GADTs meet Subtyping

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A reminder on GADTs

GADTs are algebraic data types that may carry *type equalities*. Think of the following simple type:

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  | Bool of bool
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  | Bool of bool with α = bool
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```

We can now write the following:

```
let eval : ∀α. α expr → α = function
  | Int n -> n        (* α = int *)
  | Bool b -> b       (* α = bool *)
```

Motivating variance

Subtyping: $\sigma \leq \tau$ means "all values of σ are also values of τ".
Checked by set of decidable and incomplete inference rules.

$$\frac{\sigma_1 \geq \sigma_1' \qquad \sigma_2 \leq \sigma_2'}{(\sigma_1 \to \sigma_2) \leq (\sigma_1' \to \sigma_2')}$$

Variance annotations lift subtyping to type parameters.

$$\texttt{type (-}\alpha\texttt{, =}\beta\texttt{, +}\gamma\texttt{) t = } (\alpha * \beta) \to (\beta * \gamma)$$

$$\frac{\alpha \geq \alpha' \qquad \beta = \beta' \qquad \gamma \leq \gamma'}{(\alpha, \beta, \gamma)\ \texttt{t} \leq (\alpha', \beta', \gamma')\ \texttt{t}}$$

For simple types, this is easy to check.

Variance for GADT: harder than it seems

Ok?

type $+\alpha$ expr =
  | Val : $\forall\alpha.\ \alpha \to \alpha\ \texttt{expr}$
  | Prod : $\forall\beta\gamma.\ \beta\ \texttt{expr} * \gamma\ \texttt{expr} \to (\beta * \gamma)\ \texttt{expr}$

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type $+\alpha$ expr =
  | Val : $\forall\alpha.\ \alpha \rightarrow \alpha$ expr
  | Prod : $\forall\beta\gamma.\ \beta$ expr $* \gamma$ expr $\rightarrow (\beta * \gamma)$ expr

And this one?

```
type file_descr = private int (* file_descr ≤ int *)
val stdin : file_descr

type +α t =
  | File : file_descr -> file_descr t
let o = File stdin in
let o' = (o : file_descr t :> int t)
```

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It's unsound!

let project : $\forall\alpha.\ \alpha$ t $\rightarrow (\alpha \rightarrow$ file_descr$)$ = function
  | File _ -> (fun x -> x)
project o' : int -> file_descr

Upward closure

```
type +α expr =
  | Val : ∀α. α → α expr
  | Prod : ∀βγ. β expr ∗ γ expr → (β ∗ γ) expr
```

When $\sigma \leq \sigma'$, I know it's safe to assume σ expr $\leq \sigma'$ expr.
Because I could *almost* write that conversion myself.

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let coerce (α ≤ α′) : α expr ≤ α′ expr = function
  | Val (v : α) -> Val (v :> α′)
  | Prod β γ ((b, c) : β expr ∗ γ expr) ->
      (* α = (β ∗ γ), α ≤ α′; Prod? *)
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      (* α = (β ∗ γ), α ≤ α′; Prod? *)
      (* if β ∗ γ ≤ α′, then α′ is of the form
           β′ ∗ γ′ with β ≤ β′ and γ ≤ γ′ *)
    Prod β′ γ′ ((b :> β′ expr), (c :> γ′ expr))
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Upward closure

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type +α expr =
  | Val : ∀α. α → α expr
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      (* if β * γ ≤ α', then α' is of the form
           β' * γ' with β ≤ β' and γ ≤ γ' *)
     Prod β' γ' ((b :> β' expr), (c :> γ' expr))
```

Upward closure for $\tau[\overline{\alpha}]$:

If $\tau[\overline{\sigma}] \leq \tau'$, then τ' is also of the form $\tau[\overline{\sigma}']$ for some $\overline{\sigma}'$.

Holds for $\alpha * \beta$, but fails for `file_descr`.

A conflict

Conversely, to be contravariant, a GADT parameter need be instantiated with types τ that are *downward-closed*: if $\tau' \leq \tau[\overline{\sigma}]$, then τ' is also of the form $\tau[\overline{\sigma}']$ for some $\overline{\sigma}'$.

But this will not work in presence of `private` types: for any τ we can define a distinct type $\tau' := \texttt{private}\ \tau$ with $\tau' \leq \tau$.

Are GADT contravariance and private types incompatible?

Closed-world vs. open-world

Think about a subtyping fact $\sigma \leq \tau$ as a *knowledge* about the world (of types). Private types allow to introduce, at any point in time, a new subtyping relation. You learn something new about the world.

Closure properties are a negative property: a bunch of subtyping relations *must not exist* for some variance annotation to be correct. Checking it makes a *closed world* assumption.

Closed-world vs. open-world

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Types internal to a module live in a closed world. For the exposed ones, it is also possible to resolve this tension by letting the programmer *request* that some subtyping relations *will never hold*.

```
type t = downward-closed
  | Foo ...
  | Bar ...
```

A private synonym of `t` would be rejected by the compiler.
(In other words, `t` is not privatizable. Socialist France striking again.)

An interesting problem

We have to *restrict* subtyping of private types to *enrich* it for GADTs.
A difficult design tension.

It can be compared to the use of `final` classes in object-oriented programming languages.

Going technical : how we prove these things

We must reject unsound GADT definitions. We need an algorithm to tell whether a given type is {upward,downward}-closed.

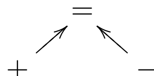
We will first explain how to check variance of type variables by a judgment $\Gamma \vdash \tau : v$.
Resembles previous work [Emir et al., 2006] and [Abel, 2006], with a twist.

We will then extend it to a judgment $\Gamma \vdash \tau : v \Rightarrow v'$ to check closure properties.

With all that and a little plumbing, we can use a soundness proof from the literature [Simonet and Pottier, 2007].
(We got the decomposability criterion by going backward from S&P.)

Variances

- $+$: only positive occurences
- $-$: only negative occurences
- $=$: both positive and negative



$$
\begin{array}{lcl}
\sigma \prec_{+} \tau & := & \sigma \leq \tau \\
\sigma \prec_{-} \tau & := & \sigma \geq \tau \\
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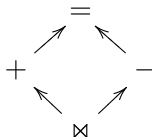

- $=$: both positive and negative
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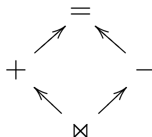

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\end{aligned}
$$

If α has variance v in $(\alpha\ \texttt{t})$, and β variance w in $(\beta\ \texttt{u})$, what is the variance of α in $((\alpha\ \texttt{t})\ \texttt{u})$?

$v.w$	$=$	$+$	$-$	$\bowtie$
$=$	$=$	$=$	$=$	$\bowtie$
$+$	$=$	$+$	$-$	$\bowtie$
$-$	$=$	$-$	$+$	$\bowtie$
$\bowtie$	$\bowtie$	$\bowtie$	$\bowtie$	$\bowtie$

(rows: v; columns: w)

$\Gamma \vdash \tau : v$

We manipulate contexts Γ of variables with variances: $(-\alpha, =\beta, +\gamma)$.
$\Gamma \vdash \tau : v$ means that "if the variables vary according to their variance, τ varies along v".

$$-\alpha, =\beta, +\gamma \;\vdash\; (\alpha * \beta) \rightarrow (\beta * \gamma) \;:\; (+)$$

$$=\alpha, =\beta, =\gamma \;\vdash\; (\alpha * \beta) \rightarrow (\beta * \gamma) \;:\; (=)$$

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$$\frac{w\alpha \in \Gamma \qquad w \geq v}{\Gamma \vdash \alpha : v} \qquad\qquad \frac{\Gamma \vdash \overline{w\alpha}\;\mathtt{t} \qquad \forall i,\ \Gamma \vdash \sigma_i : v.w_i}{\Gamma \vdash \overline{\sigma}\;\mathtt{t} : v}$$

For instance, in the arrow case:

$$\frac{\Gamma \vdash \sigma_1 : v.- \qquad \Gamma \vdash \sigma_2 : v.+}{\Gamma \vdash (\sigma_1 \to \sigma_2) : v}$$

Closure properties in depth

In our system, $\alpha * \beta$ is upward-closed. This is because the head type constructor, $(*)$, is closed.

For $\alpha\ \mathtt{t} \rightarrow (\beta * \gamma)$ to be upward-closed, $\alpha\ \mathtt{t}$ must be downward-closed. In the general case, we recursively check closure, according to variance.

What about variables?

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[Reminder] *Upward closure*: If $\tau[\overline{\sigma}] \leq \tau'$, then $\tau' = \tau[\overline{\sigma}']$ for some $\overline{\sigma}'$.

$\beta * \beta$ is not closed : $(\texttt{file_descr} * \texttt{file_descr}) \leq (\texttt{file_descr} * \texttt{int})$.
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We capture those subtleties through a partial variance operation $v_1 \curlywedgeuparrow v_2$.
Defined only when two occurences at variances v_1 and v_2 can be soundly combined.

$v \curlywedgeuparrow w$ (rows: v, columns: w)	$=$	$+$	$-$	$\bowtie$
$=$	$=$			$=$
$+$				$+$
$-$				$-$
$\bowtie$	$=$	$+$	$-$	$\bowtie$

$\Gamma \vdash \tau : v \Rightarrow v'$

We can finally extend the judgment $\Gamma \vdash \tau : v$ to capture closure properties. We want to say that $\Gamma \vdash \tau$ is v-closed if:

$$\forall \tau', \overline{\sigma},\ \tau[\overline{\sigma}] \prec_v \tau' \implies \exists \overline{\sigma}',\ \tau[\overline{\sigma}'] = \tau'$$

We need a generalization:

$$\forall \tau', \overline{\sigma},\ \tau[\overline{\sigma}] \prec_v \tau' \implies \exists \overline{\sigma}',\ \tau[\overline{\sigma}'] \prec_{v'} \tau'$$

This is our $\Gamma \vdash \tau : v \Rightarrow v'$ judgment.

Inference rules for the show

They rely on closure information for type constructors, and $\curlywedge$ to merge contexts of subterms.

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$$\dfrac{v \geq v' \qquad \Gamma \vdash \tau : v}{\Gamma \vdash \tau : v \Rightarrow v'}\;\textsc{Triv} \qquad \dfrac{w\alpha \in \Gamma \qquad w = v}{\Gamma \vdash \alpha : v \Rightarrow v'}\;\textsc{Var}$$

$$\dfrac{\Gamma \vdash \overline{w\alpha}\ \mathtt{t} : v\text{-closed} \qquad \Gamma = \curlywedge_i \Gamma_i \qquad \forall i,\ \Gamma_i \vdash \sigma_i : v.w_i \Rightarrow v'.w_i}{\Gamma \vdash \overline{\sigma}\ \mathtt{t} : v \Rightarrow v'}\;\textsc{Constr}$$

Another solution

We showed, through hard work, how to check that equality constraints are upward-closed.

With subtyping in constructor types, variance is easy to check

```
type +α expr =
| Val : ∀β ≥ α. β → α expr
| Prod : ∀βγ [α ≥ (β * γ)]. β expr * γ expr → α expr
```

Now pattern-matching only knows a subtyping relation:

```
let eval : ∀α. α expr → α = function
  | Int n -> (n :> α)          (* α ≥ int *)
  | Bool b -> (b :> α)         (* α ≥ bool *)
```

Less convenient: use of subtyping must be annotated explicitly, while equations where implicit.

Not presented here:

- GADTs with subtyping rather than equalities, less closure constraints
- What if we had a symmetric for `private`?

Future work:

- Completeness of the S&P criterion: type inhabitation.
- Verify behavior through type abstraction.
- Does this also happen with inductive dependent types?

Conclusion

GADT variance checking: suprisingly less obvious than we thought.

Not anecdotal: raises deeper design questions (open vs. closed world).

We have a sound criterion that can be implemented easily in a type checker.

Bonus Slide: Variance and the value restriction

```
type (='a) ref = { mutable contents : 'a }
```

In a language with mutable data, generalizing any expression is unsafe (because you may generalize data locations):

```
# let test = ref [];;
val test : '_a list ref
```

Solution (Wright, 1992): only generalize values (`fun () -> ref []`, or `[]`).

Painful when manipulating polymorphic data structures:

```
let test = id [] (* not generalized? *)
```

OCaml relies on variance for the *relaxed value restriction* [Garrigue, 2004]: covariant data is immutable, so covariant type variables may be safely generalized. Very useful in practice.

```
# let test = id [];;
val test : 'a list = []
```

Abel, A. (2006).
Polarized subtyping for sized types.
Mathematical Structures in Computer Science.
Special issue on subtyping, edited by Healfdene Goguen and Adriana Compagnoni.

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