COMBINING ALGEBRAIC APPROACH WITH EXTREME VALUE THEORY FOR SPIKE DETECTION

Nehla DEBBABI [1,2], *Marie KRATZ* [3], *Mamadou MBOUP* [2] *and Sadok EL ASMI* [1]

[1] Research Lab. COSIM, Engineering College of Communications, University of Carthage
[2] CReSTIC, Université de Reims Champagne Ardenne
[3] ESSEC Business School Paris
{nehla.debbabi,elasmi}@supcom.rnu.tn, kratz@essec.edu, mamadou.mboup@univ-reims.fr

ABSTRACT

This paper uses the Extreme Value Theory (EVT) for threshold selection in a previously proposed algebraic spike detection method. The algebraic method characterizes the occurrence of a spike by an irregularity in the neural signal and devises a nonlinear (Volterra) filter which enhances the presence of such irregularities. These appear as (positive) high amplitude pulses in the output signal. The pulses are isolated. We then interpret the occurrence of a spike as a rare and extreme event that we model in the framework of EVT. With this model, we derive an explicit expression of the decision threshold corresponding to a given probability of false-alarm. Simulation results show that the empirical probability of false alarm is close to the predicted one by applying the derived theoretical threshold.

***Index Terms*—** Neural spike detection, algebraic approach, Extreme Value Theory, Generalized Pareto distribution, Mean Excess Plot.

1. INTRODUCTION

The neural signal consists of a sequence of action potentials (APs) *viz* spikes representing the electrical activity of the neurons. An AP represents the polarity inversion across the neuron membrane, it has an amplitude of 100 *mv* and a duration of about 2 to 3 *ms*. This duration includes a refractory period [1] measuring roughly 1 *ms* during which a second AP cannot be initiated. Spike detection is the first mandatory step in any information processing of neural recording. It is therefore an important task in neuroscience. When the signal is recorded extracellularly, it is inevitably corrupted by several kinds of noises. The most significant one originates from the activities of remote neurons with respect to the electrode. In many situations, this background noise appears with a low Signal to Noise Ratio (SNR) making the problem of spike detection challenging. The spike detection is a classical problem which has been developed by different methods. We refer to the Nonlinear Energy Operator (NEO) method [2], as well as the Wavelet Transform one[3]. We cite likewise the algebraic approach detailed in [4], in which we are interested in this work. Using operational calculus and algebraic derivatives, this method defines a decision function for spike detection which corresponds to a Volterra filtering of the neural signal reducing the noise and highlighting spikes. A way to detect spikes is then devised by comparing the output signal with a threshold. The threshold is determined in [4] for a given probability of false alarm (pfa) after estimating the unknown cumulative distribution function (cdf) of the noise presented in the output of the Volterra filter. However, the obtained results using the determined threshold are not completely satisfactory. In order to obtain a more reliable detection, we propose in this paper a new spike decision threshold by combining the algebraic approach with the Extreme Value Theory (EVT) [5]. We characterize the spike occurrence as an extreme and rare event. Thereby, we transform the problem of spike detection into extreme value determination. Hence, we formalize a new way to determine the probability of false alarm by estimating the distribution of of extremes, *i.e*, spikes instead of estimating the unknown entire distribution of the decision function. Using the Pickands theorem [6], the extremes distribution is approximated by a Generalized Pareto Distribution (GPD). To carry out the performances of the proposed method, using simulated signals, we compare the obtained results to those achieved using the algebraic approach [4] for a fixed pfa. Our method improves the quality of the detection in terms of false alarm, providing more realistic results. The validation of the proposed method using generated signals proves its usefulness when applied to real data. The remainder of this paper is organized as follows. Section 2 develops the proposed threshold determination. Simulation results are discussed in Section 3. Finally, conclusions are drawn in section 4.

2. DESCRIPTION OF THE PROPOSED THRESHOLD

2.1. Algebraic approach

In this section we give a brief recall of the algebraic approach [4] to spike detection. A spike is the result of a sudden electrical discharge across the neural membrane. Its occurrence is interpreted in [4] as an irregularity in the neural signal, say $y(t)$. We consider that the spikes are isolated so that there is at most one spike in each interval $[\tau,\, \tau + T]$ of a given width T, representing, *e.g*, the refractory period. Then by setting $y_\tau(t) = y(\tau+t), t \in [0,T)$ and by ignoring the noise, we have $\frac{d^n}{dt^n} y_\tau(t) = [y_\tau^{(n)}](t) + \sum_{k=1}^{n} \mu_{n-k}\, \delta(t - t_\tau)^{(k-1)}$ where $[y_\tau^{(n)}]$ represents the regular part of the n^{th} signal derivative, the superscript (k) stands for differentiation of order k, t_τ is the possible irregularity point in $[\tau,\, \tau + T]$, $\mu_k = y_\tau^{(k)}(t\tau+) - y_\tau^{(k)}(t\tau-)$ and δ is the Dirac delta function. Using the algebraic framework of [7], it has been shown in [4] that the change-point (spiking instant) t_τ can be characterized by

$$a_0(\tau)t_\tau^2 - 2a_1(\tau)t_\tau + a_2(\tau) = 0, \tag{1}$$

where each coefficient $a_i(\tau)$, $i \in \{0,1,2\}$, is in the form $a_i(\tau) = \int_0^T p_i(t)y(t+\tau)dt$ for some polynomial p_i. A key point in [4] is that this representation is valid only when y has a change-point in $[\tau, \tau + T]$, otherwise the coefficients $a_i(\tau)$ vanish (in the noise free case). This observation plus the presence of noise led the authors of [4] to consider the following statistical hypothesis testing problem

$$\mathcal{H}_0: \;\; X(\tau) \leq \gamma \quad vs \quad \mathcal{H}_1: \;\; X(\tau) > \gamma$$

where the decision function $X(\tau)$ defined by the discriminant of (1), as in

$$X(\tau) = a_1(\tau)^2 - a_0(\tau)a_2(\tau). \tag{2}$$

Now, one can infer from the form of the coefficients $a_i(\tau)$, that X reads as a Volterra filtered version of the neural signal y (see [4] for more details). A spike occurrence is then meant when $X(\tau)$ is above a given threshold γ (under the alternative hypothesis $\mathcal{H}_1$). Henceforth, we model the decision function as a continuous processth $X = (X_t)_{t\in\mathbb{R}^+}$ on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and consider a n-sample $(X_i)_{i=1,\cdots,n}$. To determine the threshold γ for a given pfa p, we need to determine the expression of the probability of false alarm. In [4], it is determined as

$$p = \mathbb{P}(X > \gamma \mid \mathcal{H}_0) \tag{3}$$

Since the statistical characteristics of X are unknown, the determination of the probability (3) becomes difficult. To overcome this issue, the algebraic approach proceeds as follows: knowing that the signal under the null hypothesis $\mathcal{H}_0$ is reduced to noise, the probability (3) is calculated by estimating the noise cdf of the output Volterra filter. But as the determined probability is the noise probability, the threshold γ will depend only on the noise variance and not on signal characteristics. That causes unsatisfactory results. Hence, to improve the spike detection and to make it more reliable, we propose to combine the algebraic method with an EVT approach.

2.2. Threshold determination method for a fixed level of false alarm

In this paragraph, we present a new way to determine a decision threshold for a given pfa which does not require anymore to estimate the cdf of X. We consider the spike occurrence in the neural signal as a rare extreme event. Therefore, instead of estimating the entire distribution of X, we will restrict our attention to the estimation of the distribution of the extremes (spikes) above some high threshold u (to be determined), *i.e.* the tail distribution of X. So let Y_u define the process of exceedances of X over a given threshold u

$$Y_u(j) = (X_j - u | X_j > u), \quad \text{for } 1 \leq j \leq N_u,$$

where N_u is the number of exceedances above u. We will first focus on the estimation of the distribution of Y_u, then we will define a decision criterion for the detection of a spike. The criterion will be related on the exceedance of Y_u over a second threshold η, which will be determined using the following modeling of the appearance of a false alarm

False alarm modelling

We model the number of spike occurrences in a finite time interval of length t by a Poisson process $\{N(t); t \geq 0\}$ with intensity λ. Therefore, the waiting time, say T, between two successive spikes obeys the exponential distribution with the same parameter λ. Let $(T_i; 1 \leq i \leq k)$ be a k-sample. Unlike in [4], we choose here another way to define a false alarm. We denote by r_p the refractory period following each AP of a given neuron. Knowing that in this period no spike can be fired by the same neuron, a false alarm will be meant when the excess over u overtakes the threshold η when the condition $\{T < r_p\}$ is realized. Hence, the probability of false alarm is defined as

$$p = \mathbb{P}(Y_u > \eta, T < r_p) \tag{4}$$

The amplitude of a spike is not related to its time occurrence. Therefore, we can assume the independence between Y_u and T. Thus (4) becomes

$$p = \mathbb{P}(Y_u > \eta)\mathbb{P}(T < r_p) \tag{5}$$

GPD fit

To determine η, we need to estimate the distribution of Y_u. This distribution, named the excess distribution function, is given by

$$F_u(x) = \mathbb{P}(Y_u \leq x) = \mathbb{P}(X - u \leq x \mid X > u), \quad \forall x \geq 0.$$

Using the Pickands theorem [6], F_u can be approximated for a high threshold u by a Generalized Pareto Distribution (GPD) $G_{\xi,\sigma}$ with shape parameter ξ and scale parameter σ:

$$F_u \approx G_{\xi,\sigma}$$

with

$$G_{\xi,\sigma}(x) = \begin{cases} 1-(1+\frac{\xi}{\sigma}x)^{-\frac{1}{\xi}}, & \text{if } \xi \neq 0 \\ 1-exp(-\frac{x}{\sigma}), & \text{if } \xi = 0 \end{cases} \quad x \in \mathcal{D}(\xi,\sigma)$$

where

$$\mathcal{D}(\xi,\sigma) = \begin{cases} [0,\infty), & \text{if } \xi \geq 0 \\ [0,-\frac{\sigma}{\xi}], & \text{if } \xi < 0 \end{cases}$$

Mean Excess Plot

It remains to identify a threshold u above which this approximation holds. For this purpose, we use a graphical tool called the Mean Excess Plot (MEP) [8, 9] which helps to find an appropriate u above which the exceedances follow a GPD. Recall that the mean excess function of a random variable X is defined by $e(u) = E(X-u|X>u)$. When X follows a GPD, the mean excess function is linear in u and expressed as

$$e(u) = \frac{\sigma}{1+\xi} + \frac{\xi}{1+\xi}u$$

Hence, to determine u, we draw the empirical mean excess function defined by

$$e_n(u) = \frac{\sum_{i=1}^{n}(X_i-u)\mathbb{I}_{(X_i>u)}}{\sum_{i=1}^{n}\mathbb{I}_{(X_i>u)}}, \quad u \geq 0 \tag{6}$$

and we select the value u from which the behavior of (6) is linear. Here $\mathbb{I}_S$ denotes the indicator function of the set S.

Parameters estimation

Once the threshold u is determined, the shape and the scale parameters of the approximating GPD are estimated, using for instance the moments method [10], and are given, respectively, by

$$\widehat{\xi} = \frac{1}{2}\left(1-\frac{\overline{Y_u}^2}{S^2_{Y_u}}\right) \quad \text{and} \quad \widehat{\sigma} = \overline{Y_u}\left(\frac{1}{2}+\frac{\overline{Y_u}^2}{S^2_{Y_u}}\right),$$

where $\overline{Y_u}$ and $S^2_{Y_u}$ are the sample mean and variance of the exceedances expressed as

$$\overline{Y_u} = \frac{1}{N_u}\sum_{i=1}^{N_u} Y_u(i) \text{ and } S^2_{Y_u} = \frac{1}{N_u-1}\sum_{i=1}^{N_u}\left(Y_u(i)-\overline{Y_u}\right)^2.$$

If the condition $\xi < 1/4$ is satisfied, it can be shown by standard methods (see *e.g.* [11]) that $\widehat{\sigma}$ and $\widehat{\xi}$ are asymptotically normally distributed with covariance matrix A satisfying

$$N_u A \sim \Gamma = \frac{(1-\xi)^2}{(1-2\xi)(1-3\xi)(1-4\xi)}\,(a_{ij})_{1\leq i,j\leq 2},$$

$$\begin{aligned} \text{with} \quad a_{11} &= 2\sigma^2(1-6\xi+12\xi^2), \\ a_{22} &= (1-2\xi)^2(1-\xi+6\xi^2), \\ a_{12} &= a_{21} = \sigma(1-2\xi)(1-4\xi-12\xi^2). \end{aligned}$$

In this case, a confidence interval with asymptotic confidence level α can be deduced

$$L \leq \begin{pmatrix} \sigma \\ \xi \end{pmatrix} \leq U$$

where the lower and the upper bounds denoted by L and U respectively of the confidence interval are given by

$$L = \begin{pmatrix} \widehat{\sigma} \\ \widehat{\xi} \end{pmatrix} + \left(\frac{1}{N_u}\Gamma\right)^{1/2}\begin{pmatrix} q((1-\alpha)/2) \\ q((1-\alpha)/2) \end{pmatrix}$$

$$U = \begin{pmatrix} \widehat{\sigma} \\ \widehat{\xi} \end{pmatrix} + \left(\frac{1}{N_u}\Gamma\right)^{1/2}\begin{pmatrix} q((1+\alpha)/2) \\ q((1+\alpha)/2) \end{pmatrix}$$

Here, $q(\kappa)$ is the standard normal distribution quantile of order κ.

As well, we determine the parameter λ of the exponential distribution using the moments method. It is expressed as $\widehat{\lambda} = \overline{T}^{-1}$ with the following asymptotic confidence interval for a confidence level α

$$\left[\overline{T}-\sqrt{\frac{S^2_T}{k}}q(\alpha/2)\right]^{-1} < \lambda < \left[\overline{T}-\sqrt{\frac{S^2_T}{k}}q(1-\alpha/2)\right]^{-1}$$

where $\overline{T}$ and $S^2_T = \frac{1}{k-1}\sum_{i=1}^{k}\left(T_i-\overline{T}\right)^2$ are respectively the sample mean and variance of T.

Application for probability of false alarm determination

At this stage, we know that Y_u follows approximately a GPD with parameters $\widehat{\xi}$ and $\widehat{\sigma}$, and that T obeys an exponential law with parameter $\widehat{\lambda}$. Coming back to (5), we obtain

$$\begin{aligned} p &= (1-G_{\widehat{\xi},\widehat{\sigma}}(\eta))\mathbb{P}(T<r_p) \\ &= \left(1+\frac{\widehat{\xi}}{\widehat{\sigma}}\eta\right)^{-\frac{1}{\widehat{\xi}}}\left(1-e^{-\widehat{\lambda}r_p}\right) \end{aligned}$$

from which we deduce the threshold η, expressed as

$$\eta = \frac{\widehat{\sigma}}{\widehat{\xi}}\left[\left(\frac{p}{1-e^{-\widehat{\lambda}r_p}}\right)^{-\widehat{\xi}}-1\right]$$

Recall that Y_u models the exceedances of X above u. Thus the threshold η is obtained relatively to u and not to zero. Consequently, the final threshold $\mathcal{T}$ above which the false alarms appear with a probability p is given by

$$\mathcal{T} = \eta + u \tag{7}$$

3. APPLICATION OF THE PROPOSED THRESHOLD ON SIMULATED SIGNAL

To rigorously test the performance of the proposed threshold, we carry out simulations on synthesized signal. According to the AP morphology, we simulate a spike template. Using this template, we generate a spike train with maximal amplitude equal to M, describing a Poisson process and ensuring a refractory period of 2 *ms* as a duration after each synthesized AP. To make the simulation more realistic, we corrupt the generated signal by a noise with a fixed SNR defined as: $SNR = \frac{M}{\sigma}$, where σ is the standard deviation of the background noise. The used noise is extracted from real neural recording with high SNR [12]. Fig.1. illustrates the Volterra filtering of a generated neural signal with SNR=4. As it is noticed in this figure, using the algebraic approach, real spikes (red $*$) are successfully detected. But due to noise effect, an important number of false alarms are detected as well. Hence, the need of decision threshold determination.

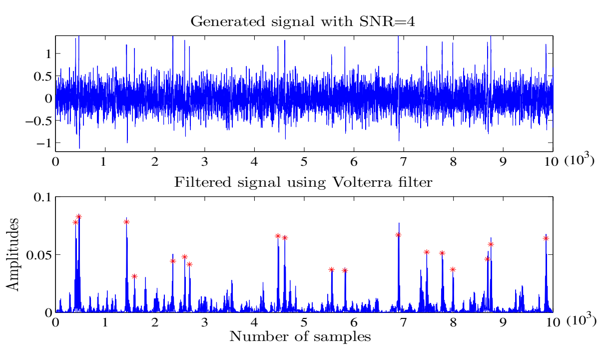

Fig. 1. Upper graph: generated neural signal with SNR=4. Lower graph: corresponding filtered signal using the algebraic approach where the real spikes are marked by red stars

3.1. MEP for GPD estimation for a certain threshold u

Fig. 2. depicts the use of the MEP method for threshold determination using two generated neural signals with different SNR. For SNR=4, the graph (c) shows the MEP where a zoomed part is given in the right top. Through this zoom we can notice that the MEP admits an oscillatory curve containing several linear behaviors. This implies that the choice of the threshold is not unique: it can be u_1 or u_2 or some other threshold. To ensure the best fit of GPD we choose the threshold that minimizes the distance d between the GPD and the empirical cdf of exceedances according to the infinity norm. Using the optimal threshold, the plot of these two distributions are shown in the graph (d) where we can see that the GPD seems to fit pretty well the empirical distribution. The analogous results for SNR=8 are presented in (a) and (b). Some results of the threshold choice in terms of distance between the estimated GPD and the corresponding empirical cdf are illustrated in the Table 1, where the optimal ones are given in the gray boxes.

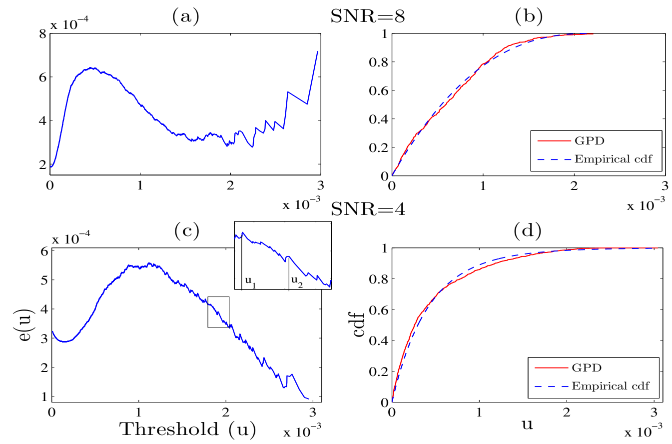

Fig. 2. MEP for GPD estimation

Table 1. Threshold determination

Thresholds	α s.t. $u = q_n(\alpha)$	N_u	$\parallel d \parallel_\infty$
Generated signal with SNR=8			
$2.4\ 10^{-4}$	84, 34%	1292	0.0832
$4.94\ 10^{-4}$	90.86%	753	0.0372
$7.57\ 10^{-4}$	93.50%	536	0.0542
Generated signal with SNR=4			
$5.27\ 10^{-4}$	84.77%	1330	0.058
$5.82\ 10^{-4}$	87.49%	1093	0.0479
$7.57\ 10^{-4}$	92.21%	680	0.06

The estimates of the parameters of the GPD and the exponential distributions are given with their confidence intervals (CI), for both signals, in Table 2.

Table 2. Simulation parameters of GPD and exponential distribution

Parameters	**Generated neural signals**	
	SNR=8	SNR=4
Samples	8247	8738
u	$4.943\ 10^{-4}$	$5.823\ 10^{-4}$
α s.t. $u = q_n(\alpha)$	90.86%	87.49%
N_u	753	1093
$\widehat{\xi}$ / $CI(\xi)$	-0.406 / $[-0.409,\ -0.403]$	0.075 / [0.073, 0.077]
$\widehat{\sigma}$ / $CI(\sigma)$	$9\ 10^{-4}$ / $[8.65,\ 9.35]\ 10^{-4}$	$4.17\ 10^{-4}$ / $[3.77,\ 4.57]\ 10^{-4}$
$\widehat{\lambda}$ / $CI(\lambda)$	0.0037 / [0.0035 0.004]	0.0054 / [0.0051 0.0057]

3.2. Interpretation and comparison

Using a generated neural signal with SNR=8, and for $p = 0.1$, we show in Fig. 3 the improvement of spike detection results using the proposed threshold. Assuming that the neural noise is Gaussian distributed, one can show [4] that X in (2) may be approximated by the sum of two χ^2 variables. Using this approximation, we deduce from (3), the threshold γ. The horizontal red solid line represents this value of γ. Next, we compute the empirical pfa, denoted by epfa$_\gamma$, according to γ. We obtain epfa$_\gamma = 0.811$: This is much higher than the prescribed value $p = 0.1$, showing that the assumed model of X under $\mathcal{H}_0$ is not adequate. For the same prescribed pfa $p = 0.1$, the black dashed line represents the proposed threshold $\mathcal{T}$ in (7). This threshold leads to a more realistic empirical pfa epfa$_\mathcal{T} = 0.14$ which is very close to the fixed value.

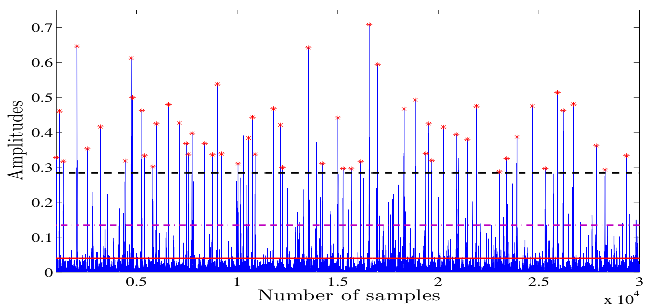

Fig. 3. Spike detection using the proposed threshold (black dashed line) compared to the algebraic approach threshold (red solid line) for $p = 0.1$. The purple point dashed line represents the threshold obtained using the MEP

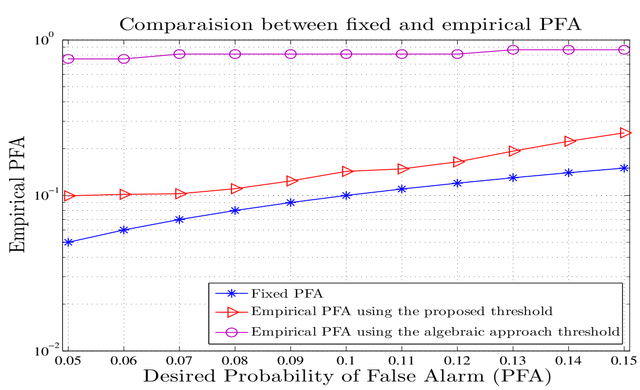

Fig. 4. Comparison between fixed pfa (blue $*$) and empirical pfa using the proposed threshold (red $\triangleright$) and the threshold described in [4] (purple $\circ$).

To highlight furthermore the performance of the proposed threshold, we calculate epfa$_\gamma$ and epfa$_\mathcal{T}$ for different values of pfa: p varies from 0.05 to 0.15. Note that the maximum value of p is obtained when $\eta = 0$, *i.e.* $\mathbb{P}(Y_u > \eta) = 1$. Therefore, from equation (5) the maximum value of p is reached for $\mathbb{P}(T \leq r_p)$. Simulation results are illustrated in the Fig. 4. We notice that the curve describing epfa$_\mathcal{T}$ (red $\triangleright$) is closer to the prescribed pfa (blue $*$), while the difference between this latter and the curve of epfa$_\gamma$ (purple $\circ$) is obvious. This improvement is due to the fact that using the proposed threshold, there is no need to go through the estimation of the unknown distribution of the decision function X. However, based on EVT, we have determined the distribution of extreme values representing spikes which is known based on the Pickands theorem. There is a minor difference between real and empirical probability of false alarm of order of 10^{-2} induced by the estimation of GPD and exponential parameters.

4. CONCLUSION

This paper develops a new spike decision threshold for the algebraic approach based on the extreme value theory. Applying the algebraic approach for detection and using the extreme value theory for making decision offers a robust and efficient spike detection. Comparing the obtained results with those achieved using the algebraic approach threshold, demonstrates the performance of the proposed threshold in terms of false alarm, giving results more realistic in accordance with what is predicted in theory.

5. REFERENCES

[1] B. S. Gutkin, B. Ermentrout, and M. Rudolph, "Spike generating dynamics and the conditions of spike-time precision in cortical neurons," *J. of Comput. Neuroscience*, vol. 15(1), pp. 91–103, 2003.

[2] K.H.Kim and S.J.Kim, "Neural spike sorting under nearly 0 db signal-to-noise ratio using nonlinear energy operator and artificial neural network classifier," *Biomedical Engineering, IEEE Transactions*, vol. 47, pp. 1406 –1411, 10 , Oct. 2000.

[3] Z. Nenadic, "Spike detection using the continuous wavelet transform," *IEEE Trans. biomedical engineering*, vol. 52, no. 1, pp. 74–87, 2005.

[4] Z. Tiganj M. Mboup, *A Volterra filter for neuronal spike detection*, prepreint version, "first version available on HAL INRIA at: http://hal.inria.fr/docs/00/67/66/06/PDF/spike_HAL.pdf.

[5] T. Mikosch P. Embrechts, C. Klppelberg, *Modelling Extremal Events for Insurance and Finance*, Springer-Verlag, 1997.

[6] J. Pickands, "Statistical inference using extreme order statistics," *Annals of Statistics*, vol. 3, pp. 119–131, 1975.

[7] M. Fliess and H. Sira-Ramírez, "An algebraic framework for linear identification," *ESAIM: COCV*, vol. 9, pp. 151–168, 2003.

[8] S. Ghosh and S. Resnick, "A discussion on mean excess plots," *Stochastic Process and their Applications*, vol. 3, pp. 1492–1517, 2010.

[9] Y. Demichel, A. Estrade, M. Kratz, and G. Samorodnitsky, "How fast can the chord-length distribution decay," *Advances in Applied Probability*, vol. 43, pp. 504–523, 2011.

[10] J. Hosking and J. Wallis, "Parameter and quantile estimation for the generalized pareto distribution funcrion," *Technometric*, vol. 29 (3), pp. 339–349, 1987.

[11] Rao. C. R, *linear statistical inference and its applications*, New York: John Wiley, 1973.

[12] Christophe Pouzat, *Methods and Models in Neurophysics. Les Houches 2003 Summer School.*, chapter Technique(s) for Spike - Sorting, pp. 729–786, Elsevier, 2005, Available from: http://fr.arxiv.org/abs/q-bio.QM/0405012.