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CURVILINEAR SCHEMES AND MAXIMUM RANK OF FORMS

EDOARDO BALLICO AND ALESSANDRA BERNARDI

ABSTRACT. We define the *curvilinear rank* of a degree d form P in $n+1$ variables as the minimum length of a curvilinear scheme, contained in the d -th Veronese embedding of \mathbb{P}^n , whose span contains the projective class of P . Then, we give a bound for rank of any homogenous polynomial, in dependance on its curvilinear rank.

INTRODUCTION

The *rank* $r(P)$ of a homogeneous polynomial $P \in \mathbb{C}[x_0, \dots, x_n]$ of degree d , is the minimum $r \in \mathbb{N}$ such that P can be written as sum of r pure powers of linear forms $L_1, \dots, L_r \in \mathbb{C}[x_0, \dots, x_n]$:

$$(1) \quad P = L_1^d + \dots + L_r^d.$$

A very interesting open question is to determine the maximum possible value that the rank of a form (i.e. a homogeneous polynomial) of given degree in a certain number of variables can have. On our knowledge, the best general achievement on this problem is due to J.M. Landsberg and Z. Teitler that in [14, Proposition 5.1] proved that the rank of a degree d form in $n+1$ variables is smaller or equal than $\binom{n+d}{d} - n$. Unfortunately this bound is sharp only for $n=1$ if $d \geq 2$; in fact, for example, if $n=2$ and $d=3, 4$, then the maximum ranks are 5 and 7 respectively (see [6, Theorem 40 and 44]).

Few more results were obtained by focusing the attention on limits of forms of given rank. When a form P is in the Zariski closure of the set of forms of rank s , it is said that P has *border rank* $\underline{r}(P)$ equal to s . For example, the maximum rank of forms of border ranks 2, 3 and 4 are known (see [6, Theorems 32 and 37] and [2, Theorem 1]). In this context, in [1] we posed the following:

Question 1 ([1]). Is it true that $r(P) \leq d(\underline{r}(P) - 1)$ for all degree d forms P ? Moreover, does the equality hold if and only if the projective class of P belongs to the tangential variety of a Veronese variety?

The Veronese variety $X_{n,d} \subset \mathbb{P}^{N_{n,d}}$, with $n \geq 1$, $d \geq 2$ and $N_{n,d} := \binom{n+d}{d} - 1$ is the classical d -uple Veronese embedding $\nu_d: \mathbb{P}^n \rightarrow \mathbb{P}^{N_{n,d}}$ and parameterizes projective classes of degree d pure powers of linear forms in $n+1$ variables. Therefore the rank $r(P)$ of $[P] \in \mathbb{P}^{N_{n,d}}$ is the minimum r for which there exists a smooth zero-dimensional scheme $Z \subset X_{n,d}$ whose span contains $[P]$

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(with an abuse of notation we are extending the definition of rank of a form P given in (1) to its projective class $[P]$). More recently, other notions of polynomial rank have been introduced and widely discussed ([8], [15], [7], [5], [3]). They are all related to the minimal length of a certain zero-dimensional schemes embedded in $X_{m,d}$ whose span contains the given form. Here we recall only the notion of *cactus rank* $\text{cr}(P)$ of a form P with $[P] \in \mathbb{P}^{N_{m,d}}$ (in [15], [7], [5] and also in [12, Definition 5.1] as “scheme length”):

$$\text{cr}(P) = \min\{\deg(Z) \mid Z \subset X_{m,d}, \dim_K Z = 0 \text{ and } [P] \in \langle Z \rangle\}.$$

With this definition, it seems more reasonable to state Question 1 as follows:

Question 2. Fix $[P] \in \mathbb{P}^{N_{m,d}}$ with $\text{r}(P) > 0$. Is it true that $\text{r}(P) \leq (\text{cr}(P) - 1)d$?

In this paper we want to deal with a more restrictive but more wieldy notion of rank, namely the “curvilinear rank”. We say that a scheme $Z \subset \mathbb{P}^N$ is *curvilinear* if it is a finite union of schemes of the form $\mathcal{O}_{C_i, P_i}/\mathfrak{m}_{P_i}^{e_i}$ for smooth points P_i on reduced curves $C_i \subset \mathbb{P}^N$, or equivalently that the tangent space at each connected component of Z supported at the P_i 's has Zariski dimension ≤ 1 . We define the *curvilinear rank* $\text{Cr}(P)$ of a degree d form P in $n + 1$ variables as:

$$\text{Cr}(P) := \min\{\deg(Z) \mid Z \subset X_{n,d}, Z \text{ curvilinear}, [P] \in \langle Z \rangle\}.$$

The main result of this paper is the following:

Theorem 1. *For any degree d form P we have that*

$$\text{r}(P) \leq (\text{Cr}(P) - 1)d + 2 - \text{Cr}(P).$$

Theorem 1 is sharp if $\text{Cr}(P) = 2, 3$ ([6, Theorem 32 and 37]).

The next question will be to understand if Theorem 1 holds even though we substitute the curvilinear rank with the cactus rank:

Question 3. Fix $[P] \in \mathbb{P}^{N_{m,d}}$ with $\text{r}(P) > 0$. Is it true that $\text{r}(P) \leq (\text{cr}(P) - 1)d + 2 - \text{cr}(P)$?

This manuscript is organized as follows: Section 1 is entirely devoted to the proof of Theorem 1 with two auxiliary lemmas; in Section 2 we study the case of ternary forms and we prove that, in such a case, Question 2 has an affirmative answer.

1. PROOF OF THEOREM 1

Let us begin this section with some Lemmas that will allow us to give a lean proof of the main theorem.

We say that an irreducible curve T is *rational* if its normalization is a smooth rational curve.

Lemma 1. *Let $Z \subset \mathbb{P}^N$ be a zero-dimensional curvilinear scheme of degree k . Then there is an irreducible and rational curve $T \subset \mathbb{P}^N$ such that $\deg(T) \leq k - 1$ and $Z \subset T \subseteq \langle Z \rangle$.*

Proof. If the scheme Z is in linearly general position, namely $\langle Z \rangle \simeq \mathbb{P}^{k-1}$, then there always exists a rational normal curve of degree $k - 1$ passing through it (this is a classical fact, see for instance [11, Theorem 1]). If Z is not in linearly general position, consider $\mathbb{P}(H^0(Z, \mathcal{O}_Z(1))) \simeq \mathbb{P}^{k-1}$. In such a \mathbb{P}^{k-1} there exists a curvilinear scheme W of degree k in linearly general position such that the projection $\ell_V : \mathbb{P}^{k-1} \setminus V \rightarrow \langle Z \rangle$ from a $(k - \dim(\langle Z \rangle) - 1)$ -dimensional vector space V induces an isomorphism between W and Z . Consider now the degree $k - 1$ rational normal curve $C \subset \mathbb{P}^{k-1}$ passing through W , its projection $\ell_V(C)$ contains Z and it is irreducible and rational since C is irreducible and rational and, by construction, $\deg(\ell_V(C)) \leq \deg(C) = k - 1$. \square

In the following lemma we will use the notion of X -rank of a point $P \in \langle X \rangle$ with respect to a variety X ; we indicate it with $r_X(P)$ and it represents the minimum number of points $P_1, \dots, P_s \in X$ whose span contains P and we will say that the set $\{P_1, \dots, P_s\}$ evinces P .

Lemma 2. *Let $Y \subset \mathbb{P}^N$ be an integral and rational curve of degree d . Fix $P \in \langle Y \rangle$ and assume the existence of a curvilinear degree k scheme $Z \subset Y$, with $d \geq k \geq 2$, such that $P \in \langle Z \rangle$ and $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$. If $k \leq (d + 2)/2$, then $r_Y(P) \leq d + 2 - k$, otherwise $r_Y(P) \leq k$.*

Proof. If Y is a rational normal curve, then this is weak version of a celebrated theorem of Sylvester (cfr. [10], [14, Theorem 5.1], [6, Theorem 23]). Hence we may assume $d > \dim(Y)$. Observe that the hypothesis $P \in \langle Z \rangle$ and $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$, allows to say that the dimension of $\langle Z \rangle$ is $k - 1$, i.e. Z is linearly independent, therefore $\dim \langle Z' \rangle = \deg(Z') - 1$ for every $Z' \subset Z$. This allows us to consider a $(d - \dim \langle Y \rangle - 1)$ -dimensional linear subspace $V \subset \mathbb{P}^d$ and a rational normal curve $C \subset \mathbb{P}^d$ of degree d such that $V \cap C = \emptyset$ and the linear projection $\ell_V : \mathbb{P}^d \rightarrow \langle Y \rangle$ from a V is surjective. Moreover it also assures the existence of a scheme $U \subset C$ such that $\ell_V(U) = Z$ is a degree k effective divisor of C that spans a \mathbb{P}^{k-1} which doesn't intersect V . Hence ℓ_V induces an isomorphism $\phi : \langle U \rangle \rightarrow \langle Z \rangle$. Let $O \in \langle U \rangle$ be the only point such that $\phi(O) = P$. Let $S_1 \subset C$ be the set of points evincing $r_C(O)$ and set $S := \ell_V(S_1) \subset Y$. Now, the crucial observations are that $\sharp(S) \leq \sharp(S_1)$ and $P \in \langle S_1 \rangle$. Therefore $r_Y(P) \leq r_C(O)$. Now, by [6, Theorem 23], we have that if $k \leq (d + 2)/2$ then $r_C(O) = d + 2 - k$, if $k > (d + 2)/2$ then either $r_C(O) = d + 2 - k$ or $r_C(O) = k$. \square

We are now ready to prove the main theorem of this paper.

Proof of Theorem 1: Let $Z \subset X_{n,d}$ be a minimal degree curvilinear scheme such that $P \in \langle Z \rangle$, and let $U \subset \mathbb{P}^n$ be the curvilinear scheme such that $\nu_d(U) = Z$. Say that of degree $\text{Cr}(P) = \deg(Z) = \deg(U) := k \geq 2$

By Lemma 1, there exists a rational curve $T \subset \mathbb{P}^n$ such that $U \subset T$ and $\deg(T) \leq k - 1$. The curve $\nu_d(T)$ is an irreducible rational curve of degree $d \cdot \deg(T) \leq d(k - 1)$, and obviously $P \in \langle \nu_d(T) \rangle$, hence the integer $r_{\nu_d(T)}(P)$ is well-defined. Now, since $\nu_d(T)$ is an integral curve of degree $\leq d(k - 1)$ it spans a projective space of dimension $\leq d(k - 1)$ (this is a weak form of Riemann-Roch), therefore P , which belongs to this span, has

$$(2) \quad r_{\nu_d(T)}(P) \leq \dim \langle \nu_d(T) \rangle \leq d(k - 1)$$

([14, Proposition 4.1] or [9, Lemma 8.2]).

Since $k \geq 2$, the function $t \mapsto d(t-1) + 2 - t$ is increasing for $t > 0$ and every subscheme of a curvilinear scheme is curvilinear, we may assume $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$.

To conclude our prove it is sufficient to apply Lemma 2 to the integral rational curve $\nu_d(T)$ and get

$$r_{\nu_d(T)}(P) \leq d(k-1) + 2 - k.$$

Now the rank $r(P)$ that we want to estimate is nothing else than $r_{X_{n,d}}(P)$, and, since $\nu_d(T) \subset X_{n,d}$, we obviously have that $r(P) \leq r_{\nu_d(T)}(P)$. \square

2. SUPERFICIAL CASE

In this section we show that Question 2 has an affirmative answer in the case $m = 2$ of ternary forms. More precisely, we prove the following result.

Proposition 1. *Let P be a ternary form of degree d with $\text{Cr}(P) \geq 2$. Then $r(P) \leq (\text{Cr}(P) - 1)d$.*

Before giving the proof of Proposition 1, we need the following result.

Proposition 2. *Let $Z \subset \mathbb{P}^2$ be a degree $k \geq 4$ zero-dimensional scheme. There is an integral curve $C \subset \mathbb{P}^2$ such that $\deg(C) = k - 1$ and $Z \subset C$ if and only if Z is not contained in a line.*

Proof. First of all, if Z is contained in a line D , we may even find a smooth curve $C \subset \mathbb{P}^2$ such that $C \cap D = Z$ as schemes (this is easy to check by using the homogeneous equations of D and C). We assume therefore that D is not contained in a line.

Claim 1. The linear system $|\mathcal{I}_Z(k-1)|$ has no base points outside Z_{red} .

Proof of Claim 1. Fix $P \in \mathbb{P}^2 \setminus Z_{\text{red}}$. Since $\deg(Z \cup \{P\}) = k+1$, we have $h^1(\mathcal{I}_{Z \cup \{P\}}(k-1)) > 0$ if and only if there is a line D containing $Z \cup \{P\}$, but, since in our case Z is not contained in line, we get $h^1(\mathcal{I}_{Z \cup \{P\}}(k-1)) = 0$. Hence $h^0(\mathcal{I}_{Z \cup \{P\}}(k-1)) = h^0(\mathcal{I}_Z(k-1)) - 1$, i.e. P is not a base point of $|\mathcal{I}_Z(k-1)|$.

By Claim 1, the linear system $|\mathcal{I}_Z(k-1)|$ induces a morphism $\psi : \mathbb{P}^2 \setminus Z_{\text{red}} \rightarrow \mathbb{P}^x$.

Claim 2. We have $\dim(\psi) = 2$.

Proof of Claim 2. It is sufficient to prove that the differential $d\psi(Q)$ of ψ has rank 2 for a general $Q \in \mathbb{P}^2$. Assume that $d\psi(Q)$ has rank ≤ 1 , i.e. assume the existence of a tangent vector \mathbf{v} at Q in the kernel of the linear map $d\psi(Q)$. Since $h^1(\mathcal{I}_{Z \cup \{P\}}(k-1)) = 0$ (see proof of Claim 1), this is equivalent to $h^1(\mathcal{I}_{Z \cup \mathbf{v}}(k-1)) > 0$. Since $\deg(Z \cup \mathbf{v}) = k+2 \leq 2(k-1) + 1$, there is a line $D \subset \mathbb{P}^2$ such that $\deg(D \cap (Z \cup \mathbf{v})) \geq k+1$. Hence $\deg(Z \cap D) \geq k-1$. Since $k \geq 4$ there are at most finitely many lines D_1, \dots, D_s such that $\deg(D_i \cap Z) \geq k-1$ for all i . If $Q \notin D_1 \cup \dots \cup D_s$, then $\deg(D \cap (Z \cup \mathbf{v})) \leq k$ for every line D .

By Claim 2 and Bertini's second theorem ([13, Part 4 of Theorem 6.3]) a general $C \in |\mathcal{I}_Z(k-1)|$ is irreducible. \square

Any degree 2 zero-dimensional scheme $Z \subset \mathbb{P}^n$, $n \geq 2$ is contained in a unique line and hence it is contained in a unique irreducible curve of degree 2. Now we check that in case our form has curvilinear rank equal to 3, then Proposition 2 fails in a unique case.

Remark 1. Let $Z \subset \mathbb{P}^2$ be a zero-dimensional scheme such that $\deg(Z) = 3$. Since $h^1(\mathcal{I}_Z(2)) = 0$ ([6], Lemma 34), we have $h^0(\mathcal{I}_Z(2)) = 3$. A dimensional count gives that Z is not contained in a smooth conic if and only if there is $P \in \mathbb{P}^2$ with $Z = 2P$ (in this case $|\mathcal{I}_Z(2)|$ is formed by the unions $R \cup L$ with R and L lines through P).

We conclude our paper with the Proof of Proposition 1.

Proof of Proposition 1. If $\text{Cr}(P) = 2, 3$, then the statement is true by [6, Theorems 32 and 37]. If $\text{Cr}(P) \geq 4$, then we can repeat the proof of Theorem 1 until (2) by using as curve T appearing in Theorem 1, the curve C of Proposition 2. \square

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