

HOMOLOGICAL TECHNIQUES FOR THE ANALYSIS OF THE DIMENSION OF TRIANGULAR SPLINE SPACES

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ABSTRACT. The spline space $C_k^r(\Delta)$ attached to a subdivided domain Δ of \mathbb{R}^d is the vector space of functions of class C^r which are polynomials of degree $\leq k$ on each piece of this subdivision. Classical splines on planar rectangular grids play an important role in Computer Aided Geometric Design, and spline spaces over arbitrary subdivisions of planar domains are now considered for isogeometric analysis applications. We address the problem of determining the dimension of the space of bivariate splines $C_k^r(\Delta)$ for a triangulated region Δ in the plane. Using the homological introduced by Billera (1988), we number the vertices and establish a formula for an upper bound on the dimension. There is no restriction on the ordering and we obtain more accurate approximations to the dimension than previous methods and furthermore, in certain cases even an exact value can be found. The construction makes also possible to get a short proof for the dimension formula when $k \geq 4r + 1$, and the same method we use in this proof yields the dimension straightaway for many other cases.

1. INTRODUCTION

Let Δ be a connected, finite two dimensional simplicial complex, supported on $|\Delta| \subset \mathbb{R}^2$, with $|\Delta|$ homotopy equivalent to a disk. We denote by $C_k^r(\Delta)$ the vector space of all C^r functions on Δ that, restricted to any simplex in Δ , are given by polynomials of degree less or equal than k . These functions are called splines, and they have many practical applications, including the finite element method for solving differential equations (Fix and Strang (2008)). Recently they have also been considered for isogeometric analysis applications (Cottrell et al. (2009)), in computer-aided design for modeling surfaces of arbitrary topology (Farin (1993)). In these application areas, spline functions of low degree are of particular interest, and the degree of smoothness attainable is an important design consideration.

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A fundamental problem is to determine the dimension of this vector space as a function of known information about the subdivision Δ . A serious difficulty for solving this problem is that the dimension of the space $C_k^r(\Delta)$ can depend not only on the combinatorics of the subdivision, but also on the geometry of the triangulation i.e., how $|\Delta|$ is embedded in \mathbb{R}^2 . In Schumaker (1984), the author presented a lower and an upper bound on the dimension of spline spaces of arbitrary degree and smoothness for general triangulations; using Bernstein-Bézier methods, a formula for the dimension for $k \geq 4r + 1$ was obtained by Alfeld and Schumaker (1987). The result was extended to $k \geq 3r + 2$ in Hong (1991), and in Alfeld and Schumaker (1990) a dimension formula is proved for almost all triangulations for $k = 3r + 1$. However, there are no explicit formulas for the dimension of the spline spaces $C_k^r(\Delta)$ for degree $k < 3r + 2$ for general triangulations. The use of homological algebra in spline theory was introduced by Billera (1988). He obtained the dimension of $C_k^1(\Delta)$ for all k for generic triangulations. Schenck and Stillman (1997a), introduced a chain complex different from that used by Billera; this complex was studied by Schenck and Stillman (1997a), Schenck and Stillman (1997b), Schenck (1997), and Geramita and Schenck (1998). The lower homology modules of the chain complex in this construction, differ from the one introduced by Billera, and they have nicer properties. The connection between fat points and the spline space defined on Δ , allows to give a complete characterization of the free resolutions of the ideals generated by power of linear forms appearing in the chain complex, and so to prove the dimension formula for $C_k^r(\Delta)$ for sufficiently high degree (Schenck and Stillman (1997b)).

The main contribution of the paper is a new formula for an upper bound on the dimension of the spline space. The formula applies to any ordering established on the interior vertices of the partition, contrarily to the upper bound formulas in Schumaker (1984), Lai and Schumaker (2007). Having no restriction on the ordering makes it possible to obtain accurate approximation to the dimension and even exact value in many cases. As a consequence, we give a simple proof for dimension formula when $k \geq 4r + 1$.

The paper is structured as follows. In Section 2 we recall the construction of this chain complex and some of the properties of the homology modules. We describe the dimension as a sum of a combinatorial part and the dimension of an homology module. This latter term which happens to be zero in high degree is always non-negative, so the formula gives a lower bound for the dimension of the spline space for any degree k . We present the main result in Section 3, which is an upper bound on the dimension of the spline space. In Section 4 we

compare the formulas of the lower and the upper bound with those appearing in Schumaker (1984). The result about the exact dimension for $k \geq 4r + 1$ is proved in Section 5. This latter result and some other examples that we present in Section 6, illustrate the interest of the homology construction for proving exact dimension formulas.

2. CONSTRUCTION OF THE CHAIN COMPLEX

We reproduce some notations and definitions presented in Schenck and Stillman (1997a), restricting them to the case where Δ is a planar simplicial complex supported on a disk.

Denote by Δ^0 the set of interior faces of Δ , and by Δ_i^0 ($i = 0, 1, 2$) the set of i -dimensional interior faces whose support is not contained in the boundary of $|\Delta|$. We denote by f_i^0 the cardinality of these sets, and by $\partial\Delta$ the complex consisting of all 1-faces lying on just one 2-face as well as all subsets of them. As it will be convenient to study the dimension of the vector space $C_k^r(\Delta)$, we embed Δ in the plane $\{z = 1\} \subseteq \mathbb{R}^3$ and form the cone $\hat{\Delta}$ over Δ with vertex at the origin. Denote by $C_k^r(\hat{\Delta})$ the set of splines on $\hat{\Delta}$ of C^r -smoothness and degree exactly k . Then $C^r(\hat{\Delta}) := \bigoplus_{\geq 0} C_k^r(\hat{\Delta})$ is a graded \mathbb{R} -algebra and there is an isomorphism of \mathbb{R} -vector spaces between $C_k^r(\Delta)$ and the elements in $C^r(\hat{\Delta})$ of degree exactly k (Billera and Rose (1991)), in particular

$$\dim C_k^r(\Delta) = \dim C_k^r(\hat{\Delta}).$$

Define $R := \mathbb{R}[x, y, z]$. For an edge $\tau \in \Delta_1^0$, let ℓ_τ denote a non-zero homogeneous linear form vanishing on $\hat{\tau}$, and define the ideal $\mathcal{J}(\beta)$ of R for each simplex $\beta \in \Delta^0$ as follows:

$$\begin{aligned} \mathcal{J}(\sigma) &= \langle 0 \rangle & \text{for each } \sigma \in \Delta_2^0 \\ \mathcal{J}(\tau) &= \langle \ell_\tau^{r+1} \rangle & \text{for each } \tau \in \Delta_1^0 \\ \mathcal{J}(\gamma) &= \langle \ell_\tau^{r+1} \rangle_{\tau \ni \gamma} & \text{for each } \gamma \in \Delta_0^0, \tau \in \Delta_1^0. \end{aligned}$$

Consider the chain complex \mathcal{R} defined on Δ^0 as $\mathcal{R}_i = R^{f_i^0}$, with the usual simplicial boundary maps $\overline{\partial}_i$ used to compute the relative (modulo $\partial\Delta$) homology with coefficients in R as in Billera (1988). Let \mathcal{R}/\mathcal{J} be the chain complex obtained as the quotient of \mathcal{R} by \mathcal{J} ,

$$0 \longrightarrow \bigoplus_{\sigma \in \Delta_2} \mathcal{R} \xrightarrow{\partial_2} \bigoplus_{\tau \in \Delta_1^0} \mathcal{R}/\mathcal{J}(\tau) \xrightarrow{\partial_1} \bigoplus_{\gamma \in \Delta_0^0} \mathcal{R}/\mathcal{J}(\gamma) \longrightarrow 0$$

where the maps ∂_i are the induced by the simplicial boundary maps $\overline{\partial}_i$. The complex \mathcal{R}/\mathcal{J} was introduced by Schenck and Stillman (1997a), and agrees with the complex studied by Billera except at the vertices.

It was shown in Billera (1988) that $C^r(\hat{\Delta})$ is isomorphic to the top homology module of \mathcal{R}/\mathcal{J}

$$H_2(\mathcal{R}/\mathcal{J}) := \ker(\partial_2).$$

Let us consider the short exact sequence of complexes

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{R} \longrightarrow \mathcal{R}/\mathcal{J} \longrightarrow 0$$

that gives rise to the long exact sequence of homology modules

$$\begin{aligned} 0 \rightarrow H_2(\mathcal{R}) \rightarrow H_2(\mathcal{R}/\mathcal{J}) \rightarrow H_1(\mathcal{J}) \rightarrow H_1(\mathcal{R}) \\ (1) \quad \rightarrow H_1(\mathcal{R}/\mathcal{J}) \rightarrow H_0(\mathcal{J}) \rightarrow H_0(\mathcal{R}) \rightarrow H_0(\mathcal{R}/\mathcal{J}) \rightarrow 0 \end{aligned}$$

Since Δ is supported on a disk, both $H_0(\mathcal{R})$ and $H_1(\mathcal{R})$ are zero. Hence, by the long exact sequence, it implies that $H_0(\mathcal{R}/\mathcal{J})$ is also zero and $H_1(\mathcal{R}/\mathcal{J})$ is isomorphic to $H_0(\mathcal{J})$. Applying the Euler characteristic equation to the chain complex \mathcal{R}/\mathcal{J}

$$\chi(H(\mathcal{R}/\mathcal{J})) = \chi(\mathcal{R}/\mathcal{J}),$$

and considering the modules in degree exactly k , leads to the formula

$$(2) \quad \dim C_k^r(\Delta) = \sum_{i=0}^2 (-1)^i \sum_{\beta \in \Delta_{2-i}^0} \dim \mathcal{R}/\mathcal{J}(\beta)_k + \dim H_0(\mathcal{J})_k.$$

The aim is to determine the modules in the previous formula as functions of known information about the subdivision Δ .

We know that

$$(3) \quad \sum_{\sigma \in \Delta_2^0} \dim \mathcal{R}_k = f_2^0 \binom{k+2}{2}$$

$$(4) \quad \sum_{\tau \in \Delta_1^0} \dim \mathcal{R}/\mathcal{J}(\tau)_k = f_1^0 \left[\binom{k+2}{2} - \binom{k+2-(r+1)}{2} \right].$$

For the computation of $\dim \mathcal{R}/\mathcal{J}(\gamma)_k$, Schenck and Stillman (1997b) proposed the following resolution for $\mathcal{R}/\mathcal{J}(\gamma)_k$. Without loss of generality we translate γ to the origin and assume that the linear forms in $\mathcal{J}(\gamma)$ involve only variables x, y . Letting $\ell_1^{r+1}, \dots, \ell_t^{r+1}$ be a minimal generating set for $\mathcal{J}(\gamma)$, then a free resolution for $\mathcal{R}/\mathcal{J}(\gamma)_k$ is given by:

$$(5) \quad 0 \rightarrow \mathcal{R}(-\Omega-1)^a \oplus \mathcal{R}(-\Omega)^b \rightarrow \oplus_{j=1}^t \mathcal{R}(-r-1) \rightarrow \mathcal{R} \rightarrow \mathcal{R}/\mathcal{J}(\gamma) \rightarrow 0$$

where $\Omega-1$ is the socle degree of $\mathcal{R}/\mathcal{J}(\gamma)$; Ω and the multiplicities a and b are given by

$$(6) \quad \Omega = \left\lfloor \frac{tr}{t-1} \right\rfloor + 1, \quad a = t(r+1) + (1-t)\Omega, \quad b = t-1-a.$$

In the case $t = 1$, we take $A = b = \Omega = 0$ so that $\mathcal{R}(-\Omega - 1)^a \oplus \mathcal{R}(-\Omega)^b = 0$. Applying this to each vertex $\gamma_i \in \Delta_0^0$, and letting t_i , Ω_i , a_i and b_i denote the values for t , Ω , a and b at γ_i respectively, leads to

$$(7) \quad \dim \bigoplus_{\gamma_i \in \Delta_0^0} \mathcal{R}/\mathcal{J}(\gamma_i)_k = \sum_{i=1}^{f_0^0} \left[\binom{k+2}{2} - t_i \binom{k+2-(r+1)}{2} + b_i \binom{k+2-\Omega_i}{2} + a_i \binom{k+2-(\Omega_i+1)}{2} \right].$$

Let us notice that the previous formula does not change when we take t_i as the number of different slopes of the edges containing the vertex γ_i . Then (3), (4), and (7), and the fact that $\dim H_1(\mathcal{R}/\mathcal{J})_k \geq 0$, yields the following theorem:

Theorem 2.1. *The dimension of $C_k^r(\Delta)$ is bounded below by*

$$\dim C_k^r(\Delta) \geq \binom{k+2}{2} + f_1^0 \binom{k+2-(r+1)}{2} - \sum_{i=1}^{f_0^0} \left[t_i \binom{k+2-(r+1)}{2} - b_i \binom{k+2-\Omega_i}{2} - a_i \binom{k+2-(\Omega_i+1)}{2} \right],$$

where t_i is the number of different slopes of the edges containing the vertex γ_i , and

$$\Omega_i = \left\lfloor \frac{t_i r}{t_i - 1} \right\rfloor + 1, \quad a_i = t_i(r+1) + (1-t_i)\Omega_i, \quad b_i = t_i - 1 - a_i.$$

In Schenck (1997), it was proved that the homology module $H_1(\mathcal{R}/\mathcal{J})_k$ vanishes for sufficiently high degree, thus the lower bound in the latter theorem is actually the exact dimension formula for $C_k^r(\Delta)$ when $k \gg 0$ (Geramita and Schenck, 1998, Theorem 4.2).

3. AN UPPER BOUND ON THE DIMENSION OF $C_k^r(\Delta)$

Let us fix an ordering $\gamma_1, \dots, \gamma_{f_0^0}$ for the vertices in Δ_0^0 . For each vertex γ_i , denote by $N(\gamma_i)$ the set edges that contain this vertex, and define \tilde{t}_i as the number of different slopes of the edges connecting γ_i to one of the first $i-1$ vertices in the list or to a vertex on the boundary.

Theorem 3.1. *The dimension of $C_k^r(\Delta)$ is bounded by*

$$\dim C_k^r(\Delta) \leq \binom{k+2}{2} + f_1^0 \binom{k+2-(r+1)}{2} - \sum_i \left[\tilde{t}_i \binom{k+2-(r+1)}{2} - \tilde{b}_i \binom{k+2-\tilde{\Omega}_i}{2} - \tilde{a}_i \binom{k+2-(\tilde{\Omega}_i+1)}{2} \right]$$

with \tilde{t}_i as we have defined above and

$$\tilde{\Omega}_i = \left\lfloor \frac{\tilde{t}_i r}{\tilde{t}_i - 1} \right\rfloor + 1, \quad \tilde{a}_i = \tilde{t}_i (r + 1) + (1 - \tilde{t}_i) \tilde{\Omega}_i, \quad \tilde{b}_i = \tilde{t}_i - 1 - \tilde{a}_i.$$

if $\tilde{t}_i > 1$ and $\tilde{a}_i = \tilde{b}_i = \tilde{\Omega}_i = 0$ if $\tilde{t}_i = 1$.

Proof. By the long exact sequence (1), and the fact that $H_1(\mathcal{R}) = 0$, we have the short exact sequence

$$0 \longrightarrow H_2(\mathcal{R}) \longrightarrow H_2(\mathcal{R}/\mathcal{J}) \longrightarrow H_1(\mathcal{J}) \longrightarrow 0$$

The Euler characteristic equation applied to the complex \mathcal{R} implies that $H_2(\mathcal{R})_k = R_k$. Hence the isomorphism between $C^r(\hat{\Delta})$ and the homology module $H_2(\mathcal{R}/\mathcal{J})$ implies that

$$\dim C_k^r(\Delta) = \dim R_k + \dim H_1(\mathcal{J})_k$$

where the complex of ideals \mathcal{J} (as defined above) is given by

$$0 \longrightarrow \bigoplus_{\tau \in \Delta_1^0} \mathcal{J}(\tau) \xrightarrow{\partial_1} \bigoplus_{\gamma \in \Delta_0^0} \mathcal{J}(\gamma) \longrightarrow 0$$

and where $H_1(\mathcal{J}) = \ker \partial_1 = K_1$. Define $W_1 := \text{Im}(\partial_1)$. By the exact sequence

$$0 \longrightarrow K_1 \longrightarrow \bigoplus_{\tau \in \Delta_1^0} \mathcal{J}(\tau) \longrightarrow W_1 \longrightarrow 0$$

we get

$$\dim C_k^r(\Delta) = \dim R_k + \sum_{\tau \in \Delta_1^0} \dim \mathcal{J}(\tau)_k - \dim(W_1)_k$$

Therefore, to find an upper bound on $\dim C_k^r(\Delta)$ it is enough to find a lower bound on the dimension of $W_1 = \text{Im} \partial_1$ in degree k . Let us consider the map

$$\delta_1 : \bigoplus_{\tau=(\gamma, \gamma') \in \Delta_1^0} \mathcal{J}(\tau) [\tau] \rightarrow \bigoplus_{\gamma_i \in \Delta_0^0} \bigoplus_{\tau \in N(\gamma)} R[\tau|\gamma]$$

such that $\delta_1([\tau]) = [\tau|\gamma] - [\tau|\gamma']$ for $\tau = (\gamma, \gamma') \in \Delta_1^0$, and the map

$$\varphi_1 : \bigoplus_{\gamma_i \in \Delta_0^0} \bigoplus_{\tau \in N(\gamma)} R[\tau|\gamma] \rightarrow \bigoplus_{\gamma \in \Delta_0^0} R[\gamma]$$

with

$$\begin{aligned} \varphi_1([\tau|\gamma]) &= [\gamma] \text{ if } \gamma \in \Delta_0^0, \\ &= 0 \text{ if } \gamma \notin \Delta_0^0. \end{aligned}$$

Then, we have $\partial_1 = \varphi_1 \circ \delta_1$. We consider now the map

$$\pi_1 : \bigoplus_{\gamma \in \Delta_0^0} \bigoplus_{\tau \in N(\gamma)} R[\tau|\gamma] \rightarrow \bigoplus_{\gamma \in \Delta_0^0} \bigoplus_{\tau \in N(\gamma)} R[\tau|\gamma]$$

with $\pi_1([\tau|\gamma]) = 0$ if γ is the end point of biggest index of τ and $\pi_1([\tau|\gamma]) = [\tau|\gamma]$ otherwise. We denote by $\tilde{\partial}_1 = \varphi_1 \circ \pi_1 \circ \delta_1$.

For $\gamma \in \Delta_0^0$, we denote by $\tilde{N}(\gamma)$ the set of interior edges τ connecting γ to another vertex which is not of bigger index. Let $\tilde{\mathcal{J}}(\gamma) = \sum_{\tau \in \tilde{N}(\gamma)} R \ell_\tau^{r+1} \subset \mathcal{J}(\gamma)$. By construction, we have

$$\text{Im } \tilde{\partial}_1 = \bigoplus_{\gamma \in \Delta_0^0} \tilde{\mathcal{J}}(\gamma) [\gamma]$$

and $\dim(W_1)_k = \dim(\text{Im } \partial_1)_k \geq \dim(\text{Im } \tilde{\partial}_1)_k$.

Using formulas (3), (4) and the resolution (5) applied to $\tilde{\mathcal{J}}(\gamma)$, we obtain the upper bound

$$\begin{aligned} \dim C_k^r(\Delta) &\leq \binom{k+2}{2} + f_1^0 \binom{k+2-(r+1)}{2} \\ &- \sum_{i=1}^{f_0^0} \left[\tilde{t}_i \binom{k+2-(r+1)}{2} - \tilde{b}_i \binom{k+2-\tilde{\Omega}_i}{2} - \tilde{a}_i \binom{k+2-(\tilde{\Omega}_i+1)}{2} \right], \end{aligned}$$

with $\tilde{t}_i = |\tilde{N}(\gamma_i)|$ and $\tilde{\Omega}_i, \tilde{a}_i, \tilde{b}_i$ defined as in (6).

As an immediate consequence of this theorem we mention the following result.

Corollary 3.2. *If for a numbering on the vertices in Δ_0^0 , either $t_i = \tilde{t}_i$ or $\tilde{t}_i \geq r+1$ for every vertex γ_i , then the upper bound we get, equals the lower bound, obtaining so the exact dimension formula for the spline space.*

Proof. We compare the terms corresponding to each interior vertex in both formulas. If $t_i = \tilde{t}_i$ then they are trivially the same. By definition (6), for $t \geq r+1$, Ω and a are both constant equal to $r+1$. Hence if $r+1 \leq \tilde{t}_i (\leq t_i)$, the terms in the binomials are the same when we compare them in the formulas in Theorem 2.1 and 3.1. Since the respective terms in t_i and \tilde{t}_i cancel out then we get the equality of the bounds.

4. THE BOUNDS ON $\dim C_k^r(\Delta)$ GIVEN BY SCHUMAKER (1984)

In this section we compare the bounds on $\dim C_k^r(\Delta)$ found by Schumaker (1984), with the lower and upper bounds given in the previous

two sections. With the notation as before, the upper bound presented by Schumaker can be stated as follows.

Theorem 4.1. (*Schumaker, 1984, Theorem 2.1*) Suppose that the vertices $\gamma_i \in \Delta_0^0$ of the partition are numbered in such a way that each pair of consecutive vertices in the list are corners of a common triangle in Δ . For each γ_i define \tilde{t}_i as the number of edges with different slopes joining the vertex γ_i to a vertex in the boundary of Δ or to one of the first $i - 1$ vertices. Then

$$(8) \quad \dim C_k^r(\Delta) \leq \binom{k+2}{2} + f_1^0 \binom{k-r+1}{2} - f_0^0 \left[\binom{k+2}{2} - \binom{r+2}{2} \right] + \sum_{i=1}^{f_0^0} \sum_{j=1}^{k-r} (r+j+1-j \cdot \tilde{t}_i)_+.$$

In the same article, the author also presented a lower bound on $\dim C_k^r(\Delta)$. The formula can be obtained by replacing \tilde{t}_i by t_i in (8). We shall prove that that formula for the lower bound is the same as the formula we presented in Theorem 2.1. Essentially the same proof shows that the upper bound formulas, the one in the previous theorem and the one we presented in Theorem (3.1), coincide. With the exception that the upper bound presented by Schumaker can only be applied for certain numberings on the vertices. That restriction makes sometimes not possible to find an upper bound. In Section 6 we will include some examples of those situations, and also some cases when not having that restriction leads to find the exact dimension of the space.

Remark. The above cited article defined \tilde{t}_i as the number of slopes of the edges containing γ_i but not containing any of the $i - 1$ vertices in the list. This coincides with the definition of \tilde{t}_i we consider here, except that the reverse counting on the vertices in Δ_0^0 is used.

Lemma 4.2. Let $t \geq 2$ and Ω defined as in (6) above. The notations in the upper bound formulas (and the lower bounds, respectively) are linked in the following way: if $\Omega = r + \ell$ then $r + j + 1 - j \cdot t > 0$ only when $j \leq \ell - 1$.

Proof. For any value of $t \geq 2$ we have

$$1 < \frac{t}{t-1} \leq 2.$$

Thus, $r + 1 \leq \Omega \leq 2r + 1$, and we can write $\Omega = r + \ell$ for some integer ℓ between 1 and $r + 1$. The interval of t where the value of Ω

is $r + \ell$ is given as follows:

$$(9) \quad \begin{cases} \frac{r+\ell}{\ell} < t \leq \frac{r+(\ell-1)}{\ell-1} & \text{when } \ell \geq 2 \\ t > r + 1 & \text{when } \ell = 1. \end{cases}$$

On the other hand, for fixed r and t , the condition

$$(10) \quad r + j + 1 - j \cdot t \geq 1$$

is satisfied if and only if $t \leq (r + j)/j$. Thus, the biggest number j subject to condition (10) must satisfy

$$\frac{r + (j + 1)}{j + 1} < t \leq \frac{r + j}{j}.$$

From (9), the previous relation holds if and only if $\Omega = r + (j + 1)$. Let $t \geq 2$ and Ω , a and b be defined as in (6). We consider the following two formulas:

$$(11) \quad t \binom{k+2-(r+1)}{2} - b \binom{k+2-\Omega_i}{2} - a \binom{k+2-(\Omega+1)}{2}$$

$$(12) \quad \binom{k+2}{2} - \binom{r+2}{2} - \sum_{j=1}^{k-r} (r + j + 1 - j \cdot t)_+.$$

Lemma 4.3. *If $t > r + 1$ and $k \geq r$ then the formulas (11) and (12) are equal.*

Proof. By (9) we know that if $t > r + 1$ one has $\Omega = r + 1$. From the definition in (6), we have $a = r + 1$ and $b = t - (r + 2)$. Thus, (11) reduces to

$$\begin{cases} 0 & \text{if } k + 1 - r < 2 \quad (k = r) \\ r + 2 & \text{if } k - r = 1 \\ \binom{k+2}{2} - \binom{r+2}{2} & \text{otherwise.} \end{cases}$$

This can be simplified to $\binom{k+2}{2} - \binom{r+2}{2}$ for any value of k and r . By Lemma 4.2, the relation (10) in this case is not satisfied by any value of j , then also (12) simplifies to that expression.

Lemma 4.4. *Let t be an integer ≥ 2 . If $t \leq r + 1$ and $k \geq r$ then the formulas (11) and (12) are equal.*

Proof. By Lemma 4.2 we know that $\Omega = r + \ell$ for some integer $\ell \geq 2$. Depending on the value of ℓ the binomials in (11) could be automatically zero. We need to consider three possible situations $k -$

$r \geq \ell + 1$, $k - r = 1$ and $k - r < \ell$. The formula reduces to a different expression in each case, but it is not difficult to check that they are respectively equivalent to the expressions we get from (12).

Proposition 4.5. *The lower bound on $\dim C_k^r(\Delta)$ given in Theorem 2.1 coincides with the lower bound in Schumaker (1984)[Theorem 3.1 p. 256].*

Proof. Since any interior vertex of Δ is contained in at least two edges with different slopes, then $t_i \geq 2$ for every $i = 1, 2, \dots, f_0^0$. Collecting the vertices γ_i with the same value t_i , the statement follows directly applying Lemmas 4.3 and 4.4 for those values of t_i .

Proposition 4.6. *If the vertices in Δ^0 are numbered as in Theorem (4.1), then the upper bound on $\dim C_k^r(\Delta)$ given in Theorem 3.1 coincides with the bound (8) in Schumaker (1984)[Theorem 2.1 p. 252].*

Proof. This follows similarly as in the latter proposition. By collecting the vertices in Δ^0 with the same value \tilde{t}_i , we apply Lemmas 4.3 and 4.4 with the values \tilde{t}_i . The only case that remains to be considered is $\tilde{t}_i = 1$. It corresponds to the third term in the formula in Theorem (3.1). Since

$$\binom{k+2-(r+1)}{2} = \binom{k+2}{2} - \binom{r+2}{2} - \sum_{j=1}^{k-r} (r+1)$$

the statement follows.

We summarize the results in the following theorem. Let us denote by LBH, UBH and LBS, UBS the respective lower and upper bounds obtained by Theorems 2.1 and 3.1 and the ones obtained in Schumaker (1984).

Theorem 4.7. *Let Δ be a connected, finite two dimensional simplicial complex, supported on a disk. Then,*

$$\begin{aligned} \text{LBH}(\dim C_k^r(\Delta)) &= \text{LBS}(\dim C_k^r(\Delta)) \\ \text{UBH}(\dim C_k^r(\Delta)) &\leq \text{UBS}(\dim C_k^r(\Delta)). \end{aligned}$$

Proof. This follows from Propositions 4.5 and 4.6, and the fact that the formula in Theorem 3.1 can be applied to any numbering on the interior vertices of Δ .

5. DIMENSION FORMULA FOR DEGREE $k \geq 4r + 1$

In this section we present an alternative proof for the dimension formula of the spline space $C_k^r(\Delta)$ for $k \geq 4r + 1$. The proof is substantially shorter than the one presented in Alfeld and Schumaker (1987).

We include the following notation: let Δ_i and Δ_i^∂ be (respectively) the sets of i -dimensional faces and i -dimensional boundary faces. Denote by $f_i(\Delta)$ and $f_i^\partial(\Delta)$ the cardinality of the preceding sets, respectively. We begin by stating the following lemma which we will need later on.

Lemma 5.1. *(Schenck and Stillman, 1997a, Lemma 3.3) If Δ is a triangulated region in \mathbb{R}^2 , then there exists a total order on Δ_0 such that for every γ in Δ_0^∂ , there exist vertices γ' , γ'' adjacent to γ , with $\gamma \succ \gamma', \gamma''$, and such that $\overline{\gamma\gamma'}$, $\overline{\gamma\gamma''}$ have distinct slopes.*

For an ordering on Δ_0 as in the previous lemma, we assign indices to the vertices in such a way that $\gamma_i \succ \gamma_j$ when $i > j$. The first indices $1, 2, \dots, f_0^\partial$ are assigned to the vertices lying on the boundary. To those interior vertices which are joined to the boundary by two or more edges of distinct slope we assign the indices $f_0^\partial + 1, \dots, n$. Let us recall the following notation and remarks from Schenck and Stillman (1997a). For each interior vertex γ , and each $f \in \mathcal{J}(\gamma)$, let $f[\gamma]$ denote the corresponding element in $H_0(\mathcal{J})$. Then $H_0(\mathcal{J})$ is generated by $\{f[\gamma] \mid f \in \mathcal{J}(\gamma)\}$. By definition of $\mathcal{J}(\gamma)$, we know that

$$(13) \quad f[\gamma] = \sum_{\tau \ni \gamma} \ell_\tau^{r+1} f_\tau[\gamma]$$

for some polynomials $f_\tau \in R$. Notice that if τ is an edge whose vertices are γ and γ' then the

$$(14) \quad \ell_\tau^{r+1} f[\gamma] = \ell_\tau^{r+1} f[\gamma']$$

in $H_0(\mathcal{J})$; in particular $\ell_\tau^{r+1} f[\gamma] = 0$ when τ is an edge connecting γ to the boundary.

Here is another lemma that we need to prove that $H_0(\mathcal{J}_k) = 0$ for degree $k \geq 4r + 1$:

Lemma 5.2. *Let ℓ_1, ℓ_2 and ℓ_3 be three equations of distinct lines through a point p and L the equation of a line that does not contain the point p . Then for any polynomial g of degree $d \geq r + 1$, there exist $u, v \in R$ of degree d and $w \in R$ of degree $r - 1$ such that*

$$\ell_3^{r+1} g = \ell_1^{r+1} u + \ell_2^{r+1} v + L^{d-r+1} \ell_3^{r+1} w,$$

with $w = 0$ in the case $r = 0$.

Proof. The case $r = 0$ is direct from the linear dependency of ℓ_1, ℓ_2 and ℓ_3 . Suppose that $r > 0$. Since ℓ_1, ℓ_2 and L are linearly independent, we can make the following change of coordinates:

$$\begin{aligned} \ell_1 &= x; & \ell_3 &= x + ay \quad \text{for some } a \neq 0; \\ \ell_2 &= y; & L &= z. \end{aligned}$$

Let $g = x^i y^j z^k$ be a monomial such that $i + j + k = d$. If $k \leq d - r$ then $i + j \geq r$ and the polynomial $\ell_2^{r+1} g$ is in the ideal $\in \langle x^{r+1}, y^{r+1} \rangle$. If $k \geq d - r + 1$ then g is a multiple of $z^{d-r+1} = L^{d-r+1}$. Thus we can write

$$\ell_3^{r+1} g = \ell_1^{r+1} u' + \ell_2^{r+1} v' + L^{d-r+1} w',$$

for some polynomials $u', v' \in R$ of degree d and $w' \in R$ of degree $2r$. As $L^{d-r+1} w'$ is in the ideal $\langle \ell_1^{r+1}, \ell_2^{r+1}, \ell_3^{r+1} \rangle$ and as L is a non-zero divisor modulo this ideal, we deduce that $w' = \ell_1^{r+1} u'' + \ell_2^{r+1} v'' + \ell_3^{r+1} w''$, with $u'', v'', w'' \in R$ of degree $r - 1$. Collecting the coefficients of $\ell_1^{r+1}, \ell_2^{r+1}$, we obtain the desired decomposition:

$$\ell_3^{r+1} g = \ell_1^{r+1} (u' + L^{d-r+1} u'') + \ell_2^{r+1} (v' + L^{d-r+1} v'') + L^{d-r+1} \ell_3^{r+1} w''.$$

We use this lemma to prove the following result:

Theorem 5.3. *The dimension of $C_k^r(\Delta)$ when $k \geq 4r + 1$ is given by the lower bound formula of Theorem 2.1.*

Proof. From (2), and the formulas for the dimension of the modules (3), (4) and (7), it suffices to show that $H_0(\mathcal{J})_k = 0$ for $k \geq 4r + 1$. Equivalently, we need to show that $f[\gamma] = 0$ in $H_0(\mathcal{J})_k$ for all γ and $f \in \mathcal{J}(\gamma)_k$. Ordering the vertices as in Lemma 5.1, we consider the first interior vertex in the ordering, and denote it by γ . Let $\tau_1, \tau_2, \dots, \tau_t$ be the edges (not necessarily with different slopes) containing γ and $\omega_1, \omega_2, \dots, \omega_t$ be respectively the end points of τ_i , which are distinct from γ . We number first the edges τ_1 and τ_2 that connect γ to the boundary, and the remaining edges counterclockwise, starting from τ_2 . Let us denote by $\ell_1, \ell_2, \dots, \ell_t$ (respectively) equations of these edges and by L_i equations of the edges connecting ω_i to ω_{i-1} for $i = 2, \dots, t$.

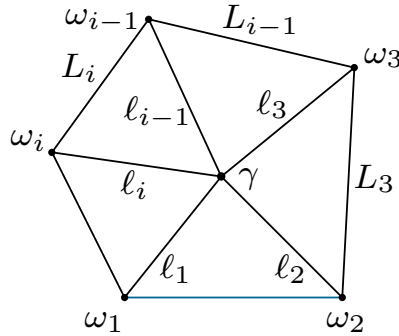


FIGURE 1. Notation in Theorem 5.3.

We are going to prove by induction on $j > 2$ that for any homogeneous polynomial f of degree $\geq 3r$, we have

- $\ell_j^{r+1} f[\omega_j] = \ell_j^{r+1} f[\gamma] = 0$,
- $L_j^{r+1} f[\omega_j] = 0$

in $H_0(\mathcal{J})$.

Let us prove it first for $j = 3$. Let $f \in R_k$ with $k \geq 3r$. By construction ω_2 is on the boundary, thus we have $L_3^{r+1} f[\omega_3] = L_3^{r+1} f[\omega_2] = 0$. By Lemma 5.2,

$$\ell_3^{r+1} f = \ell_1^{r+1} u + \ell_2^{r+1} v + L_3^{2r+1} \ell_3^{r+1} w$$

for some polynomials $u, v, w \in R$. Then we have

$$\begin{aligned} \ell_3^{r+1} f[\gamma] &= \ell_1^{r+1} u[\gamma] + \ell_2^{r+1} v[\gamma] + L_3^{2r+1} \ell_3^{r+1} w[\gamma] \\ &= L_3^{2r+1} \ell_3^{r+1} w[\omega_3] = 0. \end{aligned}$$

Let us take now $i > 2$ and assume that the induction hypothesis is true for $3 \leq j < i$. We consider first $\ell_i^{r+1} f[\omega_i]$ with f homogeneous of degree $\geq 3r$. By Lemma 5.2 applied to ℓ_1, ℓ_2, ℓ_3 and L_i , we have

$$\ell_i^{r+1} f = \ell_1^{r+1} u + \ell_2^{r+1} v + L_i^{2r+1} \ell_i^{r+1} w,$$

for some polynomials $u, v, w \in R$. Then we have

$$\begin{aligned} \ell_i^{r+1} f[\gamma] &= \ell_1^{r+1} u[\gamma] + \ell_2^{r+1} v[\gamma] + L_i^{2r+1} \ell_i^{r+1} w[\gamma] \\ &= L_i^{2r+1} \ell_i^{r+1} w[\omega_i] = L_i^{2r+1} \ell_i^{r+1} w[\omega_{i-1}]. \end{aligned}$$

As L_i^{2r+1} is in the ideal generated by ℓ_{i-1}^{r+1} , L_{i-1}^{r+1} , by the induction hypothesis, we have

$$L_i^{2r+1} \ell_i^{r+1} w[\omega_{i-1}] = 0.$$

This proves our claim for $\ell_i^{r+1} f[\omega_i] = 0$.

Let us consider now $L_i^{r+1} f[\omega_i] = L_i^{r+1} f[\omega_{i-1}]$ with f homogeneous of degree $\geq 3r$. By Lemma 5.2,

$$L_i^{r+1} f = \ell_{i-1}^{r+1} u + L_{i-1}^{r+1} v + \ell_i^{r+1} L_i^{r+1} w,$$

for some polynomials $u, v, w \in R$. We deduce that

$$\begin{aligned} L_i^{r+1} f[\omega_{i-1}] &= \ell_{i-1}^{r+1} u[\omega_{i-1}] + L_{i-1}^{r+1} v[\omega_{i-1}] + \ell_i^{r+1} L_i^{r+1} w[\omega_{i-1}] \\ &= \ell_i^{r+1} L_i^{r+1} w[\omega_i] = \ell_i^{r+1} L_i^{r+1} w[\gamma] = 0 \end{aligned}$$

by the induction hypothesis and the previous computation.

This concludes the induction proof and shows that for any $f \in \mathcal{J}(\gamma)_{4r+1}$, we have

$$\begin{aligned} f[\gamma] &= \sum_{i=1}^t \ell_i^{r+1} f_i[\gamma] = \sum_{i=3}^t \ell_i^{r+1} f_i[\gamma] \\ &= 0. \end{aligned}$$

Therefore $H_0(\mathcal{J}) = 0$ and the dimension of $C_k^r(\Delta)$ is given by the lower bound when $k \leq 4r + 1$.

6. EXAMPLES AND REMARKS

In this section we compute the dimension formula for some complexes. We prove that the dimension formula is in fact given by the corresponding lower bound given in Theorem 2.1. In the first example we show directly that $H_0(\mathcal{J})_k = 0$. In the second example we number the vertices in such a way that the upper bound we obtain agrees with the lower one. The subdivisions we consider are two of the so-called Powell-Sabin subdivisions (Powell and Sabin (1977)).

Example 6.1. *Powell-Sabin 12-split.*

Let Δ be the simplicial complex supported on a triangle $|\Delta|$ subdivided into twelve smaller triangles, as in Figure 2.

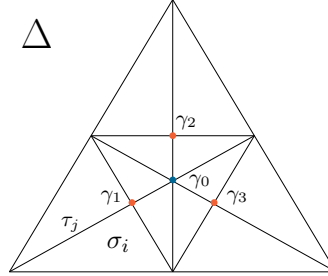


FIGURE 2. Powell-Sabin 12-split.

Let us note that by using any numbering on Δ_0^0 , the upper bound we get from Theorem 3.1 equals the lower bound, leading easily to the dimension formula. On the other hand, it is not possible to find a numbering on the interior vertices of Δ with the condition required to apply Theorem 4.1. Hence it is not possible to find an upper bound for this spline space by using the formula of Schumaker.

By the remarks in Section 5, it is easy to check that $\ell_j^{r+1} f[\gamma_i] = 0$ for every γ_i and $\ell_j^{r+1} f \in \mathcal{J}(\gamma_i)$. Thus, for any k and any $r \leq k$, the

homology module $H_0(\mathcal{J})_k$ is zero and the dimension of the spline space $C_k^r(\Delta)$ is given by the formula in Theorem 2.1.

This example is an instance of a triangulation where all edges are pseudoboundaries (in the terminology of Schenck and Stillman (1997b)), and the computation also follows by Lemma 2.5 of Schenck and Stillman (1997b).

Example 6.2. *Powell-Sabin 6-split.*

Let Δ be the simplicial complex supported on a simply connected triangulated region $|\Delta|$ in \mathbb{R}^2 . Assume the triangulation consists of f_0 vertices (interior and exterior) and f_i^0 i -dimensional interior faces ($i = 0, 1, 2$), where f_2^0 is taken as the number of all the triangles σ_i in the subdivision. We refine this triangulation by subdividing each triangle into 6 triangles to get $\tilde{\Delta}$, as follows.

- (1) In each triangle $\sigma_i \in \Delta_2^0$ choose an interior point ν_i , in such a way that if two triangles σ_i, σ_j have an edge τ in common, then the line joining ν_i and ν_j intersects τ at an interior point μ_{ij} .
- (2) Join each new point ν_i to the vertices of the triangle σ_i , and to the points μ_{ij} on the edges of σ_i .
- (3) For the triangles having an edge (or more) on the boundary, choose a point in the interior of such edge and join it with ν_i .

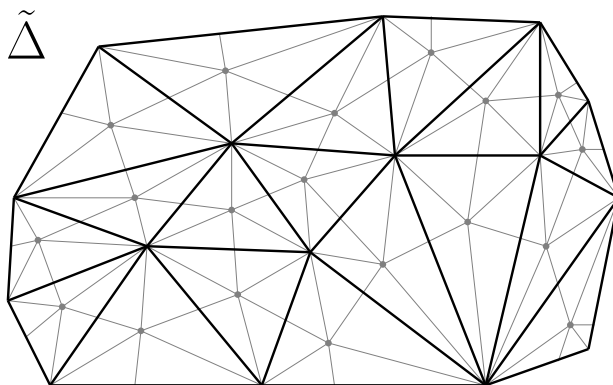


FIGURE 3. Powell-Sabin 6-split of Δ .

We want to apply Corollary 3.2 and give a formula for $\dim C_k^1(\tilde{\Delta})$.

Let us consider the following numbering on the vertices in $\tilde{\Delta}_0^0$. We take first a triangle $\sigma_i \in \Delta$ with (at least) one of its edges on the boundary, denote by γ_i the vertex in σ_i that is in the interior of Δ . With the notation as above, we assign the index 1 to the vertex ν_i , the indices 2 and 3 to the vertices μ_{ij} on the edges of σ_i , and finally the

index 4 to the vertex γ_i . After this, we consider the edges of σ_i as if they were all part of the boundary and iterate the previous process. We choose another triangle in Δ having (at least) one edge on the “new” boundary. We index its vertices following the same order as we did for the ones in σ_i . We continue this numbering until we have considered all the triangles in Δ and hence have index all the vertices in $\tilde{\Delta}_0^0$. Figure 4 shows the triangles that could be considered first, and the different colors refer to the values of \tilde{t}_i for the interior vertices in each case.

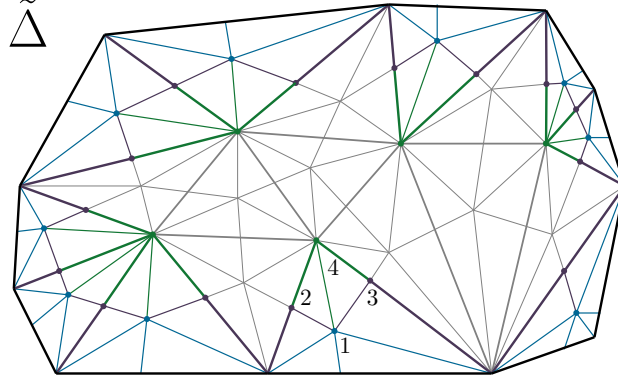


FIGURE 4. Numbering on the vertices in $\tilde{\Delta}_0^0$.

Let us notice that for this numbering, the values for \tilde{t}_i corresponding to these vertices are either ≥ 3 , as is the case of the vertices ν_i and γ_i , or $\tilde{t}_i = t_i = 2$, as it happens for the vertices μ_{ij} . Thus, we arrive to the following result.

Proposition 6.3. *For any $k \geq 3$:*

$$\dim C_k^1(\tilde{\Delta}) = \binom{k+2}{2} + f_1^0 \binom{k-2}{2} + 2f_0^0 \binom{k-1}{2} + \binom{2k-1}{2} (f_0 - 2).$$

For $k = 2$:

$$\dim C_2^1(\tilde{\Delta}) = 3f_0.$$

Proof. It follows directly from the remarks we made above about \tilde{t}_i for the counting, and Corollary 3.2.

For the numbering we defined on $\tilde{\Delta}_0^0$, it is not possible to apply Theorem 4.1 to find an exact value for the dimension. In fact, when the initial subdivision consists of more than one triangle, by using the numberings allowed in that theorem, we get upper bounds which are strictly bigger than the actual dimension of the space.

Remark 1. In this paper we have confined our attention to triangular partitions but it is easy to check that our proofs of the lower and the

upper bounds can be extended to rectilinear partitions. The construction also allows to consider regions subdivided by curved boundaries, but then the ideals must be considered individually. The boundary of $|\Delta|$ itself can, of course, be curved.

Remark 2. The resolution for ideals generated by power of linear forms presented in the paper by Geramita and Schenck (1998), applied also when the power of the linear forms is different. It makes possible to apply the ideas we present in this paper to mixed splines, i.e., splines where the order of smoothness may differ on the various edges of the subdivision.

Remark 3. The proof of the exact dimension formula when the degree of the spline space k is $\geq 4r + 1$ cannot be directly extended to spaces of polynomials of degree $k \geq 3r + 1$. With some restrictions associated to the number of different slopes of the edges containing each vertex it is possible to prove the result for this degree.

Remark 4. We hope that the two methods we used in the examples to find the exact dimension, namely, by showing that the lower and the upper bounds coincide for certain numbering on the vertices, or that $H_0(\mathcal{J}) = 0$ directly by considering the equations of the edges, illustrate the way to easily find the dimension of the spline space for many particular triangulations.

Remark 5. There is a version of the upper bound given by Schumaker (1984) in Lai and Schumaker (2007), where the constraint on the numbering of the vertices is relaxed. However, for this new version of the theorem, there are still examples for which the upper bound formula does not give the correct result, see for instance the example p. 243 of Lai and Schumaker (2007).

Remark 6. For the three dimensional version of the problem, it is necessary to analyse ideals generated by powers of linear forms in three variables. The formula of the dimension, analogous to formula (2) in Section 2, have two homology modules, and in order to approximate the dimensions the dimension of these two modules must be bound. Some issues and positive results about this problem will be in an upcoming paper.

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