$ . However, the fact that  $e$  is continuously differentiable at least guarantees that the densities exist. In the case  $b \equiv 0$  we refer to [13, Theorem 14.4].

In the general case existence of these densities will be guaranteed in the next section by Lemma 5.3. The next section also contains the key first hitting time/killed process density estimates, which will allow us to prove Theorem 5.1. We would like to emphasize that these estimates are not trivial, but we chose to collect all the technical proofs in a separate set of notes [6] for two reasons: firstly so as not to distract attention away from the main difficulty of the novel nonlinearity in the current problem, and secondly since the notes present stand-alone estimates for the density of the first-hitting time of a constant level by a non-homogeneous diffusion process, that we hope can be reused in other contexts.

**5.2. General bounds for the density of the first hitting time for the non-homogeneous diffusion process.** Fix  $T > 0$ , and for  $e \in \mathcal{C}^1[0, T]$  consider the stochastic processes  $(\chi_t^e)_{0 \leq t \leq T}$  which satisfies

$$d\chi_t^e = b(\chi_t^e)dt + \alpha e'(t)dt + dW_t, \quad t \in [0, T]; \quad \chi_0 < 1 \text{ a.s.}, \quad (5.2)$$

together with the stopping time

$$\tau^e := \inf\{t \in [0, T] : \chi_t^e \geq 1\}, \quad (\inf\{\emptyset\} = \infty).$$

Here  $\alpha \in (0, 1)$  and the drift  $b$  is globally Lipschitz, exactly as above.

**Lemma 5.3.** *Let  $e \in \mathcal{C}^1[0, T]$ . Suppose there exist  $\beta, \epsilon > 0$  such that  $\mathbb{P}(\chi_0 \in dx) \leq \beta(1-x)dx$  for any  $x \in (1-\epsilon, 1]$  and that the density of  $\chi_0$  on the interval  $(1-\epsilon, 1]$  is differentiable at point 1. Then:*

(i) *For any  $t \in (0, T]$ , the law of the diffusion  $\chi_t^e$  killed at the threshold is absolutely continuous with respect to the Lebesgue measure.*

(ii) *Denoting the density of  $\chi_t^e$  killed at the threshold by*

$$p_e(t, y) := \frac{1}{dy} \mathbb{P}(\chi_t^e \in dy, t < \tau), \quad t \in [0, T], \quad y \leq 1, \quad (5.3)$$

*$p_e(t, y)$  is continuous in  $(t, y)$  and continuously differentiable in  $y$  on  $(0, T] \times (-\infty, 1]$  and admits Sobolev derivatives of order 1 in  $t$  and of order 2 in  $y$  in any  $L^\varsigma$ ,  $\varsigma \geq 1$ , on any compact subset of  $(0, T] \times (-\infty, 1)$ . When  $\chi_0 \leq 1 - \epsilon$  a.s. it is actually continuous and continuously differentiable in  $y$  on any compact subset of  $([0, T] \times (-\infty, 1]) \setminus (\{0\} \times (-\infty, 1 - \epsilon])$ .*

(iii) *The density  $p_e$  satisfies the Fokker-Planck equation:*

$$\partial_t p_e(t, y) + \partial_y [(b(y) + \alpha e'(t))p_e(t, y)] - \frac{1}{2} \partial_{yy}^2 p_e(t, y) = 0, \quad t \in (0, T], \quad y < 1, \quad (5.4)$$

*with the Dirichlet boundary condition  $p_e(t, 1) = 0$  and the measure-valued initial condition  $p_e(0, y)dy = \mathbb{P}(\chi_0 \in dy)$ , both  $p_e(t, y)$  and  $\partial_y p_e(t, y)$  decaying to 0 as  $y \rightarrow -\infty$ .*

(iv) The first hitting time,  $\tau^e$  has a density on  $[0, T]$ , given by

$$\frac{d}{dt}\mathbb{P}(\tau^e \leq t) = -\frac{1}{2}\partial_y p_e(t, 1), \quad t \in [0, T], \quad (5.5)$$

the mapping  $[0, T] \ni t \mapsto \partial_y p_e(t, 1)$  being continuous and its supremum norm being bounded in terms of  $T$ ,  $\alpha$ ,  $\|e'\|_{\infty, T}$ ,  $\beta$  and  $b$  only.

Lemma 5.3 is quite standard. Anyhow, it is difficult to find a complete proof of it under our assumptions. For this reason, we provide a complete proof in [6].

When  $\chi_0 = x_0$  for some  $x_0 < 1$  i.e. the initial distribution is a Dirac measure at  $x_0$ , the conditions of the above Lemma are certainly satisfied. Therefore, for  $e \in \mathcal{C}^1[0, T]$  it makes sense to consider the density  $p_e(t, y)$ ,  $t \in (0, T]$ ,  $y \leq 1$  of the process killed at 1 started at  $x_0$ . We will write  $p_e(t, y) = p_e^{x_0}(t, y)$  in this case.

The following are the two key results proved in [6] (where they appear as Corollary 4.3 and Corollary 5.3) that we will need.

**Proposition 5.4.** *Let  $e \in \mathcal{C}^1[0, T]$ . Then there exists a constant  $\kappa(T)$  (depending only on  $T$  and the drift function  $b$ ) which increases with  $T$  such that for all  $x_0 < 1$ ,*

$$|\partial_y p_e^{x_0}(t, 1)| \leq \kappa(T)(\|e'\|_{\infty, T} + 1) \frac{1}{t} \exp\left(-\frac{(1-x_0)^2}{\kappa(T)t}\right)$$

for all  $t \leq \min\{[(\|e'\|_{\infty, T} + 1)\kappa(T)]^{-2}, T\}$ . In particular  $\kappa(T)$  is independent of  $e$ .

**Proposition 5.5.** *Let  $e_1, e_2 \in \mathcal{C}^1[0, T]$  and let  $A = \max\{\|e'_1\|_{\infty, T}, \|e'_2\|_{\infty, T}\}$ . Then there exists a constant  $\kappa(T)$  (depending only on  $T$  and the drift function  $b$ ) which increases with  $T$  such that for all  $x_0 < 1$ ,*

$$|\partial_y p_{e_1}^{x_0}(t, 1) - \partial_y p_{e_2}^{x_0}(t, 1)| \leq \kappa(T)(A + 1) \frac{1}{\sqrt{t}} \exp\left(-\frac{(1-x_0)^2}{\kappa(T)t}\right) \|e'_1 - e'_2\|_{\infty, t},$$

for all  $t \leq \min\{[(A + 1)\kappa(T)]^{-2}, T\}$ . In particular  $\kappa(T)$  is independent of  $e_1$  and  $e_2$ .

**5.3. Application to  $\Gamma$ :** In this section we return to the setting of Section 4, and apply the results of the previous subsection to complete the proof of Theorem 5.1.

The first result ensures the differentiability of  $\Gamma(e)$  whenever  $e \in \mathcal{L}(T, A)$ , which is the first step in showing that  $\Gamma$  is stable on the space  $\mathcal{H}(T, A)$  for some  $A$  (recall that  $\mathcal{H}$  is simply a growth controlled subspace of  $\mathcal{L}$ ).

**Proposition 5.6.** *Let  $e \in \mathcal{L}(T, A)$  and  $X_0$  be such that there exist  $\beta, \epsilon > 0$  with  $\mathbb{P}(X_0 \in dx) \leq \beta(1-x)dx$  for any  $x \in (1-\epsilon, 1]$  and that the density of  $X_0$  on the interval  $(1-\epsilon, 1]$  is differentiable at point 1. Then the mapping  $[0, T] \ni t \mapsto \Gamma(e)(t)$  is continuously differentiable. Moreover,*

$$\frac{d}{dt}[\Gamma(e)](t) = -\int_0^t \frac{1}{2}\partial_y p_e^{(0,s)}(t-s, 1) \frac{d}{ds}[\Gamma(e)](s) ds - \frac{1}{2}\partial_y p_e(t, 1), \quad t \in [0, T], \quad (5.6)$$

where  $p_e$  represents the density of the process  $X^e$  killed at 1 and  $p_e^{(0,s)}$  represents the density of the process  $X^{e^{\#s}}$  killed at 1 with  $X_0^{e^{\#s}} = 0$ .

*Proof.* We first check that  $\Gamma(e)$  is Lipschitz continuous on  $[0, T]$ . Considering a finite difference in (5.1) and using (5.5), we get, for  $t, t+h \in [0, T]$ ,

$$\begin{aligned} \Gamma(e)(t+h) - \Gamma(e)(t) &= \sum_{k \geq 1} \int_t^{t+h} \mathbb{P}(\tau_1^{e^{\sharp s}} \leq t+h-s | X_0^{e^{\sharp s}} = 0) \mathbb{P}(\tau_k^e \in ds) \\ &\quad - \frac{1}{2} \sum_{k \geq 1} \int_0^t \int_{t-s}^{t+h-s} \partial_y p_e^{(0,s)}(r, 1) dr \mathbb{P}(\tau_k^e \in ds) \\ &\quad - \frac{1}{2} \int_t^{t+h} \partial_y p_e(s, 1) ds. \end{aligned} \quad (5.7)$$

By Lemma 5.3, we can handle the two last terms in the above to find a constant  $C > 0$  ( $C$  depending such that

$$\begin{aligned} \Gamma(e)(t+h) - \Gamma(e)(t) &\leq \sum_{k \geq 1} \int_t^{t+h} \mathbb{P}(\tau_1^{e^{\sharp s}} \leq t+h-s | X_0^{e^{\sharp s}} = 0) \mathbb{P}(\tau_k^e \in ds) \\ &\quad + Ch(1 + \Gamma(e)(T)), \end{aligned}$$

the last term in the right-hand side being finite thanks to Proposition 4.1. Moreover, from Proposition 4.1 again, we deduce that

$$\begin{aligned} &\lim_{h \searrow 0} \sup_{0 \leq s \leq T-h} \mathbb{P}(\tau_1^{e^{\sharp s}} \leq h | X_0^{e^{\sharp s}} = 0) \\ &= \lim_{h \searrow 0} \sup_{0 \leq s \leq T-h} \mathbb{P} \left( \sup_{0 \leq r \leq h} Z_r^{e^{\sharp s}} \geq 1 | X_0^{e^{\sharp s}} = 0 \right) = 0, \end{aligned} \quad (5.8)$$

where  $Z^{e^{\sharp s}}$  is given by (4.6), the second line following from (4.9). Therefore, there exists a mapping  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  matching 0 at 0 and continuous at 0 such that

$$\Gamma(e)(t+h) - \Gamma(e)(t) \leq \eta(h) [\Gamma(e)(t+h) - \Gamma(e)(t)] + Ch(1 + \Gamma(e)(T)).$$

Choosing  $h$  small enough, Lipschitz continuity easily follows.

As a consequence, we can divide both sides of (5.7) by  $h$  and then let  $h$  tend to 0. By (5.8), we have for a given  $t \in [0, T]$ ,

$$\begin{aligned} &\lim_{h \searrow 0} h^{-1} \sum_{k \geq 1} \int_t^{t+h} \mathbb{P}(\tau_1^{e^{\sharp s}} \leq t+h-s | X_0^{e^{\sharp s}} = 0) \mathbb{P}(\tau_k^e \in ds) \\ &\leq \lim_{h \searrow 0} \left[ \sup_{0 \leq s \leq T-h} \mathbb{P}(\tau_1^{e^{\sharp s}} \leq h | X_0^{e^{\sharp s}} = 0) \frac{\Gamma(e)(t+h) - \Gamma(e)(t)}{h} \right] = 0. \end{aligned}$$

Handling the second term in (5.7) by Lemma 5.3 and using the Lebesgue Dominated Convergence Theorem, we deduce that

$$\frac{d}{dt} \Gamma(e)(t) = - \sum_{k \geq 1} \int_0^t \frac{1}{2} \partial_y p_e^{(0,s)}(t-s, 1) \mathbb{P}(\tau_k^e \in ds) - \frac{1}{2} \partial_y p_e(t, 1).$$

By Lemma 5.3, we know that  $\partial_y p_e^{(0,s)}(\cdot, 1)$  and  $\partial_y p_e(\cdot, 1)$  are continuous (in  $t$ ). This proves that  $(d/dt)\Gamma(e)$  is continuous as well.

Formula (5.6) follows from the relationship

$$\Gamma(e)(t) = \sum_{k \geq 1} \int_0^t \mathbb{P}(\tau_k^e \in ds), \quad t \in [0, T]. \quad (5.9)$$

□

The second point is to show that the difference between the derivatives of  $\Gamma(e_1)$  and  $\Gamma(e_2)$  is uniformly small in terms of the distance between two functions  $e_1$  and  $e_2$  in the space  $\mathcal{H}(T, A)$  in small time.

**Proposition 5.7.** *Let  $T > 0$  and  $X_0$  be such that there exist  $\beta, \epsilon > 0$  with  $\mathbb{P}(X_0 \in dx) \leq \beta(1-x)dx$  for any  $x \in (1-\epsilon, 1]$  and that the density of  $X_0$  on the interval  $(1-\epsilon, 1]$  is differentiable at point 1.*

*Suppose  $e_1, e_2 \in \mathcal{H}(T, A)$  for some  $A \geq 0$ . Then there exists a constant  $\kappa(T)$ , independent of  $A, \beta$  and  $\epsilon$ , and increasing in  $T$ , and a constant  $\tilde{\kappa}(T, \beta, \epsilon)$ , independent of  $A$  and increasing in  $T$ , such that for any  $e_1, e_2 \in \mathcal{H}(T, A)$ ,*

$$\sup_{0 \leq s \leq t} \left| \frac{d}{ds} \Gamma(e_1)(s) - \frac{d}{ds} \Gamma(e_2)(s) \right| \leq (A+1) \tilde{\kappa}(T, \beta, \epsilon) \sqrt{t} \|e'_1 - e'_2\|_{\infty, t},$$

for  $t \leq \min\{[(A+1)\kappa(T)]^{-2}, T\}$ .

*Proof.* We have by (5.6)

$$\begin{aligned} & \left| \frac{d}{dt} \Gamma(e_1)(t) - \frac{d}{dt} \Gamma(e_2)(t) \right| \\ & \leq \frac{1}{2} \int_{-\infty}^1 |\partial_y p_{e_1}^x(t, 1) - \partial_y p_{e_2}^x(t, 1)| \mathbb{P}(X_0 \in dx) \\ & \quad + \frac{1}{2} \int_0^t |\partial_y p_{e_1}^{(0,s)}(t-s, 1) - \partial_y p_{e_2}^{(0,s)}(t-s, 1)| \frac{d}{ds} \Gamma(e_1)(s) \\ & \quad + \frac{1}{2} \int_0^t |\partial_y p_{e_2}^{(0,s)}(t-s, 1)| \left| \frac{d}{ds} \Gamma(e_1)(s) - \frac{d}{ds} \Gamma(e_2)(s) \right| ds \\ & := \frac{1}{2} (L_1 + L_2 + L_3). \end{aligned} \quad (5.10)$$

Suppose  $t \leq T$  and  $\sqrt{t} \leq [(A+1)\kappa(T)]^{-1}$ , where  $\kappa(T)$  is as in Proposition 5.5. The value of  $\kappa(T)$  will be allowed to increase when necessary below. Considering the first term only, we can use Proposition 5.5 to see that

$$\begin{aligned} L_1 & \leq (A+1)\beta\kappa(T) \left( \int_{1-\epsilon}^1 \frac{1}{\sqrt{t}} \exp\left(-\frac{(1-x)^2}{\kappa(T)t}\right) (1-x) dx \right) \|e'_1 - e'_2\|_{\infty, t} \\ & \quad + (A+1)\kappa(T) \left( \int_{-\infty}^{1-\epsilon} \frac{1}{\sqrt{t}} \exp\left(-\frac{(1-x)^2}{\kappa(T)t}\right) \mathbb{P}(X_0 \in dx) \right) \|e'_1 - e'_2\|_{\infty, t}. \end{aligned}$$

We deduce that there exists a constant  $\tilde{\kappa}(T, \beta, \epsilon) > 0$ , which is independent of  $A$  and which is allowed to increase as necessary from line to line below, such that

$$\begin{aligned} L_1 &\leq (A+1)\beta\kappa(T)\sqrt{t} \left( \int_0^\infty z \exp\left(-\frac{z^2}{\kappa(T)}\right) dz \right) \|e'_1 - e'_2\|_{\infty,t} \\ &\quad + (A+1)\kappa(T)\frac{1}{\sqrt{t}} \exp\left(-\frac{\epsilon^2}{\kappa(T)t}\right) \|e'_1 - e'_2\|_{\infty,t} \\ &\leq (A+1)\tilde{\kappa}(T, \beta, \epsilon)\sqrt{t}\|e'_1 - e'_2\|_{\infty,t}. \end{aligned} \quad (5.11)$$

We can then use Proposition 5.5 again to see that

$$L_2 \leq (A+1)\kappa(T) \sup_{0 < s \leq t} \left[ s^{-1/2} \exp\left(-\frac{1}{\kappa(T)s}\right) \right] \Gamma(e_1)(t) \|e'_1 - e'_2\|_{\infty,t}.$$

By Proposition 4.1 (since  $e_1 \in \mathcal{H}(T, A)$ ), we deduce that

$$L_2 \leq (A+1)\kappa(T)\sqrt{t}\|e'_1 - e'_2\|_{\infty,t}, \quad (5.12)$$

where  $\kappa(T)$  has been increased as necessary, and we have used the elementary inequality  $\exp(-1/v) \leq v$  for all  $v \geq 0$ . We finally turn to  $L_3$  in (5.10). By Proposition 5.4, we have that

$$\begin{aligned} |\partial_y p_{e_2}^{(0,s)}(t-s, 1)| &\leq \kappa(T)(A+1) \frac{1}{(t-s)} \exp\left(-\frac{1}{\kappa(T)(t-s)}\right) \\ &\leq \kappa(T)(A+1), \end{aligned} \quad (5.13)$$

again by increasing  $\kappa(T)$ . Thus, from (5.10), (5.11), (5.12) and (5.13), we deduce

$$\begin{aligned} \left| \frac{d}{dt} \Gamma(e_1)(t) - \frac{d}{dt} \Gamma(e_2)(t) \right| &\leq (A+1)\tilde{\kappa}(T, \beta, \epsilon)\sqrt{t}\|e'_1 - e'_2\|_{\infty,t} \\ &\quad + (A+1)\kappa(T) \int_0^t \left| \frac{d}{ds} \Gamma(e_1)(s) - \frac{d}{ds} \Gamma(e_2)(s) \right| ds. \end{aligned}$$

By taking the supremum over all  $s \leq t$  in the above, we have, for  $t \leq (2\kappa(T)(A+1))^{-1}$ , (which actually follows from the aforementioned condition  $t \leq (\kappa(T)(A+1))^{-2}$  by assuming w.l.o.g.  $\kappa(T) \geq 2$ ),

$$\sup_{0 \leq s \leq t} \left| \frac{d}{ds} \Gamma(e_1)(s) - \frac{d}{ds} \Gamma(e_2)(s) \right| \leq 2(A+1)\tilde{\kappa}(T, \beta, \epsilon)\sqrt{t}\|e'_1 - e'_2\|_{\infty,t}.$$

□

We can then finally complete this section with the proof of Theorem 5.1.

*Proof.* [**Proof of Theorem 5.1**] Choose

$$A_1 = 2 \sup_{0 \leq t \leq 1} \left| \frac{d}{dt} \Gamma(0)(t) \right| + 1.$$

Note that  $A_1$  depends on  $\beta$ . Then choose  $T_1 \leq \min\{[(A_1+1)\kappa(1)]^{-2}, 1\}$  such that

$$\sqrt{T_1}\tilde{\kappa}(1, \beta, \epsilon)(A_1+1) \leq \frac{1}{4}, \quad (5.14)$$

where  $\kappa(1)$  and  $\tilde{\kappa}(1, \beta, \epsilon)$  are as in Proposition 5.7. By that result, if  $e \in \mathcal{H}(A_1, T_1)$  then

$$\left| \frac{d}{dt} \Gamma(e)(t) \right| \leq \sqrt{t} \tilde{\kappa}(T_1, \beta, \epsilon) (A_1 + 1) A_1 + \left| \frac{d}{dt} \Gamma(0)(t) \right|$$

for all  $t \leq \min\{[(A_1 + 1)\kappa(T_1)]^{-2}, T_1\} = T_1$ . By definition, we have  $T_1 \leq 1$  so that  $\kappa(T_1) \leq \kappa(1)$  and  $\tilde{\kappa}(T_1, \beta, \epsilon) \leq \tilde{\kappa}(1, \beta, \epsilon)$ . Therefore

$$\left| \frac{d}{dt} \Gamma(e)(t) \right| \leq \sqrt{t} \tilde{\kappa}(1, \beta, \epsilon) (A_1 + 1) A_1 + \left| \frac{d}{dt} \Gamma(0)(t) \right|$$

for all  $t \leq T_1$ . Hence for all  $t \leq T_1$

$$\left| \frac{d}{dt} \Gamma(e)(t) \right| \leq \frac{A_1}{2} + \sup_{0 \leq t \leq 1} \left| \frac{d}{dt} \Gamma(0)(t) \right| \leq A_1$$

by (5.14), so that  $\Gamma(e) \in \mathcal{H}(A_1, T_1)$ .

To prove that  $\Gamma$  is a contraction on  $\mathcal{H}(A_1, T_1)$ , first note that for  $e \in \mathcal{H}(A_1, T_1)$

$$\|e'\|_{\infty, T_1} \leq \|e\|_{\mathcal{H}(A, T_1)} \leq 2\|e'\|_{\infty, T_1}$$

by the mean-value theorem, since  $e(0) = 0$  and  $T_1 \leq 1$ . Thus for any  $e_1, e_2 \in \mathcal{H}(A_1, T_1)$

$$\begin{aligned} \|\Gamma(e_1) - \Gamma(e_2)\|_{\mathcal{H}(A_1, T_1)} &\leq 2 \|\Gamma(e_1)' - \Gamma(e_2)'\|_{\infty, T_1} \\ &\leq 2\sqrt{T_1} \tilde{\kappa}(T_1, \beta, \epsilon) (A_1 + 1) \|e_1' - e_2'\|_{\infty, T_1} \\ &\leq \frac{1}{2} \|e_1 - e_2\|_{\mathcal{H}(A_1, T_1)}, \end{aligned}$$

by our choice of  $T_1$  and using Proposition 5.7 once more. Since  $\mathcal{H}(A_1, T_1)$  is a closed subspace of  $\mathcal{C}^1[0, T]$  (a complete metric space), the existence of a fixed point for  $\Gamma$  follows from the Banach Fixed Point Theorem.  $\square$

## 6. LONG-TIME ESTIMATES

In order to extend the existence and uniqueness from small time to any arbitrarily prescribed interval, we need an *a priori* bound for the Lipschitz constant of  $t \mapsto \mathbb{E}(M_t)$  on any finite interval  $[0, T]$ , which is given by Theorem 2.3. The purpose of this section is to prove this result.

In the whole section, for a given initial condition  $X_0 = x_0 < 1$ , we thus assume that there exists a solution to (2.1) according to Definition 2.2 i.e. such that  $e : [0, T] \ni t \mapsto \mathbb{E}(M_t)$  is continuously differentiable.

### 6.1. Reformulation of the equation and *a priori* bounds for the solution.

In the whole proof, we shall use a reformulated version of the nonlinear equation (2.1). Indeed, given a solution  $(X_t, M_t)_{0 \leq t \leq T}$  to (2.1) on some interval  $[0, T]$  according to Definition 2.2, we set

$$Z_t = X_t + M_t, \quad t \in [0, T]. \tag{6.1}$$

Then, the dynamics of  $(Z_t)_{0 \leq t \leq T}$  have the form:

$$Z_t = X_0 + \int_0^t b(X_s) ds + \alpha \mathbb{E}(M_t) + W_t, \quad t \in [0, T], \quad (6.2)$$

so that, by continuity of the mapping  $[0, T] \ni t \mapsto e(t) = \mathbb{E}(M_t)$ , the process  $(Z_t)_{0 \leq t \leq T}$  has continuous paths. Moreover, as noted above in Proposition 4.1,  $M_t$  is completely defined in terms of  $Z_t$ , since it holds that

$$M_t = \lfloor \left( \sup_{0 \leq s \leq t} Z_s \right)_+ \rfloor = \sup_{0 \leq s \leq t} \lfloor (Z_s)_+ \rfloor. \quad (6.3)$$

The following is easily proved by adapting the proof of Proposition 4.1:

**Lemma 6.1.** *There exists a constant  $B(T, \alpha, b)$ , only depending upon  $T$ ,  $\alpha$ ,  $b$  and non-decreasing in  $\alpha$ , such that*

$$\sup_{0 \leq t \leq T} e(t) = e(T) \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} (Z_t)_+ \right] \leq B(T, \alpha, b). \quad (6.4)$$

A possible choice for  $B$  is

$$B(T, \alpha, b) = \frac{\mathbb{E}[(X_0)_+] + 4T^{1/2} + \Lambda T}{1 - \alpha} \exp \left( \frac{2\Lambda T}{1 - \alpha} \right).$$

**6.2. Local Hölder bound of the solution.** We now turn to the critical point of the proof. Indeed, in the next subsection, we shall prove that, for  $\alpha$  small enough, the function  $t \mapsto e(t) = \mathbb{E}(M_t)$  generated by some solution to (2.1) according to Definition 2.2 (so that  $e$  is continuously differentiable) satisfies an *a priori*  $1/2$ -Hölder bound, with an explicit Hölder constant. This acts as the keystone of the argument to extend the local existence and uniqueness result into a global one. As a first step, the proof consists in establishing a local Hölder bound for  $e$  in the case when the probability that the process  $X$  lies in the neighbourhood of 1 is not too large.

**Lemma 6.2.** *Keep the same notation as in the previous subsection and consider in particular a solution  $(X_t)_{0 \leq t \leq T}$  to (2.1) on some interval  $[0, T]$ , with  $T > 0$  and initial condition  $X_0 = x_0 < 1$ . Assume in addition that there exists some time  $t_0 \in [0, T]$  and two constants  $\epsilon \in (0, 1)$  and  $c \in (0, 1/\alpha)$  such that for any Borel subset  $A \subset [1 - \epsilon, 1]$ ,*

$$\mathbb{P}(X_{t_0} \in A) \leq c|A|, \quad (6.5)$$

where  $|A|$  stands for the Lebesgue measure of  $A$ . Then, with

$$\mathcal{B}_0 = \frac{\exp(2\Lambda)[(8 + 5c + 8\epsilon^{-1})\Lambda + 4(2 + c + \epsilon^{-1})]}{1 - c\alpha},$$

it holds that, for any  $h \in (0, 1)$ ,

$$\left. \begin{array}{l} \mathcal{B}_0 \exp(2\Lambda h) h^{1/2} \leq \epsilon/2 \\ t_0 + h \leq T \end{array} \right\} \Rightarrow e(t_0 + h) - e(t_0) \leq \mathcal{B}_0 h^{1/2}.$$

*Proof.* By the Markov property, we can assume  $t_0 = 0$ , with  $T$  being understood as  $T - t_0$ . Indeed, setting

$$X_t^{\#t_0} := X_{t_0+t}, \quad t \in [0, T - t_0], \quad (6.6)$$

we observe that, for  $t \in [0, T - t_0]$ ,

$$X_t^{\#t_0} = X_{t_0} + \int_0^t b(X_r^{\#t_0}) dr + \alpha \mathbb{E}(M_{t+t_0} - M_{t_0}) + W_{t+t_0} - W_{t_0} - (M_{t+t_0} - M_{t_0}). \quad (6.7)$$

Here  $M_{t+t_0} - M_{t_0}$  represents the number of times the process  $X$  reaches 1 within the interval  $(t_0, t+t_0]$ . Therefore, this also matches the number of times the process  $X^{\#t_0}$  hits 1 within the interval  $(0, t]$ , so that  $X^{\#t_0}$  indeed satisfies the nonlinear equation (2.1) on  $[0, T - t_0]$ , with  $X_0^{\#t_0} = X_{t_0}$  as initial condition and with respect to the shifted Brownian motion  $(W_t^{\#t_0} := W_{t_0+t} - W_{t_0})_{0 \leq t \leq T-t_0}$ . In what follows,  $t_0$  is thus assumed to be zero, the new  $T$  standing for the previous  $T - t_0$  and the new  $X_0$  matching the previous  $X_{t_0}$  and thus satisfying (6.5).

For a given  $h \in (0, 1)$ , such that  $h \leq T$ , and a given  $\mathcal{B}_0 > 0$  (the value of which will be fixed later), we then define the deterministic hitting time:

$$R = \inf \{ t \in [0, h] : \mathbb{E}(M_t) = e(t) \geq \mathcal{B}_0 h^{1/2} \}.$$

Following the proof of (4.9) (see more specifically (4.7)), we have, for any  $t \in [0, h \wedge R]$ ,

$$\begin{aligned} M_t &\leq \sup_{0 \leq s \leq t} (Z_s)_+ \leq (X_0)_+ + \Lambda \int_0^t (1 + (Z_s)_+ + M_s) ds + \alpha e(t) + 2 \sup_{0 \leq s \leq t} |W_s| \\ &\leq (X_0)_+ + 2\Lambda \int_0^t (1 + M_s) ds + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leq s \leq t} |W_s| \\ &\leq (X_0)_+ + 2\Lambda h + 2\Lambda \int_0^t M_s ds + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leq s \leq t} |W_s|, \end{aligned}$$

where we have used (6.3) to pass from the first to the second line. By Gronwall's Lemma, we obtain

$$\begin{aligned} M_t &\leq \exp(2\Lambda h) \left[ (X_0)_+ + 2\Lambda h + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leq s \leq h} |W_s| \right] \\ &\leq (X_0)_+ + \exp(2\Lambda h) \left[ 4\Lambda h + \alpha \mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leq s \leq h} |W_s| \right], \end{aligned} \quad (6.8)$$

as  $\exp(2\Lambda h) \leq 1 + 2\Lambda h \exp(2\Lambda h)$  and  $(X_0)_+ \leq 1$ .

Assume that  $\mathcal{B}_0 \exp(2\Lambda h) h^{1/2} \leq \epsilon/2 \leq 1/2$ . Then, by Doob's  $L^2$  inequality for martingales,

$$\begin{aligned} \sum_{k \geq 2} \mathbb{P}(M_t \geq k) &\leq \sum_{k \geq 2} \mathbb{P} \left( \exp(2\Lambda h) \left[ 4\Lambda h + 2 \sup_{0 \leq s \leq h} |W_s| \right] \geq k - 3/2 \right) \\ &\leq 2 \exp(2\Lambda h) \mathbb{E} [4\Lambda h + 4|W_h|] \leq \exp(2\Lambda h) [8\Lambda h + 8h^{1/2}]. \end{aligned} \quad (6.9)$$

Moreover,

$$\begin{aligned}
& \mathbb{P}(M_t \geq 1) \\
& \leq \mathbb{P}\left((X_0)_+ + \exp(2\Lambda h) \left[4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leq s \leq h} |W_s|\right] \geq 1\right) \\
& \leq \mathbb{P}\left(X_0 \in [1 - \epsilon, 1], X_0 + \exp(2\Lambda h) \left[4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leq s \leq h} |W_s|\right] \geq 1\right) \\
& \quad + \mathbb{P}\left(\exp(2\Lambda h) \left[4\Lambda h + 2 \sup_{0 \leq s \leq h} |W_s|\right] \geq \epsilon/2\right) \\
& := I_1 + I_2,
\end{aligned}$$

where we have used  $\mathcal{B}_0 \exp(2\Lambda h) h^{1/2} \leq \epsilon/2$  in the third line.

By Doob's maximal inequality, we deduce that

$$I_2 \leq 2 \exp(2\Lambda h) \epsilon^{-1} \mathbb{E}[4\Lambda h + 2|W_h|] \leq \exp(2\Lambda h) \epsilon^{-1} [8\Lambda h + 4h^{1/2}]. \quad (6.10)$$

We now switch to  $I_1$ . By independence of  $X_0$  and  $(W_s)_{0 \leq s \leq T}$  and by (6.5),

$$\begin{aligned}
I_1 & \leq c \int_0^\epsilon \mathbb{P}\left(\exp(2\Lambda h) \left[4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leq s \leq h} |W_s|\right] \geq x\right) dx \\
& \leq c \int_0^{+\infty} \mathbb{P}\left(\exp(2\Lambda h) \left[4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leq s \leq h} |W_s|\right] \geq x\right) dx \\
& = c \exp(2\Lambda h) \mathbb{E}\left[4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 2 \sup_{0 \leq s \leq h} |W_s|\right].
\end{aligned}$$

By Doob's  $L^2$  inequality,

$$I_1 \leq c \exp(2\Lambda h) [4\Lambda h + \alpha\mathcal{B}_0 h^{1/2} + 4h^{1/2}].$$

Together with (6.10), we deduce that

$$\mathbb{P}(M_t \geq 1) \leq \exp(2\Lambda h) [4(c + 2\epsilon^{-1})\Lambda h + 4(c + \epsilon^{-1})h^{1/2} + c\alpha\mathcal{B}_0 h^{1/2}].$$

From (6.9), we finally obtain, for  $t \leq R \wedge h$ ,

$$\begin{aligned}
\mathbb{E}(M_t) & = \sum_{k \geq 1} \mathbb{P}(M_t \geq k) \\
& \leq \exp(2\Lambda h) [4(2 + c + 2\epsilon^{-1})\Lambda h + 4(2 + c + \epsilon^{-1})h^{1/2} + c\alpha\mathcal{B}_0 h^{1/2}] \\
& \leq \exp(2\Lambda h) [(8 + 5c + 8\epsilon^{-1})\Lambda h + 4(2 + c + \epsilon^{-1})h^{1/2}] + c\alpha\mathcal{B}_0 h^{1/2},
\end{aligned}$$

provided  $\mathcal{B}_0 \exp(2\Lambda h) h^{1/2} \leq \epsilon/2 \leq 1/2$ , which implies

$$c\alpha\mathcal{B}_0 \exp(2\Lambda h) h^{1/2} \leq c\alpha\mathcal{B}_0 h^{1/2} + c\Lambda h,$$

using the fact that  $\exp(2\Lambda h) \leq 1 + 2\Lambda h \exp(2\Lambda h)$ . Therefore, if  $R \leq h$ , then we can choose  $t = R$  in the left-hand side above. By continuity of  $e$  on  $[0, T]$ , it then holds  $e(R) = \mathcal{B}_0 h^{1/2}$ , so that

$$\begin{aligned}
(1 - c\alpha)\mathcal{B}_0 h^{1/2} & \leq \exp(2\Lambda h) [(8 + 5c + 8\epsilon^{-1})\Lambda h + 4(2 + c + \epsilon^{-1})h^{1/2}] \\
& < \exp(2\Lambda h) [(8 + 5c + 8\epsilon^{-1})\Lambda + 4(2 + c + \epsilon^{-1})] h^{1/2},
\end{aligned}$$

which is not possible when

$$\mathcal{B}_0 = \frac{\exp(2\Lambda)[(8 + 5c + 8\epsilon^{-1})\Lambda + 4(2 + c + \epsilon^{-1})]}{1 - c\alpha}.$$

Precisely, with  $\mathcal{B}_0$  as above and  $\mathcal{B}_0 \exp(2\Lambda h)h^{1/2} \leq \epsilon/2$  it cannot hold  $R \leq h$ .  $\square$

**6.3. Global Hölder bound.** In this subsection, we shall prove:

**Proposition 6.3.** *For a given  $\epsilon \in (0, 1)$ , there exists a positive constant  $\alpha_0 \in (0, 1]$ , only depending upon  $\epsilon$ ,  $K$  and  $\Lambda$ , such that, for any  $\alpha < \alpha_0$ , there exists a constant  $\mathcal{B}$ , only depending on  $\alpha$ ,  $\epsilon$ ,  $K$  and  $\Lambda$ , such that, for any positive time  $T > 0$  and any initial condition  $X_0 = x_0 \leq 1 - \epsilon$ , any solution to (2.1) according to Definition 2.2, so that the mapping  $e : [0, T] \ni t \mapsto \mathbb{E}(M_t)$  is continuously differentiable, satisfies, for any  $h \in (0, 1)$  and  $t_0 \in [0, T]$ ,*

$$\left. \begin{array}{l} \mathcal{B}h^{1/2} \leq \epsilon/2 \\ t_0 + h \leq T \end{array} \right\} \Rightarrow e(t_0 + h) - e(t_0) \leq \mathcal{B}h^{1/2}.$$

Note that  $\mathcal{B}$  above may differ from  $\mathcal{B}_0$  in the statement of Lemma 6.2. The constant  $\alpha_0$  can be described as follows. Defining  $T_0$  as the largest time less than 1 such that

$$(1 - \epsilon) \exp(\Lambda T_0) \leq 1 - 7\epsilon/8, \quad \Lambda T_0 \exp(\Lambda T_0) \leq \epsilon/8,$$

then  $\alpha_0$  can be chosen as the largest (positive) real satisfying (with  $B(T_0, \alpha_0, b)$  as in Lemma 6.1)

$$\begin{aligned} \alpha_0 B(T_0, \alpha_0, b) &\leq \epsilon/4, \\ \alpha_0 2^{3/2} (c')^{3/2} \exp\left(-\frac{1}{2}\right) \left[\epsilon^{-1} + B(T_0, \alpha_0, b)\right] &\leq 1, \\ \alpha_0 \left[c' T_0^{-1/2} + 2^{3/2} (c')^{3/2} \exp\left(-\frac{1}{2}\right) B(T_0, \alpha_0, b)\right] &\leq 1, \end{aligned}$$

where the constant  $c'$  is defined by the following property:  $c' > 0$ , depending on  $K$  only, is such that for any diffusion process  $(U_t)_{0 \leq t \leq 1}$  satisfying

$$dU_t = F(t, U_t)dt + dW_t, \quad t \in [0, 1],$$

where  $U_0 = 0$  and  $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is  $K$ -Lipschitz in  $x$  and satisfies  $F(t, 0) = 0$  for any  $t \in [0, 1]$ , it holds that

$$\frac{1}{dx} \mathbb{P}(U_t \in dx) \leq \frac{c'}{\sqrt{t}} \exp\left(-\frac{x^2}{c't}\right), \quad x \in \mathbb{R}, \quad t \in (0, 1].$$

The proof relies on the following

**Lemma 6.4.** *Given an initial condition  $X_0 = x_0 \leq 1 - \epsilon$ , with  $\epsilon \in (0, 1)$ , and a solution  $(X_t)_{0 \leq t \leq T}$  to (2.1) on some interval  $[0, T]$ , the random variable  $X_t$  has a density on  $(-\infty, 1]$ , for any  $t \in (0, T]$ . Moreover, defining  $T_0$  as in the statement of Proposition 6.3 and choosing  $\alpha \leq \alpha_1$  satisfying*

$$\alpha_1 B(T_0, \alpha_1, b) \leq \epsilon/4,$$

it holds, for  $x \in [1 - \epsilon/4, 1)$ ,

$$\begin{aligned} \frac{1}{dx} \mathbb{P}(X_t \in dx) &\leq 2^{3/2} (c')^{3/2} \exp\left(-\frac{1}{2}\right) \left[\epsilon^{-1} + B(T_0, \alpha, b)\right] \quad \text{if } t \leq T_0 \\ \frac{1}{dx} \mathbb{P}(X_t \in dx) &\leq c' T_0^{-1/2} + 2^{3/2} (c')^{3/2} \exp\left(-\frac{1}{2}\right) B(T_0, \alpha, b) \quad \text{if } t > T_0, \end{aligned}$$

where the constant  $c'$  is as in the statement of Proposition 6.3.

Before we prove Lemma 6.4, we introduce some materials. As usual, we set  $e(t) = \mathbb{E}(M_t)$ , for  $t \in [0, T]$ , the mapping  $e$  being assumed to be continuously differentiable on  $[0, T]$ . Moreover, with  $(X_t)_{0 \leq t \leq T}$ , we associate the sequence of hitting times  $(\tau_k)_{k \geq 0}$  given by (2.3). We then investigate the marginal distributions of  $(X_t)_{0 \leq t \leq T}$ . Given a Borel subset  $A \subset (-\infty, 1]$ , we write in the same way as in the proof of (5.1):

$$\begin{aligned} \mathbb{P}(X_t \in A) &= \mathbb{P}(X_t \in A, \tau_1 > t) \\ &\quad + \sum_{k \geq 1} \int_0^t \mathbb{P}(X_t \in A, \tau_{k+1} > t | \tau_k = s) \mathbb{P}(\tau_k \in ds), \end{aligned} \quad (6.11)$$

where the notation  $\mathbb{P}(\cdot | \tau_k = s)$  stands for the conditional law given  $\tau_k = s$ . Following (6.6) and (6.7), we can shift the system by length  $s \in [0, T]$ . Precisely, we know that  $(X_r^{\sharp s} := X_{s+r})_{0 \leq r \leq T-s}$  satisfies

$$X_r^{\sharp s} = X_s + \int_0^r b(X_u^{\sharp s}) du + \alpha e^{\sharp s}(r) + W_{s+r} - W_s - M_r^{\sharp s}, \quad (6.12)$$

with

$$\begin{aligned} e^{\sharp s}(r) &:= e(s+r) - e(s) \quad ; \quad M_r^{\sharp s} := \sum_{k \geq 1} \mathbf{1}_{(0,r]}(\tau_k^{\sharp s}) \quad ; \\ \tau_k^{\sharp s} &:= \inf\{u > \tau_{k-1}^{\sharp s} : X_{s+u-} \geq 1\}, \quad k \geq 1 \quad ; \quad \tau_0^{\sharp s} := 0. \end{aligned} \quad (6.13)$$

Conditionally on  $\tau_k = s$ , the law of  $(X_r^{\sharp s})_{0 \leq r \leq T-s}$  until  $\tau_1^{\sharp s}$  coincides with the law of  $(\hat{Z}_r^{\sharp s, 0})_{0 \leq r \leq T-s}$  until the first time it reaches 1, where, for a given  $\mathcal{F}_0$ -measurable initial condition  $\zeta$  with values in  $(-\infty, 1)$ ,  $(\hat{Z}_r^{\sharp s, \zeta})_{0 \leq r \leq T-s}$  stands for the solution of the SDE:

$$\hat{Z}_r^{\sharp s, \zeta} = \zeta + \int_0^r b(\hat{Z}_u^{\sharp s, \zeta}) du + \alpha e^{\sharp s}(r) + W_r, \quad r \in [0, T-s]. \quad (6.14)$$

Below, we will write  $\hat{Z}_r^\zeta$  for  $\hat{Z}_r^{\sharp_0, \zeta}$ . In the end, we thus have (by (6.11)):

$$\begin{aligned}
\mathbb{P}(X_t \in A) &= \mathbb{P}(X_t \in A, \tau_1 > t) \\
&+ \sum_{k \geq 1} \int_0^t \mathbb{P}(X_{t-s}^{\sharp_s} \in A, \tau_1^{\sharp_s} > t-s | \tau_k = s) \mathbb{P}(\tau_k \in ds) \\
&\leq \mathbb{P}(\hat{Z}_t^{X_0} \in A) + \sum_{k \geq 1} \int_0^t \mathbb{P}(\hat{Z}_{t-s}^{\sharp_s, 0} \in A) \mathbb{P}(\tau_k \in ds) \\
&= \mathbb{P}(\hat{Z}_t^{X_0} \in A) + \int_0^t \mathbb{P}(\hat{Z}_{t-s}^{\sharp_s, 0} \in A) e'(s) ds,
\end{aligned} \tag{6.15}$$

for any Borel set  $A \subset (-\infty, 1]$ , the passage from the third to the fourth line following from (5.9).

We can now turn to:

*Proof of Lemma 6.4.* Given an initial condition  $x_0 \in (-\infty, 1 - \epsilon]$  for  $\epsilon \in (0, 1)$ , we know from Delarue and Menozzi [7] that  $\hat{Z}_t^{x_0}$  has a density for any  $t \in (0, T]$  (and thus  $\hat{Z}_{t-s}^{\sharp_s, 0}$  as well for  $0 \leq s < t$ ). From (6.15), we deduce that the law of  $X_t$  has a density on  $(-\infty, 1]$  since  $\mathbb{P}(X_t \in A) = 0$  when  $|A| = 0$ , where  $|A|$  stands for the Lebesgue measure of  $A$ . Moreover, there exists a constant  $c' \geq 1$ , depending on  $K$  only, such that, for any  $t \in [0, T \wedge 1]$ :

$$\frac{1}{dx} \mathbb{P}(\hat{Z}_t^{x_0} \in dx) \leq \frac{c'}{\sqrt{t}} \exp\left(-\frac{[x - \vartheta_t^{x_0}]^2}{c't}\right), \tag{6.16}$$

where  $\vartheta_t^{x_0}$  is the solution of the ODE:

$$\frac{d}{dt} \vartheta_t = b(\vartheta_t) + \alpha e'(t), \quad t \in [0, T], \tag{6.17}$$

with  $\vartheta_0^{x_0} = x_0$ . Above, the function  $[0, T] \ni t \mapsto e(t)$  represents  $[0, T] \ni t \mapsto \mathbb{E}(M_t)$  given  $X_0 = x_0$ , which means that the initial condition  $x_0$  of  $X_0$  upon which  $e$  depends is fixed once and for all, independently of the initial condition of  $\vartheta$ . In particular, as the initial condition of  $\vartheta$  varies, the function  $e$  does not. We emphasize that  $c'$  is independent of  $e$  and can be taken to be that defined in Proposition 6.3. Indeed, we can write  $\mathbb{P}(\hat{Z}_t^{x_0} \in dx)$  as  $\mathbb{P}(\hat{Z}_t^{x_0} - \vartheta_t^{x_0} \in d(x - \vartheta_t^{x_0}))$ , with

$$\begin{aligned}
d(\hat{Z}_t^{x_0} - \vartheta_t^{x_0}) &= F(t, \hat{Z}_t^{x_0} - \vartheta_t^{x_0}) dt + dW_t, \quad t \in [0, T]; \quad \hat{Z}_0^{x_0} - \vartheta_0^{x_0} = 0; \\
F(t, x) &= b(x + \vartheta_t^{x_0}) - b(\vartheta_t^{x_0}), \quad t \in [0, T], \quad x \in \mathbb{R}.
\end{aligned}$$

We then notice that  $F(t, \cdot)$  is  $K$ -Lipschitz continuous (since  $b$  is) and satisfies  $F(t, 0) = 0$ , so that, referring to [7], all the parameters involved in the definition of the constant  $c'$  are independent of  $e$ . The fact that  $c'$  is independent of  $e$  is crucial. As a consequence, we can bound  $(d/dx)\mathbb{P}(\hat{Z}_{t-s}^{\sharp_s, 0} \in dx)$  in a similar way, that is, with the same constant  $c'$  as in (6.16): for any  $0 \leq s < t \leq T$ , with  $t - s \leq 1$ ,

$$\frac{1}{dx} \mathbb{P}(\hat{Z}_{t-s}^{\sharp_s, 0} \in dx) \leq \frac{c'}{\sqrt{t-s}} \exp\left(-\frac{[x - \vartheta_{t-s}^{\sharp_s, 0}]^2}{c'(t-s)}\right), \tag{6.18}$$

where  $\vartheta_t^{\sharp s,0}$  is the solution of the ODE:

$$\frac{d}{dt}\vartheta_t^{\sharp s} = b(\vartheta_t^{\sharp s}) + \alpha \frac{d}{dt}e^{\sharp s}(t), \quad t \in [0, T-s],$$

with  $\vartheta_0^{\sharp s,0} = 0$  as initial condition.

*Bound of the density in small time.* Keep in mind that  $X_0 = x_0 \leq 1-\epsilon$ . Therefore, by the comparison principle for ODEs,  $\vartheta_t^{x_0} \leq \vartheta_t^{1-\epsilon}$  for any  $t \in [0, T]$ , so that by Gronwall's Lemma

$$\vartheta_t^{x_0} \leq \vartheta_t^{1-\epsilon} \leq (1 - \epsilon + \Lambda T + \alpha e(T)) \exp(\Lambda T).$$

By Lemma 6.1, we know that  $e(T) \leq B(T, \alpha, b)$ , so that

$$\vartheta_t^{x_0} \leq (1 - \epsilon + \Lambda T + \alpha B(T, \alpha, b)) \exp(\Lambda T). \quad (6.19)$$

Now choose  $T_0$  as in Proposition 6.3, i.e.  $T_0 \leq 1$  such that

$$(1 - \epsilon) \exp(\Lambda T_0) \leq 1 - 7\epsilon/8, \quad \Lambda T_0 \exp(\Lambda T_0) \leq \epsilon/8,$$

and then take  $\alpha_1 \in (0, 1)$  such that

$$\alpha_1 B(T_0, \alpha_1, b) \exp(\Lambda T_0) \leq \epsilon/4.$$

Then, whenever  $\alpha \leq \alpha_1$ , it holds that

$$\vartheta_t^{x_0} \leq 1 - \epsilon/2, \quad t \in [0, T_0 \wedge T].$$

Therefore, for  $x \geq 1 - \epsilon/4$ ,

$$\exp\left(-\frac{[x - \vartheta_t^{x_0}]^2}{c't}\right) \leq \exp\left(-\frac{\epsilon^2}{16c't}\right), \quad t \in [0, T_0 \wedge T]. \quad (6.20)$$

Similarly,

$$\vartheta_{t-s}^{\sharp s,0} \leq 3\epsilon/8 \leq 3/8, \quad 0 \leq s \leq t \leq T_0 \wedge T.$$

Indeed,  $e^{\sharp s}(T-s) \leq e(T)$  for  $s \in [0, T]$ , so that (6.19) applies to  $\vartheta_{t-s}^{\sharp s,0}$  with  $1-\epsilon$  therein being replaced by 0. Therefore, for  $x \geq 1 - \epsilon/4$ , it holds that  $x - \vartheta_{t-s}^{\sharp s,0} \geq 3/4 - 3/8 = 3/8 \geq 1/4$ , so that

$$\exp\left(-\frac{[x - \vartheta_{t-s}^{\sharp s,0}]^2}{c'(t-s)}\right) \leq \exp\left(-\frac{1}{16c'(t-s)}\right), \quad 0 \leq s < t \leq T_0 \wedge T. \quad (6.21)$$

In the end, for  $x \in (1 - \epsilon/4, 1)$  and  $t \leq T_0 \wedge T$ , we deduce from (6.15), (6.16), (6.18), (6.20), (6.21) and Lemma 6.1 again, that

$$\frac{1}{dx} \mathbb{P}(X_t \in dx) \leq c' \varpi_0 [\epsilon^{-1} + e(T \wedge T_0)] \leq c' \varpi_0 [\epsilon^{-1} + B(T_0, \alpha, b)], \quad (6.22)$$

where

$$\varpi_0 = \sup_{t>0} \left[ t^{-1/2} \exp\left(-\frac{1}{16c't}\right) \right] = 4\sqrt{c'} \sup_{u>0} [u \exp(-u^2)] = 2^{3/2} \sqrt{c'} \exp\left(-\frac{1}{2}\right).$$

*Bound of the density in long time.* We now discuss what happens for  $T > T_0$  and  $t \in [T_0, T]$ . Then,

$$\begin{aligned} \frac{1}{dx} \mathbb{P}(X_t \in dx) &\leq \frac{1}{dx} \mathbb{P}\left(X_t \in dx, \tau_1^{\sharp t-T_0} \leq T_0\right) + \frac{1}{dx} \mathbb{P}\left(X_t \in dx, \tau_1^{\sharp t-T_0} > T_0\right) \\ &= \pi_1 + \pi_2, \end{aligned} \quad (6.23)$$

with  $\tau_1^{\sharp t-T_0} = \inf\{u > 0 : X_{t-T_0+u-} \geq 1\} = \inf\{u > 0 : X_{u-}^{\sharp t-T_0} \geq 1\}$ . The above expression says that we split the event ( $X_t$  is in the neighbourhood of  $x$ ) into two disjoint parts according to the fact that  $X$  reaches the threshold or not within the time window  $[t - T_0, t]$ . We have chosen this interval to be of length  $T_0$  in order to apply the results in small time.

We first investigate  $\pi_2$ . The point is that, on the event that  $\tau_1^{\sharp t-T_0} > T_0$  and within the time window  $[t - T_0, t]$ ,  $X$  behaves as a standard diffusion process without any jumps, namely as a process with the same dynamics as  $\hat{Z}_{T_0, X_{t-T_0}}^{\sharp t-T_0}$ . Following (6.16), we then have

$$\begin{aligned} \pi_2 &= \frac{1}{dx} \mathbb{P}\left(\hat{Z}_{T_0}^{\sharp t-T_0, X_{t-T_0}} \in dx, \tau_1^{\sharp t-T_0} > T_0\right) \\ &\leq \frac{1}{dx} \mathbb{P}\left(\hat{Z}_{T_0}^{\sharp t-T_0, X_{t-T_0}} \in dx\right) \leq \sup_{z \leq 1} \frac{1}{dx} \mathbb{P}\left(\hat{Z}_{T_0}^{\sharp t-T_0, z} \in dx\right) \leq c'T_0^{-1/2}. \end{aligned} \quad (6.24)$$

We now turn to  $\pi_1$ . Here we write that

$$\begin{aligned} \pi_1 &= \frac{1}{dx} \mathbb{P}\left(X_t \in dx, \tau_1^{\sharp t-T_0} \leq T_0\right) = \sum_{k \geq 1} \frac{1}{dx} \mathbb{P}\left(X_t \in dx, \tau_k^{\sharp t-T_0} \leq T_0 < \tau_{k+1}^{\sharp t-T_0}\right) \\ &= \sum_{k \geq 1} \int_0^{T_0} \frac{1}{dx} \mathbb{P}\left(X_t \in dx, \tau_k^{\sharp t-T_0} \leq T_0 < \tau_{k+1}^{\sharp t-T_0} \mid \tau_k^{\sharp t-T_0} = s\right) \mathbb{P}(\tau_k^{\sharp t-T_0} \in ds) \\ &= \sum_{k \geq 1} \int_0^{T_0} \frac{1}{dx} \mathbb{P}\left(\hat{Z}_{T_0-s}^{\sharp s+t-T_0, 0} \in dx, T_0 < \tau_{k+1}^{\sharp t-T_0}\right) \mathbb{P}(\tau_k^{\sharp t-T_0} \in ds), \end{aligned}$$

since on the event  $\{\tau_k^{\sharp t-T_0} \leq T_0 < \tau_{k+1}^{\sharp t-T_0}\}$ , given that the  $k$ -th (and last) jump of  $X$  in the interval  $[t - T_0, t]$  occurs at time  $t - T_0 + s$  with  $s \in [0, T_0]$ , we have that the process  $X_r$  for  $r \in [t - T_0 + s, t]$  coincides with the process  $\hat{Z}_u^{\sharp s+t-T_0, 0}$  for  $u \in [0, T_0 - s]$ . Thus

$$\begin{aligned} \pi_1 &\leq \sum_{k \geq 1} \int_0^{T_0} \frac{1}{dx} \mathbb{P}\left(\hat{Z}_{T_0-s}^{\sharp s+t-T_0, 0} \in dx\right) \mathbb{P}(\tau_k^{\sharp t-T_0} \in ds) \\ &= \int_0^{T_0} \frac{1}{dx} \mathbb{P}\left(\hat{Z}_{T_0-s}^{\sharp s+t-T_0, 0} \in dx\right) e'(s + t - T_0) ds. \end{aligned} \quad (6.25)$$

By (6.18), we have

$$\begin{aligned} & \int_0^{T_0} \frac{1}{dx} \mathbb{P} \left( \hat{Z}_{T_0-s}^{\#s+t-T_0,0} \in dx \right) e'(s+t-T_0) ds \\ & \leq \int_0^{T_0} \frac{c'}{\sqrt{T_0-s}} \exp \left( -\frac{[x - v_{T_0-s}^{\#s+t-T_0,0}]^2}{c'(T_0-s)} \right) e'(s+t-T_0) ds. \end{aligned}$$

Following the shift in (6.12), it is well seen that the mapping  $[0, T_0] \ni s \mapsto e^{\#t-T_0}(s)$  satisfies Lemma 6.1, that is

$$\sup_{0 \leq s \leq T_0} e^{\#t-T_0}(s) = \sup_{0 \leq s \leq T_0} [e(s+t-T_0) - e(t-T_0)] = e(t) - e(t-T_0) \leq B(T_0, \alpha, b).$$

Therefore, we can follow the same strategy as in short time, see (6.21) and (6.22). Indeed, for  $\alpha \leq \alpha_1$ , by the choice of  $T_0$  as before, it holds that

$$\pi_1 \leq c' \varpi_0 B(T_0, \alpha, b),$$

for  $x \in [1-\epsilon/4, 1)$ . Using (6.24) and the above bound, we deduce that, for  $t \in [T_0, T]$ ,

$$\frac{1}{dx} \mathbb{P}(X_t \in dx) \leq c' [T_0^{-1/2} + \varpi_0 B(T_0, \alpha, b)].$$

□

*Proof of Proposition 6.3.* Proposition 6.3 follows from the combination of Lemmas 6.2 and 6.4. Indeed, given  $T_0$  and  $\alpha_0$  as defined in Proposition 6.3, then by Lemma 6.4 it follows that  $\mathbb{P}(X_t \in A) < (1/\alpha)|A|$  for any Borel subset  $A \subset [1-\epsilon/4, 1]$ , any  $\alpha < \alpha_0$  and any  $t \in [0, T]$ . The result follows by Lemma 6.2, with  $\mathcal{B}$  being given by  $\mathcal{B}_0 \exp(2\Lambda)$  with  $\epsilon$  in  $\mathcal{B}_0$  replaced by  $\epsilon/4$ . □

**6.4. Estimate of the density of the killed process.** In light of the previous subsection, for a solution  $(X_t)_{0 \leq t \leq T}$  to (2.1) such that the mapping  $[0, T] \ni t \mapsto e(t) = \mathbb{E}(M_t)$  is continuously differentiable, we here investigate

$$\frac{1}{dx} \mathbb{P}(X_t \in dx, t < \tau_1), \quad t \in [0, T], \quad x \leq 1,$$

where  $\tau = \inf\{t > 0 : X_{t-} \geq 1\}$  as usual. This is the density of the killed process  $(X_{t \wedge \tau_1})_{0 \leq t \leq T}$ , which makes sense because of Lemma 5.3.

Here is the main result of this subsection:

**Lemma 6.5.** *For any  $\epsilon \in (0, 1)$ , any  $T > 0$  and any  $\mathcal{B} > 0$ , there exist two constants  $\mu_T$  and  $\eta_T$ , only depending upon  $T, \mathcal{B}, \epsilon, K$  and  $\Lambda$ , such that, for any initial condition  $x_0 \leq 1 - \epsilon$  and any continuously differentiable non-decreasing deterministic mapping  $[0, T] \ni t \mapsto e(t)$  satisfying*

$$e(0) = 0 \quad ; \quad e(t) - e(s) \leq \mathcal{B}(t-s)^{1/2}, \quad 0 \leq s \leq t \leq T,$$

if  $(\chi_t)_{0 \leq t \leq T}$  denotes the solution of the SDE

$$d\chi_t = b(\chi_t)dt + \alpha e'(t)dt + dW_t, \quad t \in [0, T]; \quad \chi_0 = x_0,$$

then, with  $p(t, y)$  denoting the density of  $\chi_t$  killed at 1 as in (5.3), we have

$$p(t, y) \leq \mu_T (1-y)^{\eta_T}, \quad t \in [0, T], \quad y \in [1-\epsilon/4, 1]. \quad (6.26)$$

*Proof. First Step.* The first step is to provide a probabilistic representation for  $p$ . For a given  $(T, x) \in (0, +\infty) \times (-\infty, 1)$ , we consider the solution to the SDE:

$$dY_t = -[b(Y_t) + \alpha e'(T-t)]dt + dW_t, \quad t \in [0, T]; \quad Y_0 = y, \quad (6.27)$$

together with some stopping time  $\rho \leq \rho_0 \wedge T$ , where  $\rho_0 = \inf\{t \in [0, T] : Y_t \geq 1\}$  (with  $\inf \emptyset = +\infty$ ). Then, by Lemma 5.3 and the Itô-Krylov formula (see [10, Chapter II, Section 10]),

$$\begin{aligned} & d(p(T-t, Y_t)) \\ &= -\partial_t p(T-t, Y_t)dt - [b(Y_t) + \alpha e'(T-t)]\partial_y p(T-t, Y_t)dt + \frac{1}{2}\partial_{yy}^2 p(T-t, Y_t)dt \\ & \quad + \partial_y p(T-t, Y_t)dW_t \\ &= b'(Y_t)p(T-t, Y_t)dt + \partial_y p(T-t, Y_t)dW_t, \end{aligned}$$

for  $0 \leq t \leq \rho$ . Therefore, the Feynman-Kac formula yields

$$p(T, y) = \mathbb{E} \left[ p(T-\rho, Y_\rho) \mathbf{1}_{\{Y_\rho \neq 1\}} \exp \left( - \int_0^\rho b'(Y_s) ds \right) \middle| Y_0 = y \right], \quad (6.28)$$

the indicator function following from the Dirichlet boundary condition satisfied by  $p(\cdot, 1)$ .

*Second Step.* We now specify the choice of  $\rho$ . Given some free parameters  $L \geq 1$  and  $\delta \in (0, \epsilon/4)$  such that  $L\delta \leq \epsilon/4$ , we assume that the initial condition  $y$  in (6.27) is in  $(1-\delta, 1)$  and then consider the stopping time

$$\rho = \inf\{t \in [0, T] : Y_t \notin (1-L\delta, 1)\} \wedge \delta^2. \quad (6.29)$$

Assume that  $\delta^2 \leq T$ . By (6.28), we deduce that

$$p(T, y) \leq \exp(K\delta^2)(1 - \mathbb{P}(Y_\rho = 1)) \sup_{(t,z) \in \mathcal{Q}(\delta, L)} p(t, z), \quad (6.30)$$

with

$$\mathcal{Q}(\delta, L) = \{(t, z) \in [T-\delta^2, T] \times [1-L\delta, 1]\}.$$

The point is then to give a lower bound for  $\mathbb{P}(Y_\rho = 1)$ . By assumption, we know that  $e$  is  $(1/2)$ -Hölder continuous on  $[0, T]$ . Therefore, since  $Y_0 = y \in (1-\delta, 1)$ , we have, for any  $t \in [0, \rho]$ ,

$$Y_t \geq 1 - \delta - m\delta^2 - \alpha\mathcal{B}\delta + W_t,$$

with

$$m = \sup_{0 \leq z \leq 1} |b(z)|. \quad (6.31)$$

Therefore, for  $m\delta \leq 1$ ,

$$Y_t \geq 1 - 2\delta - \alpha\mathcal{B}\delta + W_t, \quad t \in [0, \rho],$$

so that

$$\{Y_\rho = 1\} \supset \left\{ \sup_{0 \leq t \leq \delta^2} W_t > (2 + \alpha\mathcal{B})\delta \right\} \cap \left\{ \inf_{0 \leq t \leq \delta^2} W_t > (2 + \alpha\mathcal{B} - L)\delta \right\}. \quad (6.32)$$

Choosing  $L = 3 + \alpha\mathcal{B}$  and applying a scaling argument, we deduce that

$$\begin{aligned} & \mathbb{P} \left( \left\{ \sup_{0 \leq t \leq \delta^2} W_t > (2 + \alpha\mathcal{B})\delta \right\} \cap \left\{ \inf_{0 \leq t \leq \delta^2} W_t > (2 + \alpha\mathcal{B} - L)\delta \right\} \right) \\ &= \mathbb{P} \left( \left\{ \sup_{0 \leq t \leq 1} W_t > (2 + \alpha\mathcal{B}) \right\} \cap \left\{ \inf_{0 \leq t \leq 1} W_t > -1 \right\} \right) =: c'' \in (0, 1). \end{aligned} \quad (6.33)$$

We note that the above quantity  $c''$  is independent of  $\delta$  and  $T$ . Moreover, we deduce from (6.32) that  $\mathbb{P}(Y_\rho = 1) \geq c''$  and therefore, from (6.30), that

$$p(T, y) \leq (1 - c'') \exp(K\delta^2) \sup_{z \in \mathcal{I}(L\delta)} \sup_{t \in [0, T]} p(t, z),$$

with  $\mathcal{I}(r) = [1 - r, 1]$ , for  $r > 0$ . Choosing  $\delta$  small enough such that  $(1 - c'') \exp(K\delta^2) \leq (1 - c''/2)$ , we obtain

$$p(T, y) \leq \left(1 - \frac{c''}{2}\right) \sup_{z \in \mathcal{I}(L\delta)} \sup_{t \in [0, T]} p(t, z), \quad y \in \mathcal{I}(\delta).$$

Modifying  $c''$  if necessary ( $c''$  being chosen as small as needed), we can summarize the above inequality as follows: for  $\delta \leq c''$ ,

$$p(T, y) \leq (1 - c'') \sup_{z \in \mathcal{I}(L\delta)} \sup_{t \in [0, T]} p(t, z), \quad y \in \mathcal{I}(\delta). \quad (6.34)$$

We now look at what happens when  $T \leq \delta^2$  in (6.30). In such a case  $\rho$  in the previous argument must be replaced by  $\rho \wedge T$ . Observing that  $p(T - \rho \wedge T, Y_{\rho \wedge T}) = 0$  on the event  $\{\rho \geq T\} \cup \{Y_{\rho \wedge T} = 1\}$  (since  $p(0, \cdot) = 0$  on  $[1 - \epsilon/4, 1]$ ) and following (6.30), we obtain, for  $y \in \mathcal{I}(\delta)$ ,

$$p(T, y) \leq \exp(K\delta^2) [1 - \mathbb{P}(\{Y_{\rho \wedge T} = 1\} \cup \{\rho \geq T\})] \sup_{(t, z) \in \mathcal{Q}'(\delta, L)} p(t, z), \quad (6.35)$$

with  $\mathcal{Q}'(\delta, L) = \{(t, z) \in [0, T] \times [1 - L\delta, 1]\}$ . Now, the right-hand side of (6.32) is included in the event  $\{Y_{\rho \wedge T} = 1\} \cup \{\rho \geq T\}$  so that (6.33) yields a lower bound for  $\mathbb{P}(\{Y_{\rho \wedge T} = 1\} \cup \{\rho \geq T\})$ . Therefore, we can repeat the previous arguments in order to prove that (6.34) also holds when  $T \leq \delta^2$ , which means that (6.34) holds true in both cases.

Therefore, by replacing  $T$  by  $t$  in the left-hand side in (6.34) and by letting  $t$  vary within  $[0, T]$ , we have in any case,

$$\sup_{y \in \mathcal{I}(\delta)} \sup_{t \in [0, T]} p(t, y) \leq (1 - c'') \sup_{z \in \mathcal{I}(L\delta)} \sup_{t \in [0, T]} p(t, z).$$

By induction, for any integer  $n \geq 1$  such that  $L^n \delta \leq r_0$ , with  $r_0 = c'' \wedge (\epsilon/4)$ ,

$$\sup_{y \in \mathcal{I}(\delta)} \sup_{t \in [0, T]} p(t, y) \leq (1 - c'')^n \sup_{z \in \mathcal{I}(L^n \delta)} \sup_{t \in [0, T]} p(t, z).$$

Given  $\delta \in (0, r_0/L)$ , the maximal value for  $n$  is  $n = \lfloor \ln[r_0/\delta] / \ln L \rfloor$ . We deduce that, for any  $\delta \in (0, r_0/L)$ ,

$$\sup_{y \in \mathcal{I}(\delta)} \sup_{t \in [0, T]} p(t, y) \leq (1 - c'')^{(\ln[r_0/\delta] / \ln L) - 1} \sup_{z \in \mathcal{I}(\epsilon/4)} \sup_{t \in [0, T]} p(t, z). \quad (6.36)$$

Following (6.16), we know that

$$\sup_{z \in \mathcal{I}(\epsilon/4)} \sup_{t \in [0, T]} p(t, z) \leq \sup_{z \in \mathcal{I}(\epsilon/4)} \sup_{t \in [0, T]} \left[ \frac{c_T}{\sqrt{t}} \exp \left( -\frac{[z - \vartheta_t^{x_0}]^2}{c_T t} \right) \right], \quad (6.37)$$

for some constant  $c_T$  only depending upon  $T$  and  $K$  and where  $(\vartheta_t^{x_0})_{0 \leq t \leq T}$  stands for the solution of the ODE

$$\frac{d\vartheta}{dt} = b(\vartheta_t) + \alpha e'(t), \quad t \in [0, T]; \quad \vartheta_0 = x_0.$$

Pay attention that we here use the same notation as in (6.17) for the solution of the above ODE but here  $e(t)$  is not given as some  $\mathbb{E}(M_t)$ . Actually, we feel that there is no possible confusion here. Notice also that  $e$  is fixed and does not depend upon the initial condition  $x_0$ .

By the comparison principle for ODEs and then by Gronwall's Lemma, we deduce from the fact that  $e$  is  $(1/2)$ -Hölder continuous that

$$\vartheta_t^{x_0} \leq \vartheta_t^{1-\epsilon} \leq [1 - \epsilon + \Lambda t + \mathcal{B}t^{1/2}] \exp(\Lambda t), \quad t \in [0, T].$$

Using the above inequality, we can bound the right-hand side in (6.37). Precisely, the above inequality says that the exponential term in the supremum decays exponentially fast as  $t$  tends to 0 so that the term inside the supremum can be bounded when  $t$  is small; when  $t$  is bounded away from 0, the term inside the supremum is bounded by  $c_T/\sqrt{t}$ . It is plain to deduce that

$$\sup_{z \in \mathcal{I}(\epsilon/4)} \sup_{t \in [0, T]} p(t, z) \leq c_T, \quad (6.38)$$

for a new value of  $c_T$ , possibly depending on  $\epsilon$  as well. Therefore, for  $\delta \in (0, r_0/L)$ , (6.36) yields

$$\sup_{y \in \mathcal{I}(\delta)} \sup_{t \in [0, T]} p(t, y) \leq \frac{c_T}{(1 - c'')} \left( \frac{\delta}{r_0} \right)^\eta,$$

with  $\eta = -\ln(1 - c'')/\ln L$ . This proves (6.26) for  $y \in (1 - r_0/L, 1)$ . Note that  $\eta$  is here independent of  $T$ , contrary to what is indicated in the statement of Lemma 6.5. However, we feel it is simpler to indicate  $T$  in  $\eta_T$  as the constant  $\mathcal{B}$  in the sequel will be chosen in terms of  $T$  thus making  $\eta$  depend on  $T$ . Using (6.38), we can easily extend the bound to any  $y \in (1 - \epsilon/4, 1)$  by modifying if necessary the parameters  $\mu_T$  and  $\eta_T$  therein. This completes the proof.  $\square$

**6.5. Bound for the gradient.** Here is the final step to complete the proof of Theorem 2.3:

**Proposition 6.6.** *For any  $\epsilon \in (0, 1)$ , any  $\mathcal{B} > 0$  and any  $T > 0$ , there exists a constant  $\mathcal{M}_T > 0$ , only depending upon  $T$ ,  $\mathcal{B}$ ,  $\epsilon$ ,  $K$  and  $\Lambda$ , such that, for any initial condition  $x_0 \leq 1 - \epsilon$ , any continuously differentiable non-decreasing deterministic mapping  $[0, T] \ni t \mapsto e(t)$  satisfying*

$$e(0) = 0 \quad ; \quad e(t) - e(s) \leq \mathcal{B}(t - s)^{1/2}, \quad 0 \leq s \leq t \leq T,$$

if  $(\chi_t)_{0 \leq t \leq T}$  denotes the solution of the SDE

$$d\chi_t = b(\chi_t)dt + \alpha e'(t)dt + dW_t, \quad t \in [0, T]; \quad \chi_0 = x_0,$$

then, for any integer  $n$  such that  $n \geq [4/\epsilon]$ ,

$$|\partial_y p(t, 1)| \leq \frac{\mathcal{M}_T n^{-\eta_T}}{1 - \exp[-\mathcal{M}_T^{-1}(1 + \alpha C_T)n^{-1}]}(1 + \alpha C_T), \quad t \in [0, T],$$

where  $p(t, y)$  is the density of  $\chi_t$  killed at 1 as in (5.3),  $\eta_T$  is as in Lemma 6.5, and

$$C_T = \sup_{0 \leq t \leq T} e'(t).$$

*Proof.* We consider the barrier function

$$q(t, y) = \Theta \exp(Kt) [1 - \exp(\gamma(y - 1))], \quad t \geq 0, y \in \mathbb{R}, \quad (6.39)$$

where  $\gamma$  and  $\Theta$  are free nonnegative parameters. Then, for  $t > 0$  and  $y < 1$ ,

$$\begin{aligned} \partial_t q(t, y) + (b(y) + \alpha e'(t)) \partial_y q(t, y) - \frac{1}{2} \partial_{yy}^2 q(t, y) \\ = \Theta \exp(Kt) \exp(\gamma(y - 1)) \left( -(b(y) + \alpha e'(t)) \gamma + \frac{1}{2} \gamma^2 \right) + Kq(t, y). \end{aligned}$$

Keeping in mind that

$$\sup_{0 \leq t \leq T} e'(t) = C_T,$$

and choosing

$$\gamma = 2(\max(m, 1) + \alpha C_T), \quad (6.40)$$

where  $m = \sup_{0 \leq z \leq 1} |b(z)|$  as before, we obtain, for  $t \in [0, T]$  and  $y \in (0, 1)$ ,

$$-(b(y) + \alpha e'(t)) \gamma + \frac{1}{2} \gamma^2 \geq -2(\max(m, 1) + \alpha C_T)^2 + 2(\max(m, 1) + \alpha C_T)^2 = 0.$$

Thus, for  $t \in [0, T]$  and  $y \in (0, 1)$ ,

$$\partial_t q(t, y) + (b(y) + \alpha e'(t)) \partial_y q(t, y) - \frac{1}{2} \partial_{yy}^2 q(t, y) \geq Kq(t, y) \geq -b'(y)q(t, y),$$

which reads

$$\partial_t q(t, y) + \partial_y [(b(y) + \alpha e'(t))q(t, y)] - \frac{1}{2} \partial_{yy}^2 q(t, y) \geq 0. \quad (6.41)$$

For a given integer  $n \geq [4/\epsilon]$ , we choose  $\Theta$  as the solution of

$$\Theta \left[ 1 - \exp \left( -\frac{2(\max(m, 1) + \alpha C_T)}{n} \right) \right] = \mu_T n^{-\eta_T}, \quad (6.42)$$

with  $\mu_T$  and  $\eta_T$  as in the statement of Lemma 6.5. Pay attention that the factor in the left-hand side cannot be 0 as  $\max(m, 1) > 0$ . Notice also  $q$  thus depends upon  $n$ . By Lemma 6.5, we deduce that

$$q \left( t, 1 - \frac{1}{n} \right) \geq p \left( t, 1 - \frac{1}{n} \right), \quad 0 \leq t \leq T.$$

Now, we can apply the comparison principle for PDEs. Indeed, we also observe that  $q(0, y) \geq p(0, y) = 0$  for  $y \in [1 - 1/n, 1]$  and  $q(t, 1) = p(t, 1) = 0$  for  $t \in [0, T]$ . Therefore, by (6.41), we have

$$p(t, y) \leq q(t, y), \quad t \in [0, T], y \in \left[ 1 - \frac{1}{n}, 1 \right]. \quad (6.43)$$

Since  $p(t, 1) = 0 = q(t, 1)$ , we deduce

$$|\partial_y p(t, 1)| \leq |\partial_y q(t, 1)| = \frac{2\mu_T(\max(m, 1) + \alpha C_T)n^{-\eta_T}}{1 - \exp[-2(\max(m, 1) + \alpha C_T)/n]} \exp(Kt). \quad (6.44)$$

□

We now complete the proof of Theorem 2.3. We make use of Proposition 5.6. Recall (5.6)

$$e'(t) = - \int_0^t \frac{1}{2} \partial_y p^{(0,s)}(t-s, 1) e'(s) ds - \frac{1}{2} \partial_y p(t, 1), \quad t \in [0, T],$$

where  $p$  represents the density of the process  $X$  killed at 1 and  $p^{(0,s)}$  represents the density of the process  $X^{\#s}$  driven by  $e^{\#s} = e(\cdot + s) - e(s)$  (see (6.12)) killed at 1 with  $X_0^{\#s} = 0$  as initial condition.

By Proposition 6.3 and Lemma 6.1, we know that, for a given  $s \in [0, T]$  and for the prescribed values of  $\alpha$ , the mapping  $[0, T-s] \ni r \mapsto e^{\#s}(r)$  is  $1/2$ -Hölder continuous, the Hölder constant only depending upon  $T$ ,  $\alpha$ ,  $\epsilon$ ,  $K$  and  $\Lambda$  (Proposition 6.3 permits to bound the increments of  $e^{\#s}$  on small intervals and Lemma 6.1 gives a trivial bound for the increments of  $e^{\#s}$  on large intervals). Therefore, by Proposition 6.6, we know that

$$|\partial_y p^{(0,s)}(t-s, 1)| \leq \frac{\mathcal{M}_T n^{-\eta_T}}{1 - \exp[-\mathcal{M}_T^{-1}(1 + \alpha C_T)n^{-1}]} (1 + \alpha C_T), \quad t \in [s, T], \quad (6.45)$$

for  $n \geq \lceil 4/\epsilon \rceil$  and for some constant  $\mathcal{M}_T$  only depending upon  $T$ ,  $\alpha$ ,  $\epsilon$ ,  $K$  and  $\Lambda$ . The same bound also holds true for  $\partial_y p(t, 1)$ .

We deduce that, for any  $t \in [0, T]$  and any  $n$  such that  $n \geq \lceil 4/\epsilon \rceil$ ,

$$e'(t) \leq \frac{\mathcal{M}_T n^{-\eta_T}}{1 - \exp[-\mathcal{M}_T^{-1}(1 + \alpha C_T)n^{-1}]} (1 + \alpha C_T) \frac{e(T) + 1}{2}.$$

By Lemma 6.1, we have a bound for  $e(T) = \mathbb{E}(M_T)$ , which means that we can bound  $(e(T) + 1)/2$  in the right-hand side above by modifying the constant  $\mathcal{M}_T$ . Recalling

$$C_T = \sup_{0 \leq t \leq T} e'(t),$$

we deduce that

$$C_T (1 - \exp[-\mathcal{M}_T^{-1}(1 + \alpha C_T)n^{-1}]) \leq \mathcal{M}_T (1 + \alpha C_T) n^{-\eta_T}. \quad (6.46)$$

Choosing  $n$  large enough such that the right hand side is less than  $(1 + \alpha C_T)/2$  (so that  $n$  depends on  $T$ ) and multiplying by  $\alpha$ , we get (since  $\alpha \in (0, 1)$ ):

$$\frac{\alpha C_T}{2} \leq \frac{1}{2} + \alpha C_T \exp[-\mathcal{M}_T^{-1}(1 + \alpha C_T)n^{-1}].$$

This shows that  $\alpha C_T$  must be bounded in terms of  $\mathcal{M}_T$  and  $n$ . Precisely, we have

$$\alpha C_T \leq 1 + 2 \sup_{r \geq 0} [r \exp[-\mathcal{M}_T^{-1}(1 + r)n^{-1}]] := R < +\infty.$$

By (6.46), we deduce that

$$C_T \leq \sup_{0 \leq r \leq R} \left[ \frac{\mathcal{M}_T(1+r)n^{-\eta r}}{1 - \exp[-\mathcal{M}_T^{-1}(1+r)n^{-1}]} \right],$$

which is independent of  $\alpha$  (for  $\alpha \in (0, \alpha_0]$ ), as required.  $\square$

## 7. PROOFS OF THEOREM 2.4 AND REMARK 2.6

In this section we put everything together to arrive at our goal, which is the proof of Theorem 2.4. We first need the following lemma, which is a corollary of Theorem 2.3. The point is that the result will allow us to re-apply the fixed point result on successive time intervals, since it guarantees that the conditions of the fixed point result are satisfied at the final point of any interval on which we know there is a solution.

**Lemma 7.1.** *For any  $T > 0$ , initial condition  $X_0 = x_0 < 1$ , and  $\alpha < \alpha_0$ , where  $\alpha_0 = \alpha_0(x_0)$  is as in Theorem 2.3, there exists a constant  $C_{\text{den}}(T)$  depending only on  $T$ ,  $x_0$ ,  $K$  and  $\Lambda$  such that any solution to (2.1) on  $[0, T]$  satisfies*

$$\frac{1}{dy} \mathbb{P}(X_t \in dy) \leq C_{\text{den}}(T)(1-y),$$

for all  $y \in (1 - \epsilon/8, 1)$  and  $t \in [0, T]$ , with  $\epsilon = \min(1, 1 - x_0)$ .

*Proof.* We assume that  $(X_t)_{0 \leq t \leq T}$  is a solution to (2.1) with  $X_0 = x_0$  up until time  $T$ , and set  $e(t) = \mathbb{E}(M_t)$ . Following the notation of Section 5 (see also the last part of the proof of Theorem 2.3), for  $y \leq 1$  and  $t \leq T$ , let

$$\begin{aligned} p(t, y) &:= \frac{1}{dy} \mathbb{P}(X_t \in dy, t < \tau_1), \\ p^{(0,s)}(t, y) &:= \frac{1}{dy} \mathbb{P}\left(X_t^{\#s} \in dy, t < \tau_1^{\#s} \mid X_0^{\#s} = 0\right). \end{aligned}$$

By Theorem 2.3, we know that  $e$  is  $\mathcal{M}_T$ -Lipschitz continuous, so that by (6.43),

$$p(t, y) \leq q(t, y), \quad t \in [0, T], \quad y \in \left[1 - \frac{1}{n}, 1\right],$$

where  $n$  stands for  $\lceil 4/\epsilon \rceil$  and  $q$  is given by (6.39), with  $\gamma$  and  $\Theta$  being fixed by (6.40) and (6.42), with  $C_T = \mathcal{M}_T$ . By the specific form of  $q$ , this says that there exists a constant  $C'_T$ , depending only on  $T$ ,  $x_0$ ,  $K$  and  $\Lambda$ , such that

$$p(t, y) \leq C'_T(1-y), \quad t \in [0, T], \quad y \in \left[1 - \frac{\epsilon}{8}, 1\right],$$

using the elementary inequality  $1 - \exp(-x) \leq x$  for  $x \in \mathbb{R}$ . Clearly, the same argument applies to  $p^{(0,s)}(t-s, y)$ , i.e.

$$p^{(0,s)}(t-s, y) \leq C'_T(1-y), \quad 0 \leq s < t \leq T, \quad y \in \left[1 - \frac{\epsilon}{8}, 1\right].$$

Now, following the proof of (6.11), we get for  $t \in [0, T]$  and  $y \in [1 - \epsilon/8, 1]$ ,

$$\begin{aligned} \frac{1}{dy} \mathbb{P}(X_t \in dy) &= p(t, y) + \int_0^t p^{(0,s)}(t-s, y) e'(s) ds \\ &\leq C'_T(1 + e(T))(1 - y), \end{aligned} \quad (7.1)$$

where we use Lemma 5.3 for justifying the passage to the density in (6.11). By Lemma 6.1, this completes the proof.  $\square$

Finally, we can then prove the main result of the present paper:

*Proof of Theorem 2.4:* We would like a solution up until fixed time  $T > 0$ . The idea is to iterate the fixed point result (Theorem 5.1), which is possible thanks to Lemma 7.1. Indeed, by Theorem 5.1, we have that there exists a solution to (2.1) with  $X_0 = x_0$  up until some small time  $T_1 > 0$ . By Lemma 7.1, we therefore have that

$$\frac{1}{dy} \mathbb{P}(X_{T_1} \in dy) \leq C_{\text{den}}(T_1)(1 - y), \quad y \in \left[1 - \frac{\epsilon}{8}, 1\right],$$

where  $\epsilon = \min(1 - x_0, 1)$ . If  $T_1 \geq T$  we are done. If not, we have the above density bound for  $(d/dy)\mathbb{P}(X_{T_1} \in dy)$ . We also know from (7.1) and Lemma 5.3 that the density of  $X_{T_1}$  is differentiable at  $y = 1$ . Therefore, we can apply Theorem 5.1 again to see that there exists a solution to (2.1) on some interval  $[T_1, T_1 + T_2]$  starting from  $X_{T_1}$ . As  $T_2$  depends upon  $X_{T_1}$  through  $\epsilon$  and  $C_{\text{den}}(T_1)$  only and as these quantities can be bounded in terms of  $T, \epsilon, K, \Lambda$  only, we then see that

$$T_2 \geq \phi(T)$$

for some constant  $\phi(T)$  that refers to  $T, \alpha, \epsilon, K, \Lambda$  only. Now we know that there exists a solution to (2.1) with  $X_0 = x_0$  on  $[0, T_1 + T_2]$ . If  $T_1 + T_2 > T$  we are done. If not, by Lemma 7.1 once again,

$$\frac{1}{dy} \mathbb{P}(X_{T_1+T_2} \in dy) \leq C_{\text{den}}(T_1 + T_2)(1 - y), \quad y \in \left[1 - \frac{\epsilon}{8}, 1\right],$$

and we can then repeat the argument  $n$  times to get a solution up until time  $T_1 + \dots + T_n$ , where all  $T_k \geq \phi(T)$  for  $k \geq 2$  i.e. each time step is of size at least  $\phi(T)$ . It is then clear that there exists  $n \geq 1$  such that  $T_1 + \dots + T_n \geq T$ , and so we are done for the existence of a solution.

Uniqueness of the solution proceeds in the same way. Given another solution  $(X'_t, M'_t)_{0 \leq t \leq T}$  on the interval  $[0, T]$  in the sense of Definition 2.2, it must satisfy the *a priori* estimates in the statements of Theorem 2.3 and Lemmas 6.1 and 7.1. In particular, dividing the interval  $[0, T]$  into subintervals of length  $\phi(T)$  (except for the last interval the length of which might be less than  $\phi(T)$ ), with the same  $\phi(T)$  as above, we can apply the contraction property in Theorem 5.1 on each subinterval iteratively. Precisely, choosing  $A_1$  accordingly in Theorem 5.1, we prove by induction that the two solutions coincide on  $[0, \phi(T)]$ ,  $[0, 2\phi(T)]$ , and so on.  $\square$

It remains to prove:

*Proof of Remark 2.6.* For some  $X_0 = x_0 < 1$  and for  $\alpha < \alpha_0$ , we consider a solution to (2.1).

For  $\mu > 0$  and  $x \in \mathbb{R}$ , let  $\varphi(x) = \exp(\mu x)$ . Since  $e(t)$  is continuously differentiable, by the generalized Itô formula we see that for all  $t \geq 0$ ,

$$\begin{aligned} \mathbb{E}[\varphi(\exp(\lambda t)X_t)] &= \mathbb{E}(\varphi(X_0)) + \int_0^t \alpha\mu \exp(\lambda s)e'(s)\mathbb{E}[\varphi(\exp(\lambda s)X_s)]ds \\ &\quad + \frac{\mu^2}{2} \int_0^t \exp(2\lambda s)\mathbb{E}[\varphi(\exp(\lambda s)X_s)]ds \\ &\quad + \int_0^t [\varphi(0) - \varphi(\exp(\lambda s))]e'(s)ds. \end{aligned}$$

Differentiating with respect to  $t$  and writing  $z(t) = \mathbb{E}[\varphi(\exp(\lambda t)X_t)]$  yields

$$\frac{d}{dt}z(t) = \left[ \alpha\mu \exp(\lambda t)e'(t) + \frac{\mu^2}{2} \exp(2\lambda t) \right] z(t) + [\varphi(0) - \varphi(\exp(\lambda t))]e'(t)$$

for all  $t > 0$ , with initial condition  $z(0) = \exp(\mu x_0)$ . Writing  $\eta(t) := \alpha\mu \exp(\lambda t)e'(t) + (\mu^2/2) \exp(2\lambda t)$  and  $\nu(t) := [\varphi(\exp(\lambda t)) - \varphi(0)]e'(t) \geq 0$ , we can solve this first-order ordinary differential equation in the standard way to see that

$$z(t) = \exp\left(\int_0^t \eta(s)ds\right) \left[ z(0) - \int_0^t \nu(s) \exp\left(-\int_0^s \eta(u)du\right) ds \right].$$

Now, by the definition of  $\varphi$ , we can note that  $z(0) \geq 0$ , and that

$$z(t) \exp(-\mu \exp(\lambda t)) \leq 1, \quad t \geq 0,$$

since  $X_t < 1$ . Setting

$$\gamma(t) := \exp\left(-\mu \exp(\lambda t) + \int_0^t \eta(s)ds\right),$$

it follows that

$$0 \leq \gamma(t) \left[ z(0) - \int_0^t \nu(s) \exp\left(-\int_0^s \eta(u)du\right) ds \right] \leq 1, \quad (7.2)$$

for all  $t \geq 0$ . But we can see that  $\gamma(t) \rightarrow \infty$  as  $t \rightarrow \infty$  for all  $\mu > 0$ . Indeed,

$$\begin{aligned} \gamma(t) &= \exp\left(-\mu \exp(\lambda t) + \int_0^t [\alpha\mu \exp(\lambda s)e'(s) + (\mu^2/2) \exp(2\lambda s)] ds\right) \\ &\geq \exp\left(-\mu \exp(\lambda t) + \frac{\mu^2}{2} \int_0^t \exp(2\lambda s) ds\right) \\ &= \begin{cases} \exp\left(\mu \exp(\lambda t) \left[\frac{\mu}{4\lambda} \exp(\lambda t) - 1\right] - \frac{\mu^2}{4\lambda}\right) & \text{if } \lambda > 0 \\ \exp\left(-\mu + \frac{\mu^2}{2}t\right) & \text{if } \lambda = 0 \end{cases} \rightarrow \infty. \end{aligned}$$

Thus for the bound (7.2) to remain true when  $t \rightarrow \infty$ , we must have

$$z(0) - \int_0^\infty \nu(s) \exp\left(-\int_0^s \eta(u) du\right) ds = 0, \quad \forall \mu > 0,$$

which provides a surprising closed formula for  $e'$ . We can then perform an integration by parts. Indeed, by writing  $\hat{e}(s) = \int_0^s \exp(\lambda u) e'(u) du$  for all  $s \geq 0$ , the above equality yields

$$\begin{aligned} 0 &= z(0) - \int_0^\infty [\varphi(\exp(\lambda s)) - \varphi(0)] e'(s) \exp\left(-\alpha \mu \hat{e}(s) - \frac{\mu^2}{4\lambda} [\exp(2\lambda s) - 1]\right) ds \\ &= z(0) - \int_0^\infty \hat{\nu}(s) \alpha \mu e'(s) \exp(-\alpha \mu \hat{e}(s)) ds, \end{aligned}$$

with the convention that  $[\exp(2\lambda s) - 1]/(4\lambda) = s/2$  if  $\lambda = 0$ , and where

$$\hat{\nu}(s) = \frac{\varphi(\exp(\lambda s)) - \varphi(0)}{\alpha \mu} \exp\left(-\frac{\mu^2}{4\lambda} [\exp(2\lambda s) - 1]\right) \exp(-\lambda s).$$

Since it is clear that  $\hat{\nu}(\infty) = 0$ , we can perform an integration by parts, so that

$$0 = z(0) - \hat{\nu}(0) - \int_0^\infty \hat{\nu}'(s) \exp(-\alpha \mu \hat{e}(s)) ds.$$

This gives (2.5). □

## 8. EXPLICIT BOUND FOR $\alpha_0$ : BROWNIAN CASE

In this section we will consider equation (2.1) on  $[0, T]$  in the case  $b \equiv 0$ . For  $x_0 < 1$ , we look for an explicit value  $\alpha(x_0)$  such that, *a priori*, the bound (2.4) holds i.e.

$$\frac{1}{dx} \mathbb{P}(X_t \in dx) < \frac{1}{\alpha}, \quad t \in [0, T], \quad (8.1)$$

holds in a neighbourhood of 1 for all  $\alpha \in (0, \alpha_0(x_0))$ , whenever  $(X_t)_{0 \leq t \leq T}$  is a potential solution to (2.1). As noted in Remark 2.5, such an  $\alpha_0$  can then be used in Theorems 2.3 and 2.4 to guarantee the existence of a global-in-time solution to (2.1). Note in this framework that a calculation of  $\alpha_0(0.8)$  would also complete Figure 2. For that reason, we only carry out the calculations for an  $x_0$  sufficiently close to 1.

Suppose that  $\alpha \leq 1 - \eta$  for  $\eta \in (0, 1)$  with  $\eta$  to be chosen later and  $x_0 < 1 - \alpha$ . Then consider  $x \in [x_0/(1 - \alpha) + \varepsilon, 1]$  with  $\varepsilon = (1/2)[1 - x_0/(1 - \alpha)]$ . The following simple Gaussian estimate will be our starting point:

$$\begin{aligned} \frac{1}{dx} \mathbb{P}(X_t \in dx) &= \frac{1}{dx} \mathbb{P}(Z_t - M_t \in dx) = \sum_{k=0}^{\infty} \frac{1}{dx} \mathbb{P}(Z_t - k \in dx, M_t = k) \\ &\leq \frac{1}{\sqrt{2\pi t}} \sum_{k=0}^{\infty} \exp\left(-\frac{(x + k - x_0 - \alpha e(t))^2}{2t}\right), \end{aligned} \quad (8.2)$$

where we have set  $Z_t := X_t + M_t$ ,  $(Z_t = x_0 + \alpha e(t) + W_t)_{0 \leq t \leq T}$  thus being a drifted Brownian motion.

We will need an *a priori* estimate of  $e(t)$  (this will be similar to the one given in Proposition 4.1, but we can do better in this simple case when  $b \equiv 0$ ). Indeed, using the fact that  $M_t = \lfloor \sup_{s \leq t} (Z_s)_+ \rfloor$  (see (4.6)), we have that

$$\begin{aligned} e(t) &= \mathbb{E}(M_t) = \mathbb{E} \left( \lfloor \sup_{s \leq t} (Z_s)_+ \rfloor \right) \\ &\leq x_0 + \alpha e(t) + \mathbb{E} \left( \sup_{s \leq t} W_s \right) \leq x_0 + \alpha e(t) + \left[ \frac{2t}{\pi} \right]^{1/2}, \end{aligned} \quad (8.3)$$

the last inequality following from the fact that

$$\mathbb{E} \left( \sup_{0 \leq s \leq t} W_s \right) = \sqrt{\frac{2t}{\pi}}, \quad (8.4)$$

which is a consequence of the reflection principle. This yields

$$e(t) \leq \frac{1}{1-\alpha} (x_0 + C\sqrt{t}) \quad (8.5)$$

where  $C = \sqrt{2/\pi}$ . We first look at what happens in small time  $t$ .

*Small  $t$ :* By (8.5), we have that, for any  $x \in [x_0/(1-\alpha) + \varepsilon, 1]$ ,

$$\begin{aligned} x - x_0 - \alpha e(t) &\geq x - \frac{x_0}{1-\alpha} - \frac{C\alpha}{1-\alpha} \sqrt{t} \geq \varepsilon - \frac{C\alpha}{1-\alpha} \sqrt{t} \\ &\geq \frac{\varepsilon}{2} \end{aligned} \quad (8.6)$$

when  $\sqrt{t} \leq \varepsilon(1-\alpha)/(2C\alpha)$ . Let  $t_0 = [\varepsilon\eta/(2C(1-\eta))]^2$ . Then since  $\alpha \leq 1-\eta$  we have that

$$t \leq t_0 \Rightarrow \sqrt{t} \leq \frac{\varepsilon\eta}{2C(1-\eta)} \leq \frac{\varepsilon(1-\alpha)}{2C\alpha}$$

so that (8.6) holds for all  $t \leq t_0$ . At this point we define  $\beta := C(1-\eta)/\eta$ , so that  $\sqrt{t_0} = \varepsilon/(2\beta)$ . From (8.6), we thus have that, for all  $t \in [0, t_0]$ ,

$$\exp \left( -\frac{(x - x_0 - \alpha e(t))^2}{2t} \right) \leq \exp \left( -\frac{\varepsilon^2}{8t} \right)$$

and

$$\exp \left( -\frac{(x + k - x_0 - \alpha e(t))^2}{2t} \right) \leq \exp \left( -\frac{k^2}{2t} \right)$$

for  $k \geq 1$ . Hence by (8.2)

$$\frac{1}{dx} \mathbb{P}(X_t \in dx) \leq \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{\varepsilon^2}{8t} \right) + \frac{1}{\sqrt{2\pi t}} \sum_{k=1}^{\infty} \exp \left( -\frac{k^2}{2t} \right), \quad t \in [0, t_0].$$

Note that

$$\frac{1}{\sqrt{2\pi t}} \sum_{k=1}^{\infty} \exp \left( -\frac{k^2}{2t} \right) \leq \frac{1}{\sqrt{2\pi t}} \sum_{k=1}^{\infty} \int_{k-1}^k \exp \left( -\frac{x^2}{2t} \right) dx = \frac{1}{2}.$$

Moreover

$$\sup_{t \geq 0} \left[ \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{\varepsilon^2}{8t}\right) \right] = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}\right) \frac{1}{\varepsilon},$$

and the supremum is attained when  $t = \varepsilon^2/4$ .

We thus arrive at

$$\frac{1}{dx} \mathbb{P}(X_t \in dx) \leq \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}\right) \frac{1}{\varepsilon} + \frac{1}{2}, \quad t \in [0, t_0]. \quad (8.7)$$

*Large  $t$  (i.e.  $t \geq t_0$ ):* For fixed  $x \in [x_0/(1-\alpha) + \varepsilon, 1]$  and  $t > 0$  define

$$N(t, x) = (\lfloor \alpha e(t) + x_0 - x \rfloor)_+.$$

Then, by (8.5) again, we have that

$$\begin{aligned} N(t, x) &\leq (\alpha e(t) + x_0 - x)_+ \leq \left( \frac{C\alpha}{1-\alpha} \sqrt{t} + \frac{1}{1-\alpha} x_0 - x \right)_+ \\ &\leq \left( -\varepsilon + \frac{C\alpha}{1-\alpha} \sqrt{t} \right)_+ \leq \frac{C\alpha}{1-\alpha} \sqrt{t} \end{aligned} \quad (8.8)$$

for all  $t > 0$ . Moreover, for  $t \geq t_0$ ,

$$\begin{aligned} &\frac{1}{\sqrt{2\pi t}} \sum_{k=0}^{\infty} \exp\left(-\frac{(x+k-x_0-\alpha e(t))^2}{2t}\right) \\ &\leq \frac{1}{\sqrt{2\pi t}} \sum_{\substack{k \geq 0 \\ k: x+k-x_0-\alpha e(t) \leq 1}} \exp\left(-\frac{(x+k-x_0-\alpha e(t))^2}{2t}\right) + \frac{1}{\sqrt{2\pi t}} \sum_{k \geq 1}^{\infty} \exp\left(-\frac{k^2}{2t}\right) \\ &\leq \frac{1}{\sqrt{2\pi t}} \#\{k \in \mathbb{N} \cup \{0\} : k \leq 1 + x_0 + \alpha e(t) - x\} + \frac{1}{2} \\ &\leq \frac{1}{\sqrt{2\pi t}} [N(t, x) + 2] + \frac{1}{2} \\ &\leq \frac{2}{\sqrt{2\pi t_0}} + \frac{C\alpha}{(1-\alpha)\sqrt{2\pi}} + \frac{1}{2} \leq 2\sqrt{\frac{2}{\pi}} \beta \frac{1}{\varepsilon} + \frac{\beta}{\sqrt{2\pi}} + \frac{1}{2}, \end{aligned} \quad (8.9)$$

using the symbol  $\#$  for the cardinal in the third line, and the fact that  $\sqrt{t_0} = \varepsilon/(2\beta)$  and  $\beta = C(1-\eta)/\eta \geq C\alpha/(1-\alpha)$ .

We now discuss when the RHS of (8.7) is bigger than the RHS of (8.9). This occurs if

$$\begin{aligned} 2\sqrt{\frac{2}{\pi}} \beta \frac{1}{\varepsilon} + \frac{\beta}{\sqrt{2\pi}} &\leq \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}\right) \frac{1}{\varepsilon} \Leftrightarrow \frac{1}{2}(4+\varepsilon)\beta \leq \exp\left(-\frac{1}{2}\right) \\ &\Leftrightarrow 1-\eta \leq \frac{2}{2 + C \exp(1/2)(4+\varepsilon)}. \end{aligned}$$

We therefore fix  $\eta$  such that  $1 - \eta = 2/(2 + C \exp(1/2)(4 + \varepsilon))$ . Thus we have a set of constraints on  $\alpha$  in order to achieve (8.1), which are (recall (8.7))

$$\alpha < \left( \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}\right) \frac{1}{\varepsilon} + \frac{1}{2} \right)^{-1}$$

$$\alpha \leq 1 - \eta = \frac{2}{2 + C \exp(1/2)(4 + \varepsilon)},$$

subject to  $\alpha < 1 - x_0$ . Substituting in the value of  $\varepsilon$ , the constraints become

$$\alpha < \left( \frac{2C \exp(-1/2)(1 - \alpha)}{1 - \alpha - x_0} + \frac{1}{2} \right)^{-1} = f_1^{x_0}(\alpha) \quad (8.10)$$

$$\alpha \leq \frac{2}{2 + (4 + [1 - \alpha - x_0]/[2(1 - \alpha)]) C \exp(1/2)} = f_2^{x_0}(\alpha), \quad (8.11)$$

still subject to  $\alpha < 1 - x_0$ , with  $C = \sqrt{2/\pi}$ . The first thing to note is that  $f_1^{x_0}(\alpha)$  decreases to 0 on the interval  $\alpha \in [0, 1 - x_0)$ . By plotting the curves of  $f_1^{x_0}(\alpha)$  and  $f_2^{x_0}(\alpha)$  ( $\alpha \in [0, 1 - x_0)$ ) for different values of  $x_0$ , we see that (8.10) is more stringent than (8.11) for large values of  $x_0$ , while the reverse holds for small values of  $x_0$ . Indeed the critical value of  $x_0$  is approximately 0.51.

Hence for  $x_0 \gtrsim 0.51$ , if (8.10) is satisfied subject to  $\alpha < 1 - x_0$ , then we achieve (8.1). Thus for  $x_0 \gtrsim 0.51$ , we choose  $\alpha_0(x_0) = \min[1 - x_0, \tilde{\alpha}_0(x_0)]$  where  $\tilde{\alpha}_0(x_0)$  is such that

$$1 = \tilde{\alpha}_0(x_0) \left( \frac{2C \exp(-1/2)(1 - \tilde{\alpha}_0(x_0))}{1 - \tilde{\alpha}_0(x_0) - x_0} + \frac{1}{2} \right),$$

which solves as a second-order equation. For instance, this yields  $\alpha_0(0.8) \approx 0.104$ , which completes Figure 2 above.

Now, if  $x_0 \lesssim 0.51$  then (8.11) is the more stringent. In this case we must solve  $\tilde{\alpha}_0(x_0) = f_2^{x_0}(\tilde{\alpha}_0(x_0))$  and then choose  $\alpha_0(x_0) = \min[1 - x_0, \tilde{\alpha}_0(x_0)]$ . For example,  $\alpha_0(x_0) \approx 0.264$  when  $x_0 = 0.4$ . We note here, however, that as  $x_0 \downarrow 0$ ,  $f_2^{x_0}(\alpha) \rightarrow 4/[4 + 9C \exp(1/2)] \approx 0.2525$  for all  $\alpha \in (0, 1)$ . We could actually do better when  $x_0$  is small, but we refrain from doing it in order to keep the paper at a reasonable length.

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