

Church-Rosser Properties of Normal Rewriting

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CSL, Fontainebleau, September 5, 2012

An **abstract rewriting framework** that

- captures the main existing forms of rewriting
- provides with a notion of critical pair
- yields a Church-Rosser result in the terminating case ...
- that captures the existing Church-Rosser results for (first/higher)-order rewriting
- provides flexibility in the higher-order case

Rewriting by examples – Plain

$$(x + y)^{-1} \rightarrow y^{-1} + x^{-1}$$

$$x + 0 \rightarrow x$$

$$(x + y) + z \rightarrow x + (y + z)$$

Plain rewriting:

$$(1 + 2)^{-1} + 0 \rightarrow (2^{-1} + 1^{-1}) + 0 \rightarrow$$
$$2^{-1} + (1^{-1} + 0) \rightarrow 2^{-1} + 1^{-1}$$

$$(x + y)^{-1} \rightarrow y^{-1} + x^{-1}$$

$$0 + x \rightarrow x$$

$$(x + y) + z = x + (y + z)$$

$$x + y = y + x$$

Rewriting modulo:

$$(1 + 2)^{-1} + 0 \rightarrow (2^{-1} + 1^{-1}) + 0 \rightarrow 2^{-1} + 1^{-1}$$

Rewriting by examples – Normal

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$$0 + x \rightarrow x$$

$$(x + y) + z = x + (y + z)$$

$$x + y = y + x$$

$$0 + x = x$$

Normal rewriting:

$$(1 + 2)^{-1} + 0 \rightarrow (1 + 2)^{-1} \rightarrow 2^{-1} + 1^{-1}$$

[Barendregt and Klop]:

$$\begin{aligned}\omega 1 &= (\lambda x.(x x)) \lambda s.\lambda z.(s z) \\ &\longrightarrow (\lambda s.\lambda z.(s z) \lambda s.\lambda z.(s z)) \\ &\longrightarrow \lambda z.(\lambda s.\lambda z.(s z) z) \\ &\longrightarrow \lambda z.\lambda z.(z z) \quad \text{— wrong}\end{aligned}$$

$$\begin{aligned}&\xrightarrow[\alpha]{(\geq 1)^*} \lambda z.(\lambda s'.\lambda z'.(s' z') z) \\ &\xrightarrow[\beta]{1} \lambda z.\lambda z'.(z z')\end{aligned}$$

β -reduction is modulo α -conversion

[Barendregt and Klop]:

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β -reduction is modulo α -conversion

$$\begin{aligned} \mathit{rec}(0, u, f) &\rightarrow u \\ \mathit{rec}(s(y), u, f) &\rightarrow @(f, y, \mathit{rec}(y, u, f)) \\ @(\lambda z. u, v) &\rightarrow u\{z \mapsto v\} \end{aligned}$$

rewrite:

$$\begin{aligned} \mathit{rec}(s(0), 1, \lambda xy. + (x, y)) &\rightarrow \\ @(\lambda xy. + (x, y), 0, \mathit{rec}(0, 1, \lambda xy. + (x, y))) &\rightarrow \\ @(\lambda xy. + (x, y), 0, 1) \rightarrow + (0, 1) \rightarrow 1 \end{aligned}$$

Mix of plain rewriting for recursors
and rewriting modulo for binders

Higher-order rewriting [Nipkow]

rules (differentiation):

$$\text{diff}(\lambda x. \sin(f(x))) \rightarrow \lambda x. \cos(f(x)) * \text{diff}(f)$$

$$\text{diff}(\lambda x. x) \rightarrow \lambda x. 1$$

rewrite:

$$\text{diff}(\lambda x. \sin(x)) \xrightarrow[\beta]{1 \cdot 1 \cdot 1} \text{diff}(\lambda x. \sin(\lambda y. y x)) \longrightarrow$$

$$\lambda x. \cos(\lambda y. y x) * \text{diff}(\lambda y. y)$$

$$\longrightarrow \lambda x. \cos(x) * \text{diff}(\lambda y. y)$$

$$\longrightarrow \lambda x. \cos(x) * \lambda x. 1$$

$$\longrightarrow \lambda x. \cos(x)$$

Higher-order rewriting is nothing but
normal rewriting modulo alpha, beta and eta.

- 1 is my rewriting calculus terminating ?
- 2 is my rewriting calculus confluent ?

We focus on:

- Confluence assuming termination
- General abstract results
- Application to first-order rewriting
- Application to higher-order rewriting
- A treatment of binders as a particular case
- Flexibility of higher-order definitions

Main definitions/result for abstract plain rewriting

Conversion: $u \overset{*}{\longleftrightarrow} v$

Local peak: $u \longleftarrow s \longrightarrow v$

Joinability: $u \overset{*}{\longrightarrow} t \overset{*}{\longleftarrow} v$

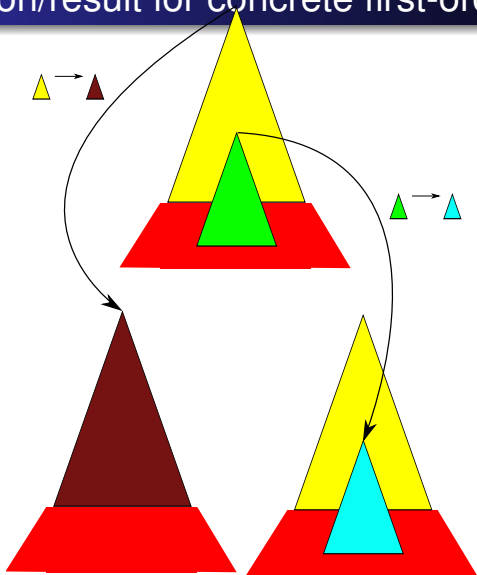
Church-Rosser:

convertible pairs are joinable.

Newmann:

Assume plain rewriting terminates. Then it is Church-Rosser iff local peaks are joinable.

Main definition/result for concrete first-order rewriting



Knuth-Bendix: joinability of critical peaks is just enough for terminating plain rewriting

- To AC-rewriting: [Peterson & Stickel]
- To rewriting modulo: [Jouannaud & Kirchner]
- To higher-order rewriting: [Nipkow & Mayr]
- etc ...

Normal rewriting with (R,S,E)

Definition: $s \xrightarrow{R \downarrow_{S_E}} t$ iff $s = s \downarrow_{S_E} \xrightarrow[R_{SE}]{p} u \xrightarrow[S_E]{!} u \downarrow_{S_E} = t$

General Assumptions

- (a) S is Church-Rosser modulo E
- (b) $R_{SE} \cup S_E$ is E -terminating,
- (c) Rules in R are S_E -normalized,
- (d) Equations in E are regular.

For Nipkow's higher-order rewriting:

E is α -conversion

S is made of β -reduction and η -expansion

R is made of rules $l \rightarrow r$ such that l and r have the same base type and l is a pattern. [Miller]

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Main property expected from normal rewriting

Conversion: $t_1 \xleftrightarrow[RUSUE]{*} t_2$

Joinability: $t_1 \xrightarrow[SE]{!} u \xrightarrow[R \downarrow_{SE}]{*} v \xrightarrow[E]{*} t_2 \xleftarrow[R \downarrow_{SE}]{*} t_1 \xleftarrow[SE]{!} t_2$

Church-Rosser: every conversion is joinable.

Theorem

Let (R, S, E) satisfy (a,b,c,d), and critical cases be joinable. Then normal rewriting is CR.

Further requirements:

- First and higher-order rewriting as instances;
- A proof independent from any term structure.

Positional (plain) rewriting

an abstract set of terms \mathcal{T}

an abstract set of positions \mathcal{P} equipped with a partial order $>_{\mathcal{P}}$ and a minimum \wedge

A **domain** is any downward closed set of positions P_p with minimum p

Rewrite relations become ternary:

$$u \xrightarrow{P_p} v$$

Successor **below** p of s :

$$s \xrightarrow{\geq_{\mathcal{P}p}} t$$

Normal form **below** p of s :

$$s \xrightarrow{(\geq_{\mathcal{P}p})^*} s \downarrow^p$$

In normal form **below** p :

$$s = s \downarrow^p$$

Rewriting with R modulo E at p

$$\frac{p}{R_E} \rightarrow := \left(\frac{>_{\mathcal{P}\mathcal{P}}}{E} \right)^* \frac{p}{R}$$

Disjoint redexes axiom

$$\frac{p}{X} \frac{q}{Y} \rightarrow \subseteq \frac{p}{Y} \frac{q}{X} \rightarrow \quad \text{if } p \# q$$

Ancestor redex axiom

$$\frac{P_p}{X} \frac{q}{Y} \rightarrow \subseteq \left(\frac{>_{\mathcal{P}\mathcal{P}}}{Y} \right)^* \frac{q}{X} \frac{>_{\mathcal{P}P_p}}{Y} \rightarrow \quad \text{if } q >_{\mathcal{P}} P_p$$

where X, Y are arbitrary rewriting relations.

Positional normal rewriting

Normal rewriting
terms in $\downarrow_{S_E}^p$ normal form at q below p :

$$s \xrightarrow[R_{S_E \downarrow}^p]{} t := s \xrightarrow[R_{S_E}^{q \geq p}]{p} u \downarrow_{S_E}^p t$$

Critical local rewrite peak:

$$v \xleftarrow[R]{P_p} u \xrightarrow[R_{SE}]{q} w \quad \text{s.t.} \quad q \in P_p \text{ and } u = u \downarrow_{SE}^p$$

Critical local simplification peaks:

$$v \xleftarrow[R]{P_p} u \xrightarrow[S_E]{q} w \quad \text{s.t.} \quad q \in P_p$$

Critical local cliffs:

$$s \xleftarrow[E]{P_p} u \xrightarrow[R_{SE}]{q} t \quad \text{s.t.} \quad q \in P_p \setminus \{p\}$$

Critical local simplification cliffs:

$$v \xleftarrow[S]{P_p} u \xrightarrow[R_{SE}]{q} w \quad \text{s.t.} \quad q \in P_p \setminus \{p\} \text{ and } u = u \downarrow_{SE}^q$$

- 1 Plain first-order rewriting
- 2 First-order rewriting modulo
- 3 First-order normal rewriting – new
- 4 Plain higher-order rewriting (modulo α)
- 5 Higher-order rewriting (modulo $\alpha\beta\eta$) in λ^{\rightarrow} :
 - with $\beta\eta^{-1}$ at base types
 - with $\beta\eta^{-1}$ at arbitrary arrow types
(may need β -extensions) – new
 - with $\beta\eta$ at arbitrary arrow types
(may need extensions) – new
- 6 HOR with AC-operators (modulo $AC\alpha\beta\eta$)
– to be carried out

A clean, flexible framework
for normal rewriting
obtained via
a novel abstract notion of
positional rewriting.

Further work:
formalization in Coq
extension to CIC

Current limitation:
applies to tree rewriting

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