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Exact and approximate algorithms for computing the hyperbolicity of large-scale graphs

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Abstract: Let G be a connected graph, and let $d(a, b)$ denotes the shortest path distance between vertices a and b of G . The graph G is δ -hyperbolic if for any vertices a, b, c, d of G , the two largest of the three sums $S_1 = d(a, b) + d(c, d)$, $S_2 = d(a, c) + d(b, d)$, and $S_3 = d(a, d) + d(b, c)$ differ by at most 2δ . This can be determined in time $O(n^4)$ which could be prohibitive for large graphs. In this document, we propose an exact algorithm for determining the hyperbolicity of a graph that is scalable for large graphs. The time complexity of this algorithm is a function of the size of the largest bi-connected component of the graph, of the shortest path distance distribution in this component and of the value of the hyperbolicity. In the worst case, the time complexity remains in $O(n^4)$, but it is much faster in practice. We also propose both a multiplicative factor and an additive constant approximation algorithms. We then analyze further the time complexity of our exact algorithm for several class of graphs, and present some computational results on large-scale graphs.

Key-words: Hyperbolicity, algorithm, graph, approximation.

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Algorithmes exact et approché pour le calcul de l'hyperbolicité d'un graphe

Résumé : Soit G un graph connexe et soit $d(a, b)$ la distance entre les sommets a et b dans le graphe. Le graphe G est dit δ -hyperbolic si pour tout quadruplet a, b, c, d de sommets de G , les deux plus grandes des sommes $S_1 = d(a, b) + d(c, d)$, $S_2 = d(a, c) + d(b, d)$, et $S_3 = d(a, d) + d(b, c)$ diffèrent d'au plus 2δ . Cette valeur peut-être déterminé en temps $O(n^4)$, ce qui est souvent inaccessible pour les grands graphes.

Nous proposons un nouvel algorithme exact pour calculer l'hyperbolicité d'un graphe, permettant de traiter des grands graphes. La complexité temporelle de cet algorithme est fonction de la taille de la plus grande composante bi-connexe du graphe, de la distribution des plus courts chemins dans cette composante, et de la valeur de l'hyperbolicité. Au pire, cet algorithme prendra un temps en $O(n^4)$. Cependant, l'algorithme est bien plus efficace en pratique. Nous proposons également un algorithme approché à un facteur multiplicatif ou à une constante additive donné en entrée. Nous analysons la complexité temporelle de notre algorithme pour des classes particulières de graphes et présentons des résultats expérimentaux sur des grands graphes.

Mots-clés : Hyperbolicité, algorithme, graphe, approximation.

1 Introduction

The (Gromov) *hyperbolicity* of a graph reflects how the metric (distances) of the graph is close to the metric of a tree. Gromov [12] defines the notion of δ -hyperbolic metric spaces using the notion of δ -thin triangles. Given any three points x, y , and z of a hyperbolic metric space, the triangle (x, y, z) is δ -thin if any point of the geodesic joining x and y is at distance at most δ of one of the geodesics joining x to z or y to z . A δ -hyperbolic space is a geodesic metric space in which every geodesic triangle is δ -thin. In other words, a graph has hyperbolicity $\leq \delta$ if, for any $u, v, w \in V(G)$ and for any shortest paths P_{uv}, P_{vw}, P_{uw} between these three vertices, any vertex in P_{uv} is at distance at most δ from $P_{vw} \cup P_{uw}$ [12]. Intuitively, in a graph with small hyperbolicity, any two shortest paths between the same pair of vertices are close to each other. For instance, a graph has hyperbolicity 0 if and only if it is acyclic.

An alternative definition given by Gromov [12], and called the *4-points condition*, is the following. Let $d(u, v)$ denotes the distance between vertices u and v . A metric space is δ -hyperbolic if for any four points u, v, w, x the two larger of the distance sums $d(u, v) + d(w, x)$, $d(u, w) + d(v, x)$, $d(u, x) + d(v, w)$ differ by at most 2δ . These notions extend to connected graphs, and we say that a connected graph $G = (V, E)$ equipped with its standard graph metric d_G (shortest path distance) is δ -hyperbolic if the metric space (V, d_G) is δ -hyperbolic. In other words, a connected graph $G = (V, E)$ is δ -hyperbolic if it satisfies the 4-points condition. The hyperbolicity of a graph measures its tree-likeness. The less the value of δ is, the more the graph looks like a tree. Moreover, it can be observed that the hyperbolicity of a connected graph is the maximum value of the hyperbolicity of its biconnected components.

Determining the hyperbolicity δ of a graph of order n can be done in time $O(n^4)$, testing all 4-tuples of vertices of the graph. An implementation of the naive algorithm for determining the hyperbolicity of a graph as been included into the *distory* (Distance Between Phylogenetic Histories) package [8] of the CRAN (The Comprehensive R Archive Network) project [6]. This package is devoted to the study of geodesic distance between phylogenetic trees and associated functions. The implementation uses the “revolving doors Gray code” principle [13] for visiting all 4-tuples of the input graph. Unfortunately, the proposed implementation is impracticable for graphs with large biconnected components. For instance, for a graph with a biconnected component of order 10 000, the number of 4-tuples to consider is higher than $4 \cdot 10^{14}$, which requires approximately 100 hours of computation assuming that 10^9 4-tuples are evaluated per second. More recently, an algorithm with time complexity in $O(n^{3.69})$ has been proposed in [11]. This algorithm is based on the fast (max,min)-matrix multiplication algorithm proposed in [9]. However, the computation time for large-scale graphs remains prohibitive.

A heuristic algorithm for determining the hyperbolicity of CAIDA AS maps has been used in [7]. Other heuristics based on sampling were used in [16, 18] for random Erdős-Renyi GNP graphs generated using probability $p = c/n$, where c is a small constant (e.g., $c = 1.5$ or $c = 2$), and so the resulting graphs are almost trees having very small biconnected components. In addition, a 2-approximation algorithm, with running time in $O(n^3)$, is obtained fixing one vertex and evaluating all possible 4-tuples containing that vertex [5].

The need for more efficient algorithms has been expressed by many authors, for instance it has recently been mentioned in the conclusion of [4] that “*exact computation of δ by its definition takes $O(n^4)$ time, which is not scalable to large graphs, and thus the design of more efficient exact or approximation algorithms would be of interest*”.

In this paper, we present a new exact algorithm for computing the hyperbolicity of a graph. It is the first exact algorithm scalable for large graphs. Furthermore, the proposed algorithm can be turned into an approximation algorithm with tunable multiplicative factor or additive constant. The worst case time complexity of the proposed algorithm is in $O(n^4)$ but it is much

faster in practice.

Outline of this document. We start in Section 2 with some definitions and notations. Then we formally describe our algorithm for computing the hyperbolicity of a graph in Section 3 and give hints on its time complexity. We also explain how to turn this algorithm into an approximation algorithm. In Section 4, we evaluate experimentally the performances of our exact algorithm. Next, in Section 5, we report on the computation of the hyperbolicity of large-scale graphs, namely CAIDA maps of the Internet autonomous systems topology. Last, we list some simple results and proofs on the hyperbolicity of particular graph classes in Section A.

2 Definitions and known results

In this section, we first recall the definition of the hyperbolicity of a graph and fix some notations. Then, we survey some known results on this graph property, in particular the value of the hyperbolicity of some families of graphs, and general lower and upper bounds.

Let G be a connected graph, and let $d(a, b)$ denotes the shortest path distance between vertices a and b of G . The hyperbolicity of a graph has been defined by Gromov [12] as follows.

Definition 1 ([12]). *The graph G is δ -hyperbolic if for any vertices a, b, c, d of G , the two largest of the three sums $S_1 = d(a, b) + d(c, d)$, $S_2 = d(a, c) + d(b, d)$, and $S_3 = d(a, d) + d(b, c)$ differ by at most 2δ .*

In this paper, we consider unweighted graphs only. Therefore, the possible values of δ are half-integers, that is $\delta \in \{\frac{i}{2}, i \in \mathbb{N}\}$. Furthermore, it is easy to check that $\delta \leq \frac{D}{2}$, where D is the diameter of the graph. This result also follows from Lemma 5 that we will prove in next section.

The hyperbolicity of a graph is the maximum of the hyperbolicity of its biconnected components. To see that, let x be a cut-vertex of the graph G and w.l.o.g. let B_1 and B_2 be the two components of G separated by x . Let now $a, b, c \in B_1$ and $d \in B_2$. We have $S_1 = d(a, b) + d(c, d) = d(a, b) + d(c, x) + d(x, d)$, $S_2 = d(a, c) + d(b, d) = d(a, c) + d(b, x) + d(x, d)$, and $S_3 = d(a, d) + d(b, c) = d(a, x) + d(x, d) + d(b, c)$. Obviously, the computed value for the 4-tuple a, b, c, d is the same than for the 4-tuple a, b, c, x . Moreover, when $a, b \in B_1$ and $c, d \in B_2$, the computed value will always be 0.

For some families of graphs it is possible to give either the value of their hyperbolicity, or tight bounds. In particular, we have:

- *Block* graphs (i.e., connected graph in which every 2-connected subgraph is a clique) are 0-hyperbolic, and so are trees and cliques;
- Cycles of order $n = 4p + \varepsilon$, with $p \geq 1$ and $\varepsilon \in \{0, 1, 2, 3\}$, are $(p - 1/2)$ -hyperbolic when $\varepsilon = 1$, and p -hyperbolic otherwise (see proof in Appendix A);
- $n \times m$ grids, with $2 \leq n \leq m$, are $(n - 1)$ -hyperbolic (see proof in Appendix A);
- d -dimensional grids of side s are $(s - 1) \lfloor d/2 \rfloor$ -hyperbolic (see proof in Appendix A);
- The hypercube $\mathcal{H}_2(n)$ of dimension n and order 2^n is $\lfloor n/2 \rfloor$ -hyperbolic (see proof in Appendix A);
- Chordal graphs are (≤ 1) -hyperbolic and 1-hyperbolic chordal graphs have been characterized [1, 2, 14];

- k -chordal graphs with $k \geq 4$ are $(\leq \lfloor k/4 \rfloor)$ -hyperbolic [20].

Using above results, we can establish a lower bound on the hyperbolicity of any graph. More precisely, let $g(G)$ denotes the *girth* of a graph (i.e., length of the shortest cycle of the graph). We have $\left\lfloor \frac{g(G)}{4} \right\rfloor - \varepsilon_g \leq \delta$, where $\varepsilon_g = \frac{1}{2}$ when $g(G) \equiv 1 \pmod{4}$ and 0 otherwise (this bound was falsely established in [15] since they have forgotten the ε_g). An upper bound has been established similarly in [3] using the *circumference* $c(G)$ of the graph, that is the maximum length of its cycles. A tighter upper bound follows from the chordality $k(G)$ of the graph since for any k -chordal graph we have $k(G) \leq c(G)$. Summarizing,

Theorem 2. *For every graph G , we have $\left\lfloor \frac{g(G)}{4} \right\rfloor - \varepsilon_g \leq \delta(G) \leq \left\lfloor \frac{k(G)}{4} \right\rfloor$, where $\varepsilon_g = \frac{1}{2}$ when $g(G) \equiv 1 \pmod{4}$ and 0 otherwise.*

Other bounds on the hyperbolicity of a graph have been established. For instance, let $\beta(G)$ be the *independence number* of G (i.e., cardinality of the largest independent set) and let $\gamma(G)$ be the *domination number* of G (i.e., size of the smallest dominating set). We have,

Theorem 3 ([17]). *For every graph G of order n , we have $\delta(G) \leq \min \left\{ \beta(G), \frac{n - \beta(G) + 2}{2} \right\}$.*

Theorem 4 ([17]). *For every graph G , we have $\delta(G) \leq \frac{3}{2}\gamma(G)$.*

3 Exact algorithm for computing the hyperbolicity

In this section, we formally describe a new exact algorithm for computing the hyperbolicity of graphs. We then give some hints on its time-complexity. Last but not least, we explain how to turn this algorithm into an approximation algorithm.

Our new algorithm is based on the following lemma, in which $d(x, y)$ denotes the graph distance between vertices x and y , and $h(a, b, c, d)$ denotes the hyperbolicity of the 4-tuple (a, b, c, d) in the graph G .

Lemma 5. *Let $G = (V, E)$ be a connected graph, let $a, b, c, d \in V$, let $S_1 = d(a, b) + d(c, d)$, $S_2 = d(a, c) + d(b, d)$, and $S_3 = d(a, d) + d(b, c)$, and assume w.l.o.g. that $S_1 \geq \max \{S_2, S_3\}$. We have $h(a, b, c, d) \leq \min \{d(a, b), d(c, d)\}$.*

Proof. We have $S_2 + S_3 = d(a, c) + d(b, d) + d(a, d) + d(b, c) = (d(a, c) + d(b, c)) + (d(a, d) + d(b, d))$. Using the triangular inequality, we deduce $S_2 + S_3 \leq 2 \cdot d(a, b)$. Since S_1 is the largest sum, we have $h(a, b, c, d) = S_1 - \max \{S_2, S_3\} \leq S_1 - \max \{S_2, S_3\} / 2 = S_1 - d(a, b) = d(c, d)$.

We obtain similarly that $h(a, b, c, d) \leq d(a, b)$. □

To make use of Lemma 5, we construct in Algorithm 1 the list SLL of 3-tuples $(S_1 = \ell_1 + \ell_2, \ell_1, \ell_2)$, for $1 \leq \ell_2 \leq \ell_1 \leq D$, sorted in decreasing lexicographic order (lines 3-4). For instance, if $D = 4$, the 3-tuples are sorted in the following order:

$(8, 4, 4), (7, 4, 3), (6, 4, 2), (6, 3, 3), (5, 4, 1), (5, 3, 2), (4, 3, 1), (4, 2, 2), (3, 2, 1), (2, 1, 1)$.

We then compute for each 3-tuple $(S_1 = \ell_1 + \ell_2, \ell_1, \ell_2)$ the values $h(a, b, c, d) = S_1 - \max \{d(a, c) + d(b, d), d(a, d) + d(b, c)\}$ for all 4-tuple (a, b, c, d) such that $d(a, b) = \ell_1$ and $d(c, d) = \ell_2$. Thus, the algorithm considers the 4-tuples (a, b, c, d) in such a way that either S_1 is the largest sum, or, in case S_2 (resp. S_3) is larger than S_1 we know that the 4-tuple has previously been considered with S_2 (resp. S_3) as maximum value, and so we have $h(a, b, c, d) \leq 0$. Then, thanks to Lemma 5

Algorithm 1 Hyperbolicity**Require:** $G = (V, E)$ is a 2-connected graph.**Ensure:** δ , the hyperbolicity of G (observe that $\delta = 2h^*$).

```

1: Let  $P[\ell]$  be the list of pairs  $(a, b) \in V \times V$  with  $a < b$  such that  $d(a, b) = \ell$ 
2: Sort  $P[\ell]$  in increasing lexicographic order
3: Let  $SLL$  be the list of triples  $(\ell_1 + \ell_2, \ell_1, \ell_2)$  for  $1 \leq \ell_2 \leq \ell_1 \leq D$ 
4: Sort  $SLL$  by decreasing lexicographic order
5: Let  $h^* := 0$ 
6: for all  $(S_1, \ell_1, \ell_2)$  in  $SLL$  such that  $\ell_2 > h^*$  do
7:   for all  $(a, b) \in P[\ell_1]$ , if  $\ell_2 > h^*$  do
8:     for all  $(c, d) \in P[\ell_2]$ , if  $\ell_2 > h^*$  do {When  $\ell_1 = \ell_2$ , we ensure that  $(a, b) < (c, d)$  in  $P[\ell_1]$ }
9:        $h^* := \max\{h^*, S_1 - \max\{d(a, c) + d(b, d), d(a, d) + d(b, c)\}\}$ 
10: return  $h^* / 2$ 

```

we know that at any step of the algorithm the value of ℓ_2 is an upper bound on the value of $h(a, b, c, d)$. Also, if the current lower bound is h^* , none of the 4-tuples such that $d(c, d) \leq h^*$ can be used to improve the lower bound. We can thus cut exploration (lines 6-8).

Since we have $\binom{n}{2}$ pairs, and that Algorithm 1 considers pairs of pairs, the worst case time complexity of the algorithm is in $O(n^4)$. However, with a more careful analysis we observe that we can parameterize the time complexity with the optimal value of the hyperbolicity and the distribution of the path lengths. That is,

Proposition 6. *Given a δ -hyperbolic graph G of diameter D and the sets $P[\ell]$ of pairs of vertices at distance ℓ from each other, the time complexity of lines 6-9 of Algorithm 1 is in*

$$O\left(\sum_{\ell_1=2\delta}^D |P[\ell_1]| \left(\frac{|P[\ell_1]| - 1}{2} + \sum_{\ell_2=\max\{1, 4\delta-\ell_1\}}^{\ell_1-1} |P[\ell_2]| \right)\right)$$

Proof. Let $P[\ell]$ be the list of pairs $(a, b) \in V \times V$ with $a < b$ such that $d(a, b) = \ell$. For a δ -hyperbolic graph, Algorithm 1 will consider in the worst case all the triples $(\ell_1 + \ell_2, \ell_1, \ell_2)$ such that $\ell_1 + \ell_2 \geq 4\delta$ and $D \geq \ell_1 \geq \ell_2 \geq 1$. Furthermore, when $\ell_1 = \ell_2$, line 8 ensures that pair $(a, b) < (c, d)$ in $P[\ell_1]$. Since we are not interested in the computation time of lines 1-4 (which is clearly dominated by the computation of the distances between each pairs of vertices), the result follows. \square

For instance, if the input graph is a $n \times n$ grid, with diameter $2n-2$ and hyperbolicity $\delta = n-1$ (so $h^* = 2n-2$), the value of the hyperbolicity will be obtained with the first considered 4-tuple. Lines 6-9 of Algorithm 1 will thus be executed in constant time. On the other hand, if the input graph is a $n \times 2$ grid, with diameter n and hyperbolicity $\delta = 1$ (so $h^* = 2$), almost all 4-tuples will be considered and so the running time will be in $O(n^4)$. Chordal graphs, which have hyperbolicity at most 1 [2], are worst case instances for this algorithm since its running time increases with the gap between the diameter and the hyperbolicity.

Algorithm 1 requires to compute the distances between all pairs of vertices (line 1), to sort pairs of vertices at equal distances in lexicographic order (line 2), and also some negligible operations (lines 3-4). Therefore we have

Corollary 7. *Given a δ -hyperbolic graph G of diameter D and the sets $P[\ell]$ of pairs of vertices*

at distance ℓ from each other, the time complexity of Algorithm 1 is in

$$O \left(\max \left\{ n(n+m), n^2 \log n, \sum_{\ell_1=2\delta}^D |P[\ell_1]| \left(\frac{|P[\ell_1]|-1}{2} + \sum_{\ell_2=\max\{1, 4\delta-\ell_1\}}^{\ell_1-1} |P[\ell_2]| \right) \right\} \right)$$

Turning Algorithm 1 into an approximation algorithm. Observe that at any step of the algorithm, the value of ℓ_1 is an upper bound for the hyperbolicity of G . Therefore, for large graphs inducing an excessive running time (some instances may require weeks of computations), Algorithm 1 can be turned into an approximation algorithm with approximation ratio $\frac{\ell_1}{h^*}$. It suffices for instance to stop the execution of the algorithm after a given time (for instance one hour) and to return the values h^* and ℓ_1 . More precisely, we can insert one of the following test between line 6 and line 7 of the algorithm:

- “If computation time is larger than allowed computation time, then stop computations and return h^* and ℓ_1 ”. We get $\frac{h^*}{2} \leq \delta \leq \frac{\ell_1}{2}$;
- “If $\frac{\ell_1}{h^*} \leq apx$, then stop computations and return $\frac{h^*}{2}$.” We get an approximation of the value δ of the hyperbolicity with proven approximation factor apx (i.e., $\frac{h^*}{2} \leq \delta \leq apx \cdot \frac{h^*}{2}$);
- “If $\frac{\ell_1}{2} - \frac{h^*}{2} \leq apx$, then stop computations and return $\frac{h^*}{2}$.” We get an approximation of the value δ of the hyperbolicity with proven additive approximation constant apx (i.e., $\frac{h^*}{2} \leq \delta \leq \frac{h^*}{2} + apx$).

As we show in the next section, the main part of the running time of the algorithm consists in closing the small gap between lower bounds, that are generally found very quickly, and upper bounds. Therefore, depending on the expected result, it may be appropriate to use the above rules that may allow to save a lot of time while preserving a sufficient precision.

4 Experimental performances

In this section, we evaluate the performances of Algorithm 1 described above. This evaluation is done by comparing the running time of Algorithm 1 with the time of the naive algorithm. Our objective is to determine the topologies for which our algorithm is better than the naive one. Indeed, Algorithm 1 appears to be worse than the naive one for some pathological topologies such as grids. The good news are that its running-time outperforms the one of known algorithms in the case of Internet-like graphs. In particular, Algorithm 1 allows us to compute the hyperbolicity of the last CAIDA AS maps ($n = 25815$ for the largest bi-connected component of the CAIDA map of September 2012) in two weeks.

We have reported in Table 1 the evolution of the computation time of Algorithm 1 on $N \times M$ grids such that $N * M = 2^2 * 3^2 * 4^2 = 576$ and $N * M = 2^2 * 3^2 * 5^2 = 900$. The running times are averages over 10 executions of the algorithm. The hyperbolicity of a $N \times M$ grid is $\delta_{N \times M} = \min\{N, M\} - 1$. As expected, the computation time on square grids is way smaller than for grids with sides of very different sizes. In Section 3, we said that rectangular grids are some of the worst case instances for Algorithm 1. To verify this claim, we have plotted in Figure 1 the relative number of visited 4-tuples with Algorithm 1 compared to the total number of 4-tuples in the graph. Recall that a 4-tuple may be visited up to three times by the algorithm. In these plots, we observe that as soon as the sides of the grids differ by a factor at least 6, then the computation time of Algorithm 1 is larger than the naive algorithm, but it is much faster when the sides of the grid differ by a factor less than 6.

N	M	δ	time
24	24	23	0.08
18	32	17	0.16
16	36	15	1.13
12	48	11	15.01
9	64	8	40.33
8	72	7	50.79
6	96	5	71.80
4	144	3	90.60
3	192	2	97.08
2	288	1	101.82

(a) $N * M = 576$

N	M	δ	time
30	30	29	0.16
25	36	24	0.18
20	45	19	5.98
18	50	17	23.10
15	60	14	86.94
12	75	11	201.71
10	90	9	297.54
9	100	8	349.51
6	150	5	496.64
5	180	4	537.42
4	225	3	574.37
3	300	2	599.19
2	450	1	622.15

(b) $N * M = 900$

Table 1: Computation time in secondes of the hyperbolicity of $N \times M$ grids such that $N * M = 2^2 * 3^2 * 4^2 = 576$ (Table 1a) and $N * M = 2^2 * 3^2 * 5^2 = 900$ (Table 1b).

Next, we have reported in Figure 2 the running time of Algorithm 1 on Barabasi-Albert (BA) graphs. We have generated graphs with number of nodes in the range [1 000..10 000] and with different values of the degree k of newly added nodes. We have generated 100 graphs for each couple number of nodes and parameter k , and reported the average values. Figure 2a reports the average values of the hyperbolicity of these graphs. We observe that the hyperbolicity of BA graphs decreases with the degree of newly added nodes. Figure 2b represents the average computation time of Algorithm 1 on these graphs. We observe a slow increase of the computation time with the increase of the parameter k . This slow increase was expected since the time complexity of both the all-pairs shortest path algorithm and the decomposition into bi-connected components algorithm depend on the number of edges of the graph and so on the parameter k . We also observe in Figure 2b some pics in the computation times for particular combinations of the number of nodes and the parameter k . However, at the time of writing this deliverable we

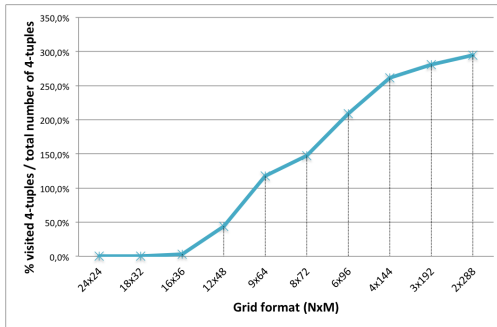
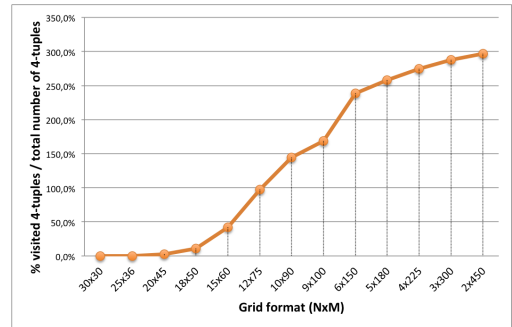
(a) $N * M = 576$ (b) $N * M = 900$

Figure 1: Relative performance of Algorithm 1 *vs.* the naive algorithm on $N \times M$ grids.

have not been able to explain this surprising behavior. Last, we have reported in Figure 2c the ratio of the average number of 4-tuples visited by Algorithm 1 over the total number of 4-tuples of the graphs. This highlights the drastic running time improvement of Algorithm 1 over the naive algorithm to compute the hyperbolicity of BA graphs. More precisely, Algorithm 1 is between 10^3 and 10^{11} times faster than the naive algorithm.

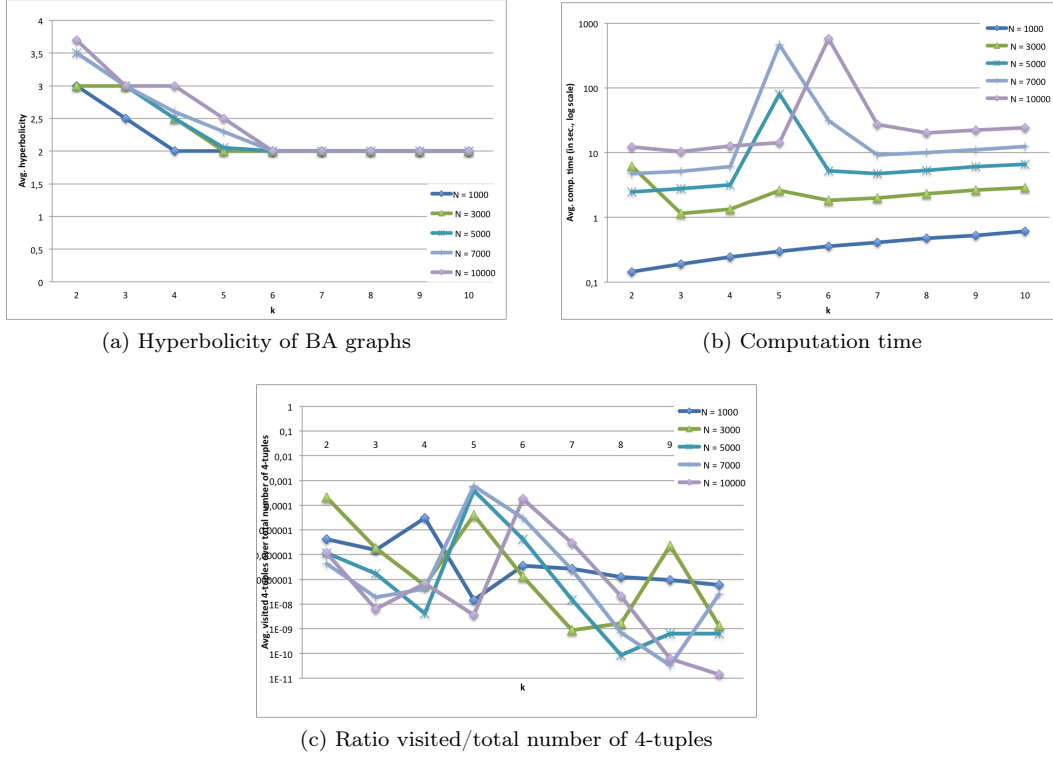


Figure 2: Computation time of the hyperbolicity of Barabasi-Albert graphs.

Last, we have reported in Table 2 the size of the largest bi-connected component (BCC), the hyperbolicity, the running time and the number of visited 4-tuples by Algorithm 1 when computing the hyperbolicity of some CAIDA maps. We have selected CAIDA maps with different values of the hyperbolicity to highlight the running time improvement of Algorithm 1. As expected with the time complexity expressed in Proposition 6, the running time improvement is correlated to the hyperbolicity of the CAIDA maps. Indeed, Algorithm 1 is more than 100 times faster than the naive algorithm for these maps. Nevertheless, the running times remains large. However, as explained in Section 3, at each step of the execution of Algorithm 1 we get proven lower and upper bounds for the hyperbolicity of the considered graph. Also, we have plotted in Figure 3 the time at which new lower and upper bounds are obtained when computing the hyperbolicity of the maps of Table 2. We observe that a few minutes of computations is sufficient for the lower bound to reach the optimal value but that the time to decrease the upper bounds to the optimal value could be very long. In fact, for maps with low hyperbolicity, almost all computation time is spent to prove the optimality of the lower bound.

In fact, the computational complexity of Algorithm 1 given in Proposition 6 is in the worst case. More precisely, it is assumed that the certificate for the hyperbolicity of the input graph

AS map name	Largest BCC	Hyperbolicity	Time	Visited 4-tuples	Tot. 4-tuples
2004/01/05	10 424	2.5	336s	$3.8 \cdot 10^{10}$	$\simeq 5 \cdot 10^{14}$
2004/06/07	11 100	2.0	18h50m	$8.8 \cdot 10^{12}$	$\simeq 6.3 \cdot 10^{14}$
2005/09/05	12 957	3.0	58s	$1.2 \cdot 10^8$	$\simeq 1.2 \cdot 10^{15}$
2012/06/01	25 815	2.0	15d	$1.8 \cdot 10^{15}$	$\simeq 1.8 \cdot 10^{16}$

Table 2: Computation time of the hyperbolicity of some CAIDA maps.

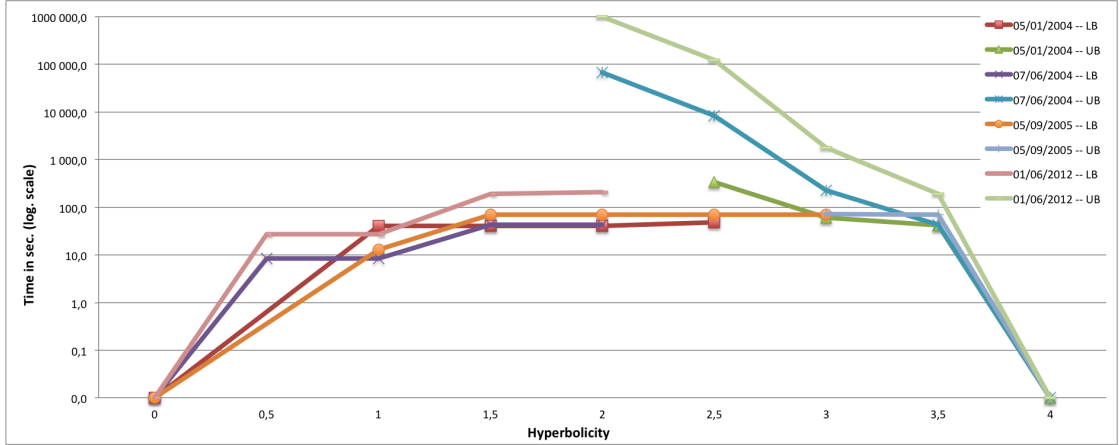


Figure 3: Time to reach lower and upper bounds of the hyperbolicity of CAIDA maps since 2004. Computation times for lower bounds are plotted from left to right, and from right to left for upper bounds.

G is found with the last visited 4-tuple. It is however likely that this certificate will be found earlier in the computations, as observed in Figure 3, hence increasing earlier the lower bound and reducing the search space. We have observed this fact in all our computations. Furthermore, we tried relabeling randomly the vertices in order to change the ordering in which pairs of vertices at distance ℓ are considered. We have observed that this experiment has no impact on the total number of visited 4-tuples. This shows again that certificates for the hyperbolicity are found very early and that the remaining computation time is devoted reducing the upper bound, hence proving the optimality.

5 Hyperbolicity of large-scale graphs

Thanks to Algorithm 1, we have been able to compute the hyperbolicity of all available CAIDA maps since 2004 (190 maps). Although the new algorithm improves upon the naive one, the overall computation time represents several months of computations. Indeed, the determination of the hyperbolicity of the 2012 CAIDA map only took 9 days of computations (but maps with hyperbolicity 3 requires only a few minutes of computations). The computed values are reported in Figure 4 where the first axis corresponds to the dates the maps were produced. We have one dot per CAIDA map and we have also reported the linear interpolation of the hyperbolicity.

Figure 4 provides several informations. First, we observe some measurement bias, for instance for the map of 07/06/2004 which has hyperbolicity 2 while maps produced in the months before



Figure 4: Evolution of the hyperbolicity of CAIDA maps since 2004.

and after have hyperbolicity 2.5. The same holds for the maps with hyperbolicity 3 (05/09/2005 and 06/02/2006). A frequent variation is also observed between consecutive maps in the period from 2007 till 2009. We still do not know whether this behavior is due to some bias of the measurement or whether it comes from some hidden fact.

The main observation resulting from this experiment is that the hyperbolicity of CAIDA maps has decreased in average from 2.5 to 2 and the hyperbolicity is stable since 2009. This is not surprising and it is clearly due to the fact that the AS network has become bigger during the last decade. In particular, many new links have appeared and have broken large cycles that made 4-tuples with bigger hyperbolicity.

We have also computed the hyperbolicity of all DIMES AS maps [19] since 2007 (62 maps). These maps are smaller than CAIDA maps and so computations are faster. The results are reported in Appendix C. We found that all these the maps have hyperbolicity 2 but the differences in running times of Algorithm 1 highlight the structural differences of these maps.

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A Simple results

Lemma 8. *Cycles of order $n = 4p + \varepsilon$, with $p \geq 1$ and $\varepsilon \in \{0, 1, 2, 3\}$, are $(p - 1/2)$ -hyperbolic when $\varepsilon = 1$, and p -hyperbolic otherwise.*

Proof of Lemma 8. Let C be a cycle of order $n = 4p + \varepsilon$, with $p \geq 1$ and $\varepsilon \in \{0, 1, 2, 3\}$, and let the vertices be indexed $\{0, 1, \dots, 4p + \varepsilon - 1\}$. Let a, b, c, d be four vertices of C . In order to break symmetries and w.l.o.g. we assume that $a = 0 < b \leq p < c \leq 2p + \lfloor \varepsilon/2 \rfloor < d \leq 3p + \lfloor \varepsilon/2 \rfloor + \lfloor \varepsilon/3 \rfloor$, and so we have

$$1 \leq d(a, b) \leq p \quad (1a)$$

$$p + 1 \leq d(a, c) \leq 2p + \lfloor \varepsilon/2 \rfloor \quad (1b)$$

$$p + \varepsilon - \lfloor \varepsilon/2 \rfloor - \lfloor \varepsilon/3 \rfloor \leq d(a, d) \leq 2p + \varepsilon - \lfloor \varepsilon/2 \rfloor - 1 \quad (1c)$$

$$1 \leq d(b, c) \leq 2p + \lfloor \varepsilon/2 \rfloor - 1 \quad (1d)$$

$$p + \lfloor \varepsilon/2 \rfloor + 1 \leq d(b, d) \leq 2p + \lfloor \varepsilon/2 \rfloor \quad (1e)$$

$$1 \leq d(c, d) \leq 2p + \lfloor \varepsilon/2 \rfloor + \lfloor \varepsilon/3 \rfloor - 1 \quad (1f)$$

and,

$$2 \leq S_1 = d(a, b) + d(c, d) \leq 3p + \lfloor \varepsilon/2 \rfloor + \lfloor \varepsilon/3 \rfloor - 1 \quad (1g)$$

$$2p + \lfloor \varepsilon/2 \rfloor + 2 \leq S_2 = d(a, c) + d(b, d) \leq 4p + 2 \lfloor \varepsilon/2 \rfloor \quad (1h)$$

$$p + \varepsilon - \lfloor \varepsilon/2 \rfloor - \lfloor \varepsilon/3 \rfloor + 1 \leq S_3 = d(a, d) + d(b, c) \leq 4p + \varepsilon - 2 \quad (1i)$$

For any values of b, c, d , we have $S_1 + S_3 = d(a, b) + d(c, d) + d(a, d) + d(b, c) = 4p + \varepsilon$, and so maximizing S_1 decreases S_3 and conversely.

Observe that the maximum value of S_1 is obtained with $b = p$, $c = p + 1$, and $d = 3p + \lfloor \varepsilon/2 \rfloor + \lfloor \varepsilon/3 \rfloor$. For these values we have $S_2 = 3p + \lfloor \varepsilon/2 \rfloor + \lfloor \varepsilon/3 \rfloor + 1 > S_1$, $S_3 = p + \varepsilon - \lfloor \varepsilon/2 \rfloor - \lfloor \varepsilon/3 \rfloor + 1 < S_1$, and so $h(a, b, c, d) = 1$. Similarly, the maximum value of S_3 is obtained with $b = 1$, $c = 2p + \lfloor \varepsilon/2 \rfloor$, and $d = 2p + \lfloor \varepsilon/2 \rfloor + 1$, which gives $S_1 = 2$, $S_2 = 4p + 2 \lfloor \varepsilon/2 \rfloor$, and so $h(a, b, c, d) = |\varepsilon - 2 - 2 \lfloor \varepsilon/2 \rfloor| \leq 2$.

Let us now assume that $S_2 = \max \{S_1, S_2, S_3\}$. Since $S_1 + S_3 = 4p + \varepsilon$, the best we can do to maximize $h(a, b, c, d)$ is to choose values of a, b, c, d such that the maximum among S_1 and S_3 is the closest possible to $2p + \varepsilon/2$, and such that S_2 is as large as possible. This can be achieved choosing $b = p$, $c = 2p + \lfloor \varepsilon/2 \rfloor$, and $d = 3p + \lfloor \varepsilon/2 \rfloor + \lfloor \varepsilon/3 \rfloor$. Indeed, with these values, we get $S_2 = 4p + 2 \lfloor \varepsilon/2 \rfloor$, so the maximum possible value, and $S_1 = 2p + \lfloor \varepsilon/3 \rfloor$ and $S_3 = 2p + \varepsilon - \lfloor \varepsilon/3 \rfloor \geq S_1$ which are the closest possible of $2p + \varepsilon/2$. We obtain $h(a, b, c, d) = 2p + 2 \lfloor \varepsilon/2 \rfloor - \varepsilon + \lfloor \varepsilon/3 \rfloor$.

Since $\varepsilon \in \{0, 1, 2, 3\}$, it follows that cycles of order $n = 4p + \varepsilon$, with $p \geq 1$, are $(p - 1/2)$ -hyperbolic when $\varepsilon = 1$, and p -hyperbolic otherwise. \square

Lemma 9. *$n \times m$ grids, with $2 \leq n \leq m$, are $(n - 1)$ -hyperbolic.*

Proof of Lemma 9. Let $\{v_{i,j}, 1 \leq i \leq n, 1 \leq j \leq m\}$ the vertices of the $n \times m$ grid.

Firstly, suppose that $n = m$, and consider the four vertices $v_{1,1}, v_{1,n}, v_{n,1}, v_{n,n}$. We have $S_1 = d(v_{1,1}, v_{1,n}) + d(v_{n,1}, v_{n,n}) = 2n - 2$, $S_2 = d(v_{1,1}, v_{n,n}) + d(v_{1,n}, v_{n,1}) = 4n - 4$, $S_3 = d(v_{1,1}, v_{n,1}) + d(v_{1,n}, v_{n,n}) = 2n - 2$, and so $h(v_{1,1}, v_{1,n}, v_{n,1}, v_{n,n}) = 2n - 2$. Since $h(a, b, c, d)$ is less or equal to the diameter of the graph for any 4-tuple (a, b, c, d) , this value is optimal.

Secondly, let $n \leq x \leq m$ and consider the four vertices $v_{1,1}, v_{1,x}, v_{n,1}, v_{n,x}$. We have $S_1 = d(v_{1,1}, v_{1,x}) + d(v_{n,1}, v_{n,x}) = 2x - 2$, $S_2 = d(v_{1,1}, v_{n,x}) + d(v_{1,x}, v_{n,1}) = 2n + 2x - 4$, $S_3 = d(v_{1,1}, v_{n,1}) + d(v_{1,x}, v_{n,x}) = 2n - 2$, and since $S_2 \geq S_1 \geq S_3$ we obtain $h(v_{1,1}, v_{1,x}, v_{n,1}, v_{n,x}) = 2n - 2$.

Let now $v_{i_1, j_1}, v_{i_2, j_2}, v_{i_3, j_3}, v_{i_4, j_4}$ be four vertices of the grid such that $1 < i_1, i_2 \leq n/2 < i_3, i_4 < n$, and $1 < j_1, j_3 \leq n/2 < j_2, j_4 < n$, and let us show that $h(v_{i_1, j_1}, v_{i_2, j_2}, v_{i_3, j_3}, v_{i_4, j_4}) \leq h(v_{i_1-1, j_1-1}, v_{i_2, j_2}, v_{i_3, j_3}, v_{i_4, j_4})$. We have $S_1 = d(v_{i_1, j_1}, v_{i_2, j_2}) + d(v_{i_3, j_3}, v_{i_4, j_4})$, $S_2 = d(v_{i_1, j_1}, v_{i_3, j_3}) + d(v_{i_2, j_2}, v_{i_4, j_4})$, and $S_3 = d(v_{i_1, j_1}, v_{i_4, j_4}) + d(v_{i_2, j_2}, v_{i_3, j_3})$.

According the shortest path distance in the grid, we have

$$d(v_{i_1, j_1}, v_{i_2, j_2}) + 1 \leq d(v_{i_1-1, j_1-1}, v_{i_2, j_2}) \leq d(v_{i_1, j_1}, v_{i_2, j_2}) + 2 \quad (2a)$$

$$d(v_{i_1, j_1}, v_{i_3, j_3}) + 1 \leq d(v_{i_1-1, j_1-1}, v_{i_3, j_3}) \leq d(v_{i_1, j_1}, v_{i_3, j_3}) + 2 \quad (2b)$$

$$d(v_{i_1-1, j_1-1}, v_{i_4, j_4}) = d(v_{i_1, j_1}, v_{i_4, j_4}) + 2 \quad (2c)$$

and so

$$S_1 + 1 \leq S'_1 = d(v_{i_1-1, j_1-1}, v_{i_2, j_2}) + d(v_{i_3, j_3}, v_{i_4, j_4}) \leq S_1 + 2 \quad (2d)$$

$$S_2 + 1 \leq S'_2 = d(v_{i_1-1, j_1-1}, v_{i_3, j_3}) + d(v_{i_2, j_2}, v_{i_4, j_4}) \leq S_2 + 2 \quad (2e)$$

$$S'_3 = d(v_{i_1-1, j_1-1}, v_{i_4, j_4}) + d(v_{i_2, j_2}, v_{i_3, j_3}) = S_3 + 2 \quad (2f)$$

From these values, we get $|S'_1 - S'_2| \leq |S_1 - S_2| + 1$, $|S'_1 - S'_3| \leq |S_1 - S_3| + 1$, and $|S'_2 - S'_3| \leq |S_2 - S_3| + 1$. Therefore, we have $h(v_{i_1, j_1}, v_{i_2, j_2}, v_{i_3, j_3}, v_{i_4, j_4}) \leq h(v_{i_1-1, j_1-1}, v_{i_2, j_2}, v_{i_3, j_3}, v_{i_4, j_4})$, and we can prove the same result when decreasing some of the values of i_1, i_2, j_1, j_3 or increasing some of the values of i_3, i_4, j_2, j_4 .

Altogether, we conclude that $n \times m$ grids, with $2 \leq n \leq m$, are $(n-1)$ -hyperbolic. \square

Lemma 10. *The d -dimensional grid of side s is $(s-1) \lfloor d/2 \rfloor$ -hyperbolic.*

Proof. The vertices of the d -dimensional grid of side s are labeled $(u_0, u_1, \dots, u_{d-1})$, with $u_i \in [0..s-1]$ for $i \in [0..d-1]$, and there is an edge between $u = (u_0, u_1, \dots, u_{d-1})$ and $v = (v_0, v_1, \dots, v_{d-1})$ if $u_i = v_i$ for $i \in [0..j-1] \cup [j+1..d-1]$ and either $v_j = u_j + 1$ if $u_j + 1 < s$ or $v_j = u_j - 1$ if $u_j - 1 \geq 0$.

Consider now the four vertices u, v, w, x such that

$$u = (0, 0, \dots, 0) \quad \text{contains only 0's} \quad (3a)$$

$$v = (s-1, s-1, \dots, s-1) \quad \text{contains only } s-1\text{'s} \quad (3b)$$

$$w = (w_0, w_1, \dots, w_{d-1}) \quad \text{contains } \lfloor d/2 \rfloor \text{ 0's and } \lceil d/2 \rceil \text{ } s-1\text{'s} \quad (3c)$$

$$x = (x_0, x_1, \dots, x_{d-1}) \quad \text{obtained from } w \text{ replacing each 0 with a } s-1 \text{ and each } s-1 \text{ with a 0} \quad (3d)$$

We have for instance $d(u, w) = (s-1) \lceil d/2 \rceil$, and so

$$S_1 = d(u, v) + d(w, x) = 2(s-1)d \quad (3e)$$

$$S_2 = d(u, w) + d(v, x) = 2(s-1) \lceil d/2 \rceil \quad (3f)$$

$$S_3 = d(u, x) + d(v, w) = 2(s-1) \lfloor d/2 \rfloor \quad (3g)$$

We obtain $h(u, v, w, x) = S_1 - S_3 = 2(s-1) \lfloor d/2 \rfloor$.

The optimality proof is similar to the proof of Lemma 9. \square

Corollary 11. *The hypercube $\mathcal{H}_2(n)$ of dimension n and order 2^n is $\lfloor n/2 \rfloor$ -hyperbolic.*

Proof of Corollary 11. The proof follows from Lemma 10 since the n -dimensional grid of side 2 is the hypercube $\mathcal{H}_2(n)$. \square

B Hyperbolicity of some CAIDA AS maps

Table 3 lists the hyperbolicity of all CAIDA AS maps since 2004 [10]. We have also reported in this table the number of 4-tuples with given hyperbolicity among a large set of randomly chosen 4-tuples. We can derive from these values the statistical distribution of the hyperbolicity of the 4-tuples. Last, we have reported some hyperbolicity *certificates*, i.e., a 4-tuple with maximum hyperbolicity.

Table 3: Hyperbolicity of CAIDA AS maps since 2004.

AS map name	δ	Certificate	Visited 4-tuples	Total number
2004/01/05	5/2	3233, 8338, 8923, 13268	$3.9 \cdot 10^{10}$	$4.9 \cdot 10^{14}$
2004/02/02	5/2	12497, 20807, 28983, 29226	$2.4 \cdot 10^{10}$	$5.1 \cdot 10^{14}$
2004/03/01	5/2	8863, 9249, 20515, 28776	$3.1 \cdot 10^{10}$	$5.3 \cdot 10^{14}$
2004/04/05	5/2	12316, 13092, 21172, 28949	$1.6 \cdot 10^{11}$	$5.9 \cdot 10^{14}$
2004/05/03	5/2	5434, 12660, 12764, 20633	$4.2 \cdot 10^{10}$	$6.2 \cdot 10^{14}$
2004/06/07	2	6802, 25454, 30838, 31318	$8.8 \cdot 10^{12}$	$6.3 \cdot 10^{14}$
2004/07/05	5/2	1955, 6802, 20685, 25454	$7.7 \cdot 10^{10}$	$6.6 \cdot 10^{14}$
2004/08/02	5/2	2148, 12644, 13105, 29335	$5.4 \cdot 10^{10}$	$7.0 \cdot 10^{14}$
2004/09/06	5/2	6802, 12615, 16010, 20816	$3.7 \cdot 10^{10}$	$7.3 \cdot 10^{14}$
2004/10/04	5/2	5434, 12550, 12644, 21341	$5.4 \cdot 10^{10}$	$7.7 \cdot 10^{14}$
2004/11/01	5/2	5434, 13042, 21341, 30768	$6.4 \cdot 10^{10}$	$8.0 \cdot 10^{14}$
2004/12/06	5/2	5379, 25454, 25496, 31340	$5.5 \cdot 10^{10}$	$8.4 \cdot 10^{14}$
2005/01/03	5/2	8880, 13121, 28903, 34214	$8.8 \cdot 10^{10}$	$8.10 \cdot 10^{14}$
2005/02/07	5/2	6802, 8880, 8923, 25496	$9.10 \cdot 10^{10}$	$8.10 \cdot 10^{14}$
2005/03/07	5/2	13302, 15378, 15775, 24825	$1.2 \cdot 10^{11}$	$9.1 \cdot 10^{14}$
2005/04/04	5/2	13105, 28903, 29632, 31258	$7.5 \cdot 10^{10}$	$9.5 \cdot 10^{14}$
2005/05/02	5/2	6850, 28903, 28949, 29233	$8.8 \cdot 10^{10}$	$1.0 \cdot 10^{15}$
2005/06/06	5/2	15378, 15775, 24607, 34784	$1.1 \cdot 10^{11}$	$1.0 \cdot 10^{15}$
2005/07/04	5/2	6736, 8226, 12880, 15595	$1.1 \cdot 10^{11}$	$1.1 \cdot 10^{15}$
2005/08/01	5/2	8471, 12983, 17480, 21103	$1.0 \cdot 10^{11}$	$1.1 \cdot 10^{15}$
2005/09/05	3	12739, 12764, 16010, 34645	$1.2 \cdot 10^8$	$1.2 \cdot 10^{15}$
2005/10/03	5/2	8825, 12507, 25306, 31549	$6.6 \cdot 10^{10}$	$1.2 \cdot 10^{15}$
2005/11/07	5/2	20702, 29390, 29663, 34645	$9.7 \cdot 10^{10}$	$1.3 \cdot 10^{15}$
2005/12/05	5/2	13365, 16070, 16138, 29442	$8.1 \cdot 10^{10}$	$1.3 \cdot 10^{15}$
2006/01/02	5/2	13071, 15682, 21011, 29390	$1.2 \cdot 10^{11}$	$1.5 \cdot 10^{15}$
2006/01/09	5/2	10747, 16070, 28733, 30952	$1.5 \cdot 10^{11}$	$1.4 \cdot 10^{15}$
2006/01/16	5/2	24416, 29632, 31258, 33818	$1.0 \cdot 10^{11}$	$1.4 \cdot 10^{15}$
2006/01/23	5/2	10687, 16070, 16130, 29663	$1.3 \cdot 10^{11}$	$1.4 \cdot 10^{15}$
2006/01/30	5/2	8825, 12507, 29630, 31589	$1.3 \cdot 10^{11}$	$1.4 \cdot 10^{15}$
2006/02/06	3	1955, 13210, 20722, 34416	$4.3 \cdot 10^{10}$	$1.4 \cdot 10^{15}$
2006/02/13	5/2	21411, 28752, 29034, 29107	$1.2 \cdot 10^{11}$	$1.5 \cdot 10^{15}$
2006/02/20	5/2	20778, 28858, 31077, 34775	$1.5 \cdot 10^{11}$	$1.5 \cdot 10^{15}$
2006/02/27	5/2	2549, 6458, 10490, 17086	$1.1 \cdot 10^{12}$	$1.5 \cdot 10^{15}$
2006/03/06	5/2	12764, 21411, 29034, 29107	$1.2 \cdot 10^{11}$	$1.5 \cdot 10^{15}$
2006/03/13	5/2	16190, 29663, 35032, 35147	$1.1 \cdot 10^{11}$	$1.4 \cdot 10^{15}$
2006/03/20	5/2	13310, 30861, 31023, 34379	$1.8 \cdot 10^{11}$	$1.5 \cdot 10^{15}$
2006/03/27	5/2	7202, 8982, 15393, 34771	$9.2 \cdot 10^{10}$	$1.5 \cdot 10^{15}$

Continued on next page

Table 3 – continued from previous page

AS map name	δ	Certificate	Visited 4-tuples	Total number
2006/04/03	5/2	15614, 20803, 31340, 38932	$1.5 \cdot 10^{11}$	$1.5 \cdot 10^{15}$
2006/04/10	5/2	15614, 20490, 31340, 38932	$9.10 \cdot 10^{10}$	$1.5 \cdot 10^{15}$
2006/04/17	5/2	13215, 23554, 28949, 31340	$1.8 \cdot 10^{11}$	$1.6 \cdot 10^{15}$
2006/04/24	5/2	15614, 20490, 31340, 38932	$1.0 \cdot 10^{11}$	$1.5 \cdot 10^{15}$
2006/05/01	5/2	12764, 15632, 16010, 16190	$1.4 \cdot 10^{11}$	$1.5 \cdot 10^{15}$
2006/05/08	5/2	2148, 8745, 29329, 31340	$1.3 \cdot 10^{11}$	$1.6 \cdot 10^{15}$
2006/05/15	5/2	2148, 23697, 24287, 34495	$1.2 \cdot 10^{11}$	$1.6 \cdot 10^{15}$
2006/05/22	5/2	2148, 23697, 24287, 34495	$7.8 \cdot 10^{10}$	$1.5 \cdot 10^{15}$
2006/05/29	5/2	3233, 9087, 12297, 30890	$1.9 \cdot 10^{11}$	$1.6 \cdot 10^{15}$
2006/06/05	5/2	12677, 19942, 28911, 34238	$1.4 \cdot 10^{12}$	$1.6 \cdot 10^{15}$
2006/06/12	5/2	5598, 13000, 13310, 31556	$1.3 \cdot 10^{11}$	$1.6 \cdot 10^{15}$
2006/06/19	5/2	5598, 12742, 21257, 24703	$2.0 \cdot 10^{11}$	$1.6 \cdot 10^{15}$
2006/06/26	5/2	12690, 18256, 20545, 24997	$1.4 \cdot 10^{11}$	$1.6 \cdot 10^{15}$
2006/07/03	5/2	12690, 20545, 24005, 24997	$1.4 \cdot 10^{11}$	$1.7 \cdot 10^{15}$
2006/07/10	5/2	9560, 20722, 35048, 39581	$1.2 \cdot 10^{11}$	$1.7 \cdot 10^{15}$
2006/07/17	5/2	13210, 34238, 34771, 39031	$1.10 \cdot 10^{11}$	$1.6 \cdot 10^{15}$
2006/07/24	5/2	5379, 16226, 28949, 33961	$1.7 \cdot 10^{11}$	$1.7 \cdot 10^{15}$
2006/07/31	5/2	5598, 13310, 30953, 41166	$1.4 \cdot 10^{11}$	$1.7 \cdot 10^{15}$
2006/08/07	5/2	8389, 12452, 12764, 16010	$1.7 \cdot 10^{11}$	$1.8 \cdot 10^{15}$
2006/08/14	5/2	12764, 16010, 25306, 29271	$1.0 \cdot 10^{11}$	$1.7 \cdot 10^{15}$
2006/08/21	5/2	6802, 15536, 29030, 30797	$1.4 \cdot 10^{11}$	$1.8 \cdot 10^{15}$
2006/08/28	5/2	12764, 15700, 16010, 29271	$2.3 \cdot 10^{11}$	$1.8 \cdot 10^{15}$
2006/09/04	5/2	5379, 12987, 28949, 39505	$9.7 \cdot 10^{10}$	$1.8 \cdot 10^{15}$
2006/09/11	5/2	8389, 12452, 12764, 16010	$1.10 \cdot 10^{11}$	$1.8 \cdot 10^{15}$
2006/09/18	5/2	10037, 12690, 20545, 28751	$1.0 \cdot 10^{11}$	$1.8 \cdot 10^{15}$
2006/09/25	5/2	12764, 16010, 25306, 29271	$1.0 \cdot 10^{11}$	$1.9 \cdot 10^{15}$
2006/10/02	5/2	16118, 34163, 34797, 35212	$2.1 \cdot 10^{11}$	$1.8 \cdot 10^{15}$
2006/10/09	2	14271, 18202, 24618, 38022	$2.8 \cdot 10^{13}$	$1.8 \cdot 10^{15}$
2006/10/16	5/2	8641, 30952, 34797, 35212	$1.6 \cdot 10^{11}$	$1.8 \cdot 10^{15}$
2006/10/23	5/2	13310, 21257, 21433, 34331	$1.4 \cdot 10^{11}$	$1.9 \cdot 10^{15}$
2006/10/30	5/2	21261, 34331, 34787, 38070	$1.0 \cdot 10^{11}$	$1.8 \cdot 10^{15}$
2006/11/06	5/2	4796, 9326, 9950, 17556	$1.1 \cdot 10^{11}$	$1.8 \cdot 10^{15}$
2006/11/13	5/2	9303, 15736, 31253, 38067	$1.7 \cdot 10^{11}$	$1.0 \cdot 10^{15}$
2006/11/20	5/2	20490, 24203, 25349, 30362	$1.5 \cdot 10^{12}$	$1.10 \cdot 10^{15}$
2006/11/27	5/2	21257, 24688, 29012, 34118	$1.0 \cdot 10^{11}$	$1.9 \cdot 10^{15}$
2006/12/04	5/2	17808, 17829, 2055, 38022	$1.8 \cdot 10^{11}$	$1.9 \cdot 10^{15}$
2006/12/11	5/2	12773, 20516, 20699, 39307	$1.0 \cdot 10^{11}$	$2.0 \cdot 10^{15}$
2006/12/18	5/2	8779, 13307, 20516, 21437	$1.9 \cdot 10^{11}$	$2.1 \cdot 10^{15}$
2006/12/25	5/2	13121, 25507, 29012, 34867	$2.5 \cdot 10^{11}$	$2.1 \cdot 10^{15}$
2007/01/01	5/2	9950, 10049, 18356, 24475	$1.4 \cdot 10^{11}$	$2.1 \cdot 10^{15}$
2007/01/08	5/2	9950, 10049, 18356, 24475	$1.4 \cdot 10^{11}$	$2.1 \cdot 10^{15}$
2007/01/15	5/2	3836, 4796, 18362, 24198	$2.1 \cdot 10^{11}$	$2.1 \cdot 10^{15}$
2007/01/22	5/2	1985, 4796, 9791, 9950	$1.4 \cdot 10^{11}$	$2.1 \cdot 10^{15}$
2007/01/29	5/2	8430, 20633, 39200, 41419	$9.2 \cdot 10^{10}$	$1.2 \cdot 10^{15}$
2007/02/05	5/2	3836, 4796, 14329, 37977	$1.3 \cdot 10^{11}$	$2.1 \cdot 10^{15}$
2007/02/12	5/2	1985, 4796, 9791, 9950	$1.1 \cdot 10^{11}$	$2.1 \cdot 10^{15}$

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Table 3 – continued from previous page

AS map name	δ	Certificate	Visited 4-tuples	Total number
2007/02/19	5/2	16309, 21488, 24618, 35483	$2.9 \cdot 10^{11}$	$2.1 \cdot 10^{15}$
2007/02/26	5/2	15458, 16231, 24618, 41308	$1.7 \cdot 10^{11}$	$2.2 \cdot 10^{15}$
2007/03/05	5/2	4855, 9881, 24490, 38296	$4.9 \cdot 10^{12}$	$2.3 \cdot 10^{15}$
2007/03/12	5/2	1727, 4796, 9791, 9950	$1.4 \cdot 10^{11}$	$2.3 \cdot 10^{15}$
2007/03/19	5/2	1727, 4796, 9791, 9950	$1.4 \cdot 10^{11}$	$2.3 \cdot 10^{15}$
2007/03/26	5/2	12790, 13107, 39528, 41684	$1.5 \cdot 10^{11}$	$2.4 \cdot 10^{15}$
2007/04/02	5/2	28858, 39528, 41390, 41684	$1.2 \cdot 10^{11}$	$2.3 \cdot 10^{15}$
2007/04/09	5/2	1955, 2549, 31218, 41832	$2.0 \cdot 10^{11}$	$2.4 \cdot 10^{15}$
2007/04/16	5/2	18082, 29012, 31347, 34870	$1.5 \cdot 10^{11}$	$2.4 \cdot 10^{15}$
2007/04/23	5/2	18082, 29012, 31347, 34870	$1.1 \cdot 10^{11}$	$2.3 \cdot 10^{15}$
2007/04/30	5/2	12790, 16070, 39528, 41684	$1.2 \cdot 10^{11}$	$2.5 \cdot 10^{15}$
2007/05/07	5/2	15700, 21257, 25386, 31023	$9.2 \cdot 10^{10}$	$2.4 \cdot 10^{15}$
2007/05/14	5/2	9881, 11189, 31253, 38155	$2.8 \cdot 10^{11}$	$2.5 \cdot 10^{15}$
2007/05/21	5/2	21257, 25386, 31023, 35339	$1.1 \cdot 10^{11}$	$2.5 \cdot 10^{15}$
2007/05/28	5/2	12374, 13107, 20699, 34792	$5.10 \cdot 10^{11}$	$2.5 \cdot 10^{15}$
2007/06/04	5/2	18393, 20459, 31253, 38155	$2.6 \cdot 10^{11}$	$2.5 \cdot 10^{15}$
2007/06/11	2	18308, 34411, 35710, 38941	$3.8 \cdot 10^{13}$	$2.6 \cdot 10^{15}$
2007/06/18	5/2	4621, 10202, 24455, 24514	$3.9 \cdot 10^{11}$	$2.5 \cdot 10^{15}$
2007/06/25	5/2	4796, 9326, 23596, 23900	$3.3 \cdot 10^{11}$	$2.6 \cdot 10^{15}$
2007/07/02	5/2	28790, 31598, 33974, 34094	$4.2 \cdot 10^{11}$	$2.6 \cdot 10^{15}$
2007/07/09	5/2	28790, 31598, 33974, 34094	$3.4 \cdot 10^{11}$	$2.7 \cdot 10^{15}$
2007/07/16	5/2	20797, 21061, 28886, 34620	$1.8 \cdot 10^{12}$	$2.6 \cdot 10^{15}$
2007/07/23	5/2	3836, 4796, 23900, 38155	$2.5 \cdot 10^{11}$	$2.6 \cdot 10^{15}$
2007/07/30	5/2	20516, 25507, 33974, 41338	$2.8 \cdot 10^{11}$	$2.7 \cdot 10^{15}$
2007/08/06	5/2	16007, 24609, 29685, 33974	$2.1 \cdot 10^{12}$	$2.7 \cdot 10^{15}$
2007/08/13	5/2	10364, 21257, 33974, 38962	$1.2 \cdot 10^{11}$	$2.7 \cdot 10^{15}$
2007/08/20	5/2	20516, 24681, 33974, 35409	$1.2 \cdot 10^{11}$	$2.6 \cdot 10^{15}$
2007/08/27	5/2	31023, 33974, 34870, 35795	$2.5 \cdot 10^{11}$	$2.6 \cdot 10^{15}$
2007/09/03	5/2	10364, 21257, 33974, 38962	$1.8 \cdot 10^{11}$	$2.7 \cdot 10^{15}$
2007/09/10	5/2	12976, 20294, 20578, 21104	$1.8 \cdot 10^{12}$	$2.7 \cdot 10^{15}$
2007/09/17	2	11025, 32262, 32703, 35183	$9.4 \cdot 10^9$	$3.1 \cdot 10^{13}$
2007/09/24	5/2	4796, 9326, 23596, 23900	$2.2 \cdot 10^{11}$	$2.8 \cdot 10^{15}$
2007/10/01	5/2	4796, 9326, 23596, 23900	$2.2 \cdot 10^{11}$	$2.9 \cdot 10^{15}$
2007/10/08	5/2	4855, 9881, 17827, 24490	$1.5 \cdot 10^{12}$	$2.8 \cdot 10^{15}$
2007/10/15	5/2	21104, 21131, 34742, 41588	$1.6 \cdot 10^{12}$	$2.8 \cdot 10^{15}$
2007/10/22	5/2	29275, 42802, 43022, 43120	$1.9 \cdot 10^{11}$	$2.9 \cdot 10^{15}$
2007/10/29	5/2	31489, 33974, 43111, 43120	$1.4 \cdot 10^{11}$	$2.9 \cdot 10^{15}$
2007/11/05	5/2	21497, 31023, 34265, 43435	$1.7 \cdot 10^{11}$	$2.9 \cdot 10^{15}$
2007/11/12	5/2	25119, 31166, 34742, 41588	$2.7 \cdot 10^{12}$	$2.9 \cdot 10^{15}$
2007/11/19	5/2	4855, 9881, 24490, 38296	$3.2 \cdot 10^{12}$	$2.10 \cdot 10^{15}$
2007/11/26	5/2	4855, 9881, 24490, 38296	$3.4 \cdot 10^{12}$	$2.10 \cdot 10^{15}$
2007/12/03	5/2	21497, 31023, 34265, 43435	$1.9 \cdot 10^{11}$	$2.10 \cdot 10^{15}$
2007/12/10	5/2	21497, 31023, 34265, 43435	$1.6 \cdot 10^{11}$	$2.10 \cdot 10^{15}$
2007/12/17	5/2	4855, 9881, 24490, 38296	$2.7 \cdot 10^{12}$	$2.9 \cdot 10^{15}$
2007/12/24	5/2	4855, 9881, 24490, 38296	$3.1 \cdot 10^{12}$	$3.3 \cdot 10^{15}$
2007/12/31	5/2	21497, 31023, 34265, 43435	$1.5 \cdot 10^{11}$	$3.2 \cdot 10^{15}$

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Table 3 – continued from previous page

AS map name	δ	Certificate	Visited 4-tuples	Total number
2008/01/07	5/2	21497, 25533, 31023, 43435	$1.1 \cdot 10^{11}$	$3.1 \cdot 10^{15}$
2008/01/14	5/2	20837, 21497, 25533, 31023	$9.7 \cdot 10^{10}$	$3.1 \cdot 10^{15}$
2008/01/21	5/2	21497, 25533, 31023, 43435	$1.2 \cdot 10^{11}$	$3.2 \cdot 10^{15}$
2008/01/28	5/2	25533, 31023, 43266, 43435	$8.2 \cdot 10^{10}$	$3.1 \cdot 10^{15}$
2008/02/04	5/2	4796, 9326, 23596, 24337	$1.3 \cdot 10^{11}$	$3.1 \cdot 10^{15}$
2008/02/11	5/2	31023, 34265, 43266, 43435	$1.3 \cdot 10^{11}$	$3.2 \cdot 10^{15}$
2008/02/18	5/2	31117, 33851, 42379, 42510	$3.3 \cdot 10^{12}$	$3.3 \cdot 10^{15}$
2008/02/25	5/2	15696, 20545, 42109, 44395	$1.1 \cdot 10^{11}$	$3.3 \cdot 10^{15}$
2008/03/03	5/2	11317, 13856, 23919, 38600	$1.10 \cdot 10^{11}$	$3.3 \cdot 10^{15}$
2008/03/10	5/2	11317, 13856, 23919, 38600	$1.10 \cdot 10^{11}$	$3.4 \cdot 10^{15}$
2008/03/17	5/2	4796, 7588, 9326, 23900	$1.9 \cdot 10^{11}$	$3.4 \cdot 10^{15}$
2008/03/24	5/2	9207, 15577, 29091, 42109	$5.5 \cdot 10^{10}$	$1.3 \cdot 10^{15}$
2008/03/31	5/2	15696, 20545, 42109, 44395	$1.1 \cdot 10^{11}$	$3.3 \cdot 10^{15}$
2008/04/07	5/2	12742, 39863, 43129, 44340	$1.1 \cdot 10^{11}$	$3.3 \cdot 10^{15}$
2008/04/14	2	10044, 24620, 24703, 34620	$4.4 \cdot 10^{13}$	$3.4 \cdot 10^{15}$
2008/04/21	5/2	15577, 17800, 27841, 29599	$1.6 \cdot 10^{11}$	$3.4 \cdot 10^{15}$
2008/04/28	5/2	9326, 23900, 23919, 24514	$2.2 \cdot 10^{11}$	$3.6 \cdot 10^{15}$
2008/05/05	5/2	14085, 29453, 33924, 34771	$2.0 \cdot 10^{11}$	$3.6 \cdot 10^{15}$
2008/05/12	2	12369, 23948, 42109, 43256	$4.4 \cdot 10^{13}$	$3.6 \cdot 10^{15}$
2008/05/19	5/2	7588, 10044, 18362, 38005	$1.3 \cdot 10^{11}$	$3.6 \cdot 10^{15}$
2008/05/26	5/2	12742, 39863, 43129, 44340	$1.0 \cdot 10^{11}$	$3.6 \cdot 10^{15}$
2008/06/02	5/2	15696, 17726, 24475, 24489	$7.6 \cdot 10^{10}$	$6.10 \cdot 10^{14}$
2008/06/09	5/2	12742, 42991, 43129, 44340	$1.2 \cdot 10^{11}$	$3.6 \cdot 10^{15}$
2008/06/16	5/2	9326, 23900, 24487, 38517	$2.0 \cdot 10^{12}$	$3.7 \cdot 10^{15}$
2008/06/23	2	12742, 25747, 42719, 44340	$4.6 \cdot 10^{13}$	$3.6 \cdot 10^{15}$
2008/06/30	2	13548, 34776, 40981, 44340	$4.8 \cdot 10^{13}$	$3.6 \cdot 10^{15}$
2008/07/07	2	4796, 9528, 10044, 25533	$4.5 \cdot 10^{13}$	$3.7 \cdot 10^{15}$
2008/07/14	2	4796, 9528, 10044, 35444	$4.7 \cdot 10^{13}$	$3.7 \cdot 10^{15}$
2008/07/21	5/2	24489, 34776, 35644, 43050	$2.6 \cdot 10^{12}$	$3.9 \cdot 10^{15}$
2008/07/28	5/2	12742, 15458, 39027, 43050	$1.7 \cdot 10^{11}$	$3.7 \cdot 10^{15}$
2008/08/04	2	14085, 16384, 30696, 36269	$1.8 \cdot 10^{10}$	$3.9 \cdot 10^{13}$
2008/08/11	5/2	2614, 20770, 34776, 43930	$1.9 \cdot 10^{11}$	$3.9 \cdot 10^{15}$
2008/08/18	2	20770, 25533, 34639, 34950	$3.6 \cdot 10^{13}$	$2.5 \cdot 10^{15}$
2008/08/25	5/2	12742, 20770, 30790, 34425	$2.0 \cdot 10^{11}$	$3.9 \cdot 10^{15}$
2008/09/01	5/2	12997, 25349, 43030, 43060	$3.2 \cdot 10^{11}$	$4.7 \cdot 10^{15}$
2008/09/10	2	8863, 10044, 12764, 42713	$6.4 \cdot 10^{13}$	$5.2 \cdot 10^{15}$
2008/10/20	5/2	17542, 36970, 36996, 38900	$1.7 \cdot 10^{11}$	$5.4 \cdot 10^{15}$
2008/11/10	5/2	15775, 34881, 41390, 41918	$2.7 \cdot 10^{11}$	$5.5 \cdot 10^{15}$
2008/12/01	5/2	20516, 41540, 42546, 48127	$2.5 \cdot 10^{11}$	$5.6 \cdot 10^{15}$
2008/12/22	2	19956, 22306, 27266, 42136	$7.6 \cdot 10^{13}$	$5.8 \cdot 10^{15}$
2009/01/05	2	8643, 15416, 17641, 29663	$7.5 \cdot 10^{13}$	$5.7 \cdot 10^{15}$
2009/01/10	5/2	18547, 18941, 30952, 47376	$2.5 \cdot 10^{11}$	$5.8 \cdot 10^{15}$
2009/01/22	2	11840, 41641, 42266, 65123	$8.0 \cdot 10^{13}$	$5.8 \cdot 10^{15}$
2009/02/01	2	8643, 10044, 36999, 44036	$7.9 \cdot 10^{13}$	$5.10 \cdot 10^{15}$
2009/02/20	2	15536, 40322, 44036, 45152	$7.8 \cdot 10^{13}$	$6.1 \cdot 10^{15}$
2009/03/11	2	10965, 17542, 28719, 39863	$8.1 \cdot 10^{13}$	$6.2 \cdot 10^{15}$

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Table 3 – continued from previous page

AS map name	δ	Certificate	Visited 4-tuples	Total number
2009/04/29	2	10200, 17739, 43060, 47254	$9.1 \cdot 10^{13}$	$3.10 \cdot 10^{15}$
2009/05/10	5/2	15440, 15536, 35358, 48053	$3.1 \cdot 10^{11}$	$6.5 \cdot 10^{15}$
2009/05/20	2	10078, 21274, 34312, 34776	$9.6 \cdot 10^{13}$	$6.6 \cdot 10^{15}$
2009/06/15	5/2	5530, 24609, 25029, 44784	$3.2 \cdot 10^{11}$	$6.7 \cdot 10^{15}$
2009/07/01	2	25035, 25503, 43030, 43934	$9.2 \cdot 10^{13}$	$6.8 \cdot 10^{15}$
2009/07/20	2	17641, 34543, 41281, 47791	$9.6 \cdot 10^{13}$	$6.8 \cdot 10^{15}$
2009/08/10	5/2	5379, 6906, 12345, 15507	$2.9 \cdot 10^{11}$	$6.9 \cdot 10^{15}$
2009/08/30	5/2	5379, 6906, 12345, 15507	$2.6 \cdot 10^{11}$	$6.10 \cdot 10^{15}$
2009/09/20	5/2	3344, 20545, 27933, 42991	$3.1 \cdot 10^{11}$	$7.1 \cdot 10^{15}$
2009/10/20	2	10097, 11001, 17668, 43030	$1.0 \cdot 10^{14}$	$7.4 \cdot 10^{15}$
2009/11/20	2	10044, 20803, 30928, 48302	$9.8 \cdot 10^{13}$	$7.5 \cdot 10^{15}$
2009/12/15	2	4382, 19616, 38140, 45146	$1.1 \cdot 10^{14}$	$7.8 \cdot 10^{15}$
2010/01/20	2	10200, 18061, 25319, 38140	$5.6 \cdot 10^{13}$	$8.0 \cdot 10^{15}$
2011/01/16	2	10044, 28750, 43111, 44379	$4.4 \cdot 10^{13}$	$1.2 \cdot 10^{16}$
2012/06/01	2	132106, 24489, 41068, 58445	$1.0 \cdot 10^{14}$	$1.9 \cdot 10^{16}$

C Hyperbolicity of some DIMES AS maps

Table 4 lists the hyperbolicity of all DIMES AS maps since 2007 [19], with the corresponding certificates. We have also reported in this table the number of visited 4-tuples for Algorithm 1 with and without good-core decomposition, as well as the total number of 4-tuples in the largest biconnected components of these graphs. We observe that Algorithm 1 out-performs the basic algorithm by a factor of at least 10^4 , and that the good-core decomposition provides in average a extra factor 10.

Table 4: Hyperbolicity of DIMES AS maps since 2007.

AS map name	δ	Certificate	Visited 4-tuples With	Without	Total number of 4-tuples
ASEdges1_2007	2	9814, 10063, 26415, 38019	$3.9 \cdot 10^{10}$	$4.2 \cdot 10^{11}$	$8.7 \cdot 10^{14}$
ASEdges2_2007	2	11546, 24398, 38012, 38018	$6.9 \cdot 10^{10}$	$6.5 \cdot 10^{11}$	$6.8 \cdot 10^{14}$
ASEdges3_2007	2	8809, 9791, 17423, 23647	$6.4 \cdot 10^{10}$	$8.5 \cdot 10^{11}$	$1.7 \cdot 10^{15}$
ASEdges4_2007	2	80, 6820, 8701, 9814	$9.4 \cdot 10^{10}$	$1.1 \cdot 10^{12}$	$1.2 \cdot 10^{15}$
ASEdges5_2007	2	2884, 7588, 10755, 40127	$3.3 \cdot 10^{10}$	$4.4 \cdot 10^{11}$	$1.8 \cdot 10^{15}$
ASEdges6_2007	2	6764, 8517, 9812, 15836	$5.6 \cdot 10^{10}$	$7.3 \cdot 10^{11}$	$1.4 \cdot 10^{15}$
ASEdges7_2007	2	10022, 27691, 28871, 32341	$1.0 \cdot 10^{10}$	$2.9 \cdot 10^{10}$	$2.5 \cdot 10^{15}$
ASEdges8_2007	2	8973, 29389, 34870, 35349	$3.1 \cdot 10^9$	$4.7 \cdot 10^{10}$	$2.3 \cdot 10^{15}$
ASEdges9_2007	2	841, 9575, 10550, 22684	$1.3 \cdot 10^{10}$	$1.9 \cdot 10^{11}$	$2.3 \cdot 10^{15}$
ASEdges10_2007	2	80, 8511, 34870, 39608	$2.0 \cdot 10^{10}$	$2.5 \cdot 10^{11}$	$1.9 \cdot 10^{15}$
ASEdges11_2007	2	80, 5719, 20661, 35349	$1.4 \cdot 10^{10}$	$1.4 \cdot 10^{11}$	$1.7 \cdot 10^{15}$
ASEdges12_2007	2	13079, 20661, 34205, 34400	$1.1 \cdot 10^{10}$	$1.1 \cdot 10^{11}$	$1.6 \cdot 10^{15}$
ASEdges1_2008	2	7588, 9781, 16905, 37977	$3.6 \cdot 10^{10}$	$3.9 \cdot 10^{11}$	$1.3 \cdot 10^{15}$
ASEdges2_2008	2	18353, 38018, 38319, 41050	$6.4 \cdot 10^{10}$	$4.4 \cdot 10^{11}$	$3.9 \cdot 10^{14}$
ASEdges3_2008	2	9270, 10063, 19008, 38511	$3.6 \cdot 10^{10}$	$4.3 \cdot 10^{11}$	$1.3 \cdot 10^{15}$
ASEdges4_2008	2	7588, 24040, 24287, 42918	$3.3 \cdot 10^{10}$	$4.2 \cdot 10^{11}$	$1.2 \cdot 10^{15}$
ASEdges5_2008	2	8226, 12497, 12987, 37971	$3.1 \cdot 10^{10}$	$3.6 \cdot 10^{11}$	$1.2 \cdot 10^{15}$
ASEdges6_2008	2	28947, 29394, 31094, 34875	$1.6 \cdot 10^{10}$	$1.4 \cdot 10^{11}$	$1.3 \cdot 10^{15}$
ASEdges7_2008	2	5572, 23030, 30214, 33962	$3.7 \cdot 10^{10}$	$9.1 \cdot 10^{10}$	$3.5 \cdot 10^{15}$
ASEdges8_2008	2	5572, 23030, 30214, 33962	$2.5 \cdot 10^{10}$	$1.2 \cdot 10^{12}$	$4.4 \cdot 10^{15}$
ASEdges9_2008	2	5572, 23030, 30214, 33962	$6.0 \cdot 10^{10}$	$2.7 \cdot 10^{11}$	$4.5 \cdot 10^{15}$
ASEdges10_2008	2	2588, 12261, 24042, 24475	$2.0 \cdot 10^{10}$	$8.0 \cdot 10^{10}$	$4.6 \cdot 10^{13}$
ASEdges11_2008	2	3836, 36996, 36982, 38600	$4.7 \cdot 10^{10}$	$4.6 \cdot 10^{11}$	$2.3 \cdot 10^{15}$
ASEdges12_2008	2	2012, 33852, 40981, 42688	$1.2 \cdot 10^{10}$	$1.4 \cdot 10^{11}$	$3.0 \cdot 10^{15}$
ASEdges1_2009	2	10063, 25253, 29673, 30947	$1.6 \cdot 10^{10}$	$1.3 \cdot 10^{11}$	$2.5 \cdot 10^{15}$
ASEdges2_2009	2	4667, 8701, 17752, 38018	$5.0 \cdot 10^9$	$4.1 \cdot 10^{10}$	$2.8 \cdot 10^{15}$
ASEdges3_2009	2	41831, 43040, 47360, 47392	$6.1 \cdot 10^9$	$7.8 \cdot 10^{10}$	$3.3 \cdot 10^{15}$
ASEdges4_2009	2	11182, 21719, 47392, 47732	$1.4 \cdot 10^{10}$	$1.6 \cdot 10^{11}$	$2.5 \cdot 10^{15}$
ASEdges5_2009	2	34205, 34974, 44674, 48858	$2.8 \cdot 10^9$	$2.1 \cdot 10^{10}$	$4.7 \cdot 10^{15}$
ASEdges6_2009	2	12304, 15895, 47326, 48008	$1.4 \cdot 10^{10}$	$5.9 \cdot 10^{10}$	$5.0 \cdot 10^{15}$
ASEdges7_2009	2	2148, 9519, 21418, 47645	$1.6 \cdot 10^{10}$	$2.7 \cdot 10^{10}$	$4.7 \cdot 10^{15}$
ASEdges8_2009	2	4796, 10063, 17585, 35518	$1.6 \cdot 10^9$	$1.6 \cdot 10^{10}$	$4.9 \cdot 10^{15}$
ASEdges9_2009	2	8462, 8940, 25432, 48965	$5.4 \cdot 10^9$	$1.9 \cdot 10^{11}$	$4.5 \cdot 10^{15}$
ASEdges10_2009	2	4667, 6342, 10131, 17484	$5.5 \cdot 10^9$	$5.6 \cdot 10^{10}$	$4.4 \cdot 10^{15}$
ASEdges11_2009	2	13247, 29259, 43910, 47310	$2.3 \cdot 10^{11}$	$5.4 \cdot 10^{10}$	$4.0 \cdot 10^{15}$

Continued on next page

Table 4 – continued from previous page

AS map name	δ	Certificate	Visited		Total
ASEdges12_2009	2	16333, 47360, 48011, 48265	$5.3 \cdot 10^{10}$	$7.2 \cdot 10^{10}$	$3.7 \cdot 10^{15}$
ASEdges1_2010	2	24783, 32277, 34205, 44674	$6.2 \cdot 10^9$	$6.5 \cdot 10^{10}$	$4.4 \cdot 10^{15}$
ASEdges2_2010	2	7588, 10004, 17823, 38550	$7.6 \cdot 10^9$	$5.1 \cdot 10^{10}$	$4.6 \cdot 10^{15}$
ASEdges3_2010	2	15236, 28509, 42688, 65090	$3.3 \cdot 10^9$	$4.5 \cdot 10^{10}$	$4.9 \cdot 10^{15}$
ASEdges4_2010	2	9038, 28666, 29238, 48832	$5.6 \cdot 10^9$	$6.0 \cdot 10^{10}$	$4.8 \cdot 10^{15}$
ASEdges5_2010	2	14328, 19915, 22512, 33738	$5.5 \cdot 10^{10}$	$6.2 \cdot 10^{10}$	$5.2 \cdot 10^{15}$
ASEdges6_2010	2	9082, 45012, 45844, 47442	$6.2 \cdot 10^{10}$	$2.7 \cdot 10^{11}$	$4.8 \cdot 10^{15}$
ASEdges7_2010	2	8601, 39644, 48759, 50164	$2.4 \cdot 10^{10}$	$4.6 \cdot 10^{11}$	$4.6 \cdot 10^{15}$
ASEdges8_2010	2	16030, 28951, 29015, 35444	$9.4 \cdot 10^{10}$	$3.9 \cdot 10^{11}$	$5.7 \cdot 10^{15}$
ASEdges9_2010	2	8395, 18239, 23850, 42136	$1.5 \cdot 10^{10}$	$1.1 \cdot 10^{11}$	$5.2 \cdot 10^{15}$
ASEdges10_2010	2	10150, 18252, 23940, 38195	$2.7 \cdot 10^{10}$	$8.9 \cdot 10^{10}$	$5.7 \cdot 10^{15}$
ASEdges11_2010	2	10282, 16292, 196840, 39425	$5.6 \cdot 10^9$	$5.2 \cdot 10^{10}$	$5.3 \cdot 10^{15}$
ASEdges12_2010	2	8557, 18352, 38437, 55454	$2.5 \cdot 10^{10}$	$7.6 \cdot 10^{11}$	$5.2 \cdot 10^{15}$
ASEdges1_2011	2	2503, 18088, 36790, 38043	$9.2 \cdot 10^{10}$	$7.9 \cdot 10^{11}$	$5.9 \cdot 10^{15}$
ASEdges2_2011	2	12576, 29032, 29039, 36892	$3.8 \cdot 10^{10}$	$1.7 \cdot 10^{11}$	$6.8 \cdot 10^{15}$
ASEdges3_2011	2	15633, 16030, 25247, 31483	$9.0 \cdot 10^{10}$	$1.5 \cdot 10^{11}$	$4.9 \cdot 10^{15}$
ASEdges4_2011	2	8193, 34557, 41904, 43246	$1.6 \cdot 10^9$	$2.3 \cdot 10^{10}$	$5.0 \cdot 10^{15}$
ASEdges5_2011	2	10207, 11879, 262815, 28174	$1.4 \cdot 10^{10}$	$6.2 \cdot 10^{10}$	$4.8 \cdot 10^{15}$
ASEdges6_2011	2	841, 6853, 8338, 25103	$4.0 \cdot 10^{10}$	$6.3 \cdot 10^{11}$	$4.3 \cdot 10^{15}$
ASEdges7_2011	2	2874, 29498, 44192, 45289	$1.1 \cdot 10^{11}$	$8.9 \cdot 10^{11}$	$4.7 \cdot 10^{15}$
ASEdges8_2011	2	23817, 25669, 33516, 40335	$3.9 \cdot 10^{10}$	$1.0 \cdot 10^{11}$	$4.1 \cdot 10^{15}$
ASEdges9_2011	2	32017, 37187, 37245, 49283	$6.8 \cdot 10^9$	$8.7 \cdot 10^{10}$	$4.0 \cdot 10^{15}$
ASEdges10_2011	2	12705, 29304, 34875, 53603	$1.4 \cdot 10^{10}$	$2.0 \cdot 10^{11}$	$3.6 \cdot 10^{15}$
ASEdges1_2012	2	10110, 18021, 35794, 38305	$2.6 \cdot 10^{10}$	$2.5 \cdot 10^{11}$	$3.1 \cdot 10^{15}$
ASEdges2_2012	2	24490, 28249, 45344, 45410	$2.7 \cdot 10^{10}$	$3.2 \cdot 10^{11}$	$3.7 \cdot 10^{15}$
ASEdges3_2012	2	7491, 41579, 47654, 56384	$3.3 \cdot 10^{10}$	$4.4 \cdot 10^{11}$	$3.7 \cdot 10^{15}$
ASEdges4_2012	2	24337, 38296, 38738, 47652	$4.6 \cdot 10^{10}$	$5.6 \cdot 10^{11}$	$3.4 \cdot 10^{15}$



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