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Edoardo Ballico, Alessandra Bernardi

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# CURVILINEAR SCHEMES AND MAXIMUM RANK OF FORMS

EDOARDO BALLICO AND ALESSANDRA BERNARDI

ABSTRACT. We define the *curvilinear rank* of a degree  $d$  form  $P$  in  $n+1$  variables as the minimum length of a curvilinear scheme, contained in the  $d$ -th Veronese embedding of  $\mathbb{P}^n$ , whose span contains the projective class of  $P$ . Then, we give a bound for rank of any homogenous polynomial, in dependance on its curvilinear rank.

## INTRODUCTION

The *rank*  $r(P)$  of a homogeneous polynomial  $P \in \mathbb{C}[x_0, \dots, x_n]$  of degree  $d$ , is the minimum  $r \in \mathbb{N}$  such that  $P$  can be written as sum of  $r$  pure powers of linear forms  $L_1, \dots, L_r \in \mathbb{C}[x_0, \dots, x_n]$ :

$$(1) \quad P = L_1^d + \dots + L_r^d.$$

A very interesting open question is to determine the maximum possible value that the rank of a form (i.e. a homogeneous polynomial) of given degree in a certain number of variables can have. On our knowledge, the best general achievement on this problem is due to J.M. Landsberg and Z. Teitler that in [14, Proposition 5.1] proved that the rank of a degree  $d$  form in  $n+1$  variables is smaller or equal than  $\binom{n+d}{d} - n$ . Unfortunately this bound is sharp only for  $n=1$  if  $d \geq 2$ ; in fact, for example, if  $n=2$  and  $d=3, 4$ , then the maximum ranks are 5 and 7 respectively (see [6, Theorem 40 and 44]).

Few more results were obtained by focusing the attention on limits of forms of given rank. When a form  $P$  is in the Zariski closure of the set of forms of rank  $s$ , it is said that  $P$  has *border rank*  $\underline{r}(P)$  equal to  $s$ . For example, the maximum rank of forms of border ranks 2, 3 and 4 are known (see [6, Theorems 32 and 37] and [2, Theorem 1]). In this context, in [1] we posed the following:

**Question 1** ([1]). Is it true that  $r(P) \leq d(\underline{r}(P) - 1)$  for all degree  $d$  forms  $P$ ? Moreover, does the equality hold if and only if the projective class of  $P$  belongs to the tangential variety of a Veronese variety?

The Veronese variety  $X_{n,d} \subset \mathbb{P}^{N_{n,d}}$ , with  $n \geq 1$ ,  $d \geq 2$  and  $N_{n,d} := \binom{n+d}{d} - 1$  is the classical  $d$ -uple Veronese embedding  $\nu_d: \mathbb{P}^n \rightarrow \mathbb{P}^{N_{n,d}}$  and parameterizes projective classes of degree  $d$  pure powers of linear forms in  $n+1$  variables. Therefore the rank  $r(P)$  of  $[P] \in \mathbb{P}^{N_{n,d}}$  is the minimum  $r$  for which there exists a smooth zero-dimensional scheme  $Z \subset X_{n,d}$  whose span contains  $[P]$

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(with an abuse of notation we are extending the definition of rank of a form  $P$  given in (1) to its projective class  $[P]$ ). More recently, other notions of polynomial rank have been introduced and widely discussed ([8], [15], [7], [5], [3]). They are all related to the minimal length of a certain zero-dimensional schemes embedded in  $X_{m,d}$  whose span contains the given form. Here we recall only the notion of *cactus rank*  $\text{cr}(P)$  of a form  $P$  with  $[P] \in \mathbb{P}^{N_{m,d}}$  (in [15], [7], [5] and also in [12, Definition 5.1] as “scheme length”):

$$\text{cr}(P) = \min\{\deg(Z) \mid Z \subset X_{m,d}, \dim_K Z = 0 \text{ and } [P] \in \langle Z \rangle\}.$$

With this definition, it seems more reasonable to state Question 1 as follows:

**Question 2.** Fix  $[P] \in \mathbb{P}^{N_{m,d}}$  with  $\text{r}(P) > 0$ . Is it true that  $\text{r}(P) \leq (\text{cr}(P) - 1)d$  ?

In this paper we want to deal with a more restrictive but more wieldy notion of rank, namely the “curvilinear rank”. We say that a scheme  $Z \subset \mathbb{P}^N$  is *curvilinear* if it is a finite union of schemes of the form  $\mathcal{O}_{C_i, P_i}/\mathfrak{m}_{P_i}^{e_i}$  for smooth points  $P_i$  on reduced curves  $C_i \subset \mathbb{P}^N$ , or equivalently that the tangent space at each connected component of  $Z$  supported at the  $P_i$ 's has Zariski dimension  $\leq 1$ . We define the *curvilinear rank*  $\text{Cr}(P)$  of a degree  $d$  form  $P$  in  $n + 1$  variables as:

$$\text{Cr}(P) := \min\{\deg(Z) \mid Z \subset X_{n,d}, Z \text{ curvilinear}, [P] \in \langle Z \rangle\}.$$

The main result of this paper is the following:

**Theorem 1.** *For any degree  $d$  form  $P$  we have that*

$$\text{r}(P) \leq (\text{Cr}(P) - 1)d + 2 - \text{Cr}(P).$$

Theorem 1 is sharp if  $\text{Cr}(P) = 2, 3$  ([6, Theorem 32 and 37]).

The next question will be to understand if Theorem 1 holds even though we substitute the curvilinear rank with the cactus rank:

**Question 3.** Fix  $[P] \in \mathbb{P}^{N_{m,d}}$  with  $\text{r}(P) > 0$ . Is it true that  $\text{r}(P) \leq (\text{cr}(P) - 1)d + 2 - \text{cr}(P)$  ?

This manuscript is organized as follows: Section 1 is entirely devoted to the proof of Theorem 1 with two auxiliary lemmas; in Section 2 we study the case of ternary forms and we prove that, in such a case, Question 2 has an affirmative answer.

## 1. PROOF OF THEOREM 1

Let us begin this section with some Lemmas that will allow us to give a lean proof of the main theorem.

We say that an irreducible curve  $T$  is *rational* if its normalization is a smooth rational curve.

**Lemma 1.** *Let  $Z \subset \mathbb{P}^N$  be a zero-dimensional curvilinear scheme of degree  $k$ . Then there is an irreducible and rational curve  $T \subset \mathbb{P}^N$  such that  $\deg(T) \leq k - 1$  and  $Z \subset T \subseteq \langle Z \rangle$ .*

*Proof.* If the scheme  $Z$  is in linearly general position, namely  $\langle Z \rangle \simeq \mathbb{P}^{k-1}$ , then there always exists a rational normal curve of degree  $k - 1$  passing through it (this is a classical fact, see for instance [11, Theorem 1]). If  $Z$  is not in linearly general position, consider  $\mathbb{P}(H^0(Z, \mathcal{O}_Z(1))) \simeq \mathbb{P}^{k-1}$ . In such a  $\mathbb{P}^{k-1}$  there exists a curvilinear scheme  $W$  of degree  $k$  in linearly general position such that the projection  $\ell_V : \mathbb{P}^{k-1} \setminus V \rightarrow \langle Z \rangle$  from a  $(k - \dim(\langle Z \rangle) - 1)$ -dimensional vector space  $V$  induces an isomorphism between  $W$  and  $Z$ . Consider now the degree  $k - 1$  rational normal curve  $C \subset \mathbb{P}^{k-1}$  passing through  $W$ , its projection  $\ell_V(C)$  contains  $Z$  and it is irreducible and rational since  $C$  is irreducible and rational and, by construction,  $\deg(\ell_V(C)) \leq \deg(C) = k - 1$ .  $\square$

In the following lemma we will use the notion of  $X$ -rank of a point  $P \in \langle X \rangle$  with respect to a variety  $X$ ; we indicate it with  $r_X(P)$  and it represents the minimum number of points  $P_1, \dots, P_s \in X$  whose span contains  $P$  and we will say that the set  $\{P_1, \dots, P_s\}$  evinces  $P$ .

**Lemma 2.** *Let  $Y \subset \mathbb{P}^N$  be an integral and rational curve of degree  $d$ . Fix  $P \in \langle Y \rangle$  and assume the existence of a curvilinear degree  $k$  scheme  $Z \subset Y$ , with  $d \geq k \geq 2$ , such that  $P \in \langle Z \rangle$  and  $P \notin \langle Z' \rangle$  for any  $Z' \subsetneq Z$ . If  $k \leq (d + 2)/2$ , then  $r_Y(P) \leq d + 2 - k$ , otherwise  $r_Y(P) \leq k$ .*

*Proof.* If  $Y$  is a rational normal curve, then this is weak version of a celebrated theorem of Sylvester (cfr. [10], [14, Theorem 5.1], [6, Theorem 23]). Hence we may assume  $d > \dim(Y)$ . Observe that the hypothesis  $P \in \langle Z \rangle$  and  $P \notin \langle Z' \rangle$  for any  $Z' \subsetneq Z$ , allows to say that the dimension of  $\langle Z \rangle$  is  $k - 1$ , i.e.  $Z$  is linearly independent, therefore  $\dim \langle Z' \rangle = \deg(Z') - 1$  for every  $Z' \subset Z$ . This allows us to consider a  $(d - \dim(Y) - 1)$ -dimensional linear subspace  $V \subset \mathbb{P}^d$  and a rational normal curve  $C \subset \mathbb{P}^d$  of degree  $d$  such that  $V \cap C = \emptyset$  and the linear projection  $\ell_V : \mathbb{P}^d \rightarrow \langle Y \rangle$  from a  $V$  is surjective. Moreover it also assures the existence of a scheme  $U \subset C$  such that  $\ell_V(U) = Z$  is a degree  $k$  effective divisor of  $C$  that spans a  $\mathbb{P}^{k-1}$  which doesn't intersect  $V$ . Hence  $\ell_V$  induces an isomorphism  $\phi : \langle U \rangle \rightarrow \langle Z \rangle$ . Let  $O \in \langle U \rangle$  be the only point such that  $\phi(O) = P$ . Let  $S_1 \subset C$  be the set of points evincing  $r_C(O)$  and set  $S := \ell_V(S_1) \subset Y$ . Now, the crucial observations are that  $\sharp(S) \leq \sharp(S_1)$  and  $P \in \langle S_1 \rangle$ . Therefore  $r_Y(P) \leq r_C(O)$ . Now, by [6, Theorem 23], we have that if  $k \leq (d + 2)/2$  then  $r_C(O) = d + 2 - k$ , if  $k > (d + 2)/2$  then either  $r_C(O) = d + 2 - k$  or  $r_C(O) = k$ .  $\square$

We are now ready to prove the main theorem of this paper.

*Proof of Theorem 1:* Let  $Z \subset X_{n,d}$  be a minimal degree curvilinear scheme such that  $P \in \langle Z \rangle$ , and let  $U \subset \mathbb{P}^n$  be the curvilinear scheme such that  $\nu_d(U) = Z$ . Say that of degree  $\text{Cr}(P) = \deg(Z) = \deg(U) := k \geq 2$

By Lemma 1, there exists a rational curve  $T \subset \mathbb{P}^n$  such that  $U \subset T$  and  $\deg(T) \leq k - 1$ . The curve  $\nu_d(T)$  is an irreducible rational curve of degree  $d \cdot \deg(T) \leq d(k - 1)$ , and obviously  $P \in \langle \nu_d(T) \rangle$ , hence the integer  $r_{\nu_d(T)}(P)$  is well-defined. Now, since  $\nu_d(T)$  is an integral curve of degree  $\leq d(k - 1)$  it spans a projective space of dimension  $\leq d(k - 1)$  (this is a weak form of Riemann-Roch), therefore  $P$ , which belongs to this span, has

$$(2) \quad r_{\nu_d(T)}(P) \leq \dim \langle \nu_d(T) \rangle \leq d(k - 1)$$

([14, Proposition 4.1] or [9, Lemma 8.2]).

Since  $k \geq 2$ , the function  $t \mapsto d(t-1) + 2 - t$  is increasing for  $t > 0$  and every subscheme of a curvilinear scheme is curvilinear, we may assume  $P \notin \langle Z' \rangle$  for any  $Z' \subsetneq Z$ .

To conclude our prove it is sufficient to apply Lemma 2 to the integral rational curve  $\nu_d(T)$  and get

$$r_{\nu_d(T)}(P) \leq d(k-1) + 2 - k.$$

Now the rank  $r(P)$  that we want to estimate is nothing else than  $r_{X_{n,d}}(P)$ , and, since  $\nu_d(T) \subset X_{n,d}$ , we obviously have that  $r(P) \leq r_{\nu_d(T)}(P)$ .  $\square$

## 2. SUPERFICIAL CASE

In this section we show that Question 2 has an affirmative answer in the case  $m = 2$  of ternary forms. More precisely, we prove the following result.

**Proposition 1.** *Let  $P$  be a ternary form of degree  $d$  with  $\text{Cr}(P) \geq 2$ . Then  $r(P) \leq (\text{Cr}(P) - 1)d$ .*

Before giving the proof of Proposition 1, we need the following result.

**Proposition 2.** *Let  $Z \subset \mathbb{P}^2$  be a degree  $k \geq 4$  zero-dimensional scheme. There is an integral curve  $C \subset \mathbb{P}^2$  such that  $\deg(C) = k - 1$  and  $Z \subset C$  if and only if  $Z$  is not contained in a line.*

*Proof.* First of all, if  $Z$  is contained in a line  $D$ , we may even find a smooth curve  $C \subset \mathbb{P}^2$  such that  $C \cap D = Z$  as schemes (this is easy to check by using the homogeneous equations of  $D$  and  $C$ ). We assume therefore that  $D$  is not contained in a line.

*Claim 1.* The linear system  $|\mathcal{I}_Z(k-1)|$  has no base points outside  $Z_{\text{red}}$ .

*Proof of Claim 1.* Fix  $P \in \mathbb{P}^2 \setminus Z_{\text{red}}$ . Since  $\deg(Z \cup \{P\}) = k+1$ , we have  $h^1(\mathcal{I}_{Z \cup \{P\}}(k-1)) > 0$  if and only if there is a line  $D$  containing  $Z \cup \{P\}$ , but, since in our case  $Z$  is not contained in line, we get  $h^1(\mathcal{I}_{Z \cup \{P\}}(k-1)) = 0$ . Hence  $h^0(\mathcal{I}_{Z \cup \{P\}}(k-1)) = h^0(\mathcal{I}_Z(k-1)) - 1$ , i.e.  $P$  is not a base point of  $|\mathcal{I}_Z(k-1)|$ .

By Claim 1, the linear system  $|\mathcal{I}_Z(k-1)|$  induces a morphism  $\psi : \mathbb{P}^2 \setminus Z_{\text{red}} \rightarrow \mathbb{P}^x$ .

*Claim 2.* We have  $\dim(\psi) = 2$ .

*Proof of Claim 2.* It is sufficient to prove that the differential  $d\psi(Q)$  of  $\psi$  has rank 2 for a general  $Q \in \mathbb{P}^2$ . Assume that  $d\psi(Q)$  has rank  $\leq 1$ , i.e. assume the existence of a tangent vector  $\mathbf{v}$  at  $Q$  in the kernel of the linear map  $d\psi(Q)$ . Since  $h^1(\mathcal{I}_{Z \cup \{P\}}(k-1)) = 0$  (see proof of Claim 1), this is equivalent to  $h^1(\mathcal{I}_{Z \cup \mathbf{v}}(k-1)) > 0$ . Since  $\deg(Z \cup \mathbf{v}) = k+2 \leq 2(k-1) + 1$ , there is a line  $D \subset \mathbb{P}^2$  such that  $\deg(D \cap (Z \cup \mathbf{v})) \geq k+1$ . Hence  $\deg(Z \cap D) \geq k-1$ . Since  $k \geq 4$  there are at most finitely many lines  $D_1, \dots, D_s$  such that  $\deg(D_i \cap Z) \geq k-1$  for all  $i$ . If  $Q \notin D_1 \cup \dots \cup D_s$ , then  $\deg(D \cap (Z \cup \mathbf{v})) \leq k$  for every line  $D$ .

By Claim 2 and Bertini's second theorem ([13, Part 4 of Theorem 6.3]) a general  $C \in |\mathcal{I}_Z(k-1)|$  is irreducible.  $\square$

Any degree 2 zero-dimensional scheme  $Z \subset \mathbb{P}^n$ ,  $n \geq 2$  is contained in a unique line and hence it is contained in a unique irreducible curve of degree 2. Now we check that in case our form has curvilinear rank equal to 3, then Proposition 2 fails in a unique case.

**Remark 1.** Let  $Z \subset \mathbb{P}^2$  be a zero-dimensional scheme such that  $\deg(Z) = 3$ . Since  $h^1(\mathcal{I}_Z(2)) = 0$  ([6], Lemma 34), we have  $h^0(\mathcal{I}_Z(2)) = 3$ . A dimensional count gives that  $Z$  is not contained in a smooth conic if and only if there is  $P \in \mathbb{P}^2$  with  $Z = 2P$  (in this case  $|\mathcal{I}_Z(2)|$  is formed by the unions  $R \cup L$  with  $R$  and  $L$  lines through  $P$ ).

We conclude our paper with the Proof of Proposition 1.

*Proof of Proposition 1.* If  $\text{Cr}(P) = 2, 3$ , then the statement is true by [6, Theorems 32 and 37]. If  $\text{Cr}(P) \geq 4$ , then we can repeat the proof of Theorem 1 until (2) by using as curve  $T$  appearing in Theorem 1, the curve  $C$  of Proposition 2.  $\square$

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DEPT. OF MATHEMATICS, UNIVERSITY OF TRENTO, 38123 POVO (TN), ITALY  
*E-mail address:* edoardo.ballico@unitn.it

GALAAD, INRIA MÉDITERRANÉE, BP 93, 06902 SOPHIA ANTIPOLIS, FRANCE.  
*E-mail address:* alessandra.bernardi@inria.fr