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A COMPARISON OF DIFFERENT NOTIONS OF RANKS OF SYMMETRIC TENSORS

ALESSANDRA BERNARDI & JÉRÔME BRACHAT & BERNARD MOURRAIN

Abstract. We introduce various notions of rank for a symmetric tensor, namely: rank, border rank, catalecticant rank, generalized rank, scheme length, border scheme length, extension rank and smoothable rank. We analyze the stratification induced by these ranks. The mutual relations between these stratifications, allow us to describe the hierarchy among all the ranks. We show that strict inequalities are possible between rank, border rank, extension rank and catalecticant rank. Moreover we show that scheme length, generalized rank and extension rank coincide.

Introduction

The tensor decomposition problem arises in many applications (see [25] and references therein). This decomposition is often used as a way to recover geometric or intrinsic information, "hidden" in the tensor data. For a given tensor with a certain structure, this problem consists in finding the minimal decomposition into indecomposable tensors with the same structure. The best known and studied case is the one of completely symmetric tensors (see examples in [11], [16], [17]), ie. homogeneous polynomials. The minimum number r of indecomposable symmetric tensors $v_i^{\otimes d}$'s (pure powers of linear forms l_i's) needed to write a given symmetric tensor T of order d (that is a homogeneous polynomial f of degree d) is called the *rank* $r(T)$ of T (the rank $r(f)$ of f):

$$T = \sum_{i=1}^{r} v_i^{\otimes d}; \quad f = \sum_{i=1}^{r} l_i^d.$$

From a geometric point of view, this decomposition shows that f is in the r-th secant of the Veronese variety in the projective space of polynomials of degree d. The order $r_\sigma(f)$ of the smallest secant variety to the Veronese variety containing a given f is called the *border rank* of the homogeneous polynomial f and may differs from the rank of f (see Example 2.2).

A first method to decompose a symmetric tensor is classically attributed to Sylvester and it works for tensors $T \in V^{\otimes d}$ with $\dim V = 2$ (ie. for binary forms). Such a method (see for a modern reference [15]) is based on the analysis of the kernel of so-called catalecticant matrices associated to the tensor. This leads to the notion of *catalecticant rank* $r_H(f)$ of a tensor f, which is also called the differential length in [24][Definition 5.66, p.198].

Extending the apolarity approach of Sylvester, an algorithm to compute the decomposition and the rank of a symmetric tensor f in any dimension was described in [6]. The main ingredient of this work is an algebraic characterization

of the property of flat extension of a catalecticant matrix. This extension property is not enough to characterize tensors with a given rank, since the underlying scheme associated to the catalecticant matrix extension should also be reduced. To get a better insight on this difference, we introduce hereafter the notions of *extension rank* $r_{\mathcal{E}^0}(f)$ and *border extension rank* $r_{\mathcal{E}}(f)$ of f, and analyze the main properties.

Another approach leading to a different kind of algorithm is proposed in [4] and it is developed for some cases. The idea there, is to classify all the possible ranks of the polynomials belonging to certain secant varieties of Veronese varieties in relation with the structure of the embedded non reduced zero-dimensional schemes whose projective span is contained in that secant variety. In [8], the authors clarify the structure of the embedded schemes whose span is contained in the secant varieties of the Veronese varieties. Moreover they introduce an algebraic variety, namely the r-th cactus variety $\mathcal{K}_r^d$. This lead us to the notion of what we will call the *border scheme length* $r_{\mathrm{sch}}(f)$ of a polynomial f. We will show that this notion is related to the *scheme length* associated to f defined in ([24][Definition 5.1, p. 135, Definition 5.66, p. 198]), we will call it the *scheme length* which is sometimes called the *cactus rank* of a homogeneous polynomial f (see [30] for a first definition of it).

Another notion related to the scheme length and called the *smoothable rank* $r_{\mathrm{smooth}^0}(f)$ of a homogeneous polynomial f is also used in [24][Definition 5.66, p. 198] or [30]. Instead of considering all the schemes of length r apolar to f, one considers only the smoothable schemes, that are the schemes which are the limits of smooth schemes of r simple points. Analogously we can define the *border smoothable rank* $r_{\mathrm{smooth}}(f)$ of a homogeneous polynomial f, as the smallest r such f belongs to the closure of the set of tensors of smoothable rank r.

In relation with the "generalized additive decomposition" of a homogeneous polynomial f, there is the so called length of f: it was introduced for binary forms in [24][Definition 1.30, p. 22], and extended to any form in [24][Definition 5.66, p. 198]. In this paper we describe a new generalization of the notion of *generalized affine decomposition* of a homogeneous polynomial f (see Definition 2.16) and study the corresponding *generalized rank* $r_{\mathcal{G}^0}(f)$. Again there is a notion of *border generalized rank* $r_{\mathcal{G}}(f)$.

As in the classical tensor decomposition problem, the decompositions associated to these different notions of rank can be useful to analyse geometric information "hidden" in a tensor. The purpose of this paper is to relate all these notions of rank.

In Corollary 3.9 we will show that the generalized rank, the scheme length and the flat extension rank coincide:

$$r_{\mathcal{G}^0}(f) = r_{\mathrm{sch}^0}(f) = r_{\mathcal{E}^0}(f).$$

and hence their respective "border versions": $r_{\mathcal{G}}(f) = r_{\mathrm{sch}}(f) = r_{\mathcal{E}}(f)$.

We can summarize the relations among the ranks in the following table:

$$r_H(f) \leq \left\{ \begin{array}{ccc} r_{\mathcal{G}}(f) & \leq & r_{\mathcal{G}^0}(f) \\ \| & & \| \\ r_{\mathrm{sch}}(f) & \leq & r_{\mathrm{sch}^0}(f) \\ \| & & \| \\ r_{\mathcal{E}}(f) & \leq & r_{\mathcal{E}^0}(f) \\ \wedge\!| & & \wedge\!| \\ r_{\mathrm{smooth}}(f) & \leq & r_{\mathrm{smooth}^0}(f) \\ \| & & \\ r_{\sigma}(f) & & \end{array} \right\} \leq r(f). \tag{1}$$

Let $\mathcal{G}_r^{d,0}$, $\mathcal{K}_r^{d,0}$ and $\mathcal{E}_r^{d,0}$ be the sets of homogeneous polynomial of degree d in a given number of variables of generalized rank, scheme length and extension rank respectively less than or equal to r and let $\mathcal{G}_r^d$, $\mathcal{K}_r^d$ and $\mathcal{E}_r^d$ their Zariski closures. The main results of this paper is Theorem 3.7 where we show that

$$\mathcal{G}_r^{d,0} = \mathcal{K}_r^{d,0} = \mathcal{E}_r^{d,0},$$

and hence (Corollary 3.8) that

$$\mathcal{G}_r^d = \mathcal{K}_r^d = \mathcal{E}_r^d.$$

The paper is organized as follows. After the preliminary Section 1 where we recall all the basic definitions, we will define, in Section 2, all the definitions of rank that we want to study and for each one of them we will give detalied examples. In Sections 3 we will prove our main results.

1. Preliminaries

1.1. **Notations.** Let $S = \mathbb{K}[\mathbf{x}]$ be the graded polynomial ring in the variables $\mathbf{x} = (x_0, \ldots, x_n)$ over an algebraically closed field $\mathbb{K}$ of characteristic 0. For $d \in \mathbb{N}$, let S^d be the the vector space spanned by the homogeneous polynomials of degree d in S. We denote by $R = \mathbb{K}[\underline{\mathbf{x}}]$ the ring of polynomials in the variables $\underline{\mathbf{x}} = (x_1, \ldots, x_n)$ and by $R^{\leq d}$ the vector space of polynomials in R of degree $\leq d$. An ideal $I \subset S$ is *homogeneous* if it can be generated by homogeneous elements.

For $f \in S^d$, we denote by $\underline{f} = f(1, x_1, \ldots, x_n) \in R^{\leq d}$ the polynomial obtained by substituting x_0 by 1. This defines a bijection between S^d and $R^{\leq d}$, which depends on the system of coordinates chosen to represent the polynomials. For $f \in R$, we define $f^h(x_0, \ldots, x_n) = x_0^{\deg(f)} f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$ and we call it the *homogenization* of f. A set B of monomials of R is *connected to* 1 if it contains 1 and if $m \neq 1 \in B$ then there exists $1 \leq i \leq n$ and $m' \in B$ such that $m = x_i m'$.

We denote by $\mathbb{P}^n := \mathbb{P}(\mathbb{K}^{n+1})$ the projective space of dimension n. A point in $\mathbb{P}^n$ which is the class of the non-zero element $\mathbf{k} = (k_0, \ldots, k_n) \in \mathbb{K}^{n+1}$ modulo the collinearity relation is denoted by $[\mathbf{k}] = (k_0 : \cdots : k_n)$. For an homogeneous ideal $I \subset S$, the set of points $[\mathbf{k}] \in \mathbb{P}^n$ such that $\forall f \in I, f(\mathbf{k}) = 0$ is denoted $V_{\mathbb{P}^n}(I)$. We say that an ideal $I \subset S$ is *zero-dimensional* if $V_{\mathbb{P}^n}(I)$ is finite and not empty. We say that $\zeta \in V_{\mathbb{P}^n}(I)$ is *simple* if the localisation $(S/I)_{\mathbf{m}_\zeta}$ of S/I at the maximal ideal $\mathbf{m}_\zeta$ associated to ζ is of dimension 1 (cf. [2]). An ideal I of S is *saturated* if $(I : S^1) = I$.

The *Hilbert function* associated to I evaluated at $d \in \mathbb{N}$ is $H_{S/I}(d) = \dim(S^d/I^d)$. When I is zero-dimensional, the Hilbert function becomes equal to a constant $r \in \mathbb{N}$ for $d \gg 0$. When moreover I is saturated, this happens when $d \geq r$ (see e.g. [23] for more details).

For a homogeneous ideal $I \subset S$, let $\underline{I}$ be the ideal of R, generated by the elements $\underline{f}$ for $f \in I$. We recall that if $H_{S/I}(d) = r$ for $d \gg 0$ and if x_0 is a non-zero divisor in S/I, then $R/\underline{I}$ is a $\mathbb{K}$-vector space of dimension r. Conversely, if $\tilde{I}$ is an ideal of R such that $\dim_{\mathbb{K}}(R/\tilde{I}) = r$ then the homogeneous ideal $I = \{f^h \mid f \in \tilde{I}\}$ is saturated, x_0 is a non-zero divisor in S/I and $H_{S/I}(d) = r$ for $d \geq r$.

Remark 1.1. *If I is a saturated zero-dimensional ideal of S of degree r and $(I : x_0) = I$, then we have the natural isomorphism for d:*

$$S^d/I^d \simeq R^{\leq d}/\underline{I}^{\leq d}.$$

For a point $\mathbf{k} = (k_0, \ldots, k_n) \in \mathbb{K}^{n+1}$, we define a corresponding element $\mathbf{k}(\mathbf{x}) \in S^1$ as $\mathbf{k}(\mathbf{x}) = k_0 x_0 + \cdots + k_n x_n$. The element $\mathbf{k}(\mathbf{x})$ is unique, up to a non-zero multiple: it corresponds to a unique element $[\mathbf{k}(\mathbf{x})]$ in $\mathbb{P}(S^1)$. In the following, we will use the same notation $\mathbf{k} = \mathbf{k}(\mathbf{x})$ to denote either an element of $\mathbb{K}^{n+1}$ or of S^1. The following product is sometimes called "Bombieri product" or "Sylvester product".

Definition 1.2. *For all $f, g \in S^d$, we define the* apolar product *on S^d as follows:*

$$\langle f, g \rangle = \sum_{|\alpha|=d} f_\alpha \, g_\alpha \binom{d}{\alpha}.$$

where $f = \sum_{|\alpha|=d} f_\alpha \binom{d}{\alpha} \mathbf{x}^\alpha$, $g = \sum_{|\alpha|=d} g_\alpha \binom{d}{\alpha} \mathbf{x}^\alpha$, $\binom{d}{\alpha} = \frac{d!}{\alpha_0! \cdots \alpha_n!}$ for $|\alpha| = \alpha_0 + \cdots + \alpha_n = d$. It can also be defined on $R^{\leq d}$ in such a way that for all $f, g \in S^d$, $\langle \underline{f}, \underline{g} \rangle = \langle f, g \rangle$ (just by replacing x_0 by 1 in the previous formula).

For any vector space E, we denote by $E^* = \mathrm{Hom}_{\mathbb{K}}(E, \mathbb{K})$ its *dual space*. Notice that the dual S^* is an S-module: $\forall \Lambda \in S^*, \forall p \in S, p \cdot \Lambda : q \mapsto \Lambda(p, q)$.

For any homogeneous polynomial $f \in S^d$, we define the element $f^* \in (S^d)^*$ as follows:

$$\forall g \in S^d, f^*(g) = \langle f, g \rangle.$$

Similarly, $\underline{f}^* \in (R^{\leq d})^*$ is defined so that $\forall g \in S^d, \underline{f}^*(\underline{g}) = \langle \underline{f}, \underline{g} \rangle = \langle f, g \rangle$.

Let I be an ideal of S. The *inverse system* $I^\perp$ of I is the S-submodule of elements of S^* that vanish on I, ie. $I^\perp = \{\Lambda \in S^* \mid \forall f \in I, \Lambda(f) = 0\}$.

For $D \subset R^*$, we define $D^\perp \subset R$ as

$$D^\perp := \{p \in R \mid \forall \Lambda \in D, \Lambda(p) = 0\}.$$

We check that if D is a R-module, then $D^\perp$ is an ideal.

When I is a homogeneous ideal, an element in $I^\perp$ is a sum (not necessarily finite) of elements in $(I^d)^\perp$.

Remark 1.3. *The dimension of the degree d part of the inverse system of an ideal $I \subset S$ is the Hilbert function of S/I in degree d:*

$$H_{S/I}(d) = \dim_K (I^d)^\perp = \mathrm{codim}(I^d).$$

We denote by $(\mathbf{d}^\alpha)_{|\alpha|=d}$ the basis of $(S^d)^*$ that is dual to the standard monomial basis $(\mathbf{x}^\beta)_{|\beta|=d}$ of S^d, more precisely $\mathbf{d}^\alpha = \mathbf{d}_0^{\alpha_0} \cdots \mathbf{d}_n^{\alpha_n}$ and $\mathbf{d}^\alpha(\mathbf{x}^\beta) = 1$ if $\alpha = \beta$ and 0 otherwise. An element in $(S^d)^*$ is represented by a homogeneous polynomial of degree d in the *dual variables* $\mathbf{d}_0, \ldots, \mathbf{d}_n$. It will also be called a *dual polynomial*.

We remark that $x_i \cdot \mathbf{d}^\alpha = \mathbf{d}^{\alpha_0} \cdots \mathbf{d}_{i-1}^{\alpha_{i-1}} \mathbf{d}_i^{\alpha_i - 1} \mathbf{d}^{\alpha_{i+1}} \cdots \mathbf{d}_n^{\alpha_n}$ if $\alpha_i > 0$ and 0 otherwise. More generally, for any $\Lambda \in (S^d)^*$ represented by a dual polynomial of degree d, we have that $x_i \cdot \Lambda$ is either 0 or a dual polynomial of degree $d-1$. It is formally obtained by multiplying by $\mathbf{d}_i^{-1}$ and by keeping the terms with positive exponents. This property explains the name of *inverse system* introduced by F.S. Macaulay [29]. The dual monomials are also called *divided powers* in some works, when a structure of ring is given to the S^* (see e.g. [24][Appendix A]), but this structure is not really needed in the following. It comes from the description of $\mathbf{d}^\alpha$ in terms of differentials: $\forall p \in S$,

$$\mathbf{d}^\alpha(p) = \frac{1}{\alpha!} \partial_0^{\alpha_0} \cdots \partial_n^{\alpha_n}(p)(0, \ldots, 0),$$

where $\alpha! = \prod_{i=0}^n \alpha_i!$.

For $D \subset S^*$, we define the *inverse system* generated by D as S-module of S^* generated by D, that is the vector space spanned by the elements of the form $\mathbf{x}^\alpha \cdot \Lambda$ for $\alpha \in \mathbb{N}^{n+1}$ and $\Lambda \in D$.

Example 1.4. *The inverse system generated by* $\mathbf{d}_0\mathbf{d}_1$ *is* $\langle \mathbf{d}_0\mathbf{d}_1, \mathbf{d}_0, \mathbf{d}_1, 1 \rangle$. *It is a vector space of dimension* 4 *in* $\mathbb{K}[\mathbf{d}_0, \mathbf{d}_1]$.

By extension, the elements of S^* can be represented by a formal power series in the variables $\mathbf{d}_0, \ldots, \mathbf{d}_n$.

By restriction, the elements R^* are represented by formal power series in the dual variables $\mathbf{d}_1, \ldots, \mathbf{d}_n$. The elements of $(R^{\leq t})^*$ are represented by polynomials of degree $\leq t$ in the variables $\mathbf{d}_1, \ldots, \mathbf{d}_n$. The structure of R-module of R^* shares the same properties as S^*: x_i acts as the "inverse" of $\mathbf{d}_i$. We define the inverse system spanned by $D \subset R^*$ as the R-module of R^* generated by D.

For a non-zero point $\mathbf{k} \in \mathbb{K}^{n+1}$, we define the evaluation $1_\mathbf{k}^d \in (S^d)^*$ at $\mathbf{k}$ as

$$\begin{array}{rcl} 1_\mathbf{k}^d : S^d & \to & \mathbb{K} \\ p & \mapsto & p(\mathbf{k}) \end{array}$$

In the following, we may drop the exponent d to simplify notations when it is implicitly defined.

To describe the dual of zero-dimensional ideals defining points with multiplicities, we need to consider differentials. For $\mathbf{k} \in \mathbb{K}^{n+1}$ and $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$, we defined

$$\begin{array}{rcl} 1_\mathbf{k} \circ \partial^\alpha : S & \to & \mathbb{K} \\ p & \mapsto & \partial_0^{\alpha_0} \cdots \partial_n^{\alpha_n}(p)(\mathbf{k}). \end{array}$$

We extend this definition by linearity, in order to define $1_\mathbf{k} \circ \phi(\partial) \in S^*$ for any polynomial $\phi(\partial)$ in the *differential variables* $\partial_0, \ldots, \partial_n$. We check that the inverse system generated by $1_\mathbf{k} \circ \phi(\partial)$ is the vector space spanned by the elements of the form $1_\mathbf{k} \circ \phi'(\partial)$ where ϕ' is obtained from ϕ by possibly several derivations with

respect to the differential variables $\partial_0, \ldots, \partial_n$. It is a finite dimensional vector space.

This leads to the following result, which characterizes the dual of a zero-dimensional (affine) ideal (see e.g. [21] or [20][Theorem 7.34, p. 185]).

Theorem 1.5. *Suppose that $I \subset R$ is such that $\dim_{\mathbb{K}}(R/I) = r < \infty$. Then $\forall \Lambda \in I^{\perp}$, there exists distinct points $\zeta_1, \ldots, \zeta_s \in V_{\mathbb{K}^n}(I)$ and differential polynomials $\phi_1, \ldots, \phi_s$ in the variables $\partial_1, \ldots, \partial_n$ such that*

$$\Lambda = \sum_{i=1}^{s} 1_{\zeta_i} \circ \phi_i(\partial).$$

As a consequence, we check that the inverse system generated by Λ is the direct sum of the inverse systems D_i generated by $1_{\zeta_i} \circ \phi_i(\partial)$ for $i = 1, \ldots, s$. The sum of the dimensions of these inverse systems is thus $\leq \dim_{\mathbb{K}}(I^{\perp}) = \dim_{\mathbb{K}}(R/I) = r$.

Proposition 1.6. *Let $\Lambda = \sum_{i=1}^{s} 1_{\zeta_i} \circ \phi_i(\partial)$, D be the inverse system (or R-module) generated by Λ and D_i be the inverse system generated by $1_{\zeta_i} \circ \phi_i(\partial)$ for $i = 1, \ldots, s$. Then $D^{\perp} = Q_i \cap \cdots \cap Q_s$ where*

- *$Q_i = D_i^{\perp}$ is a primary ideal for the maximal ideal $\mathbf{m}_{\zeta_i}$ defining ζ_i,*
- *$\mu_i = \dim_{\mathbb{K}}(D_i) = \dim_{\mathbb{K}}(R/Q_i)$ is the multiplicity of ζ_i,*
- *$\dim R/D^{\perp} = \sum_{i=1}^{s} \mu_i$.*

Example 1.7. *Let us consider the ideal $I = (x_1^2 + x_2 - 1, x_2^2 - 1)$ of $R = \mathbb{K}[x_1, x_2]$. It defines the points $(0,1)$, $(\sqrt{2}, -1)$, $(-\sqrt{2}, -1) \in \mathbb{K}^2$. An element $\Lambda \in I^{\perp}$ can be decomposed as*

$$\Lambda = 1_{(0,1)} \circ (a_1 \partial_1 + b_1) + \lambda_2 1_{(\sqrt{2},1)} + \lambda_3 1_{(-\sqrt{2},-1)}$$

where $a_1, b_1, \lambda_2, \lambda_3 \in \mathbb{K}$. If $a_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$, then the inverse system spanned by Λ is

$$\langle 1_{(0,1)} \circ \partial_1, 1_{(0,1)}, 1_{(\sqrt{2},1)}, 1_{(-\sqrt{2},-1)} \rangle.$$

Lemma 1.8. *Suppose that I is a saturated ideal defining r simple points $[\zeta_1], \ldots, [\zeta_r] \in \mathbb{P}^n$. Then $(I^d)^{\perp}$ is spanned by $\mathbf{1}_{\zeta_1}, \ldots, \mathbf{1}_{\zeta_r}$ for $d \geq r$.*

Proof. Obviously $\langle \mathbf{1}_{\zeta_1}, \ldots, \mathbf{1}_{\zeta_r} \rangle \subset (I^d)^{\perp}$. Moreover, as already observed in Remark 1.3, we have that $\dim (I^d)^{\perp} = H_{S/I}(d)$. Therefore, for $d \geq r$, $\dim (I^d)^{\perp} = r = \dim \langle \mathbf{1}_{\zeta_1}, \ldots, \mathbf{1}_{\zeta_r} \rangle$ and $(I^d)^{\perp} = \langle \mathbf{1}_{\zeta_1}, \ldots, \mathbf{1}_{\zeta_r} \rangle$. □

1.2. **Tensor decomposition problem.** The main problem we are interested in, is the problem of *decomposition of a symmetric tensor* into a sum of minimal size of indecomposable terms which are the powers of a linear forms:

Definition 1.9. *An element $f \in S^d$ has a decomposition of size r if there exists distinct non-zero elements $\mathbf{k}_1, \ldots, \mathbf{k}_r \in S^1$ such that*

$$f = \mathbf{k}_1^d + \cdots + \mathbf{k}_r^d. \tag{2}$$

This problem is also called the *Generalized Waring problem* as it generalizes the problem of Waring in arithmetic [33].

In order to find a decomposition of $f \in S^d$ as a sum of d-th powers of linear forms, we will consider the polynomials which are apolar to f and use the following result.

Lemma 1.10. *For all $g \in S^d, \mathbf{k} \in S^1$ with $\mathbf{k} = k_0x_0 + \cdots + k_nx_n$, $k_j \in \mathbb{K}$, for $j = 0, \ldots, n$, it turns out that*

$$\langle g, \mathbf{k}^d \rangle = g(\mathbf{k}),$$

where $g(\mathbf{k}) = g(k_0, \ldots, k_n)$.

Proof. By an explicit computation, we have $\mathbf{k}^d = \sum_{|\alpha|=d} \binom{d}{\alpha} \prod_{j=0}^n k_j^{\alpha_j} \prod_{j=0}^n x_j^{\alpha_j}$. Thus $\langle g, \mathbf{k}^d \rangle = \sum_{|\alpha|=d} \binom{d}{\alpha} g_\alpha \prod_{j=0}^n k_j^{\alpha_j} = g(\mathbf{k})$. □

Thus if $f = \mathbf{k}_1^d + \cdots + \mathbf{k}_r^d$ with $\mathbf{k}_i \in S^1$ and if $g \in S^k$ is such that $g(\mathbf{k}_i) = 0$ for $i = 1, \ldots, r$, then for all $h \in S^{d-k}$ we have

$$\langle g\, h, f \rangle = 0.$$

This shows that the ideal of polynomials vanishing at the points $\mathbf{k}_1, \ldots, \mathbf{k}_r \in \mathbb{P}^n$ is in the set of polynomials apolar to f. It leads us to the following definition (see also [24] for the where the same definition is given via an apolar product that differs from our Definition 1.2 only because it is not defined as an inner product but as a product between S^d and S^{d*}).

Definition 1.11 (Apolar ideal)**.** *Let $f \in S^d$. We define the* apolar ideal *of f as the homogeneous ideal of S generated by S^{d+1} and by the polynomials $g \in S^i$ ($0 \le i \le d$) such that $\langle gh, f \rangle = 0$ for all $h \in S^{d-i}$. It is denoted $(f^\perp)$.*

Example 1.12. *For $f := x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ with $\alpha_0 + \cdots + \alpha_n = d$, we have $f^* = \binom{d}{\alpha}^{-1} \mathbf{d}_0^{\alpha_0} \cdots \mathbf{d}_n^{\alpha_n}$ and $(f^\perp) = (x_0^{\alpha_0+1}, \ldots, x_n^{\alpha_n+1})$.*

Hereafter, we will need the following standard lemma.

Lemma 1.13. *For any ideal $I \subset S$, $\langle I^d, f \rangle = 0$ if and only if $I \subset (f^\perp)$.*

Proof. Clearly, if $I \subset (f^\perp)$ then $I^d \subset (f^\perp)^d$ so that $\langle I^d, f \rangle = 0$.

Let us prove the reverse inclusion. By definition of the apolar ideal $J := (f^\perp)$, we have $J^i : S^k = J^{i-k}$, $\forall\ 0 \le i \le d$, $0 \le k \le i$. We also have $I^d : S^k \supset I^{d-k}$, $\forall\ 0 \le k \le d$. The hypothesis $\langle I^d, f \rangle = 0$ implies that $I^d \subset J^d$. We deduce that $I^i \subset J^i$, $\forall\ 0 \le i \le d$. Since $J^{d+1} = S^{d+1}$, we have the inclusion $I \subset J = (f^\perp)$. □

The tensor decomposition problem can be reformulated in terms of apolarity as follows via the well known as Apolarity Lemma (cf. [24, Lemma 1.15]).

Proposition 1.14. *A symmetric tensor $f \in S^d$ has a decomposition of size $s \le r$ iff there exits an ideal $I \subset S$ such that*

(a) *$I \subset (f^\perp)$,*
(b) *I is saturated, zero dimensional, of degree $\le r$,*
(c) *I is defining simple points.*

Proof. Suppose that f has a decomposition of size $\le r$: $f = \sum_1^s \mathbf{w}_i^d$ where $\mathbf{w}_i \in S^1 - \{0\}$ and $s \le r$. Then consider the homogeneous ideal I of polynomials vanishing at the points $[\mathbf{w}_i] \in \mathbb{P}^n$, $i = 1, \ldots, s$. By construction, for all $g \in I^d$,

$$\langle f, g \rangle = \sum_1^s \langle \mathbf{w}_i^d, g \rangle = \sum_1^s g(\mathbf{w}_i) = 0$$

so that I is a saturated ideal, defining s ($\leq r$) simple points and with $I \subset (f^\perp)$.

Conversely, suppose that I is an ideal of S satisfying (a), (b), (c). Let us denote by $\mathbf{w}_1, \ldots, \mathbf{w}_s$ the $[\mathbf{w}_1], \ldots, [\mathbf{w}_s]$ the simple points of $\mathbb{P}^n$ defined by I and by $\mathbf{w}_1, \ldots, \mathbf{w}_s$ corresponding elements in S^1. Then by Lemma 1.8, $(I^d)^\perp$ is spanned by $\mathbf{1}_{\mathbf{w}_1}, \ldots, \mathbf{1}_{\mathbf{w}_s}$. As $I \subset (f^\perp)$, we have $f^* \in (I^d)^\perp$ so that there exists $\lambda_1, \ldots, \lambda_s \in \mathbb{K}$ such that

$$f^* = \sum_{i=1}^{s} \lambda_i \mathbf{1}_{\mathbf{w}_i}.$$

This implies that

$$f = \sum_{i=1}^{s} \lambda_i \mathbf{w}_i^d = \sum_{i=1}^{s} (\lambda_i^{\frac{1}{d}} \mathbf{w}_i)^d$$

and f has a decomposition of size $\leq s \leq r$. □

2. Ranks of symmetric tensors

In this section we introduce all the different notions of rank of a homogeneous polynomial $f \in S^d$, that we will use all along the paper.

2.1. Rank and border rank.

The following definition is nowadays a classical one, see e.g. [25] and references therein.

Definition 2.1 (Rank). *Let $\sigma_r^{0,d} \subset \mathbb{P}(S^d)$ be the set of projective classes of homogeneous polynomials defined by*

$$\sigma_r^{0,d} := \{[f] \in \mathbb{P}(S^d) |\ \exists\, \mathbf{k}_1, \ldots, \mathbf{k}_s \in S^1 \text{ with } s \leq r \text{ s.t. } f = \mathbf{k}_1^d + \cdots + \mathbf{k}_s^d\}.$$

For any $f \in S^d$, the minimal r such that $[f] \in \sigma_r^{0,d}$ is called the rank *of f and denoted $r(f)$.*

Example 2.2. *Let us take $f := x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ with $\alpha_0 + \cdots + \alpha_n = d$. We consider the ideal $I_\epsilon := (x_1^{\alpha_1+1} - \epsilon^{\alpha_1+1} x_0^{\alpha_1+1}, x_2^{\alpha_2+1} - \epsilon^{\alpha_2+1} x_0^{\alpha_2+1}, \ldots, x_n^{\alpha_n+1} - \epsilon^{\alpha_n+1} x_0^{\alpha_n+1})$ for some $\epsilon \in \mathbb{K} \setminus \{0\}$. It is defining $(\alpha_1+1)\cdots(\alpha_n+1)$ simple points (which i^{th} coordinates are ϵ times the (α_i+1)-roots of unity). Let us consider the element Λ of S^* defined as follows:*

$$\Lambda := \frac{1}{(\alpha_1+1)\cdots(\alpha_n+1)\binom{d}{\alpha}} \sum_{k_1=0}^{\alpha_1} \cdots \sum_{k_n=0}^{\alpha_n} \epsilon^{\alpha_0-d} \zeta_1^{k_1} \cdots \zeta_n^{k_n} \mathbf{1}_{(1,\epsilon\zeta_1^{k_1},\ldots,\epsilon\zeta_n^{k_n})}$$

where ζ_i is a primitive (α_i+1)-th root of unity for $i = 1, \ldots, n$. Then for any monomial $\mathbf{x}^\beta = x_0^{\beta_0} \cdots x_n^{\beta_n}$, we have

$$\Lambda(\mathbf{x}^\beta) = \frac{1}{(\alpha_1+1)\cdots(\alpha_n+1)\binom{d}{\alpha}} \sum_{k_1=0}^{\alpha_1} \cdots \sum_{k_n=0}^{\alpha_n} \epsilon^{\alpha_0-d+\beta_1+\cdots+\beta_n} \zeta_1^{(\beta_1+1)\,k_0} \cdots \zeta_n^{(\beta_n+1)\,k_n}$$

$$= \begin{cases} \frac{1}{\binom{d}{\alpha}} \epsilon^{\rho(l_1,\ldots,l_n)+\alpha_0-d} \text{ if } \forall i = 1, \ldots, n, \exists l_i \in \mathbb{N}_+, \beta_i + 1 = l_i(\alpha_i+1) \\ 0 \text{ otherwise.} \end{cases}$$

where $\rho(l_1,\dots,l_n) = l_1(\alpha_1+1)+\cdots+l_n(\alpha_n+1)-n$. *Its minimal value on* $\mathbb{N}_+^n$ *is* $\rho(1,\dots,1)=\alpha_1+\cdots+\alpha_n = d-\alpha_0$. *The previous computation shows that*

$$\Lambda_{|S^d} = \frac{1}{\binom{d}{\alpha}} \sum_{\mathbf{l}\in\mathbb{N}_+^n | d-\alpha_0\le\rho(\mathbf{l})\le d} \epsilon^{\alpha_0+\rho(\mathbf{l})-d}\,\mathbf{d}_0^{d-\rho(\mathbf{l})}\mathbf{d}_1^{l_1(\alpha_1+1)-1}\cdots\mathbf{d}_n^{l_n(\alpha_n+1)-1}. \tag{3}$$

Suppose that $\alpha_0 = \min_{i=0,\dots,d}\alpha_i$. *Then the ideal* I_ϵ *is included in* $(f^\perp) = (x_0^{\alpha_0+1},\dots,x_n^{\alpha_n+1})$. *By Proposition 1.14, we deduce that* f *has a decomposition of size* $\frac{\prod_0^n(\alpha_i+1)}{\min_i(\alpha_i+1)}$.

As $\rho(\mathbf{l})\le d$ *implies* $\mathbf{l}=(1,\dots,1)$, *we have*

$$\Lambda_{|S^d} = \frac{1}{\binom{d}{\alpha}}\mathbf{d}_0^{\alpha_0}\mathbf{d}_1^{\alpha_1}\cdots\mathbf{d}_n^{\alpha_n} = f^*$$

which gives the decomposition of f^*. *The corresponding decomposition of* f *in terms of* d^{th}*-powers of linear forms is*

$$x_0^{\alpha_0}\cdots x_n^{\alpha_n} = \frac{1}{(\alpha_1+1)\cdots(\alpha_n+1)\binom{d}{\alpha}}\times$$
$$\sum_{k_1=0}^{\alpha_1}\cdots\sum_{k_n=0}^{\alpha_n}\epsilon^{d-\alpha_0}\,\zeta_1^{k_1}\cdots\zeta_n^{k_n}\left(x_0+\epsilon\zeta_1^{k_1}x_1+\cdots+\epsilon\zeta_n^{k_n}x_n\right)^d.$$

It can be proved that this decomposition has a minimal size (see [12], [13]*), so that we have*

$$r(x_0^{\alpha_0}\cdots x_n^{\alpha_n}) = \frac{\prod_0^n(\alpha_i+1)}{\min_i(\alpha_i+1)}.$$

This example also shows that the decomposition is not unique, since ϵ *is any non-zero constant.*

For more details on rank of monomials see also [14], [30] and [9]; the example above was also shown with different approach in [9, §2] and in [12, Corollary 3.8].

Definition 2.3 (Border rank)**.** *The Zariski closure of* $\sigma_r^{0,d}\subset\mathbb{P}(S^d)$, *also known as the* r^{th} *secant variety of the Veronese variety of* S^d, *is denoted* σ_r^d.

The minimal r *such that* $[f]\in\sigma_r^d$ *is called the* border rank *of* f *and we denote it* $r_\sigma(f)$ *(cf.* [10,17,32]*).*

Example 2.4. *Consider again* $f := x_0^{\alpha_0}\cdots x_n^{\alpha_n}$ *with* $\alpha_0+\cdots+\alpha_n = d$. *Suppose now that* $\alpha_0 = \max_i\alpha_i$. *Then the decomposition* (3) *is of the form*

$$f_\epsilon^* = f^* + \frac{1}{\binom{d}{\alpha}}\sum_{\mathbf{l}\in\mathbb{N}_+^n | d-\alpha_0<\rho(\mathbf{l})\le d}\epsilon^{\alpha_0+\rho(\mathbf{l})-d}\,\mathbf{d}_0^{d-\rho(\mathbf{l})}\mathbf{d}_1^{l_1(\alpha_1+1)-1}\cdots\mathbf{d}_n^{l_n(\alpha_n+1)-1},$$

with possibly some terms in the sum which involves positive powers of ϵ. *This shows that* $\lim_{\epsilon\to0} f_\epsilon^* = f^*$. *As* $f_\epsilon^*\in I_\epsilon^\perp$ *and* I_ϵ *is defining simple points, the rank of* f_ϵ *is* $\le\frac{\prod(\alpha_i+1)}{\max_i(\alpha_i+1)}$.

We deduce that the border rank of f *is less than* $\frac{\prod_{i=0}^n(\alpha_i+1)}{\max_i(\alpha_i+1)}$. *In* [26] *it is shown that if* $\max\alpha_i$ *is equal to the sum of all the others* α_i*'s then such a bound is actually sharp.*

Consider eg. $f = x_0x_1^{d-1}$ *(for* $d>2$*). This is the first well known case where the*

rank and border rank are different: from Example 2.2 we get that $r(f) = d$, while here we have just seen that $r_\sigma(f) = 2$ (see also [15], [4], [26]*).*

2.2. **Smoothable rank.** Let $\mathrm{Hilb}_r^{\mathrm{red}}(\mathbb{P}^n)$ be the set of schemes of length r which are the limit of smooth schemes of r points, and let us consider the two following definitions according e.g. to [30] and [5].

Definition 2.5. *For any integers r and d, we define $\mathcal{S}_d^r \subset \mathbb{P}(S^{d,0})$, as the set*

$$\mathcal{S}_r^{d,0} := \{[f] \in \mathbb{P}(S^d) |\ \exists\, s \leq r, \exists\, I \in \mathrm{Hilb}_r^{\mathrm{red}}(\mathbb{P}^n), \langle I^d, f\rangle = 0\}.$$

This leads to the following definition.

Definition 2.6 (Smoothable rank)**.** *The smallest r such that $[f] \in \mathcal{S}_r^{d,0}$ is called the* smoothable rank *of f and it is denoted $r_{\mathrm{smooth}^0}(f)$.*

Remark 2.7. *In* [24, Lemma 5.17] *it is shown that $\mathcal{S}_r^{d,0} \subset \sigma_r^d$. This proves that*

$$r_\sigma(f) \leq r_{\mathrm{smooth}^0}(f).$$

The following example is a personal communication from W. Buczyńska and J. Buczyński ([7]*). It shows that strictly inequalities can occur.*

Example 2.8 ([7])**.** *The following polynomial has border rank ≤ 5 but smoothable rank ≥ 6:*

$$f = x_0^2 x_2 + 6x_1^2 x_3 - 3\,(x_0 + x_1)^2 x_4.$$

One can easily check that the following f_ϵ has rank 5, for $\epsilon > 0$, and that $\lim_{\epsilon\to 0} \frac{1}{3\epsilon} f_\epsilon = f$:

$$f_\epsilon = (x_0 + \epsilon x_2)^3 + 6(x_1 + \epsilon x_3)^3 - 3(x_0 + x_1 + \epsilon x_4)^3 + 3(x_0 + 2\,x_1)^3 - (x_0 + 3x_1)^3.$$

Therefore $r_b(f) \leq 5$.

An explicit computation of $(f^\perp)$ yields the following Hilbert function for $H_{R/(f^\perp)} = [1, 5, 5, 1, 0, \ldots]$. If $I \subset (f^\perp)$, then $H_{R/I}(n) \geq H_{R/(f^\perp)}(n)$ for all $n \in \mathbb{N}$. As $H_{R/I}(n)$ is an increasing function of $n \in \mathbb{N}$ with $H_{R/(f^\perp)}(n) \leq H_{R/I}(n) \leq 5$, we deduce that $H_{R/I} = [1, 5, 5, 5, \ldots]$. This shows that $R/I = \langle x_0, \ldots, x_4\rangle$ so that $I_1 = \{0\}$ and that $I_2 = (f^\perp)_2$. As I is saturated, $I_2 : (x_0, \ldots, x_4) = I_1 = \{0\}$ since $H_{R/(f^\perp)}(1) = 5$. But an explicit computation of $((f^\perp)_2 : (x_0, \ldots, x_4))$ gives $\langle x_2, x_3, x_4\rangle$. This shows that $r_s(f) \geq 6$ so that $r_b(f) < r_s(f)$.

Remark 2.9. *If we indicate with $r_{\mathrm{smooth}}(f)$ the smallest r such that $[f] \in \overline{\mathcal{S}_r^{d,0}} := \mathcal{S}_r^d$, then we can observe that $\mathcal{S}_r^d = \sigma_r^d$. Obviously $\sigma_r^d \subset \mathcal{S}_r^d$. The other inclusion follows from Remark 2.7. This shows that*

$$r_\sigma(f) = r_{\mathrm{smooth}}(f).$$

In the Introduction we called $r_{\mathrm{smooth}}(f)$ the border smoothable rank *of f.*

2.3. **Catalecticant rank.** The apolar ideal $(f^\perp)$ can be also defined via the kernel of the following operators. Let us recall the following standard definition.

Definition 2.10 (Catalecticant)**.** *Given a homogeneous polynomial $f \in S^d$ and a positive integer k such that $k \leq d$, the Catalecticant of order k of f, denoted by $H_{f^*}^{k,d-k}$, is the application:*

$$\begin{array}{rcl} H_{f^*}^{k,d-k} : S^k & \to & (S^{d-k})^* \\ p & \mapsto & p \cdot f^*. \end{array}$$

Its matrix in the monomial basis $\{\mathbf{x}^\alpha\}_{|\alpha|=k}$ of S^k and in the basis $\{\binom{d-k}{\beta}^{-1} \mathbf{d}^\beta\}_{|\beta|=d-k}$ of $(S^{d-k})^*$ is denoted $\mathtt{H}_{f^*}^{k,d-k}$.

By construction, $\ker H_{f^*}^{k,d-k}$ is the component $(f^\perp)^k$ of degree k of the apolar ideal $(f^\perp)$ of f.

Given two families of monomials $B \subset S^k$ and $B' \subset S^{d-k}$, we denote by $H_{f^*}^{B',B}$ the "restriction" of $H_{f^*}^{k,d-k}$ from the vector space spanned by B to the dual of the vector space spanned by B'.

Remark 2.11. *By symmetry of the apolar product, we have $H_{f^*}^{k,d-k} = {}^tH_{f^*}^{d-k,k}$ via the identification $S^{d-i} \simeq (S^{d-i})^*$. In terms of matrices, we have $\mathtt{H}_{f^*}^{k,d-k} = {}^t\mathtt{H}_{f^*}^{d-k,k}$. or $\mathtt{H}_{f^*}^{B',B} = {}^t\mathtt{H}_{f^*}^{B,B'}$ for all families of monomials $B \subset S^k$ and $B' \subset S^{d-k}$.*

Definition 2.12 (Catalecticant rank)**.** *Let $f \in S^d$. The maximal rank of the operators $H_{f^*}^{k,d-k}$, for $0 \leq k \leq d$, is called the* catalecticant rank *of f and it is denoted $r_H(f)$.*

This rank was already introduced in [24][Definition 5.66, p.198] where it was called the differential length of f and denoted by $l\mathrm{diff}(f)$.

Definition 2.13. *Given an integer $i \leq d \in \mathbb{N}^t$ and $r \in \mathbb{N}$, we define the variety $\Gamma_r^{i,d-i} \subset \mathbb{P}(S^d)$ as:*

$$\Gamma_r^{i,d-i} := \{[f] \in \mathbb{P}(S^d) | \text{ rank } (H_{f^*}^{i,d-i}) = \text{rank } (H_{f^*}^{d-i,i}) \leq r\}.$$

Remark 2.14. *The set $\Gamma_r^{i,d-i} \subset \mathbb{P}(S^d)$ is a not necessary reducible algebraic variety. In fact it is defined by the minors $(r+1) \times (r+1)$ of the catalecticant matrices $\mathtt{H}_{f^*}^{i,d-i}$ (or $\mathtt{H}_{f^*}^{d-i,i}$). These minors give not necessary reduced equations but they represents in $\mathbb{P}(S^d)$ the variety that is the union of linear spaces spanned by the images of the divisors (hypersurfaces in $\mathbb{P}(S^1)$) of degree r on the Veronese $\nu_d(\mathbb{P}(S^1))$ (see e.g. [4] and [22]).*
If $i = 1$, such a variety is known as the "subspace variety in $\mathbb{P}(S^d)$" $Sub_r(S^d(V)) := \mathbb{P}\{f \in S^d(V) | \exists W \subset V, \dim(W) = r, f \in S^d(W)\}$. For a generic i, it can be geometrically obtained by intersecting $\mathbb{P}(S^d)$ with the r-th secant variety of the Segre variety of $\mathbb{P}(S^a) \times \mathbb{P}(S^{d-a})$ (for a better description of subspace varieties see [25, §17.4].

Example 2.15. *For a monomial $f = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ with $\alpha_0 + \cdots + \alpha_n = d$, the apolar ideal of f is $J := (f^\perp) = (x_0^{\alpha_0+1}, \ldots, x_n^{\alpha_{n+1}})$. By Remark 1.3, the rank of $H_{f^*}^{i,d-i}$ is the dimension of S^i/J^i that is the coefficient of t^i in*

$$h(t) = \prod_{i=0}^{n} (1 + t + \cdots + t^{\alpha_i}).$$

The maximum value of these coefficients which is the catalecticant rank is reached for the coefficients of degree the closest to $\frac{1}{2}(a_1+\cdots+a_n)$ *(it is proved in* [31] *that the polynomial* $h(t)$ *is symmetric unimodal, that is its coefficients are increasing up to the median degree(s) and then decreasing symmetrically). The exact value of the maximum is not known but asymptotic equivalents are known in some cases, see eg.* [18, p. 234-240].

Consider for instance the monomial $f = x_0x_1^2x_2^2$. *The previous computation yields the following Hilbert series for the apolar ideal:*

$$H_{S/(f^\perp)}(t) = (1+t)(1+t+t^2)^2 = 1+3\,t+5\,t^2+5\,t^3+3\,t^4+t^5.$$

This shows that rank $H^{1,4}_{f^*}$ = rank $H^{4,1}_{f^*} = 3$, rank $H^{2,3}_{f^*}$ = rank $H^{3,2}_{f^*} = 5$ *and thus that* $r_H(f) = 5$.

According to Example 2.4, the border rank of $f = x_0x_1^2x_2^2$ *is* $(1+1)\times(2+1) = 6$, *which shows that the border rank of* f *is strictly bigger that its Catalecticant rank.*

In [27, Theorems 1.2.3 and 4.2.7]*), it is shown that* $\Gamma_5^{2,3}(\mathbb{P}^2)$ *has codimension* 5 *in* $\mathbb{P}(S^5)$ *while the secant variety* $\sigma_5^5(\mathbb{P}^2)$ *has codimension 6. Therefore a generic element of* $\Gamma_5^{2,3}(\mathbb{P}^2)$ *has border rank strictly bigger than 5.*

2.4. Generalized rank and border generalized rank.

Definition 2.16. *A generalized affine decomposition of size* r *of* $f \in S^d$ *is a decomposition of the form*

$$\underline{f}^* = \sum_{i=1}^m \mathbf{1}_{\zeta_i} \circ \phi_i(\partial) \text{ on } R^{\leq d}$$

where $\zeta_i \in \mathbb{K}^n$ *and* $\phi_i(\partial)$ *are differential polynomials, such that the dimension of the inverse systems spanned by* $\sum_{i=1}^m \mathbf{1}_{\zeta_i} \circ \phi_i(\partial)$ *is* r.

Notice that the inverse system generated by $\sum_{i=1}^m \mathbf{1}_{\zeta_i} \circ \phi_i(\partial)$ is the direct sum of the inverse systems generated by $\mathbf{1}_{\zeta_i} \circ \phi_i(\partial)$ for $i = 1,\ldots,m$. The inverse system generated by $\mathbf{1}_{\zeta_i} \circ \phi_i(\partial)$ is the vector space spanned by the elements $\mathbf{1}_{\zeta_i} \circ \partial_{\partial_1}^{\alpha_1}\cdots\partial_{\partial_n}^{\alpha_n}\,\phi_i(\partial)$ for all $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n$.

This decomposition generalizes the (Waring) decomposition of Definition 1.9, since when $\phi_i(\partial) = \nu_i \in \mathbb{K}$ are constant polynomials, we have the decomposition

$$\underline{f}^* = \sum_{i=1}^m \nu_i\,\mathbf{1}_{\zeta_i} \text{ iff } f = \sum_{i=1}^m \nu_i\,(1+\zeta_{i,1}x_1+\cdots\zeta_{i,n}\,x_n)^d.$$

Definition 2.17 (Generalized rank)**.** *Given two integers* r *and* d, *we define* $\mathcal{G}_r^{d,0} \subset \mathbb{P}(S^d)$ *by:*

$$\mathcal{G}_r^{d,0} := \bigcup_{[g]\in PGL(n+1)} \{[f] \in \mathbb{P}(S^d) \mid g\cdot f^* \text{ has a generalized affine decomposition of size} \leq r\}.$$

The smallest r *such that* $[f] \in \mathcal{G}_r^{d,0}$ *is called the* generalized rank *of* f *and it is denoted* $r_{\mathcal{G}^0}(f)$.

Example 2.18. *The polynomial* $f = x^3y + y^3z$ *defines an inverse system of dimension 4 obtained as* $\langle \mathbf{1}_{(1,0,0)}, \mathbf{1}_{(1,0,0)}\partial_y, \mathbf{1}_{(0,1,0)}, \mathbf{1}_{(0,1,0)}\partial_z\rangle$, *therefore* $r_{\mathcal{G}^0}(f) = 4$. *Moreover, we are in a case of a polynomial of border rank 4 and rank 7 (as described in* [4, Theorem 44]*). In this case* $r_{\mathcal{G}^0}(f) = 4 = r_\sigma(f) < r(f) = 7$.

Example 2.19. *For a monomial $f = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ with $\alpha_0 + \cdots + \alpha_n = d$, we have*

$$\underline{f}^* = \frac{1}{d!}\mathbf{1}_{(1,0,\ldots,0)} \cdot \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$

The inverse system spanned by $\mathbf{1}_{(1,0,\ldots,0)} \cdot \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ is of dimension $(\alpha_1+1) \times \cdots \times (\alpha_n+1)$. Assuming that $\alpha_0 = \max_i \alpha_i$, the previous decomposition is a generalized decomposition of minimal size (according to Corollary 3.9 and Example 2.28). Therefore we have

$$r_{\mathcal{G}^0}(x_0^{\alpha_0} \cdots x_n^{\alpha_n}) = \frac{\prod_{i=0}^n (\alpha_i + 1)}{\max_i(\alpha_i + 1)}.$$

Notice that $[f] \in \mathcal{G}_r^{d,0}$ iff there exists a change of coordinates such that in the new set of coordinates $\underline{f}$ has a generalized affine decomposition of size $\leq r$.

This notion of generalized affine decomposition and of generalized rank is related to the generalized additive decomposition introduced in [24, Definition 1.30, p. 22] for binary forms, called the length of f and denoted $l(f)$. However, the extension to forms in more variables proposed in [24, Definition 5.66, p. 198] does not correspond to the one we propose but to the border rank. For binary forms, the border rank and the generalized rank coincide as we will see.

Definition 2.20 (Border generalized rank)**.** *Given two integers r and d, we define $\mathcal{G}_r^d \subset \mathbb{P}(S^d)$ to be the Zariski closure of $\mathcal{G}_r^d$ defined above. The smallest r such that $[f] \in \mathcal{G}_r^d$ is called the* border generalized rank *of f and it is denoted $r_{\mathcal{G}}(f)$.*

2.5. **Flat extension rank and border flat extension rank.** We describe here a new notion of rank based the property of extension of bounded rank of the catalecticant matrices.

Definition 2.21. *For any integers r and d, we define $\mathcal{E}_r^{d,0} \subset \mathbb{P}(S^d)$, as the set*

$$\begin{aligned}\mathcal{E}_r^{d,0} \quad := \quad & \{[f] \in \mathbb{P}(S^d) |\ \exists\, \mathbf{u} \in S^1 \setminus \{0\}, \exists\, \tilde{f} \in \Gamma_r^{m,m'} \ \text{ with} \\ & m = \max\{r, \lceil \frac{d}{2} \rceil\},\ m' = \max\{r-1, \lfloor \frac{d}{2} \rfloor\} \text{ s.t. } \mathbf{u}^{m+m'-d} \cdot \tilde{f}^* = f^*\}.\end{aligned}$$

By definition, $m+m' \geq \lceil \frac{d}{2} \rceil + \lfloor \frac{d}{2} \rfloor = d$. Moreover, if $d \geq 2r-1$ then $\mathcal{E}_r^d = \Gamma_r^{m,m'}$ since $m = \lceil \frac{d}{2} \rceil$, $m' = \lfloor \frac{d}{2} \rfloor$ and $m + m' - d = 0$.

Definition 2.22 (Flat extension rank)**.** *The smallest r such that $[f] \in \mathcal{E}_r^{d,0}$ is called the* flat extension rank *of f and it is denoted $r_{\mathcal{E}^0}(f)$.*

A $\tilde{f} \in \Gamma_r^{m,m'}$ such that $\exists\, \mathbf{u} \in S^1 \setminus \{0\}$ with $\mathbf{u}^{m+m'-d} \cdot \tilde{f}^* = f^*$ and rank $H_{f^*}^{m,m'} = r$ *is called a* flat extension *of $f \in S^d$ of rank r.*

Example 2.23. *For a monomial $f = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ with $\alpha_0 + \cdots + \alpha_n = d$, the elements*

$$\Lambda = \frac{1}{d!}\mathbf{1}_{(1,0,\ldots,0)} \cdot \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$$

of S^*, define, by restriction, a Hankel operator $H_{\tilde{f}^*}^{r,r-1}$ from S^r to $(S^{r-1})^*$ where $r = (\alpha_1+1) \times \cdots \times (\alpha_n+1)$ and $\tilde{f}^* = \frac{1}{d!}\mathbf{d}_0^{2r-1-\alpha_1-\cdots-\alpha_n}\mathbf{d}_1^{\alpha_1} \cdots \mathbf{d}_n^{\alpha_n}$ is the restriction of Λ to S^{2r-1}. We check that the image of $H_{\tilde{f}^*}^{r,r-1}$ is the vector space of $(S^{r-1})^*$ spanned by $\mathbf{d}_0^{r-1-\beta_1-\cdots-\beta_n}\mathbf{d}_1^{\beta_1} \cdots \mathbf{d}_n^{\beta_n}$ for $0 \leq \beta_i \leq \alpha_i$. Thus $H_{\tilde{f}^*}^{r,r-1}$ is of rank

r. Assuming that $\alpha_0 = \max_i \alpha_i$, this rank is the minimal possible rank of a flat extension of f. Thus it is the flat extension rank of f:

$$r_{\mathcal{E}}(x_0^{\alpha_0} \cdots x_n^{\alpha_n}) = \frac{\prod_{i=0}^n (\alpha_i + 1)}{\max_i(\alpha_i + 1)}.$$

Notice that $[f] \in \mathcal{E}_r^d$ iff there exists a change of coordinates such that after this change of coordinates we have $\mathbf{u} = x_0$ so that $\underline{\tilde{f}}^* \in (R^{\leq m+m'})^*$ is such that

- $\underline{\tilde{f}}^*_{|R^{\leq d}} = \underline{f}^*$ and
- rank $H^{m,m'}_{\underline{\tilde{f}}^*} \leq r$.

In other words, $[f] \in \mathcal{E}_r^d$ iff after a change of coordinates, $\underline{f}^* \in (R^{\leq d})^*$ can be extended to a linear form $\underline{\tilde{f}}^* \in (R^{\leq m+m'})^*$ with rank $H^{m,m'}_{\underline{\tilde{f}}^*} \leq r$. We will see hereafter that we can choose a generic change of coordinates.

A simple way to characterize a flat extension of a given rank is given by the following result.

Theorem 2.24 ([28], [3])**.** *Let M, M' be sets of R, $B \subset M$ and $B' \subset M'$ be two monomial sets of size r connected to 1 such that $M \cdot M'$ contains $B^+ \cdot B'^+$. If $\Lambda \in \langle M \cdot M' \rangle^*$ is such that* rank $\mathtt{H}_\Lambda^{B,B'}$ = rank $\mathtt{H}_\Lambda^{M,M'} = r$*, then there exists an extension $\tilde{\Lambda} \in R^*$ of Λ such that* rank $\mathtt{H}_{\tilde{\Lambda}} = r$*. Moreover, we have* $\ker \mathtt{H}_{\tilde{\Lambda}} = (\mathtt{H}_\Lambda^{M,M'})$*.*

Definition 2.25 (border flat extension rank)**.** *For any integers r and d, we define $\mathcal{E}_r^d \subset \mathbb{P}(S^d)$, as the Zariski closure of set $\mathcal{E}_r^{d,0}$ defined above and the smallest r such that $[f] \in \mathcal{E}_r^d$ is called the* border flat extension rank *of f and it is denoted $r_{\mathcal{E}}(f)$.*

2.6. **Scheme length and border scheme length.** We recall that $\mathrm{Hilb}_s(\mathbb{P}^n)$ is the set of 0-dimensional schemes Z of s points (counted with multiplicity). It can be identified with the set of homogeneous saturated ideals $I \subset S$ such that the algebra S/I has a constant Hilbert polynomial equal to s.

Definition 2.26. *For any integers r and d, we define $\mathcal{K}_r^{d,0} \subset \mathbb{P}(S^d)$, as the set*

$$\mathcal{K}_r^{d,0} := \{[f] \in \mathbb{P}(S^d) |\ \exists\, s \leq r, \exists\, I \in \mathrm{Hilb}_s(\mathbb{P}^n), \langle I^d, f \rangle = 0\}.$$

In [8], the *r-cactus variety* $\mathcal{K}_r^d$ is defined as the closure of $\mathcal{K}_r^{d,0}$.

Definition 2.27 (Scheme length and border scheme length)**.** *The smallest r such that $[f] \in \mathcal{K}_r^{d,0}$ is called the* scheme length *(or* cactus rank *in* [30]*) of f and it is denoted $r_{\mathrm{sch}^0}(f)$. The smallest r such that $[f] \in \mathcal{K}_r^d$ is called the* border scheme length *of f and it is denoted $r_{\mathrm{sch}}(f)$.*

We have used the same definition of "scheme length" of f used in [24][Definition 5.1, p. 135, Definition 5.66, p. 198], where it is denote $l\mathrm{sch}(f)$.

Example 2.28. *For a monomial $f = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$ with $\alpha_0 + \cdots + \alpha_n = d$, the ideal $I = (x_1^{\alpha_1+1}, \ldots, x_n^{\alpha_n+1})$ is an ideal of length $(\alpha_1 + 1) \times \cdots \times (\alpha_n + 1)$, which is apolar to f. Assuming that $\alpha_0 = \max_i \alpha_i$, this length $\frac{\prod_{i=0}^n (\alpha_i+1)}{\max_i \alpha_i + 1}$ is minimal as proved in* [30, Cor. 2]*, using Bezout theorem.*

Thus, the scheme length of f is:

$$r_{\mathrm{sch}}(x_0^{\alpha_0}\cdots x_n^{\alpha_n}) = \frac{\prod_{i=0}^n(\alpha_i+1)}{\max_i(\alpha_i+1)}.$$

This is an example where the border rank and scheme length coincides but differ from the rank (see Example 2.4). In the next example, we have a case where the scheme length is strictly smaller that the border rank.

Example 2.29. *In the case of cubics polynomials, the scheme length of a generic form is smaller than its border rank for forms in 9 variables. The border rank of a generic cubic form in 9 variables is in fact 19 (this is Alexander and Hirschowitz Theorem* [1]*), while the scheme length is smaller or equal than 18 (see* [5]*).*

3. The generalized decomposition

The objective of this section is to relate the scheme length, generalized rank and flat extension rank.

Lemma 3.1. *Given two integers r and d, we have $\mathcal{G}_r^{d,0} \subset \mathcal{K}_r^{d,0}$.*

Proof. To prove this inclusion, we show that if $g\cdot \underline{f}^*$ has a generalized affine decomposition of size r of the form:

$$g\cdot \underline{f}^* = \sum_{i=1,\dots,m} \mathbf{1}_{\zeta_i}\circ\phi_i(\partial),$$

on R^d, then $f^*\in (I^d)^\perp$ for some ideal $I\in \mathrm{Hilb}_s(\mathbb{P}^n)$ with $s\le r$.

By a change of coordinates, we can assume that $g=\mathrm{Id}_{n+1}$. Then the linear form:

$$\underline{\Lambda} = \sum_{i=1,\dots,m} \mathbf{1}_{\zeta_i}\circ\phi_i(\partial)\in R^*$$

coincides with $\underline{f}^*$ on $R^{\le d}$. By Proposition 1.6, as the dimension of the inverse system generated by Λ is $s\le r$, $\underline{I}=\ker H_\Lambda\subset R$ is a zero-dimensional ideal of multiplicity $s\le r$. We denote by $I\subset S$ the homogenization of $\underline{I}$ with respect to x_0. Then,

$$I\in \mathrm{Hilb}_s(\mathbb{P}^n).$$

As $\underline{f}^*=\underline{\Lambda}$ on $R^{\le d}$, we have:

$$f^*\in (I^d)^\perp,$$

which proves the inclusion. □

Lemma 3.2. *Given integers r, d and i such that $0\le i\le d$, we have $\mathcal{K}_r^{d,0}\subset \Gamma_r^{i,d-i}$.*

Proof. Let us prove that for all $f\in S^d$ such that $\langle I^d, f\rangle = 0$ with $I\in \mathrm{Hilb}_s(\mathbb{P}^n)$ and $s\le r$, we have:

$$\mathrm{rank}(H_{f^*}^{i,d-i})\le s\le r$$

for all $0\le i\le d$. By Lemma 1.13, we have $I\subset J:=(f^\perp)$ so that $I^i\subset J^i$ for $0\le i\le d$.

As the Hilbert function of a saturated ideal $I\in \mathrm{Hilb}_s(\mathbb{P}^n)$ is increasing until the degree s and then is constant and equal to s, we have

$$\dim S^i/I^i\le s,\ \forall\ 0\le i\le d.$$

By the above inclusion, this implies that

$$\dim\ S^i/J^i \leq s,\ \forall\ 0 \leq i \leq d.$$

As $J^i := \ker\ H^{i,d-i}_{f^*}$, we deduce that

$$\text{rank}\ \ H^{i,d-i}_{f^*} \leq s,\ \forall\ 0 \leq i \leq d.$$

Consequently as $s \leq r$, $f \in \Gamma^{i,d-i}_r$ and $\mathcal{K}^{d,0}_r \subset \Gamma^{i,d-i}_r$. □

Corollary 3.3. *For any homogeneous polynomial f we have that $r_H(f) \leq r_{\text{sch}}(f) \leq r_{\text{sch}^0}(f)$.*

Proof. By Lemma 3.2 we have that $\mathcal{K}^{d,0}_r \subset \Gamma^{i,d-i}_r$. that implies that $r_H(f) \leq r_{\text{sch}^0}(f)$. Now since $\Gamma^{i,d-i}_r$ is closed by definition, we get that $\mathcal{K}^{d,0}_r \subset \overline{\mathcal{K}^{d,0}_r} = \mathcal{K}^d_r \subset \Gamma^{i,d-i}_r$ that implies that $r_H(f) \leq r_{\text{sch}}(f) \leq r_{\text{sch}^0}(f)$. □

Lemma 3.4. *Let $d \geq r$ and $E \subset S^d$ such that S^d/E is of dimension r. Then for a generic change of coordinates $g \in PGL(n+1)$, $S^d/g \cdot E$ has a monomial basis of the form $x_0 B$ with $B \subset S^{d-1}$. Moreover, $\underline{B}$ is connected to 1.*

Proof. Let $\succ$ be the lexicographic ordering such that $x_0 \succ \cdots \succ x_n$. By [19][Theorem 15.20, p. 351], after a generic change of coordinates, the initial J of the ideal $I = (E)$ for $\succ$ is Borel fixed. That is, if $x_i \mathbf{x}^\alpha \in J$ then $x_j \mathbf{x}^\alpha \in J$ for $j > i$.

To prove that there exists a subset B of monomials of degree $d-1$ such that $x_0\, B$ is a basis of S^d/I^d, we show that $J^d + x_0 S^{d-1} = S^d$. Let $J'^d = (J^d + x_0 S^{d-1})/x_0 S^{d-1}$, $S'^d = S^d/x_0 S^{d-1} = \mathbb{K}[x_1, \ldots, x_n]$ and $L = (J : x_0)$. Then we have the exact sequence

$$0 \to S^{d-1}/L^{d-1} \xrightarrow{\mu_{x_0}} S^d/J^d \to S'^d/J'^d,$$

where μ_{x_0} is the multiplication by x_0. Let us denote by $s_k = \dim S^k$ and $q(k) = s_k - r$ for $k \in \mathbb{N}$. Suppose that $\dim S'^d/J'^d > 0$, then $\dim L^{d-1} > s_{d-1} - r = q(d-1)$. As $d \geq r$ and r is the Gotzmann regularity of q, by [23, (2.10), p. 66] we have $\dim S^1\, L^{d-1} > q(d)$. As J is Borel fix, ie. $x_0\, p \in J$ implies $x_i\, p \in J$ for $i \geq 0$, we have $S^1\, L^{d-1} \subset J$, so that $\dim J^d \geq \dim S^1\, L^{d-1} > q(d) = s_d - r$. This implies that $\dim S^d/J^d = \dim S^d/I^d = \dim S^d/E < r$, which contradicts the hypothesis on E. Thus $J^d + x_0 S^{d-1} = S^d$.

Let B' be the complementary of J^d in the set of monomials of degree d. The sum $S^d = J^d + x_0 S^{d-1}$ shows us that $B' = x_0 B$ for some subset B of monomials of degree $d-1$.

As J^d is Borel fix and different from S^d, its complementary B' contains x_0^d. Similarly we check that if $x_0^{\alpha_0} \cdots x_n^{\alpha_n} \in B'$ with $\alpha_1 = \cdots = \alpha_{k-1} = 0$ and $\alpha_k \neq 0$ then $x_0^{\alpha_0+1}\, x_{k+1}^{\alpha_{k+1}} \cdots x_n^{\alpha_n} \in B'$. This shows that $\underline{B'} = \underline{B}$ is connected to 1. □

The equality $\mathcal{K}^d_r = \Gamma^{i,d-i}_r$ for $d \geq 2r$ and $r \leq i \leq d-r$, that appears in the following theorem was proved, with a different technique, in [8, Theorem 1.7].

Theorem 3.5. *For integers r, d and i such that $d \geq 2r$, $r \leq i \leq d-r$, we have*

$$\mathcal{G}^d_r = \mathcal{K}^d_r = \mathcal{E}^d_r = \Gamma^{i,d-i}_r.$$

Proof. First of all let, us observe that, if $d \geq 2r$ and $r \leq i \leq d-r$, then $\mathcal{G}_r^{d,0} = \mathcal{G}_r^d$, $\mathcal{K}_r^{d,0} = \mathcal{K}_r^d$ and $\mathcal{E}_r^{d,0} = \mathcal{E}_r^d$.

Now we prove the following inclusion:

$$\Gamma_r^{i,d-i} \subset \mathcal{G}_r^d. \tag{4}$$

Let us fix $f \in \Gamma_r^{i,d-i}$ for an integer $r \leq i \leq d-r$. Let us denote $E := \mathrm{Ker}(H_{f^*}^{i,d-i})$ and $F := \mathrm{Ker}(H_{f^*}^{d-i,i})$ and $k \leq r$ the rank of $\mathtt{H}_{f^*}^{i,d-i}$. We recall that:

$$\mathtt{H}_{f^*}^{i,d-i} = {}^t\mathtt{H}_{f^*}^{d-i,i}.$$

The quotients S^i/E and S^{d-i}/F are thus of dimension k. As $k \leq r \leq i$ and $k \leq r \leq d-i$, by Lemma 3.4 and a generic change of coordinates we may assume that there exists a familly B (resp. B') of k monomials of S^{i-1} (resp. S^{d-i-1}) such that $x_0\, B$ (resp. $x_0\, B'$) is a basis of S^i/E (resp. S^{d-i}/F) and that

$$\underline{B} \subset R^{\leq i-1} \text{ (resp. } \underline{B'} \subset R^{\leq d-i-1})$$

are connected to 1. Notice then that

$$\mathtt{H}_{\underline{f^*}}^{\underline{B'},\underline{B}} = \mathtt{H}_{f^*}^{x_0\, B', x_0\, B}$$

is an invertible matrix of size $k \times k$. As the monomials of $\underline{B}$ are in $R^{\leq i-1}$ (resp. $R^{\leq d-i-1}$), the monomials of $\underline{B}^+$ (resp. $\underline{B'}^+$) are in the set M (resp. M') of monomials of degree $\leq i$ (resp. $\leq d-i$) and $B^+ \cdot B'^+ \subset M \cdot M'$. Moreover, we have

$$\mathrm{rank}\ \ \mathtt{H}_{\underline{f^*}}^{\underline{B'},\underline{B}} = \mathrm{rank}\ \mathtt{H}_{f^*}^{x_0\, B', x_0\, B} = \mathrm{rank}\ \ \mathtt{H}_{\underline{f^*}}^{M',M} = \mathrm{rank}\ \mathtt{H}_{f^*}^{i,d-i} = k.$$

By Theorem 2.24, there exists a linear form $\Lambda \in R^*$ which extends $\underline{f}^*$ such that $\dim_{\mathbb{K}}(R/I_\Lambda) = r$ where $I_\Lambda = \ker H_\Lambda$. By Theorem 1.5, as $\Lambda \in I_\Lambda^\perp$, there exists $\zeta_1, \ldots, \zeta_m \in \mathbb{K}^n$, differential polynomials $\phi_1, \ldots, \phi_m \in \mathbb{K}[\partial_1, \ldots, \partial_n]$ such that

$$\Lambda = \sum_{i=1,\ldots,m} \mathbf{1}_{\zeta_i} \circ \phi_i(\partial).$$

As the inverse system spanned by Λ is included in $(I_\Lambda)^\perp$ which is of dimension s and as Λ coincides with $\underline{f}^*$ on $R^{\leq d}$, f has a generalized affine decomposition of size $\leq s \leq r$. This proves that

$$\Gamma_r^{i,d-i} \subset \mathcal{G}_r^d.$$

Using Lemmas 3.1 and 3.2, we have

$$\Gamma_r^{i,d-i} \subset \mathcal{G}_r^d \subset \mathcal{K}_r^d \subset \Gamma_r^{i,d-i}. \tag{5}$$

By definition, $\mathcal{E}_r^d = \Gamma_r^{r,d-r}$ when $d \geq 2\,r$. Therefore the previous inclusions show that $\mathcal{E}_r^d = \mathcal{G}_r^d = \mathcal{K}_r^d = \Gamma_r^{i,d-i}$ for $r \leq i \leq d$. □

To analyze the relationship between the sets $\mathcal{G}_r^d$, $\mathcal{E}_r^d$, $\mathcal{K}_r^d$ for general $r, d \in \mathbb{N}$, we need the following lemma.

Lemma 3.6. *Given r, $d \in \mathbb{N}$, $\mathbf{u} \in S^1$, $f \in S^d$ and $I \in \mathrm{Hilb}_r(\mathbb{P}^n)$ such that $\mathbf{u} \neq 0$ is a non-zero divisor for I, there exists a flat extension $\tilde{f} \in \Gamma_r^{m+m'}$ of f such that $\mathbf{u}^{m+m'-d} \cdot \tilde{f}^* = f^*$, and we have $\mathcal{K}_r^{d,0} \subset \mathcal{E}_r^{d,0}$.*

Proof. Let $f \in S^d$ with $f^* \in (I^d)^\perp$, $I \in \mathrm{Hilb}_r(\mathbb{P}^n)$ and $s \leq r$.

Let $\mathbf{u} \in S^1$ be a linear form such that $(I : \mathbf{u}) = I$. By a change of coordinates, we can assume that $\mathbf{u} = x_0$.

We denote by $\underline{I} \subset R$ the dehomogenization of $I \subset S$ (setting $x_0 = 1$). As $I \in \mathrm{Hilb}_r(\mathbb{P}^n)$ and $(I : x_0) = I$, the quotient algebra $R/\underline{I}$ is a $\mathbb{K}$-vector space of dimension s. By the following natural isomorphism:

$$(I^d)^\perp \simeq (S^d/I^d)^* \simeq (R^d/\underline{I}^{\leq d})^*. \tag{6}$$

a linear form $f^* \in (I^d)^\perp$ corresponds to a linear form $\underline{f}^* \in (R^d/\underline{I}^{\leq d})^*$. As

$$R^d/\underline{I}^{\leq d} \hookrightarrow R/\underline{I},$$

the linear form $\underline{f}^* \in (R^d/\underline{I}^{\leq d})^*$ can be extended (not necessarly in a unique way) to a linear form $\phi \in (R/\underline{I})^*$.

As the ideal $I^\phi = \ker H_\phi$ contains $\underline{I}$ and $\dim R/\underline{I} = s$, we deduce that $\mathtt{H}_\phi$ is of rank $\leq s \leq r$. By restriction, $\mathtt{H}^{m,m}_{\tilde{\phi}}$ is of rank $\leq s \leq r$, where $\tilde{\phi} = \phi_{|R^{\leq 2m}}$. The restriction of $\tilde{\phi}$ to $R^{\leq d}$ is $\underline{f}^*$.

For any $[f] \in \mathcal{K}_r^{d,0}$, there exists $I \in \mathrm{Hilb}_r(\mathbb{P}^n)$ such that $f^* \in (I^d)^\perp$. As I is a saturated ideal, it is always possible to find $\mathbf{u} \in S^1$ such that $\mathbf{u} \neq 0$ is non-zero divisor for I. By the previous construction, we can find a flat extension $\tilde{f}^* \in \Gamma_r^{m+m'}$ of f, which this shows that $f \in \mathcal{E}_r^{d,0}$ and that $\mathcal{K}_r^{d,0} \subset \mathcal{E}_r^{d,0}$. □

Theorem 3.7. *Let $f \in S^d$. The followings are equivalent:*

- *There exists a zero-dimensional saturated ideal I defining r points counted with multiplicity such that $I \subset (f^\perp)$;*
- *f has a generalized decomposition of size $\leq r$;*
- *f has a flat extension of size $\leq r$.*

In other words,

$$\mathcal{G}_r^{d,0} = \mathcal{K}_r^{d,0} = \mathcal{E}_r^{d,0}. \tag{7}$$

and $r_{\mathcal{G}^0}(f) = r_{\mathrm{sch}^0}(f) = r_{\mathcal{E}^0}(f)$.

Proof. By Lemma 3.1, we have $\mathcal{G}_r^{d,0} \subset \mathcal{K}_r^{d,0}$. By Lemma 3.6, we have $\mathcal{K}_r^{d,0} \subset \mathcal{E}_r^{d,0}$.

Let us prove that $\mathcal{E}_r^{d,0} \subset \mathcal{G}_r^{d,0}$. For any $f \in \mathcal{E}_r^{d,0}$, there exists $\tilde{f} \in \Gamma_r^{m,m}$ and $\mathbf{u} \in S^1$ such that $\mathbf{u}^{m-r} \cdot \tilde{f}^* = f^*$. By Theorem 3.5, $\tilde{f} \in \mathcal{G}_r^{2m,0}$. Thus after some change of coordinates, $\tilde{f}$ has an affine generalized decomposition of the form:

$$\underline{\tilde{f}}^* = \sum_{i=1,\ldots,m} \mathbf{1}_{\zeta_i} \circ \phi_i(\partial).$$

As $f^* = \mathbf{u}^{m-r} \cdot \tilde{f}^*$, we have

$$\underline{f}^* = \underline{\mathbf{u}}^{2m-r} \cdot \underline{\tilde{f}}^* = \sum_{i=1,\ldots,m} \underline{\mathbf{u}}^{2m-r} \cdot (\mathbf{1}_{\zeta_i} \circ \phi_i(\partial)) = \sum_{i=1,\ldots,m} \mathbf{1}_{\zeta_i} \circ \phi_i'(\partial)$$

where ϕ_i' is obtained from ϕ_i by derivation. As ϕ_i' is obtained from ϕ_i by derivation, the inverse system spanned by $\mathbf{1}_{\zeta_i} \circ \phi_i'(\partial)$ is included in the inverse system spanned by $\mathbf{1}_{\zeta_i} \circ \phi_i(\partial)$. Thus f has an affine decomposition $\underline{f}^* = \sum_{i=1,\ldots,m} \mathbf{1}_{\zeta_i} \circ \phi_i'(\partial)$ of size $\leq r$ and $f \in \mathcal{G}_r^{d,0}$.

This shows that

$$\mathcal{G}_r^{d,0} \subset \mathcal{K}_r^{d,0} \subset \mathcal{E}_r^{d,0} \subset \mathcal{G}_r^{d,0}.$$

and concludes the proof of the theorem. □

Corollary 3.8. $\mathcal{G}_r^d = \mathcal{K}_r^d = \mathcal{E}_r^d$ *and* $r_{\mathcal{G}}(f) = r_{\mathrm{sch}}(f) = r_{\mathcal{E}}(f)$.

Corollary 3.9. *For any homogeneous polynomial f we have that:*

$$r_H(f) \leq r_{\mathcal{G}}(f) = r_{\mathrm{sch}}(f) = r_{\mathcal{E}}(f) \leq$$

$$\leq r_{\mathcal{G}^0}(f) = r_{\mathrm{sch}^0}(f) = r_{\mathcal{E}^0}(f) \leq r(f).$$

Proof. Form Corollary 3.3 we get that $r_H(f) \leq r_{\mathrm{sch}}(f) \leq r_{\mathrm{sch}^0}(f)$. By definitions of $\mathcal{G}_r^d$, $\mathcal{K}_r^d$ and $\mathcal{E}_r^d$ we obviously have that $r_{\mathcal{G}}(f) \leq r_{\mathcal{G}^0}(f)$, $r_{\mathrm{sch}}(f) \leq r_{\mathrm{sch}^0}(f)$, $r_{\mathcal{E}}(f) \leq r_{\mathcal{E}^0}(f)$. Finally Theorem 3.7 and Corollary 3.8 end the proof. □

Remark 3.10. *Obviously $\mathcal{S}_r^d \subset \mathcal{K}_r^d$ and $\mathcal{S}_r^{d,0} \subset \mathcal{K}_r^{d,0}$. This justify $r_H(f) \leq r_{\mathcal{G}}(f) = r_{\mathrm{sch}}(f) = r_{\mathcal{E}}(f) \leq r_{\mathrm{smooth}}(f)$ and $r_{\mathcal{G}^0}(f) = r_{\mathrm{sch}^0}(f) = r_{\mathcal{E}^0}(f) \leq r_{\mathrm{smooth}^0}(f)$. Now, in order to complete table (1) one it is sufficient to use Remarks 2.7 and 2.9.*

Remark 3.11. *At the beginning of the proof of Theorem 3.5, we showed that if $d \geq 2r$ and $r \leq i \leq d-r$, then $\mathcal{G}_r^{d,0} = \mathcal{G}_r^d$, $\mathcal{K}_r^{d,0} = \mathcal{K}_r^d$ and $\mathcal{E}_r^{d,0} = \mathcal{E}_r^d$. Observe that in Example 2.8 the condition $d \geq 2r$ is not satisfied. Since, by Corollary 3.9, we have that $r_{\mathcal{G}^0}(f) = r_{\mathrm{sch}^0}(f)$, then Example 2.8 gives also an example of a polynomial f such that $r_{\mathcal{G}}(f) = r_{\mathrm{sch}}(f) = r_{\mathcal{E}}(f) < r_{\mathcal{G}^0}(f) = r_{\mathrm{sch}^0}(f) = r_{\mathcal{E}^0}(f)$.*

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