



**HAL**  
open science

## Design of One Dimensional Digital IIR Filters using Orthogonal Polynomials

Vinay Kumar, Sunil Bhooshan

► **To cite this version:**

Vinay Kumar, Sunil Bhooshan. Design of One Dimensional Digital IIR Filters using Orthogonal Polynomials. International Scholarly Research Notices, 2012, ISRN Signal Processing, Volume 2012 (2012), pp.1-7. 10.5402/2012/870276 . hal-00724688v1

**HAL Id: hal-00724688**

**<https://inria.hal.science/hal-00724688v1>**

Submitted on 22 Aug 2012 (v1), last revised 18 Oct 2012 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Design of One Dimensional Digital IIR Filters using Orthogonal Polynomials

Vinay Kumar and Sunil Bhooshan <sup>\*†</sup>

August 1, 2012

## Abstract

In this paper we discuss the method of designing a zero or linear phase 1 Dimensional Infinite Impulse Response (IIR) filter using orthogonal polynomials. The approach is easy to understand and implement. Simulation results show the excellent performance of the proposed filter design technique.

## 1 Introduction

In the past two to two and half decades a great deal of work has been carried out in the field of design of linear phase IIR filters. In general, designing exact linear phase IIR filter is not possible, schemes have been proposed to approximate pass band linearity. Conventionally, first the magnitude specifications of an IIR filter are met, and then all pass equalizers are applied to linearize the phase response [1,2]. Mostly IIR filters are designed with equiripple or maximally flat group delay [3]. But their magnitude characteristics are poor. Optimization techniques are used to simultaneously approximate magnitude and phase response characteristics [4,5]. To meet with the magnitude and phase characteristics at the same time, generally, linear programming is used [6]. To directly design a linear phase IIR filter Lu *et al* [7] give

an iterative procedure, it is based on a weighted least-squares algorithm.

Xiao *et al* [8] discusses a method to design a linear phase IIR filter with frequency weighted least square error optimization using Broyden-Fletcher-Goldfarb-Shanno (BFGS) [9] method.

The model reduction approach has also been proposed by various authors [10,11]. A procedure to design linear phase IIR filter from linear phase FIR filter has been discussed by Holford *et al* [12] using frequency weighting model reduction, for highly selective filters. Holford *et al* [12] gives good compromise for order of the filter, pass band maximum ripple and stop band minimum attenuation.

The present paper discusses a technique to design IIR filters with approximately linear phase. An algorithm is presented to design such a filter. The algorithms have been discussed stepwise to make sure that any person with basic programming capabilities can easily design them. We have not used any standard routine of any particular platform, therefore any freely available programming platform (like C, C++, Scilab, Octave, etc.) can be used to design these filters.

The paper is divided into 5 sections. Section 1 is an introduction to the already existing techniques, Section 2 discusses the preliminary discussion required to understand the complete procedure, Section 3 presents the procedure to design the proposed method, application of the proposed technique together with a discussion is demonstrated in Section 4 and finally Section 5 is gives the conclusion and future work.

---

<sup>\*</sup>Vinay Kumar is with Jaypee University of Information Technology, Solan-173215, India (e-mail:vinay.kumar@juit.ac.in, vinaykverma@gmail.com)

<sup>†</sup>Sunil Bhooshan is with Jaypee University of Information Technology, Solan-173215, India (e-mail:sunil.bhooshan@juit.ac.in)

## 2 Preliminaries

The proposed linear phase IIR filter design uses linear phase high pass and low pass FIR filters. To design a linear phase low pass IIR filter ( $H_{iirlowpass}(\omega)$ ), we divide linear phase low pass FIR filter characteristics ( $H_{LP(FIR)N}(\omega)$ , Figure 1)<sup>1</sup> with linear phase high pass FIR filter characteristics, ( $H_{HP(FIR)D}(\omega)$ , Figure 2)<sup>2</sup>. The magnitude characteristics of two FIR filters are such that the pass band of low pass FIR filter is the stop band of high pass FIR filter and vice-versa; the transition bands overlap, when one transition band goes from lower value to higher value other goes from higher to lower. Therefore, the low pass IIR filter is given by

$$\begin{aligned} H_{iirlowpass}(\omega) &= \frac{H_{LP(FIR)N}(\omega)}{H_{HP(FIR)D}(\omega)} \\ &= \frac{|H_{LP(FIR)N}| \angle H_{LP(FIR)N}}{|H_{HP(FIR)D}| \angle H_{HP(FIR)D}} \\ &= \frac{|H_{LP(FIR)N}|}{|H_{HP(FIR)D}|} \angle (H_{LP(FIR)N} - H_{HP(FIR)D}) \end{aligned}$$

It is clear from Equation (1) that at every point the amplitudes of low pass and high pass filters are divided and phase subtracted to get the amplitude and phase characteristics of the resulting low pass IIR filter at these points, respectively. If both, high pass and low pass, filters have linear phase, then, resulting IIR filter will have zero phase (if the phase response is same for both of the numerator and denominator) or linear phase.

From Equation (1) we can deduce that while designing the FIR high pass filter; that is,  $H_{HP(FIR)D}(\omega)$ , we have to ensure that this FIR filter has no zero in  $0 \leq \omega \leq \pi$ , otherwise these zeros will make the resulting IIR filter unstable.

Detailed procedure to design FIR filters using orthogonal polynomials is discussed in [13]. The out-

<sup>1</sup>The subscript indicates Low Pass FIR filter to be used as Numerator.

<sup>2</sup>The subscript indicates High Pass FIR filter to be used as Denominator.

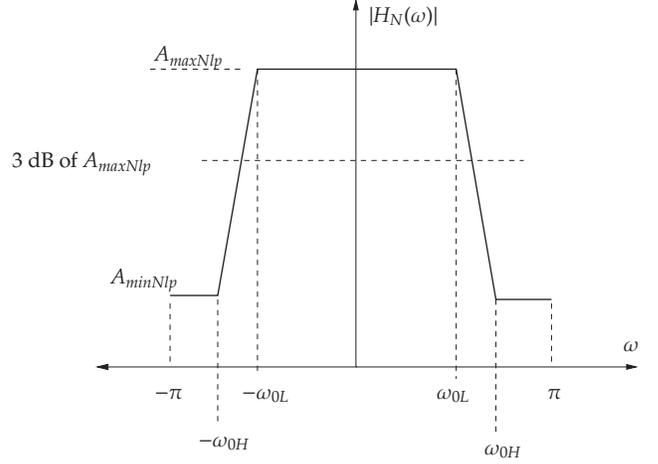


Figure 1: Desired low pass FIR filter characteristics to be used as numerator.

line of the procedure is presented here for convenience.

### 2.1 Design of linear phase FIR filter

Suppose user needs to design a linear phase FIR filter, as shown in Figure 1. The filter characteristics are first transformed to a function, which we call as object function, using the following transformation [14, 15]

$$x = x_0 \cos(\omega/2) \quad (2)$$

where,  $x_0$  is the maximum value of  $x$ .

We consider Legendre polynomials in the present discussion, though the procedure is general and can be applied on any orthogonal polynomial. Legendre polynomials are orthogonal between  $[-1, 1]$ , therefore, value of  $x_0$  is 1 in the present case.

The object function corresponding to Figure 1 is shown in Figure 3. The function thus obtained is than represented as a linear combination of even terms of orthogonal polynomials (proof in Appendix A), that is,

$$f(x) = \sum_{n=0}^{\infty} a_{2n} P_{2n}(x) \quad (3)$$

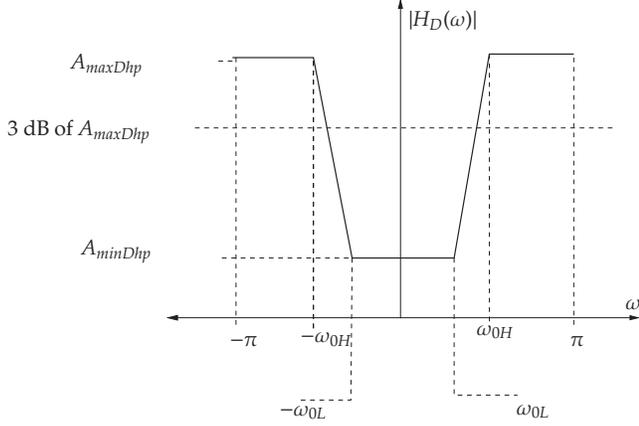


Figure 2: Desired high pass FIR filter characteristics to be used as denominator.

where,  $a_{2n}$  represent the coefficients which are required to be multiplied with the orthogonal polynomials,  $P_{2n}(x)$ , to calculate the object function.  $a_{2n}$ 's are calculate by (see Appendix A),

$$a_{2n} = \frac{\int_0^1 f(x)P_{2n}(x)dx}{\int_0^1 P_{2n}(x)P_{2n}(x)dx} \quad (4)$$

where,  $n = 0, 1, 2, \dots$

It is not possible to calculate  $f(x)$  using infinite number of orthogonal polynomial terms, therefore, we find the approximate object function  $f_a(x)$  using only first  $(N + 1)$  polynomial terms

$$f_a(x) = \sum_{n=0}^N a_{2n}P_{2n}(x) \quad (5)$$

$f_a(x)$  is then transformed to filter characteristics by using the inverse of the transformation given in Equation (2); i.e.,

$$\omega = 2 * \cos^{-1} \frac{x}{x_0}$$

We can design the linear phase high pass FIR filter in the same fashion.

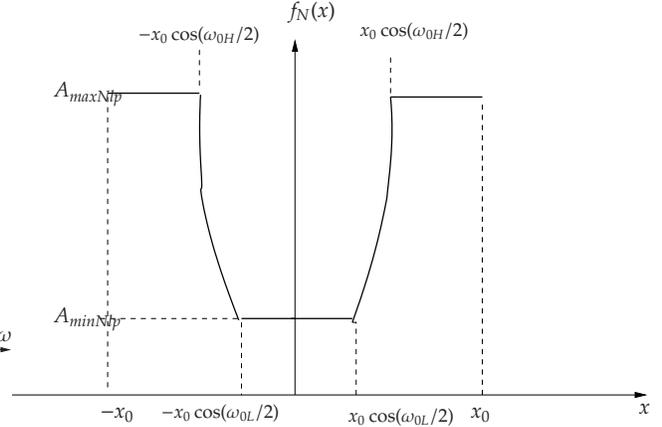


Figure 3: Object function for LPF of Figure 1, showing slight non linearity in the transition region.

Based on the method discussed above, we discuss the procedure to design the IIR filters in detail in the next section.

### 3 Procedure

From here onwards we represent  $H_{LP(FIR)N}(\omega)$  by  $H_N(\omega)$  and  $H_{HP(FIR)N}(\omega)$  by  $H_D(\omega)$  for simplicity.

As discussed in the last section low pass and high pass FIR filter characteristics are realized by their corresponding object functions –  $f_N(x)$  for  $H_N(\omega)$  and  $f_D(x)$  for  $H_D(\omega)$  – as shown in Figures 3 and 4, respectively. The object functions, corresponding to numerator and denominator object functions, respectively, are given by

$$f_N(x) = \sum_{n=0}^{\infty} a_{2n}P_{2n}(x) \quad (6)$$

$$f_D(x) = \sum_{n=0}^{\infty} b_{2n}P_{2n}(x) \quad (7)$$

where,  $P_0, P_2, P_4 \dots$  are the Legendre polynomials of the order of 0, 2, 4,  $\dots$ , and  $a_{2n}$  and  $b_{2n}$  are coefficients which are multiplied with the Legendre polynomials to approximate the required object function characteristics shown in Figures 3 and 4, respectively.

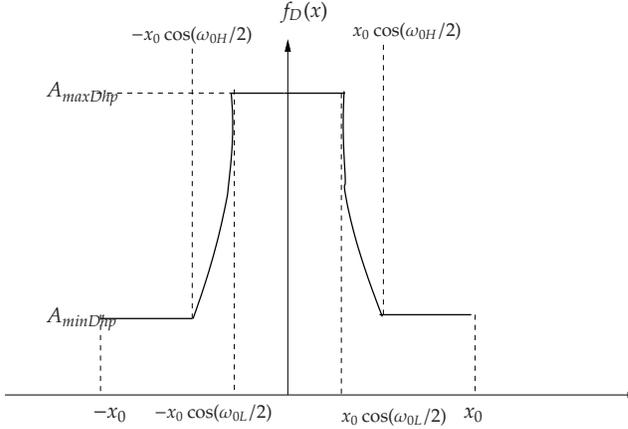


Figure 4: Object function for HPF of Figure 2, showing slight non linearity in the transition region.

The following steps outline the procedure to design the linear phase IIR filters.

**Step 1 :** Calculate the coefficients  $a_0, a_2, a_4, \dots$  and  $b_0, b_2, b_4, \dots$

$$a_{2n} = \frac{\int_0^1 f_N(x) P_{2n}(x) dx}{\int_0^1 P_{2n}(x) P_{2n}(x) dx} \quad (8)$$

$$b_{2n} = \frac{\int_0^1 f_D(x) P_{2n}(x) dx}{\int_0^1 P_{2n}(x) P_{2n}(x) dx} \quad (9)$$

where,  $n = 0, 1, 2, \dots$

**Step 2 :** We use finite number of  $a_{2n}$ 's and  $b_{2n}$ 's in Equations (6) and (7), therefore, we get the approximate object functions  $f_{Na}(x)$  and  $f_{Da}(x)$  in place of  $f_N(x)$  and  $f_D(x)$ , respectively; that is,

$$f_{Na}(x) = \sum_{n=0}^N a_{2n} P_{2n}(x) \quad (10)$$

$$f_{Da}(x) = \sum_{n=0}^M b_{2n} P_{2n}(x) \quad (11)$$

**Step 3 :** Polynomials  $f_{Na}(x)$  and  $f_{Da}(x)$  are converted to  $H_{Na}(\omega)$  and  $H_{Da}(\omega)$ , respectively, using the transformation of Equation (2).

The discussion in the previous section makes it clear that  $H_{Na}(\omega)$  and  $H_{Da}(\omega)$  represent the approximate low pass and high pass filter characteristics corresponding to  $f_{Na}(x)$  and  $f_{Da}(x)$ , respectively.

**Step 4 :** Calculate the rational function in  $\cos(\omega/2)$

$$H_{iirlowpass}(\omega) = \frac{H_{Na}(\omega)}{H_{Da}(\omega)} \quad (12)$$

Notice that the denominator must not have a zero in  $0 \leq \omega \leq \pi$  or  $0 \leq x \leq 1$ , since this will lead to instability.

**Step 5 :** Calculate zeros of the IIR filter by solving  $H_{Da}(\omega) = 0$ . They are represented by  $z_{iN} = \exp(j\omega_{iN})$  and  $z_{iD} = \exp(j\omega_{iD})$ ; where,  $i = 1, 2, 3, \dots$

**Step 6 :** The transfer function of the resulting IIR filter is

$$H_{iirlowpass}(z) = \frac{(z - z_{1N})(z - z_{2N})(z - z_{3N}) \dots}{(z - z_{1D})(z - z_{2D})(z - z_{3D}) \dots} \quad (13)$$

To design high pass IIR filter we have to divide high pass FIR filter characteristics by low pass FIR filter characteristics; that is,

$$H_{iirhighpass} = \frac{H_{Da}(\omega)}{H_{Na}(\omega)}$$

### 3.1 Pole-Zero Consideration

To realize the proposed filter it is necessary that the filter must be causal in nature. In other words, all the poles of IIR filter must lie within the unit circle. If some of the poles, Equation (13), lie outside unit circle then we have to modify the object function which is used to calculate the filter characteristics. To perform this operation we shift all the poles lying outside the unit circle to the origin.

To calculate the frequency response of a system having  $n$  poles and  $m$  zeros from its pole-zero plot we use [16] following formula

$$H(e^{j\omega}) = K \frac{r_1 * r_2 \dots * r_n}{d_1 * d_2 \dots * d_m} \quad (14)$$

where,  $K$  represents a constant,  $r_1, r_2, \dots, r_n$  represent the distance of the of  $n$ -zeros from the some complex frequency point. Similarly,  $d_1, d_2, \dots, d_m$  is the distance of  $m$ -poles from the same frequency point. To

understand it better let us look at Figure 5. In the figure point  $p$  represents the point where frequency response has to be calculated. From Figure 5 it is clear that some poles are near to the point  $p$  and some are far. From Equation (14) we know that we have to multiply various poles. We also know that if a pole is at  $\infty$  distance from the desired frequency point it will make the frequency response zero at that point. Therefore, poles which are far from the desired frequency point makes the frequency response lower and those which are near makes it high in magnitude. Once we shift the poles at origin all the poles will be at equal distance; that is, at a distance 1. Therefore, when we move along the unit circle, to calculate complete frequency response, some poles appear near the point of interest and some far away. For example in Figure 5 from point  $p$  pole with distance  $d_3$  is far, while when we move onto the unit circle and reach diagonally opposite point (say  $p'$ ) same pole will be near. Since poles are distributed around the unit circle, after we shift all the poles to origin distances will be all equal; that is, 1. It will average out the distances. We have observed that once we shift the poles to origin the effect of shift is very small on the frequency response.

This shift of poles, lying outside the unit circle, changes the phase and magnitude response slightly but on the other hand makes sure that resulting filter is stable in nature.

For clear understanding of the above procedure we design an IIR filter in the next section.

## 4 Application and Discussion

Suppose we intend to design an IIR filter with following characteristics

$$H_{iirlowpass}(\omega) = \begin{cases} 500 & \text{if } 0 \leq \omega < 2.0007 \\ \text{transition band} & \text{if } 2.0007 \leq \omega < 2.3186 \\ 0 & \text{if } 2.3186 \leq \omega < \pi \end{cases}$$

Therefore, as per the discussion of the previous section, first we have to design the low pass and high pass FIR filters. The, assumed, low pass and high pass FIR filter characteristics are as follows

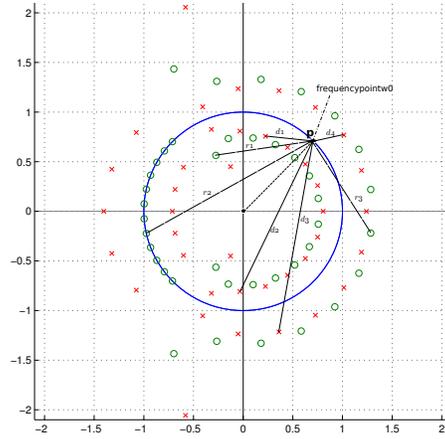


Figure 5: Calculation of transfer function from pole-zero plot at frequency  $\omega_0$ .

Low pass FIR (Figure 1) filter

$$H_{firlowpass}(\omega) = \begin{cases} 1000 & \text{if } 0 \leq \omega < 2.0007 \\ \text{transition band} & \text{if } 2.0007 \leq \omega < 2.3186 \\ 0 & \text{if } 2.3186 \leq \omega < \pi \end{cases}$$

High pass FIR (Figure 2) filter

$$H_{firhighpass}(\omega) = \begin{cases} 1 & \text{if } 0 \leq \omega < 2.0007 \\ \text{transition band} & \text{if } 2.0007 \leq \omega < 2.3186 \\ 2 & \text{if } 2.3186 \leq \omega < \pi \end{cases}$$

Note that as of now we show only positive half of the graph the negative half being a mirror image.

### 4.1.1 Design

Let us design the IIR filter with 20 Legendre polynomial terms used to approximate the object functions (both for low pass and high pass FIR filter characteristics). After following the steps outlined in Section 2, we get the approximate object functions  $f_{Na}(x)$

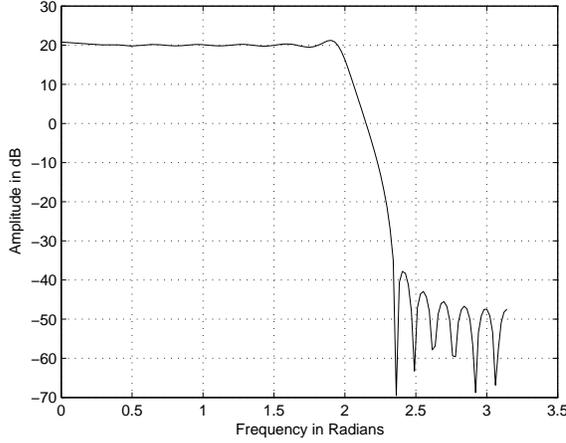


Figure 6: Magnitude response in dB of low pass IIR filter corresponding to the object function approximated using 20 orthogonal polynomial terms.

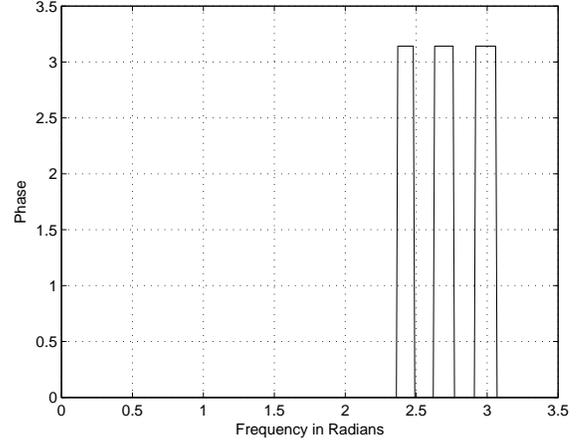


Figure 7: Phase response of low pass IIR filter corresponding to the object function approximated using 20 orthogonal polynomial terms.

and  $f_{Da}(x)$  and eventually  $H_{iir\ low\ pass}(\omega)$ . The magnitude response in dB and phase response are shown in Figures 6 and 7, respectively.

Let us look at the pole-zero distribution of this filter, which is shown in Figure 8. It is clear from the plot that some of the poles are lying outside the unit circle. Therefore, the resulting filter is unstable and hence can not be realized.

To make sure that the proposed filter is stable we shift all those poles which lie outside the unit circle to the origin.

Original distribution of poles and zeros is shown in Figure 8 and after shift it is shown in Figure 9. The magnitude response together with phase response is shown in Figure 10. It is clear from Figure 10 that the phase response is non linear during the transition band. In the pass band phase remains linear. The group delay is shown in Figure 11, it is zero during the pass band of the filter characteristics.

We observed that when we move a pole which is lying near the origin (but out of unit circle) to origin, frequency response changes at the point where transition band starts. While when we move the poles lying far away there is change near frequency zero of the frequency response. When all the poles of

the frequency response are moved to the origin the overaly effect is very small.

From the Figure 11, it is clear that the IIR filter designed with proposed design technique gives zero group delay while technique discussed in [6, 17, 18] has a group delay which is not zero. The frequency response of the proposed technique is almost same of [6, 17, 18]. The proposed technique does not use any optimization technique to design the IIR filter, which makes proposed technique mathematically simple to understand and design, while [8] uses a technique where FIR filter is approximated using computationally expensive technique.

## 5 Conclusion and Future Work

Above discussion makes it clear that a post processing IIR filter with linear phase can easily be designed by using the orthogonal polynomials. The proposed IIR filter gives good cutoff characteristics. By increasing the number of polynomial terms, we can approximate our object function very closely, which in turn will produce good frequency characteristics both in the pass band and the transition region. The

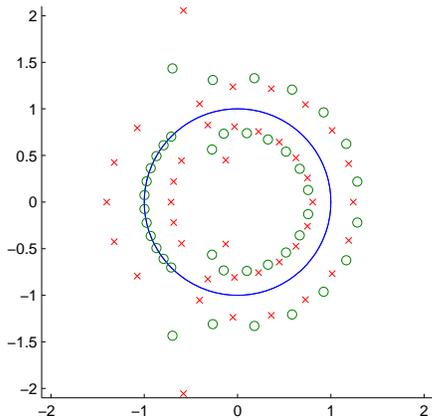


Figure 8: Pole-Zero Distribution of the IIR Filter.

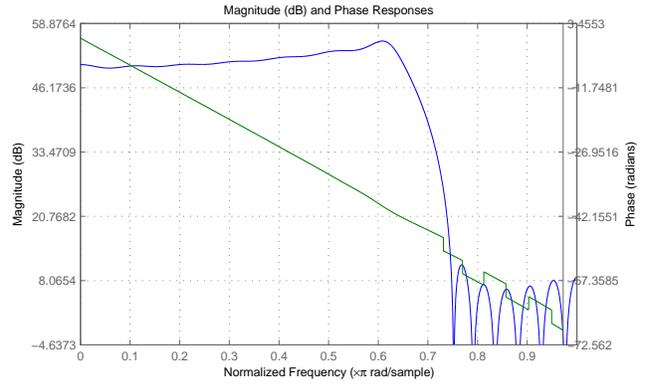


Figure 10: Magnitude and Phase Response after shifting poles at origin.

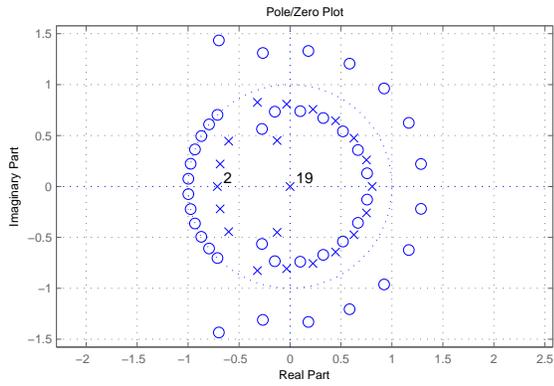


Figure 9: Pole-Zero Distribution after shifting poles at origin.

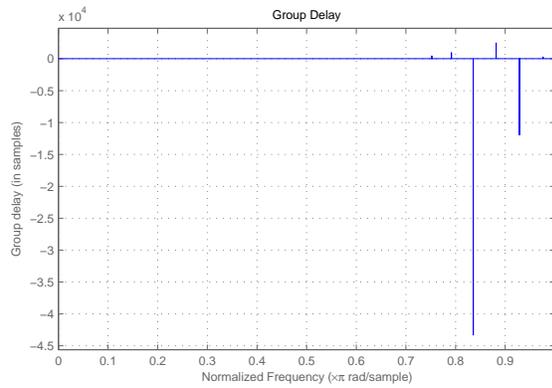


Figure 11: Group Delay after shifting poles at origin.

ripples in the pass band becomes negligible as we increase the number of terms to approximate our object function. Stop band amplitude decreases as we increase the number of terms in our object functions. In all, we may state that the alternate approach discussed in the present paper gives much easier design of IIR filter when compared with the currently available methods [6, 8, 11, 17, 18] together with absolutely zero group delay while methods discussed in [6, 8, 11, 17, 18] does not zero group delay even for higher order filters when compared with the present method. By moving the poles of the IIR filter within the unit circle, we can easily realize the IIR filter with almost same magnitude and phase characteristics as were for non causal filter.

We are working to develop a mathematical model which can be used to predict the frequency response when poles are shifted from outside the unit circle to inside.

## Appendix A

*The sum of Legendre polynomials multiplied by suitable coefficients approximates the ideal polynomial in the least mean square error sense.*

*Proof:*

Suppose a polynomial  $f(x)$  is approximated by  $f_a(x)$  and its representation is

$$f_a(x) = \sum_{n=0}^N a_{2n} P_{2n}(x)$$

The mean squared error (MSE) polynomial between the original polynomial and approximated polynomial is given by

$$E(x) = \{f(x) - f_a(x)\}^2$$

or,

$$E(x) = \left[ f(x) - \sum_{n=0}^N a_{2n} P_{2n}(x) \right]^2 \quad (15)$$

Note that  $E(x)$  is either positive or zero hence the minimum value of its integral is zero. We define

$$\epsilon(a_0, a_2, \dots) = \int_{-1}^1 E(x) dx = \int_{-1}^1 \left[ f(x) - \sum_{n=0}^N a_{2n} P_{2n}(x) \right]^2 dx \quad (16)$$

We need to find the coefficients  $a_0, a_2, \dots$  from the above equation so that this integral is minimum. The general way of solving this problem is well known, and is described below

$$\frac{\partial}{\partial a_i} \int_{-1}^1 \left[ f(x) - \sum_{n=0}^N a_{2n} P_{2n}(x) \right]^2 dx = 0 \quad i = 1 \dots N \quad (17)$$

or

$$\begin{aligned} \frac{\partial}{\partial a_i} \int_{-1}^1 \left[ \{f(x)\}^2 + \left\{ \sum_{n=0}^N a_{2n} P_{2n}(x) \right\}^2 \right. \\ \left. - 2f(x) \left\{ \sum_{n=0}^N a_{2n} P_{2n}(x) \right\} \right] dx = 0 \quad i = 1 \dots N \quad (18) \end{aligned}$$

which is

$$\begin{aligned} \int_{-1}^1 \left[ 0 + \frac{\partial}{\partial a_i} \left\{ \sum_{n=0}^N a_{2n} P_{2n}(x) \right\}^2 \right. \\ \left. - 2 \frac{\partial}{\partial a_i} f(x) \left\{ \sum_{n=0}^N a_{2n} P_{2n}(x) \right\} \right] dx = 0 \quad i = 1 \dots N \quad (19) \end{aligned}$$

spelt out this is

$$\begin{aligned} \int_{-1}^1 \left[ \frac{\partial}{\partial a_i} \{ a_0^2 P_0^2 + 2a_0 a_2 P_0 P_2 + \dots + a_i^2 P_i^2 \right. \\ \left. + 2a_i a_{i+2} P_i P_{i+2} + 2a_i a_{i+4} P_i P_{i+4} + \dots \} dx \right. \\ \left. - 2 \int_{-1}^1 f(x) a_i P_i \right] dx = 0 \quad (20) \end{aligned}$$

The orthogonality property of the Legendre polynomials reduces this integral to

$$2 \int_{-1}^1 a_i P_i^2 dx - 2 \int_{-1}^1 f(x) P_i dx = 0 \quad i = 1 \dots N \quad (21)$$

$$a_i = \frac{\int_{-1}^1 f(x)P_i dx}{\int_{-1}^1 P_i^2 dx} \quad i = 1 \dots N \quad (22)$$

## References

- [1] A. Antoniou, *Digital Filters: Analysis, Design, and Applications*. McGraw-Hill, "2nd" ed., 1993.
- [2] C. Charalambous and A. Antoniou, "Equalization of recursive digital filters," *IEE Proceedings Part G*, vol. 127, no.4, pp. 219–225, 1980.
- [3] J.P.Thiran, "Recursive digital filters with maximally flat group delay," *IEEE Transactions on Circuit Theory*, vol. CT-18, pp. 659–664, 1971.
- [4] G. Cortelazzo and M. R. Lightner, "Simultaneous design in both magnitude and group delay of iir and fir filters based on multiple criterion optimization," *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. ASSP-32, pp. 949–967, 1984.
- [5] H. Baher, "Digital filters with finite-band approximation to constant amplitude and delay, and arbitrary selectivity," *Proceeding of the IEEE International Symposium on Circuits ans Systems*, pp. 657–659, 1986.
- [6] A. T. Chottera and G. A. Jullien, "A linear programming approach to recursive digital filter design with linear phase," *IEEE Transactions on Circuits and Systems*, vol. CAS-29, pp. 139–149, 1982.
- [7] W. Lu, S. Pei, and C. Tseng, "A weighted least-squares method for the design of stable 1-d and 2-d iir digital filters," *IEEE Transactions on Signal Processing*, vol. 46, pp. 1–10, 1998.
- [8] C. Xiao, J. C. Olivier, and P. Agathoklis, "Design of linear phase iir filters via weighted least-squares approximation," *Proceedings of ICASSP*, vol. 6, pp. 3817–3820, 2001.
- [9] R. Fletcher, *Practical Methods of Optimization*. JohnWiley & Sons, 2nd ed., 1987.
- [10] B. Beliczynski, I. Kale, and G. D. Cain, "Approximation of fir by iir digital filters: an algorithm based on balanced model reduction," *IEEE Transactions on Signal Processing*, vol. 40, pp. 532–542, 1992.
- [11] M. F. Fahmy, Y. M. Yassin, G. Abdel-Raheem, and N. El-Gayed, "Design of linear phase iir filters from fir specifications," *IEEE Transactions on Signal Processing*, vol. 42, pp. 437–440, 1994.
- [12] S. Holford and P. Agathoklis, "The use of model reduction techniques for designing iir filters with linear phase in the passband," *IEEE Transactions on Signal Processing*, vol. 44, pp. 2396–2404, 1996.
- [13] S. Bhooshan and V. Kumar, "A new method to design linear phase fir filter," in *IEEE International Conference On Industrial Technology*, pp. 1–6, 21-24 April, 2008.
- [14] S. Bhooshan and V. Kumar, "A polynomial approach towards the design of linear phase fir filters," in *Proceedings of IEEE International Conference on Industrial Technology*, 2006.
- [15] S. Bhooshan and V. Kumar, "Design of chebyshev fir filter based on antenna theory approach," *WSEAS Transactions on Signal Processing*, vol. 3, no. 2, pp. 179–184, 2007.
- [16] P. Hardt, "Understanding poles and zeros." Online.
- [17] V. Sreeram and P. Agathoklis, "Design of linear-phase iir filters via impulse-response grami-ans," *IEEE Transactions on Signal Processing*, vol. 40, pp. 389–394, 1992.
- [18] L. Li, L. Xie, W.-Y. Yan, and Y. C. Soh, "Design of low-order linear-phase iir filters via orthogonal projection," *IEEE Transactions on Signal Processing*, vol. 47, pp. 448–457, 1999.