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# THE FILTERING STEP OF DISCRETE LOGARITHM AND INTEGER FACTORIZATION ALGORITHMS 

CYRIL BOUVIER


#### Abstract

The security of most current public-key cryptosystems is based on the difficulty of finding discrete logarithms in large finite fields or factoring large integers. Most discrete logarithm and interger factoring algorithms, such as the Number Field Sieve (NFS) or the Function Field Sieve (FFS), can be described in 3 main steps: data collection, filtering and linear algebra. The goal of the filtering step is to generate a small, sparse matrix over a finite field, for which one will compute the kernel during the linear algebra step. The filtering step is mainly a structured Gaussian elimination (SGE). For the current factorization records, billions of data are collected in the first step and have to be processed in the filtering step. One part of the filtering step is to remove heavy rows of the matrix. The choice of the weight function to select heavy rows is critical in order to obtain the smallest matrix possible. In this paper, several weight functions are studied in order to determine which one is more suited in the context of discrete logarithm and factorization algorithms.


Keywords: filtering, linear algebra, structured Gaussian elimination, sparse linear system, factorization, Number Field Sieve, discrete logarithm, Function Field Sieve.

## 1. Introduction

The difficulty of the discrete logarithm and integer factorization problems is the key of the security of most current public-key cryptosystems. Most of the algorithms solving these problems use the relation collection method. The filtering step is a common step of all such algorithms, like the Quadratic Sieve (QS) or the Number Field Sieve (NFS) for the factorization problem, and the Coppersmith algorithm, the Function Field Sieve (FFS) or the Number Field Sieve (NFS-DL) for the discrete logarithm problem. The main goal of the filtering step is to reduce the size of a large sparse matrix over a finite field, constructed from the data collected previously by the algorithm, in order to being able to compute its kernel.

The main difference between the discrete logarithm and the factorization context is that in the former the matrix is defined over $\operatorname{GF}(q)$, with $q$ a prime of 160 to 320 bits, with current standards of security, as in the latter the matrix is defined over GF(2).

The filtering step is mostly a structured Gaussian elimination (SGE) [12, 17]. A complete description of the state of the art of the filtering step in the factorization context can be found in Cavallar's thesis [7]. Some integer factorization tools that contain code for filtering are CADO-NFS [2], Msieve [16] and GGNFS [15]. Code for the filtering step in the discrete logarithm context is available in the development version of CADO-NFS.

During the filtering step, the "heaviest" rows are removed. The definition of heavy depends on the choice of a weight function. After giving on overview of relation collection algorithms in the rest of the introduction, and describing in detail the filtering step in Section 2, we will propose, in Section 3, new weight functions for the filtering step and present experiments on three real-case integer factorizations in Section 4 and on three real-case discrete logarithm computations in Section 6. The main contribution of this article is to present several weight functions, including ones previously described in the literature, in an unified formalism and to compare them on real-case factorizations and discrete logarithm computations. With these comparisons, better weight functions as the ones previously used will emerge.
1.1. Overview of the relation collection method. In order to understand the goal of the filtering step, we give a description of the NFS algorithm to illustrate the relation collection method in the factorization context, and a description of the FFS algorithm to illustrate the relation collection method in the discrete logarithm context. The other algorithms based on relation collection differ in some way, but the filtering step remains the same.
1.1.1. The relation collection method for factoring: the example of NFS. For a more detailed description of NFS, see [13].

The goal of the algorithm is to find an equality between two squares modulo the number $n$ that one wants to factor. The algorithm looks for two integers $x$ and $y$ such that $x^{2} \equiv y^{2}(\bmod n)$.

The NFS algorithm can be described in 3 main steps. A first step (data collection) computes lots of relations of the form

$$
\prod_{i} p_{i}^{e_{i}}=\prod_{j} q_{j}^{f_{j}}
$$

where the $p_{i}$ and the $q_{j}$ represent prime ideals in two distinct number fields. In the following, we will not make the distinction between prime ideals of the two number fields and call them ideals.

The goal is to find a subset of all the relations whose product is a square on each side of the equality. This can be rephrased as: for each ideal $p$, the sum of the exponents of $p$ in all chosen relations must be even.

This can be translated in a linear algebra problem. If one sees the relations as rows of a matrix where each column corresponds to an ideal, the coefficients of the matrix are the exponents of the ideals in the relations. As we are looking for even exponents, one can consider the matrix over GF(2). Finding a linear combination of relations such as every exponent is even is equivalent to computing the (left) kernel of the matrix.
1.1.2. The relation collection method for discrete logarithm: the example of FFS. For a more detailed description of FFS, see [1].

The goal of the algorithm is to compute logarithms of "small" elements of the field. In order to do so, there is still a data collection step that will compute relations between ideals in two distinct function fields. Then, one wants to compute "logarithms" of these ideals. This can be translated in a linear algebra problem. If, once again, one sees the relations as rows of a matrix where each column corresponds to an ideal, the coefficients of the matrix are the exponents of the ideals in the
relations. As we are looking for logarithms in a group of prime order $q$, one can consider the matrix over $\operatorname{GF}(q)$.

Finding the logarithms of the ideals is equivalent to solve this linear system, i.e. computing the (right) kernel of the matrix.
1.1.3. Common points and differences for the filtering step. In the two contexts, the goal of the filtering step is to reduce a matrix defined over a finite field in order to accelerate the linear algebra step, which consists in computing the (left or right) kernel of the matrix and constructing the solution from this kernel.

The excess is defined as the difference between the number of rows (relations) and the number of columns (ideals). The relative excess is defined as the ratio between the excess and the number of columns (ideals).

In the factorization context, the matrix is defined over GF(2). The linear algebra step computes the left kernel, so the excess should always be positive in order for a nontrivial kernel to surely exist. In practice one tries to have an excess around 150 before linear algebra to have enough vectors in the kernel. In the discrete logarithm context, the matrix is defined over $\operatorname{GF}(q)$ where $q$ is a prime of 160 to 320 bits. The linear algebra step computes the right kernel, and in practice one wants this kernel to have dimension 1, so the excess should not be negative. In practice one tries to have an excess of zero before linear algebra as having more rows than column is useless in this context.

By construction, the matrices at the beginning of the filtering step are sparse (about 20 non-zero coefficients per row). In the linear algebra step, algorithms dedicated to sparse matrices are used in order to deal with matrices from hundreds of thousands to hundreds of millions of rows and columns. The cost of those algorithms depends on the matrix dimensions and on its density. So at the end of the filtering step one should try to keep the matrix as sparse as possible (around 100-200 non-zero coefficients per row).
1.2. Usefulness of the filtering step. The goal of the filtering step is to reduce the size of the matrix that is going to be solved in the linear algebra step. In order to realize in which proportion the reduction is made, this is illustrated with data from record factorizations with NFS and record discrete logarithm computations on prime extensions of the binary field with FFS.

The current NFS record factorization is RSA-768 [11], an integer of 768 bits (232 digits). At the beginning of the filtering step, the matrix had about 47 billion rows and 35 billion columns. After the first part of the filtering step (see Section 2 for more detail), the matrix had about 2.5 billion rows and 1.7 billions columns. At the end of the filtering step, the matrix used in the linear algebra had about 193 million rows and columns. So the filtering step reduced the size of the matrix by more than $99 \%$ (it was reduced by almost $95 \%$ after the first part).

The current FFS record over $\operatorname{GF}\left(2^{n}\right)$, with $n$ prime, is computing a discrete logarithm in $\operatorname{GF}\left(2^{809}\right)$ [5]. At the beginning of the filtering step, the matrix had about 80 million rows and 40 million columns. After the first part of the filtering step, the matrix had about 10 million rows and columns. At the end of the filtering step, the matrix used in the linear algebra had about 3.6 million rows and columns. So the filtering step reduced the size of the matrix by $96 \%$ (it was reduced by almost $88 \%$ after the first part).

## 2. Description of the filtering step

The goal of the filtering step is to construct a matrix which is the smallest and the sparsest possible. Then, in the linear algebra step, one finds the kernel of that matrix. The following description of the state of the art for the filtering step is an adaptation of [7], in order to work in the factorization and the discrete logarithm context.

The filtering step is classically divided in 3 parts. The first part removes duplicate relations. It will not be studied in this article, as the set of input relations will always be a set of unique relations. The second part, called purge (Section 2.1), removes singletons (ideals appearing only once, or equivalently columns with only one non-zero coefficient) and reduces the size of the matrix thanks to the clique removal algorithm. The last part, called merge (Section 2.2), is a beginning of Gaussian elimination that reduces the size of the matrix by combining rows (and thus making the matrix less sparse).

The purge algorithm is similar to a structured Gaussian elimination (SGE) [12, 17]. SGE begins by removing columns without non-zero coefficients, which do not exist in our case by construction of the matrix. Then, SGE removes columns with exactly one non-zero coefficient and heavy rows, which corresponds to the purge algorithm. The difference between SGE and the filtering step of relation collection algorithms is that purge is followed by merge, so the best weight function adapted to these two cases may be different.

Definition 1. The weight of a row or a column of the matrix is the number of non-zero coefficients of the row or the column. By extension, one can talk about the weight of a relation or an ideal. The total weight of a matrix is the number of non-zero coefficients in the matrix.
2.1. Purge. The purge algorithm is identical in the factorization and the discrete logarithm contexts.

A singleton is a column of the matrix of weight 1 (with only one non-zero coefficient). The corresponding row and column in the matrix can be removed from the matrix without loss of any information. The first part of purge is to remove all singletons and the corresponding rows and columns. Removing all singletons gives a deterministic results, as no choice has to be made. Moreover, when one removes a singleton, one removes a row and a column, so the excess does not change (except in the very rare case where a relation contains more than one singleton).

Example 2. Let us consider the matrix

$$
M_{0}=\left(\begin{array}{llllll}
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

One can see that the first column is a singleton, so one can remove the first column and the second row. By doing this, one creates a singleton (the second column of $M_{0}$ ) and so one can remove the second column and the first row. The remaining
matrix is

$$
M_{1}=\left(\begin{array}{llll}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

with no more singleton and excess 2.
Removing excess rows is possible while one is sure that the kernel is still of desired dimension. In the factorization context, when one removes a row, one looses some information on the kernel, but as only a few vectors of the kernel are necessary, one can remove rows until the excess is around 150. In the discrete logarithm context, no information is lost while the excess in non-negative.

In particular, if there is a column of weight 2 and one of the rows corresponding to a non-zero coefficient is removed, then the column becomes a singleton and will be removed during the next singleton phase (as well as the other row corresponding to the other non-zero coefficient). Thus two rows and a column of the matrix will have been deleted, reducing the excess by 1 .

Definition 3. Let us consider the graph where the nodes are the rows of the matrix and the edges are the columns of weight 2 , connecting the two corresponding rows / nodes. A "clique" is a connected component of this graph.

Remark 4. Even though the component is not a clique in the common sense of graph theory, in the context of filtering it is traditional to call it clique [7], thus we keep that terminology.

Example 5. Let us consider the matrix $M_{1}$ at the end of Example 2. One can see that there are 3 cliques: one containing 3 rows (the first, second and third rows), one containing two rows (the fourth and the fifth) and one containing one row. The graph obtained from this matrix is presented in Figure 1. Removing one of these three cliques reduces the excess by 1. Removing the clique with three rows connects the two other cliques via the last column.


Figure 1. A graph with 3 cliques, constructed from matrix $M_{1}$ of Example $2\left(r_{i}\right.$ corresponds to the $i$ th row and $c_{j}$ to the $j$ th column).

The stage of purge that removes excess rows is called clique removal (or pruning as in [7]), as it will, once all the singletons have been removed, compute all the cliques of the matrix and remove some of them. As one can remove any row, one can choose the cliques to remove in order to obtain the smallest, sparsest matrix possible. On the clique removal stage, a weight is given to each clique and the
heaviest ones (with respect to that weight function) are deleted. The choice of the weight function associated to a clique is crucial to obtain a small and sparse matrix.

Every algorithm removing rows can be written as a clique removal algorithm with an appropriate weight function, as every row belongs in a clique (possibly a clique with only one row) and removing a row corresponds to removing the entire clique containing this row.
2.2. Merge. The merge stage is a beginning of Gaussian elimination, as one combines some rows and deletes some columns in order to reduce the size of the matrix. In this stage, the total weight of the matrix usually increases.
2.2.1. The merge algorithm in the factorization context. Recall that, in the factorization context, the matrix is defined over $\mathrm{GF}(2)$, so that adding two non-zero coefficients gives a zero coefficient.

In order to understand the idea behind the merge algorithm, let us begin with two examples. Let $p$ be an ideal representing a column of weight 2 and let $r_{1}$ and $r_{2}$ be the two rows corresponding to those two non-zero coefficients. If one replaces in the matrix the row $r_{1}$ by $r_{1}+r_{2}$, then the column corresponding to the ideal $p$ becomes a singleton and can be deleted, as well as the row $r_{2}$ which is the only remaining row containing the ideal $p$. The idea is that if the relation $r_{1}$ (resp. $r_{2}$ ) is used in the kernel, as the exponent of the ideal $p$ must be even and the only other relation with an odd exponent for $p$ is $r_{2}$ (resp. $r_{1}$ ), it must also be used. This operation is called a 2 -merge. A 2-merge removes one row and one column (the one corresponding to $p$ ) of the matrix, thus leaving the excess unchanged.

Now, if the column corresponding to the ideal $p$ has weight 3 and $r_{1}, r_{2}$ and $r_{3}$ are the rows corresponding to those 3 non-zero coefficients, then, as the previous case, if one replaces $r_{1}$ by $r_{1}+r_{3}, r_{2}$ by $r_{2}+r_{3}$, the column corresponding to $p$ becomes a singleton and can be deleted, as well as $r_{3}$. This operation is called a 3 -merge. A 3 -merge removes one row (in this case $r_{3}$ ) and one column (the one corresponding to $p$ ) of the matrix, thus leaving the excess unchanged. An important difference with a 2 -merge is that there is a choice to be made because there is more than one way of combining the 3 relations. Indeed, one can also choose to add $r_{2}$ to both $r_{1}$ and $r_{3}$ and to delete $r_{2}$, or to add $r_{1}$ to both $r_{2}$ and $r_{3}$ and to delete $r_{1}$.

Example 6. For this example, let us consider the matrix $M_{0}$ of Example 2, while forgetting that the first column is a singleton. The second column has weight 2 , so one can do a 2-merge. If one adds the second row to the first row, then column 2 becomes a singleton and the second column and the second row can be deleted.

An example of 3 -merge comes from the third column which has weight 3 . One can see that, in order to minimize the total weight of the matrix, one should add row 4 to row 1 and row 3 .

The following definition generalizes those examples to a $k$-merge, for any integer $k \geq 2$.

Definition 7. Let $k \geq 2$ be a positive integer. Let $p$ be an ideal representing a column of weight $k$ and let $r_{1}, \ldots, r_{k}$ be the $k$ rows corresponding to these $k$ nonzero coefficients. A $k$-merge is a way of performing successive rows additions of the form $r_{i} \leftarrow r_{i}+r_{j}$, with $i \neq j$ and $1 \leq i, j \leq k$, such that the column corresponding to $p$ becomes a singleton that is deleted (as well as the only remaining row containing a non-zero coefficient in the column corresponding to $p$ ).

A $k$-merge removes one row and one column, so a $k$-merge does not change the excess of the matrix (except in the very rare case where the excess can increase if doing a $k$-merge allows one to delete more than one column).

A 2-merge always reduces the total weight of the matrix. If the two relations $r_{1}$ and $r_{2}$ had respectively $w_{1}$ and $w_{2}$ non-zero coefficients, then the combined row $r_{1}+r_{2}$ will have at most $w_{1}+w_{2}-2$ non-zero coefficients. So every 2-merge reduces the number of rows by 1 and the total weight by at least 2 . For a $k$-merge $(k>2)$, the total weight of the matrix will increase in general. In order to control the increase of the total weight of the matrix, one will choose the way of combining the $k$ rows that minimize this increase. In order to do that, one can see the rows as nodes of a graph where all nodes are connected with weights corresponding to the weight of the row resulting of the addition of the two nodes. Finding the best way to combine the $k$ rows is equivalent to finding a spanning tree of minimal weight in this graph (an example is given in Figure 2).

|  | $r_{2}$ | $r_{4}$ | $r_{5}$ | $r_{7}$ | $r_{8}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $r_{1}$ | 4 | 2 | 4 | 2 | 3 |
| $r_{2}$ |  | 4 | 2 | 4 | 3 |
| $r_{4}$ |  |  | 2 | 2 | 1 |
| $r_{5}$ |  |  |  | 2 | 1 |
| $r_{7}$ |  |  |  |  | 1 |



Figure 2. Example of a 6 -merge from the sixth column of matrix $M_{0}$ of Example 2. The table on the left gives the weight of the sum of $r_{i}$ and $r_{j}, i<j$. The figure on the right shows the minimal spanning tree associated to these weights; the edges link two nodes that are going to be added in the 6 -merge.

Markovitz algorithm [14] is used to choose the merges that are going to be made. The merges are in a heap sorted by the weight increase corresponding to that merge.

The merge algorithm always reduces the number of rows but usually increases the total weight of the matrix. So one has to stop while the matrix is still sparse.
2.2.2. Adaptation to the discrete logarithm context. There is only one change is the merge algorithm in the discrete logarithm context. If $p$ is an ideal which occurs in $r_{i}$ and $r_{j}$ and if one computes $r_{i} \leftarrow r_{i}+r_{j}$, the column corresponding to $p$ does not usually disappear in the new relation. In order to achieve that, one has to compute $r_{i} \leftarrow \alpha r_{i}+\beta r_{j}$ where $\alpha$ and $\beta$ are non-zero and depend on the values of the coefficients corresponding to $p$ in $r_{i}$ and $r_{j}$. Apart from that, nothing changes in the merge algorithm.

In theory, the values of the coefficients are in $\mathrm{GF}(q)$, and when two rows are combined the computation of the new coefficients should be done modulo $q$, but in practice one considers that the coefficients lie in the ring of integers $\mathbb{Z}$. Experimental data show that the initial values of the coefficients are very small relatively to $q$ and that, at the end of merge, more than $99.5 \%$ of the coefficients are in $[-10,10]$ and more than $90 \%$ are $\pm 1$.

The problem we address. Computing the (left or right) kernel in the linear algebra step is done with algorithms dedicated to sparse matrices (as the Wiedemann [19] or Lanczos [8] algorithms). These algorithms have a time complexity which is, when one only considers the first term, proportional to the product of the number of rows and the total weight of the final matrix [18, Section 5]. To sum up, the problem of the filtering step is: given as input a set of unique relations, a target excess and a target average weight for the final matrix, what weight function should be used in the clique removal algorithm in order for the filtering step to produce the smallest matrix possible?

## 3. Several weight functions for clique removal

What one wants to improve is the size of the matrix at the end of the filtering step (for a given target excess and average weight). It means that the weight function used in the clique removal stage has to produce the smallest matrix at the end of merge, and not necessarily at the end of purge. So a good weight function should remove lots of heavy relations but also reduce the total weight of the matrix and create many 2-merges that will reduce the total weight of the matrix during merge.

Before defining the weight functions that are going to be studied, some notations are needed. At the beginning of the clique removal stage, it is assumed that there is no more singleton. The matrix has $N$ rows and the total weight is denoted as $W$ (number of non-zero coefficients). Let $c$ be a clique. The number of relations contained in the clique is denoted as $n(c)$. By definition of a clique, the $n(c)$ relations of the clique $c$ are linked by $n(c)-1$ columns of weight 2 . The weight associated to the clique is denoted as $\Omega(c)$. If $p$ is an ideal, then $w(p)$ is the weight of the column that corresponds to the ideal $p$.

One wants to delete ideals of small weight in order to obtain more singletons and 2 -merges, so the weight associated to a column appearing in the clique should be a decreasing function of $w(p)$. Also, due to the nature of the data collecting step, large ideals have very few occurrences, so the weight associated to a column appearing in the clique could also be an increasing function of the "size" of $p$.
3.1. Weight functions studied. We propose to consider weight functions of the following form: $\Omega(c)=\Lambda(c)+\nu(c)$.

The function $\Lambda$ measures the contribution of the ideals of weight greater or equal to 3 . This function will favor removing cliques with many ideals, or cliques which contain ideals that can create 2 -merges. In the experiments, 7 functions $\Lambda_{0}, \ldots, \Lambda_{6}$ are tried, they all are of the following shape:

$$
\Lambda_{i}(c)=\sum_{p \in c, w(p) \geq 3} \lambda_{i}(w(p))
$$

where the sum is taken for all ideal $p$ that appears in a relation of the clique and with weight greater or equal to 3 (an ideal appearing in more than one relation of the clique contribute for each appearance). The formulae for the $\lambda_{i}$ functions are:

$$
\begin{gathered}
\lambda_{0}(w)=1, \quad \lambda_{1}(w)=\left(\frac{2}{3}\right)^{w-2}, \quad \lambda_{2}(w)=\left(\frac{1}{2}\right)^{w-2}, \quad \lambda_{3}(w)=\left(\frac{4}{5}\right)^{w-2} \\
\lambda_{4}(w)=\frac{1}{\log _{2}(w)}, \quad \lambda_{5}(w)=\frac{2}{w}, \quad \text { and } \quad \lambda_{6}(w)=\frac{4}{w^{2}}
\end{gathered}
$$

Note that for all $\lambda_{0}, \ldots, \lambda_{6}$ functions, the value in $w=2$ is 1 . The case where the $\Lambda$ function is zero is treated in the additional weight functions described below.

The function $\nu$ measures the contribution of the ideals of weight 2 , that is the ideals that link the relations of the clique together. So the function $\nu$ is in fact a function depending of $n(c)$ and will try to remove cliques with many relations in order to reduce the size of the matrix. The following formulae for the $\nu$ function were tried: $\nu_{0}(c)=0, \nu_{1}(c)=\frac{n(c)}{4}, \nu_{2}(c)=\frac{n(c)}{2}$ and $\nu_{3}(c)=n(c)$.

With these 7 formulae for $\Lambda$ and 4 formulae for $\nu$, one can construct 28 different weight functions $\Omega_{x y}(c)=\Lambda_{x}(c)+\nu_{y}(c)$. For example,

$$
\Omega_{23}(c)=\Lambda_{2}(c)+\nu_{3}(c)=\sum_{z(p) \geq 3} \frac{1}{2^{w(p)-2}}+n(c)
$$

Remark 8. In the discrete logarithm context, the weight functions do not depend on the values of the non-zero coefficients but depend only, via $w(p)$, on the fact that the coefficients are zero or non-zero.

In addition to the 28 weight functions constructed above, 3 others were tried:
(1) The relations of the clique are linked with columns of weight 2 that can be used to perform 2-merges. The weight of the clique $c$ is the weight of the remaining relation once all the 2 -merges possible inside the clique have been made. The weight of the remaining relation is exactly the number of ideals that appear an odd number of times in the clique, so

$$
\Omega_{s 0}(c)=\sum_{w(p) \geq 2} \frac{1-(-1)^{w_{c}(p)}}{2}
$$

where $w_{c}(p)$ count the number of relations belonging to the clique $c$ that contains the ideal $p$. This function is similar to $\Omega_{00}$, if one consider that $w_{c}(p)=0$ or 1 when $w(p) \geq 3$ (which is almost always the case for "large" ideals).
(2) The weight function corresponds to the case where the $\Lambda$ function is zero:

$$
\Omega_{s 1}(c)=\nu_{3}(c)=n(c)
$$

(3) The expression of this weight function is inspired from the derivative of the product $N \times W$, which is what we are trying to minimize:

$$
\Omega_{s 2}(c)=\frac{n(c)}{N}+\frac{\Lambda_{0}(c)}{W}
$$

In total, 31 weight functions were tested, the results of their comparison are given in Section 4 in the context of integer factorization and in Section 6 in the context of discrete logarithm.
3.2. Previously used weight functions. In Cavallar's thesis, a weight function is proposed by the author [7]:
"The metric being used weighs the contribution from the small prime ideals by adding 1 for each relation in the clique and 0.5 for each free relation. The large prime ideals which occur more than twice in the relation table contribute $0.5^{f-2}$ where $f$ is the prime ideal's frequency."

Since the free relations only represent a small percentage (less than $0.5 \%$ ) of the total amount of relations, one can considered that the weight function $\Omega_{23}$ is the same as the one defined by Cavallar.

The weight function used in Msieve 1.50 is Cavallar's weight function, $\Omega_{23}$, where only ideals up to weight 15 are considered. In practice, it gives exactly the same results. Cavallar's weight function was also used in the filtering code of the CWI factoring tool.

In GGNFS 0.77.1, the weight of a clique is the sum of the memory used to store all the relations of the clique, which correspond, up to a constant, to $\Omega_{03}$.

The weight function $\Omega_{s 1}$ is, up to a constant, the default weight function used in CADO-NFS 1.0 and CADO-NFS 1.1. The weight function $\Omega_{s 2}$ was also available in the source code of CADO-NFS 1.0 and CADO-NFS 1.1 as an alternative to $\Omega_{s 1}$.

## 4. Comparison of the weight functions in a NFS SEtup

The 31 weight functions described in the Section 3 have been implemented in the development version of CADO-NFS and have been tested on data from three different real-case factorizations: RSA-155, B200 and RSA-704. The original data were used for B200 and RSA-704, and the data for RSA-155 were recomputed with CADO-NFS.

RSA-155 is a 155 -digit, 512 -bit integer that was factored in 1999 [6] using the NFS algorithm. It was part of the RSA challenge. B200 is the 200th Bernouilli number. In the factorization of the numerator of B200, a 204-digit (677-bit) composite integer remained unfactored until August 2012 [10]. RSA-704 is a 212-digit, 704-bit integer that was factored in July 2012 using CADO-NFS [3]. It was part of the RSA challenge.

The computations were done with the development version of the CADO-NFS software. As CADO-NFS 1.1, Msieve 1.50 and GGNFS 0.77.1 have different methods of combining the singleton and clique removal stages, the implementation in the development version of CADO-NFS of their weight functions (respectively $\Omega_{s 1}$, $\Omega_{23}$ and $\Omega_{03}$ ) allows us to have a fair comparison where only the weight functions differ (and not the interleaving of the singleton removal and clique removal stages).
4.1. Experiments. The purge algorithm used in these computations interleaves 50 stages of singleton removal and 50 stages of clique removal. On each singleton removal stage all the singletons are removed from the matrix. On each clique removal stage, the excess is decreased by $1 / 50$ of the difference beetwen the initial excess and the desired final excess. This strategy is used in the development version of CADO-NFS. The filtering step (purge and merge) was run for the 31 weight functions without any difference in the other parameters, shown in Table 1.

Let recall that the time of the linear algebra step is mostly proportional to the product $N \times W$. And the average weight of a row at the end of merge is the same for all weight functions, thus the product $N \times W$ depends only on the size of the matrix at the end of merge. So the smaller the size of the matrix is at the end of merge, the better the weight function is. But, in the following, when we say that a weight function is $x \%$ better than another, we mean that the product $N \times W$ is $x \%$ smaller, and not only the size of the matrix.
4.2. Results. In this section, only partial results are presented, the complete data are in the Appendix. The only weight functions that are shown are: the two best

|  |  | RSA-155 | B200 | RSA-704 |
| :---: | :--- | :---: | :---: | :---: |
| Beginning | Number of relations | 97.0 M | 702 M | 833 M |
|  | Number of ideals | 89.7 M | 667 M | 745 M |
| After first singleton removal | Number of relations | 62.8 M | 453 M | 650 M |
|  | Number of ideals | 51.9 M | 385 M | 552 M |
|  | Excess | 10.8 M | 67.4 M | 98.5 M |
|  | Relative excess | $20.8 \%$ | $17.5 \%$ | $17.9 \%$ |
| End of purge | Excess | 160 | 160 | 160 |
| End of merge | $k$-merge for $k \in$ | $[2,30]$ | $[2,30]$ | $[2,50]$ |
|  | $W / N$ | 100 | 120 | 187 |

Table 1. Common parameters for each factorization
ones, the one that gives the smallest matrix after purge, the worst one and ones of Cavallar $\left(\Omega_{23}\right)$, CADO-NFS $1.1\left(\Omega_{s 1}\right)$ and GGNFS $0.77 .1\left(\Omega_{03}\right)$ if they are not already in the table.

Summaries of the results for RSA-155, B200 and RSA704 are given, respectively, in Table 2, Table 3 and Table 4. Complete data are shown, respectively, in Table 9, Table 10 and Table 11 in the Appendix.

|  | After purge | After merge |  |  |
| ---: | :---: | :---: | :---: | :---: |
| $\Omega$ | $N$ | $N$ | $W \times N$ |  |
| (best) 11 | 20801141 | 6576610 | $4.33 \mathrm{e}+15$ |  |
| (second best) 21 | 20812812 | 6586358 | $4.34 \mathrm{e}+15$ | $+0.30 \%$ |
| (best after purge) 23 | 20300078 | 6689249 | $4.47 \mathrm{e}+15$ | $+3.45 \%$ |
| 03 | 20737766 | 6841292 | $4.68 \mathrm{e}+15$ | $+8.21 \%$ |
| (worst) s 1 | 20560066 | 6961546 | $4.85 \mathrm{e}+15$ | $+12.05 \%$ |

TABLE 2. Results for RSA-155, see Table 9 for complete data

|  | After purge | After merge |  |  |
| ---: | :---: | :---: | :---: | :---: |
| $\Omega$ | $N$ | $N$ | $W \times N$ |  |
| (best) 11 | 154087021 | 44202262 | $2.34 \mathrm{e}+17$ |  |
| (second best) 21 | 153346744 | 44237278 | $2.35 \mathrm{e}+17$ | $+0.16 \%$ |
| (best after purge) 23 | 150426986 | 45029911 | $2.43 \mathrm{e}+17$ | $+3.78 \%$ |
| 03 | 153261960 | 45536067 | $2.49 \mathrm{e}+17$ | $+6.13 \%$ |
| (worst) s 1 | 154029290 | 47030544 | $2.65 \mathrm{e}+17$ | $+13.21 \%$ |

Table 3. Results for B200, see Table 10 for complete data
4.3. Interpretation of the results. In the three cases, the best weight function for the whole filtering step (purge and merge) is $\Omega_{11}$. In all cases, $\Omega_{11}$ produces a matrix around $3.5 \%$ better than the one produced by Cavallar weight function $\Omega_{23}$, 6 to $8 \%$ better than the one produced by GGNFS 0.77 .1 weight function $\Omega_{03}$, and 11 to $13 \%$ better than the one produced by CADO-NFS 1.1 weight function $\Omega_{s 1}$ (which is also the worst one in all cases).

|  | After purge | After merge |  |  |
| ---: | :---: | :---: | :---: | :---: |
| $\Omega$ | $N$ | $N$ | $W \times N$ |  |
| (best) 11 | 312721893 | 81239661 | $1.23 \mathrm{e}+18$ |  |
| (second best) 10 | 323519043 | 81263553 | $1.23 \mathrm{e}+18$ | $+0.06 \%$ |
| (best after purge) 23 | 305215508 | 82669415 | $1.28 \mathrm{e}+18$ | $+3.55 \%$ |
| 03 | 312257402 | 83835119 | $1.31 \mathrm{e}+18$ | $+6.49 \%$ |
| (worst) s 1 | 307415189 | 85540465 | $1.37 \mathrm{e}+18$ | $+10.87 \%$ |

Table 4. Results for RSA-704, see Table 11 for complete data

These three experiments show that the best weight function $\Omega_{11}$ for the whole filtering step (purge and merge) is not the one that produces the smallest matrix at the end of purge, which is, in all cases, Cavallar's weight function $\Omega_{23}$.

The matrices produced with $\Omega_{00}$ and $\Omega_{s 0}$ are almost the same, confirming the fact that almost always, if an ideal $p$ of weight $w(p) \geq 3$ appears in a clique, then it appear only once.

The weight functions that have few or no contribution from the number of relations in the clique (or equivalently ideals of weight 2 ) are the best ones (e.g. $\Omega_{11}$, $\left.\Omega_{21}, \Omega_{10}, \Omega_{30}, \ldots\right)$. This means that the number of relations in the clique should have a small contribution in the weight of the clique.

## 5. Other experiments with RSA-155

5.1. Statistics on weight of ideals and size of cliques. In order to try to explain why weight functions with few or no contribution from the number of relations in the clique (or equivalently ideals of weight 2 ) are better, some data on the number of relations in cliques and on the number of ideals of weight 2,3 and 4 are presented for $\Omega_{11}, \Omega_{23}, \Omega_{03}$ and $\Omega_{s 1}$ for RSA- 155 .

Remember that the weight function $\Omega_{s 1}$ removes cliques with the largest number of relations in the clique and $\Omega_{03}$ and $\Omega_{23}$ have an important contribution from ideal of weight 2 (or equivalently number of relations in the clique) whereas $\Omega_{11}$ has only few contribution from the number of relations in the clique.

|  | Number of | $\Omega_{11}$ | $\Omega_{23}$ | $\Omega_{s 1}$ | $\Omega_{03}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| At the beginning of purge | ideals of weight 2 | 18214716 |  |  |  |
|  | ideals of weight 3 | 9856626 |  |  |  |
|  | ideals of weight 4 | 5208195 |  |  |  |
|  | cliques with $n(c) \in[1,5]$ | 44104433 |  |  |  |
|  | cliques with $n(c) \in[6,10]$ | 407970 |  |  |  |
|  | cliques with $n(c)>10$ | 30664 |  |  |  |
|  | ideals of weight 2 | 6.9 M | 5.7M | 4.8 M | 5.4M |
|  | ideals of weight 3 | 3.3 M | 3.7 M | 4.6 M | 4.4M |
| before last | ideals of weight 4 | 2.1 M | 2.3 M | 2.6 M | $2 . .5 \mathrm{M}$ |
| stage of purge | cliques with $n(c) \in[1,5]$ | 14M | 15M | 16M | 15M |
|  | cliques with $n(c) \in[6,10]$ | 18k | 3.1 k | 2.5 k | 5.0k |
|  | cliques with $n(c)>10$ | 9 | 2 | 4 | 2 |

Table 5. Statistics on weight of ideals and size of cliques at the first and last stages of purge for RSA-155.

Data in Table 5 show that even if the weight functions do not take into account the number of relations in the cliques, the cliques with many relations are deleted. The major difference between these weight functions is the number of possible 2merges at the beginning of the merge algorithm. The gain in term of number of relations at the end of purge, by $\Omega_{23}$ for example, is not worth the loss in term of the number of 2 -merges available at the beginning of merge.

A clique with a large number of relations will anyway be reduced in one single row during merge while decreasing the total weight of the matrix. So it is more important, during the clique removal, to create many 2-merges, or at least $k$-merges with $k$ small. This explains why the best weight functions are the ones with few or no contribution from the number of relations in the cliques.
5.2. Influence of the excess. In this experiment, we want to see what is the effect of the initial excess on the size of the final matrix and on the ratio between different weight functions. Figure 3 shows the value of the product $N \times W$ for the final matrix of RSA- 155 depending on the number of unique relations at the beginning, for the following weight functions: $\Omega_{11}$ (the best one), $\Omega_{23}$ (Cavallar), $\Omega_{s 1}$ (CADO-NFS 1.1), $\Omega_{03}$ (GGNFS 0.77.1) and $\Omega_{31}$.

Figure 3 shows that the more relations there are the better the weight function $\Omega_{11}$ is, relatively to the others. For example, the difference between $\Omega_{11}$ and $\Omega_{23}$ is less than $0.2 \%$ when there are few relations but can go as high as $20 \%$ when there are many relations. This shows that the choice of deleted cliques made with $\Omega_{11}$ is better, as the more choice (i.e. excess) there is, the better $\Omega_{11}$ is.

Figure 3 also shows that, for all the weight functions, except $\Omega_{s 1}$ the more excess there is the better the matrix is. This indicates that $\Omega_{s 1}$ is not a good weight functions because when the excess is too high, it produces worst matrices than with lower excess.

Finally, Figure 3 shows that the best weight function may change with the excess, as shown by the fact that $\Omega_{31}$ produces better matrices that $\Omega_{11}$ when there is lot of excess.

## 6. Comparison of the weight functions in a FFS setup

6.1. Experiments and results. The 31 weight functions described in the Section 3 have been implemented in the development version of CADO-NFS and have been tested on data from three computations of discrete logarithms in $\operatorname{GF}\left(2^{n}\right)$ with $n=619, n=809$ and $n=1039$. The relations used as inputs in these three cases were computed with the relation collection tools for FFS available in the development version of CADO-NFS [9].

For more details on the computation of discrete logarithms in $\operatorname{GF}\left(2^{619}\right)$, see [4]. For more details on the $\operatorname{GF}\left(2^{809}\right)$ case, see [5]. In the case of $\operatorname{GF}\left(2^{1039}\right)$, the complete computation is not finished yet as the linear algebra step has not been done.

The computations of the filtering step were done with the development version of the CADO-NFS software for the 31 weight functions as described in Section 4 with the parameters as shown in Table 6.

As in Section 4, only partial results are presented in this section in Table 7, Table 8 and Table ??. Complete data are shown in the Appendix in Table 12, Table 13 and Table ??.


Figure 3

|  |  | 619 | 809 | 1039 |
| :---: | :--- | :---: | :---: | :---: |
| Beginning | Number of relations | 20.5 M | 81.0 M | 1306 M |
|  | Number of ideals | 10.5 M | 39.4 M | 986 M |
| After first singleton removal | Number of relations | 19.7 M | 79.1 M | 1080 M |
|  | Number of ideals | 9.7 M | 37.5 M | 746 M |
|  | Excess | 10 M | 41.6 M | 334 M |
|  | Relative excess | $103 \%$ | $111 \%$ | $44.7 \%$ |
| End of purge | Excess | 0 | 0 | 0 |
| End of merge | $k$-merge for $k \in$ | $[2,30]$ | $[2,30]$ | $[2,30]$ |
|  | $W / N$ | 100 | 100 | 100 |

TABLE 6. Common parameters for each discrete logarithm computation
6.2. Interpretation of the results. The first thing to notice is that the difference between the weight functions are larger than in the NFS setup. As shown in Section 5, this is due to the fact that excess were bigger.

|  | After purge | After merge |  |  |
| ---: | :---: | :---: | :---: | :---: |
| $\Omega$ | $N$ | $N$ | $W \times N$ |  |
| (best) 31 | 1978752 | 648476 | $4.21 \mathrm{e}+13$ |  |
| (second best) 30 | 2083125 | 650626 | $4.23 \mathrm{e}+13$ | $+0.66 \%$ |
| (best after purge) 62 | 1893132 | 671975 | $4.52 \mathrm{e}+13$ | $+7.38 \%$ |
| 23 | 1934876 | 718629 | $5.16 \mathrm{e}+13$ | $+22.81 \%$ |
| 03 | 2483373 | 887541 | $7.88 \mathrm{e}+13$ | $+87.32 \%$ |
| (worst) s 1 | 2548952 | 949242 | $9.01 \mathrm{e}+13$ | $+114.27 \%$ |

Table 7. Results for discrete logarithm in $\operatorname{GF}\left(2^{619}\right)$, see Table 12
for complete data

|  | After purge | After merge |  |  |
| ---: | :---: | :---: | :---: | :---: |
| $\Omega$ | $N$ | $N$ | $W \times N$ |  |
| (best) 31 | 9644786 | 3484569 | $1.21 \mathrm{e}+15$ |  |
| (second best) 30 | 10126436 | 3493773 | $1.22 \mathrm{e}+15$ | $+0.53 \%$ |
| (best after purge) 62 | 9153959 | 3596069 | $1.29 \mathrm{e}+15$ | $+6.50 \%$ |
| 23 | 9226205 | 3822940 | $1.46 \mathrm{e}+15$ | $+20.36 \%$ |
| 03 | 12363963 | 4651334 | $2.16 \mathrm{e}+15$ | $+78.18 \%$ |
| (worst) s 1 | 22190421 | 9075847 | $8.24 \mathrm{e}+15$ | $+578.38 \%$ |

Table 8. Results for discrete logarithm in $\operatorname{GF}\left(2^{809}\right)$, see Table 13 for complete data

The best weight function is $\Omega_{31}$ which outperforms $\Omega_{23}$ by $20 \%$ to $23 \%, \Omega_{03}$ by $78 \%$ to $87 \%$ and $\Omega_{s 1}$ by $115 \%$ to $575 \%$.

The best weight function after purge is not $\Omega_{23}$ but $\Omega_{62}$. This is probably due to the large excess as experiments with RSA- 155 indicates that it can also be the case in a NFS setup with a lot of excess. The weight function $\Omega_{23}$ still produces small matrices after purge but is not competitive for the whole filtering step.

The fact that the best weight function after purge is not the best weight function for the whole filtering step is still true in a FFS setup. The fact that the best weight functions are the ones with few or no contribution from the number of relations in the clique is also still true in a FFS setup.

## 7. Conclusion

In this article, we presented several weight functions, including ones previously described in the literature, in an unified formalism. Then, we compared them on three real-case factorization and three real-case discrete logarithm computations, using the strategies of CADO-NFS, with the same parameters and where only the weight function differs.

These experiments showed that the best weight functions are the ones with few or no contribution from the ideals of weight 2 (or equivalently from the number of relations in the clique).

We also showed that the best strategy for the purge algorithm only is not the best one for the whole filtering step, this is what makes the difference between the filtering step and Structured Gaussian Elimination (SGE).

New weight functions that outperform previously known ones, in both integer factorization and discrete logarithm context, were identified.

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The appendix contains the complete data for all the weight functions for RSA$155, \mathrm{~B} 200$ and RSA-704, and the computations of discrete logarithms in GF $\left(2^{619}\right)$, $\mathrm{GF}\left(2^{809}\right)$ and $\mathrm{GF}\left(2^{1039}\right)$. It also contains more information about the parameters used in purge and merge that were not necessary for the understanding of the comparison between the weight functions.

For RSA-155, B200 and RSA-704, during merge, the 32 heaviest columns (with most non-zero coefficients) are buried because they are discarded during the linear algebra step. For the computations of discrete logarithms, no column is buried during merge.

For B200 and RSA-704, the original relations were used, for RSA-155, they were generated with CADO-NFS, with the following polynomials:

$$
\begin{aligned}
& g(x)=3216929091619 x-73059843906355468503546958700 \\
& f(x)=5256451200 x^{5}-408896411797380 x^{4}+11615227400106879434 x^{3} \\
& \quad+18952602538400986455536150 x^{2}-310117807628012039542143196507 x \\
& \\
& -123205044635594488674617705124790497
\end{aligned}
$$

For RSA-155, during purge, only ideals greater than 16 M on rational side and 32 M on algebraic side were taken into account. The complete set of data is shown in Table 9.

For B200, during purge, only ideals greater than 250 M on rational side and 250 M on algebraic side were taken into account. The complete set of data is shown in Table 10.

For RSA-704, during purge, only ideals greater than 250 M on rational side and 500 M on algebraic side were taken into account. The complete set of data is shown in Table 11.

For $\operatorname{GF}\left(2^{619}\right)$, during purge, only ideals of degree greater than 19 on rational side and 19 on algebraic side were taken into account. The complete set of data is shown in Table 12.

For $\mathrm{GF}\left(2^{809}\right)$, during purge, only ideals of degree greater than 23 on rational side and 23 on algebraic side were taken into account. The complete set of data is shown in Table 13.

For $\mathrm{GF}\left(2^{1039}\right)$, during purge, only ideals of degree greater than 25 on rational side and 25 on algebraic side were taken into account. The complete set of data is shown in Table ??.

| $\Omega$ | After purge |  | After merge |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $N$ | $W \times N$ | $N$ |  |  |
| 11 | 20801141 | $8.71 \mathrm{e}+15$ | 6576610 | $4.33 \mathrm{e}+15$ |  |
| 21 | 20812812 | $8.71 \mathrm{e}+15$ | 6586358 | $4.34 \mathrm{e}+15$ | +0.30\% |
| 61 | 20591172 | $8.53 \mathrm{e}+15$ | 6592609 | $4.35 \mathrm{e}+15$ | +0.49\% |
| 10 | 21677978 | $9.46 \mathrm{e}+15$ | 6597020 | $4.35 \mathrm{e}+15$ | +0.62\% |
| 30 | 21066248 | $8.93 \mathrm{e}+15$ | 6598208 | $4.35 \mathrm{e}+15$ | +0.66\% |
| 12 | 20525758 | $8.48 \mathrm{e}+15$ | 6605260 | $4.36 \mathrm{e}+15$ | +0.87\% |
| 50 | 20940957 | $8.83 \mathrm{e}+15$ | 6607958 | $4.37 \mathrm{e}+15$ | +0.96\% |
| 60 | 21672863 | $9.45 \mathrm{e}+15$ | 6611637 | $4.37 \mathrm{e}+15$ | +1.07\% |
| 31 | 20732159 | $8.65 \mathrm{e}+15$ | 6614847 | $4.38 \mathrm{e}+15$ | +1.17\% |
| 22 | 20437176 | $8.40 \mathrm{e}+15$ | 6616157 | $4.38 \mathrm{e}+15$ | +1.21\% |
| 51 | 20632230 | $8.57 \mathrm{e}+15$ | 6630136 | $4.40 \mathrm{e}+15$ | +1.63\% |
| 32 | 20570710 | $8.52 \mathrm{e}+15$ | 6638018 | $4.41 \mathrm{e}+15$ | +1.88\% |
| 62 | 20377654 | $8.35 \mathrm{e}+15$ | 6640471 | $4.41 \mathrm{e}+15$ | +1.95\% |
| 40 | 20855869 | $8.76 \mathrm{e}+15$ | 6657267 | $4.43 \mathrm{e}+15$ | +2.47\% |
| 52 | 20493661 | $8.45 \mathrm{e}+15$ | 6659152 | $4.43 \mathrm{e}+15$ | +2.53\% |
| 13 | 20361903 | $8.34 \mathrm{e}+15$ | 6668262 | $4.45 \mathrm{e}+15$ | +2.81\% |
| 41 | 20628073 | $8.57 \mathrm{e}+15$ | 6676570 | $4.46 \mathrm{e}+15$ | +3.06\% |
| 33 | 20433585 | $8.40 \mathrm{e}+15$ | 6686196 | $4.47 \mathrm{e}+15$ | +3.36\% |
| 23 | 20300078 | $8.29 \mathrm{e}+15$ | 6689249 | $4.47 \mathrm{e}+15$ | +3.45\% |
| 42 | 20514904 | $8.47 \mathrm{e}+15$ | 6701793 | $4.49 \mathrm{e}+15$ | +3.84\% |
| 63 | 20321565 | $8.30 \mathrm{e}+15$ | 6712215 | $4.51 \mathrm{e}+15$ | +4.17\% |
| 53 | 20400561 | $8.37 \mathrm{e}+15$ | 6713693 | $4.51 \mathrm{e}+15$ | +4.21\% |
| 43 | 20434660 | $8.40 \mathrm{e}+15$ | 6750781 | $4.56 \mathrm{e}+15$ | +5.37\% |
| 20 | 22876961 | $1.05 \mathrm{e}+16$ | 6751559 | $4.56 \mathrm{e}+15$ | +5.39\% |
| 02 | 20858120 | $8.76 \mathrm{e}+15$ | 6818390 | $4.65 \mathrm{e}+15$ | +7.49\% |
| 01 | 20860148 | $8.76 \mathrm{e}+15$ | 6818411 | $4.65 \mathrm{e}+15$ | +7.49\% |
| s0 | 21048884 | $8.92 \mathrm{e}+15$ | 6831805 | $4.67 \mathrm{e}+15$ | +7.91\% |
| 00 | 21048891 | $8.92 \mathrm{e}+15$ | 6831875 | $4.67 \mathrm{e}+15$ | +7.91\% |
| 03 | 20737766 | $8.65 \mathrm{e}+15$ | 6841292 | $4.68 \mathrm{e}+15$ | +8.21\% |
| s2 | 20496848 | $8.44 \mathrm{e}+15$ | 6883627 | $4.74 \mathrm{e}+15$ | +9.55\% |
| s1 | 20560066 | $8.49 \mathrm{e}+15$ | 6961546 | $4.85 \mathrm{e}+15$ | +12.05\% |

Table 9. Complete data for RSA-155

| $\Omega$ | After purge |  | After merge |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $N$ | $W \times N$ | $N$ |  |  |
| 11 | 154087021 | $5.23 \mathrm{e}+17$ | 44202262 | $2.34 \mathrm{e}+17$ |  |
| 21 | 153346744 | $5.18 \mathrm{e}+17$ | 44237278 | $2.35 \mathrm{e}+17$ | +0.16\% |
| 61 | 152480272 | $5.12 \mathrm{e}+17$ | 44295368 | $2.35 \mathrm{e}+17$ | +0.42\% |
| 30 | 158109764 | $5.51 \mathrm{e}+17$ | 44344210 | $2.36 \mathrm{e}+17$ | +0.64\% |
| 10 | 161480932 | $5.74 \mathrm{e}+17$ | 44359892 | $2.36 \mathrm{e}+17$ | +0.71\% |
| 50 | 156737711 | $5.41 \mathrm{e}+17$ | 44359936 | $2.36 \mathrm{e}+17$ | +0.71\% |
| 31 | 154254136 | $5.24 \mathrm{e}+17$ | 44370581 | $2.36 \mathrm{e}+17$ | +0.76\% |
| 60 | 161056386 | $5.71 \mathrm{e}+17$ | 44386107 | $2.36 \mathrm{e}+17$ | +0.83\% |
| 12 | 151985472 | $5.08 \mathrm{e}+17$ | 44398656 | $2.37 \mathrm{e}+17$ | +0.89\% |
| 51 | 153312434 | $5.18 \mathrm{e}+17$ | 44455926 | $2.37 \mathrm{e}+17$ | +1.15\% |
| 22 | 151171217 | $5.03 \mathrm{e}+17$ | 44504976 | $2.38 \mathrm{e}+17$ | +1.37\% |
| 32 | 152617177 | $5.13 \mathrm{e}+17$ | 44519799 | $2.38 \mathrm{e}+17$ | +1.44\% |
| 40 | 155854758 | $5.35 \mathrm{e}+17$ | 44624257 | $2.39 \mathrm{e}+17$ | +1.92\% |
| 62 | 150943964 | $5.01 \mathrm{e}+17$ | 44631783 | $2.39 \mathrm{e}+17$ | +1.95\% |
| 52 | 151973891 | $5.08 \mathrm{e}+17$ | 44647196 | $2.39 \mathrm{e}+17$ | +2.02\% |
| 41 | 153252776 | $5.17 \mathrm{e}+17$ | 44718043 | $2.40 \mathrm{e}+17$ | +2.35\% |
| 13 | 150807090 | $5.00 \mathrm{e}+17$ | 44822261 | $2.41 \mathrm{e}+17$ | +2.82\% |
| 33 | 151379703 | $5.04 \mathrm{e}+17$ | 44835865 | $2.41 \mathrm{e}+17$ | +2.89\% |
| 42 | 152135118 | $5.09 \mathrm{e}+17$ | 44879381 | $2.42 \mathrm{e}+17$ | +3.09\% |
| 53 | 151103756 | $5.02 \mathrm{e}+17$ | 44990483 | $2.43 \mathrm{e}+17$ | +3.60\% |
| 23 | 150426986 | $4.98 \mathrm{e}+17$ | 45029911 | $2.43 \mathrm{e}+17$ | +3.78\% |
| 20 | 167572608 | $6.18 \mathrm{e}+17$ | 45145579 | $2.45 \mathrm{e}+17$ | +4.31\% |
| 43 | 151365155 | $5.04 \mathrm{e}+17$ | 45160515 | $2.45 \mathrm{e}+17$ | +4.38\% |
| 63 | 150505303 | $4.98 \mathrm{e}+17$ | 45168479 | $2.45 \mathrm{e}+17$ | +4.42\% |
| 02 | 154067402 | $5.22 \mathrm{e}+17$ | 45495013 | $2.48 \mathrm{e}+17$ | +5.93\% |
| 01 | 155395404 | $5.32 \mathrm{e}+17$ | 45505293 | $2.48 \mathrm{e}+17$ | +5.98\% |
| 03 | 153261960 | $5.17 \mathrm{e}+17$ | 45536067 | $2.49 \mathrm{e}+17$ | +6.13\% |
| 00 | 156540087 | $5.40 \mathrm{e}+17$ | 45550198 | $2.49 \mathrm{e}+17$ | +6.19\% |
| s0 | 156540092 | $5.40 \mathrm{e}+17$ | 45550331 | $2.49 \mathrm{e}+17$ | +6.19\% |
| s2 | 151983844 | $5.08 \mathrm{e}+17$ | 45897498 | $2.53 \mathrm{e}+17$ | +7.82\% |
| s1 | 154029290 | $5.21 \mathrm{e}+17$ | 47030544 | $2.65 \mathrm{e}+17$ | +13.21\% |
|  |  | 10. C | plete da | or B200 |  |


|  | After purge |  | After merge |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $\Omega$ | $N$ | $W \times N$ | $N$ | $W \times N$ |  |
| 11 | 312721893 | $2.18 \mathrm{e}+18$ | 81239661 | $1.23 \mathrm{e}+18$ |  |
| 10 | 323519043 | $2.34 \mathrm{e}+18$ | 81263553 | $1.23 \mathrm{e}+18$ | $+0.06 \%$ |
| 21 | 312473810 | $2.18 \mathrm{e}+18$ | 81379271 | $1.24 \mathrm{e}+18$ | $+0.34 \%$ |
| 30 | 317826394 | $2.26 \mathrm{e}+18$ | 81411698 | $1.24 \mathrm{e}+18$ | $+0.42 \%$ |
| 60 | 324041496 | $2.34 \mathrm{e}+18$ | 81418192 | $1.24 \mathrm{e}+18$ | $+0.44 \%$ |
| 61 | 309891857 | $2.14 \mathrm{e}+18$ | 81427075 | $1.24 \mathrm{e}+18$ | $+0.46 \%$ |
| 50 | 316478268 | $2.24 \mathrm{e}+18$ | 81480389 | $1.24 \mathrm{e}+18$ | $+0.59 \%$ |
| 12 | 308837595 | $2.13 \mathrm{e}+18$ | 81584492 | $1.24 \mathrm{e}+18$ | $+0.85 \%$ |
| 31 | 312574204 | $2.18 \mathrm{e}+18$ | 81589561 | $1.24 \mathrm{e}+18$ | $+0.86 \%$ |
| 51 | 311175033 | $2.16 \mathrm{e}+18$ | 81724892 | $1.25 \mathrm{e}+18$ | $+1.20 \%$ |
| 22 | 307395363 | $2.11 \mathrm{e}+18$ | 81774400 | $1.25 \mathrm{e}+18$ | $+1.32 \%$ |
| 32 | 309900307 | $2.14 \mathrm{e}+18$ | 81854176 | $1.25 \mathrm{e}+18$ | $+1.52 \%$ |
| 40 | 315549828 | $2.22 \mathrm{e}+18$ | 81971998 | $1.26 \mathrm{e}+18$ | $+1.81 \%$ |
| 62 | 306584663 | $2.10 \mathrm{e}+18$ | 81998217 | $1.26 \mathrm{e}+18$ | $+1.88 \%$ |
| 52 | 308695518 | $2.13 \mathrm{e}+18$ | 82055985 | $1.26 \mathrm{e}+18$ | $+2.02 \%$ |
| 41 | 311251786 | $2.16 \mathrm{e}+18$ | 82176315 | $1.26 \mathrm{e}+18$ | $+2.32 \%$ |
| 13 | 306186056 | $2.09 \mathrm{e}+18$ | 82336697 | $1.27 \mathrm{e}+18$ | $+2.72 \%$ |
| 33 | 307411911 | $2.11 \mathrm{e}+18$ | 82425057 | $1.27 \mathrm{e}+18$ | $+2.94 \%$ |
| 42 | 309026229 | $2.13 \mathrm{e}+18$ | 82465478 | $1.27 \mathrm{e}+18$ | $+3.04 \%$ |
| 20 | 335993975 | $2.52 \mathrm{e}+18$ | 82558900 | $1.27 \mathrm{e}+18$ | $+3.27 \%$ |
| 23 | 305215508 | $2.08 \mathrm{e}+18$ | 82669415 | $1.28 \mathrm{e}+18$ | $+3.55 \%$ |
| 53 | 306704487 | $2.10 \mathrm{e}+18$ | 82735374 | $1.28 \mathrm{e}+18$ | $+3.72 \%$ |
| 63 | 305410102 | $2.08 \mathrm{e}+18$ | 82943841 | $1.29 \mathrm{e}+18$ | $+4.24 \%$ |
| 43 | 307120972 | $2.10 \mathrm{e}+18$ | 83091811 | $1.29 \mathrm{e}+18$ | $+4.61 \%$ |
| 01 | 315512310 | $2.22 \mathrm{e}+18$ | 83775945 | $1.31 \mathrm{e}+18$ | $+6.34 \%$ |
| 02 | 314374563 | $2.21 \mathrm{e}+18$ | 83783791 | $1.31 \mathrm{e}+18$ | $+6.36 \%$ |
| 03 | 312257402 | $2.18 \mathrm{e}+18$ | 83835119 | $1.31 \mathrm{e}+18$ | $+6.49 \%$ |
| s 0 | 318482486 | $2.27 \mathrm{e}+18$ | 83863260 | $1.32 \mathrm{e}+18$ | $+6.56 \%$ |
| 00 | 318482490 | $2.27 \mathrm{e}+18$ | 83863378 | $1.32 \mathrm{e}+18$ | $+6.56 \%$ |
| s2 | 307155720 | $2.10 \mathrm{e}+18$ | 85114020 | $1.35 \mathrm{e}+18$ | $+9.77 \%$ |
| s1 | 307415189 | $2.11 \mathrm{e}+18$ | 85540465 | $1.37 \mathrm{e}+18$ | $+10.87 \%$ |

Table 11. Complete data for RSA-704

THE FILTERING STEP OF SIEVING ALGORITHMS

| $\Omega$ | After purge |  | After merge |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $N$ | $W \times N$ | $N$ |  |  |
| 31 | 1978752 | $8.51 \mathrm{e}+13$ | 648476 | $4.21 \mathrm{e}+13$ |  |
| 30 | 2083125 | $9.43 \mathrm{e}+13$ | 650626 | $4.23 \mathrm{e}+13$ | +0.66\% |
| 50 | 1982615 | $8.55 \mathrm{e}+13$ | 653358 | $4.27 \mathrm{e}+13$ | +1.51\% |
| 32 | 1938808 | $8.16 \mathrm{e}+13$ | 655589 | $4.30 \mathrm{e}+13$ | +2.21\% |
| 12 | 1916454 | $7.96 \mathrm{e}+13$ | 657308 | $4.32 \mathrm{e}+13$ | +2.74\% |
| 11 | 1984857 | $8.54 \mathrm{e}+13$ | 657967 | $4.33 \mathrm{e}+13$ | +2.95\% |
| 61 | 1926763 | $8.05 \mathrm{e}+13$ | 661554 | $4.38 \mathrm{e}+13$ | +4.07\% |
| 51 | 1936270 | $8.15 \mathrm{e}+13$ | 663028 | $4.40 \mathrm{e}+13$ | +4.54\% |
| 62 | 1893132 | $7.77 \mathrm{e}+13$ | 671975 | $4.52 \mathrm{e}+13$ | +7.38\% |
| 22 | 1906373 | $7.87 \mathrm{e}+13$ | 675500 | $4.56 \mathrm{e}+13$ | +8.51\% |
| 52 | 1926009 | $8.05 \mathrm{e}+13$ | 677077 | $4.58 \mathrm{e}+13$ | +9.02\% |
| 21 | 1971971 | $8.42 \mathrm{e}+13$ | 678108 | $4.60 \mathrm{e}+13$ | +9.35\% |
| 60 | 2125923 | $9.80 \mathrm{e}+13$ | 678718 | $4.61 \mathrm{e}+13$ | +9.54\% |
| 33 | 1925359 | $8.04 \mathrm{e}+13$ | 678894 | $4.61 \mathrm{e}+13$ | +9.60\% |
| 13 | 1901372 | $7.83 \mathrm{e}+13$ | 679181 | $4.61 \mathrm{e}+13$ | +9.69\% |
| 10 | 2182072 | $1.03 \mathrm{e}+14$ | 683442 | $4.67 \mathrm{e}+13$ | +11.08\% |
| 40 | 1980510 | $8.53 \mathrm{e}+13$ | 692468 | $4.80 \mathrm{e}+13$ | +14.03\% |
| 41 | 1973283 | $8.46 \mathrm{e}+13$ | 707211 | $5.00 \mathrm{e}+13$ | +18.94\% |
| 20 | 2227058 | $1.07 \mathrm{e}+14$ | 713659 | $5.09 \mathrm{e}+13$ | +21.11\% |
| 53 | 1948261 | $8.22 \mathrm{e}+13$ | 714369 | $5.10 \mathrm{e}+13$ | +21.36\% |
| 23 | 1934876 | $8.09 \mathrm{e}+13$ | 718629 | $5.16 \mathrm{e}+13$ | +22.81\% |
| 42 | 1975296 | $8.46 \mathrm{e}+13$ | 723109 | $5.23 \mathrm{e}+13$ | +24.34\% |
| 63 | 1957756 | $8.28 \mathrm{e}+13$ | 737947 | $5.45 \mathrm{e}+13$ | +29.50\% |
| 43 | 1984188 | $8.52 \mathrm{e}+13$ | 745938 | $5.56 \mathrm{e}+13$ | +32.32\% |
| 03 | 2483373 | $1.34 \mathrm{e}+14$ | 887541 | $7.88 \mathrm{e}+13$ | +87.32\% |
| 01 | 2490814 | $1.35 \mathrm{e}+14$ | 887905 | $7.88 \mathrm{e}+13$ | +87.48\% |
| 02 | 2490830 | $1.35 \mathrm{e}+14$ | 887956 | $7.88 \mathrm{e}+13$ | +87.50\% |
| s2 | 2480183 | $1.34 \mathrm{e}+14$ | 888767 | $7.90 \mathrm{e}+13$ | +87.84\% |
| s0 | 2504747 | $1.37 \mathrm{e}+14$ | 890097 | $7.92 \mathrm{e}+13$ | +88.40\% |
| 00 | 2504738 | $1.37 \mathrm{e}+14$ | 890107 | $7.92 \mathrm{e}+13$ | +88.41\% |
| s1 | 2548952 | $1.40 \mathrm{e}+14$ | 949242 | $9.01 \mathrm{e}+13$ | +114.27\% |

Table 12. Complete data for discrete logarithm in $\operatorname{GF}\left(2^{619}\right)$

| $\Omega$ | After purge |  | After merge |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $N$ | $W \times N$ | $N$ |  |  |
| 31 | 9644786 | $2.10 \mathrm{e}+15$ | 3484569 | $1.21 \mathrm{e}+15$ |  |
| 30 | 10126436 | $2.32 \mathrm{e}+15$ | 3493773 | $1.22 \mathrm{e}+15$ | +0.53\% |
| 50 | 9761906 | $2.15 \mathrm{e}+15$ | 3515248 | $1.24 \mathrm{e}+15$ | +1.77\% |
| 32 | 9424672 | $2.00 \mathrm{e}+15$ | 3517593 | $1.24 \mathrm{e}+15$ | +1.90\% |
| 12 | 9313137 | $1.95 \mathrm{e}+15$ | 3528628 | $1.25 \mathrm{e}+15$ | +2.54\% |
| 11 | 9670420 | $2.11 \mathrm{e}+15$ | 3530193 | $1.25 \mathrm{e}+15$ | +2.64\% |
| 51 | 9454582 | $2.02 \mathrm{e}+15$ | 3548367 | $1.26 \mathrm{e}+15$ | +3.70\% |
| 61 | 9360098 | $1.97 \mathrm{e}+15$ | 3550649 | $1.26 \mathrm{e}+15$ | +3.83\% |
| 62 | 9153959 | $1.89 \mathrm{e}+15$ | 3596069 | $1.29 \mathrm{e}+15$ | +6.50\% |
| 52 | 9336976 | $1.97 \mathrm{e}+15$ | 3607460 | $1.30 \mathrm{e}+15$ | +7.18\% |
| 22 | 9238788 | $1.92 \mathrm{e}+15$ | 3614219 | $1.31 \mathrm{e}+15$ | +7.58\% |
| 60 | 10265165 | $2.37 \mathrm{e}+15$ | 3616670 | $1.31 \mathrm{e}+15$ | +7.73\% |
| 33 | 9284572 | $1.94 \mathrm{e}+15$ | 3623881 | $1.31 \mathrm{e}+15$ | +8.16\% |
| 21 | 9544594 | $2.05 \mathrm{e}+15$ | 3631009 | $1.32 \mathrm{e}+15$ | +8.58\% |
| 13 | 9159235 | $1.89 \mathrm{e}+15$ | 3634841 | $1.32 \mathrm{e}+15$ | +8.81\% |
| 10 | 10569620 | $2.51 \mathrm{e}+15$ | 3644690 | $1.33 \mathrm{e}+15$ | +9.40\% |
| 40 | 9838799 | $2.19 \mathrm{e}+15$ | 3660460 | $1.34 \mathrm{e}+15$ | +10.35\% |
| 41 | 9634081 | $2.10 \mathrm{e}+15$ | 3696603 | $1.37 \mathrm{e}+15$ | +12.54\% |
| 42 | 9535985 | $2.05 \mathrm{e}+15$ | 3747073 | $1.40 \mathrm{e}+15$ | +15.63\% |
| 53 | 9298354 | $1.95 \mathrm{e}+15$ | 3753044 | $1.41 \mathrm{e}+15$ | +16.00\% |
| 20 | 10617233 | $2.53 \mathrm{e}+15$ | 3764939 | $1.42 \mathrm{e}+15$ | +16.74\% |
| 23 | 9226205 | $1.91 \mathrm{e}+15$ | 3822940 | $1.46 \mathrm{e}+15$ | +20.36\% |
| 43 | 9477803 | $2.02 \mathrm{e}+15$ | 3846167 | $1.48 \mathrm{e}+15$ | +21.83\% |
| 63 | 9238544 | $1.92 \mathrm{e}+15$ | 3862733 | $1.49 \mathrm{e}+15$ | +22.88\% |
| s2 | 12231926 | $3.38 \mathrm{e}+15$ | 4649193 | $2.16 \mathrm{e}+15$ | +78.02\% |
| 03 | 12363963 | $3.46 \mathrm{e}+15$ | 4651334 | $2.16 \mathrm{e}+15$ | +78.18\% |
| 02 | 12442274 | $3.50 \mathrm{e}+15$ | 4652580 | $2.16 \mathrm{e}+15$ | +78.27\% |
| 01 | 12445627 | $3.50 \mathrm{e}+15$ | 4652845 | $2.16 \mathrm{e}+15$ | +78.29\% |
| 00 | 12735446 | $3.67 \mathrm{e}+15$ | 4703312 | $2.21 \mathrm{e}+15$ | +82.18\% |
| s0 | 12735441 | $3.67 \mathrm{e}+15$ | 4703398 | $2.21 \mathrm{e}+15$ | +82.19\% |
| s1 | 22190421 | $1.07 \mathrm{e}+16$ | 9075847 | $8.24 \mathrm{e}+15$ | +578.38\% |

Table 13. Complete data for discrete logarithm in $\mathrm{GF}\left(2^{809}\right)$

