Birth and death processes on certain random trees: Classification and stationary laws

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Abstract

The main substance of the paper concerns the growth rate and the classification (ergodicity, transience) of a family of random trees. In the basic model, new edges appear according to a Poisson process of parameter λ and leaves can be deleted at a rate μ. The main results lay the stress on the famous number e. A complete classification of the process is given in terms of the intensity factor $\rho = \lambda/\mu$: it is ergodic if $\rho \leq e^{-1}$, and transient if $\rho > e^{-1}$. There is a phase transition phenomenon: the usual region of null recurrence (in the parameter space) here does not exist. This fact is rare for countable Markov chains with exponentially distributed jumps. Some basic stationary laws are computed, e.g. the number of vertices and the height. Various bounds, limit laws and ergodic-like theorems are obtained, both for the transient and ergodic regimes. In particular, when the system is transient, the height of the tree grows linearly as the time $t \to \infty$, at a rate which is explicitly computed. Some of the results are extended to the so-called multiclass model.

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1 Introduction and model description

So far, very few results seem to exist for random trees as soon as insertions and deletions are simultaneously permitted (see e.g. [14]). We shall study one of the simplest models in this class, which offers both interesting and non trivial properties. Broadly speaking, one might think of a vertex as being a node of a network (e.g. the Internet) or of some general data structure. This paper is a self-contained continuation of the report [5].

Let $G = \{G(t), t \geq 0\}$ be a continuous time Markov chain with state space the set of finite directed trees rooted at some fixed vertex v_0.

Throughout the study, the *distance* between two vertices is the number of edges in the path joining them, and the *height $h(v)$* of a vertex v is the distance from the root. The set of vertices having the same height k form the k-th *level* of the tree, the root v_0 being at level 0. Hence the height of G is a stochastic process $\{H_G(t), t \geq 0\}$, where

$$H_G(t) \stackrel{\text{def}}{=} \max_{v \in G(t)} h(v).$$

$N_G(t)$ will stand for the *volume* of $G(t)$ (*i.e.* its total number of vertices).

Wherever the meaning is clear from the context, we shall omit the subscript G and simply write H or N. The *indegree* of a vertex v is the number of edges starting at v and a vertex with indegree 0 is a *leaf*. Finally, we will also need the classical notion of *subtree* with root v, which goes without saying.

At time $t = 0$, $G(0)$ consists of the single vertex v_0. Then at time $t > 0$, the transitions on G are of two types:

- **Adjunction.** At each vertex v, a new edge having its origin at v can be appended to the tree at the epochs of a Poisson process with parameter $\lambda > 0$. In this case, the *indegree* of v is increased by one and the new edge produces a new leaf.

- **Deletion.** From its birth, a leaf (but the root) can be deleted at a rate μ. In other words, a vertex *as long as it has no descendant* has an exponentially distributed lifetime with parameter $\mu \geq 0$.

1.1 Organization of the paper, results and related studies

Section 2 is devoted to the birth and death model described above, with $\lambda, \mu > 0$. An exact and complete classification of G is given. Indeed, necessary and sufficient conditions are derived for the process to be ergodic ($\mu \geq \lambda e$) or transient ($\mu < \lambda e$). A phase transition phenomenon is enlightened, which corresponds precisely to the absence of a null recurrence region.

When the system is ergodic, the stationary distributions of the volume and of the height of the tree are computed in section 3.

Section 4 deals with limit laws and scalings for $H_G(t)$ and $N_G(t)$ in the transient case. The main outcome is a kind of ergodic theorem for $H_G(t)$, valid for any $\mu \geq 0$. It allows, in the particular case $\mu = 0$ (pure birth-process), to rediscover the magic growth rate λe, originally derived e.g. in [4, 15].

Finally, section 5 proposes an extension to a multiclass model, in which the parameters of the process depend possibly on the class, the key result being a qualitative theorem for ergodicity.

Recently, the authors were made aware of a model studied in [16, 13]. The setting considered there is a contact process on a d-ary ordered tree, also known as a Catalan tree, where $d \geq 2$ is an arbitrary *finite* integer. The main difference with our model resides in the fact that each empty descendant of an occupied vertex can become occupied at a rate $\beta > 0$. The classification of the process was obtained for $d = 2$ in [16], and [13] extends the result to any finite d, but nothing was said for $d = \infty$. It turns out that most of the points presented in our study (ergodicity, zero-one laws, etc.) cannot be obtained by simply letting $d \to \infty$ in [16, 13]. Likewise, reversibility arguments (which should theoretically lead to explicit invariant measures) used in the latter papers do not seem to be effective when d is infinite (see section 3).

2 The birth and death case: $\lambda > 0, \mu > 0$

The random tree G evolves according to the rules given in the introduction, the first important question being to find exact conditions for this process

to be recurrent or transient. Main results in this respect are stated in theorem 2.1.

For convenience, we define the *lifetime* τ_v of an arbitrary vertex v, which measures the length of the time interval between the birth and the death of v (for consistency $\tau_v = \infty$ if v is never erased).

Lemma 2.1. *All vertices, but the root, have the same lifetime distribution $p(t)$, which satisfies the following system (S)*

$$\beta(t) \;=\; \mu \exp\left\{-\lambda \int_0^t (1 - p(x))dx\right\}, \tag{2.1}$$

$$\beta(t) \;=\; \frac{dp(t)}{dt} + \int_0^t \beta(t-y)dp(y), \tag{2.2}$$

with the initial condition $p(0) = 0$.

Proof. Let v be a particular vertex of $G(t)$ and consider the related random subtree with root v. Its evolution does not depend on anything below v, as long as v exists. Therefore all these subtrees are identically distributed and, accordingly, their vertices have the same lifetime distribution.

To capture more precisely the evolution of the process, we associate with each vertex v with age t its number $X_v(t)$ of direct descendants (i.e. who are located at a distance 1 from v).

At rate λ, a vertex v produces descendants whose lifetimes are independent, with the common distribution $p(t)$. As soon as $X_v(t) = 0$, v can die at rate μ, in which case the process of production stops.

It is actually useful to extend $X_v(t)$ for all $t \geq 0$ by deciding that, instead of deleting v, a *μ-event* occurs without stopping the production of descendants. With this convention, the number of descendants of the root vertex v_0 evolves as $X_{v_0}(t)$, for all $t \geq 0$. Let τ_v denote the random epoch of the first μ-event, which is distributed according to $p(t)$.

Clearly the process X_v is regenerative with respect to the μ-events. Thus the random variables $X_v(t)$ and $X_v(\tau_v + t)$ have the same distribution.

For any fixed t, we write down a sum of conditional probabilities, expressing the fact that v had exactly k descendants, who all have died in $[0, t]$, their birth-times being independent and uniformly spread over $[0, t]$. This yields

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at once equation (2.1), since

$$
\begin{aligned}
\mathsf{P}\{X_v(t) = 0\} &= \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \left(\int_0^t \frac{p(x)dx}{t} \right)^k \\
&= \exp\left\{ -\lambda \int_0^t (1 - p(x))dx \right\}.
\end{aligned}
\tag{2.3}
$$

By means of a regenerative argument, it is also possible to rewrite the above probability in another way, starting from the decomposition

$$
\mathsf{P}\{X_v(t) = 0\} = \mathsf{P}\{X_v(t) = 0, \tau_v \geq t\} + \mathsf{P}\{X_v(t) = 0, \tau_v < t\}.
\tag{2.4}
$$

In fact, we have the trite relations

$$
\begin{cases}
\dfrac{dp(t)}{dt} = \mu \mathsf{P}\{X_v(t) = 0, \tau_v \geq t\}, \\[2mm]
\mathsf{P}\{X_v(t) = 0, \tau_v < t\} = \mathsf{P}\{X_v(t - \tau_v) = 0, \tau_v < t\}, \\[2mm]
\mathsf{P}\{\tau_v \in (y, y + dy)\} = dp(y),
\end{cases}
$$

which yield in particular,

$$
\mathsf{P}\{X_v(t) = 0, \tau_v < t\} = \int_0^t \mathsf{P}\{X_v(t - y) = 0\}dp(y).
$$

Hence, putting $\beta(t) \stackrel{\text{def}}{=} \mu \mathsf{P}\{X_v(t) = 0\}$, one sees that (2.4) corresponds term by term to (2.2). The proof of the lemma is concluded. ■

It is convenient to introduce now $\tau \stackrel{\text{def}}{=} \tau_{v_0}$, which is the random variable representing the epoch of the first μ-event for the root of the tree. Nonetheless in the sequel, especially in sections 3 and 4, τ will also often refer to the lifetime of an arbitrary generic vertex v, owing to the fact that all these quantities have the same distributions.

We are ready to state the main result of this section.
Theorem 2.1.

(A) The Markov chain G is ergodic if, and only if,

$$
\rho \stackrel{\text{def}}{=} \frac{\lambda}{\mu} \leq \frac{1}{e}.
\tag{2.5}
$$

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(B) *When the system is ergodic, the mean lifetime $m \stackrel{\mathrm{def}}{=} \mathsf{E}\tau$ is given by*

$$m = \frac{r}{\lambda},$$

where $r \leq 1$ denotes the smallest root of the equation

$$re^{-r} = \rho \qquad\qquad (2.6)$$

and represents the mean number of descendants of an arbitrary vertex at steady state.

(C) *When $\rho > \dfrac{1}{e}$, then the system is transient. In this case,*

$$\lim_{t \to \infty} p(t) \stackrel{\mathrm{def}}{=} \ell < 1.$$

As a rule, x being the positive root of $xe^x = \rho^{-1}$, we have for any ρ

$$x \leq \ell \leq \min\left(1, \frac{1}{\rho}\right) \quad \text{and} \quad \lim_{\rho \to \infty} \rho\ell = 1.$$

The proof of the theorem is spread over the next two subsections.

2.1   Ergodicity

Relying on standard theory of Markov chains with countable state space (see [6, vol. I]), we claim the system ergodic if, and only if, $m < \infty$. As a matter of fact, the μ-events are regeneration points for the process $X(t)$, which represents exactly the number of descendants of the *root v_0*. Hence when $\mathsf{E}\tau < \infty$ (i.e. $\beta(\infty) > 0$), the event $\{X(t) = 0\}$ has a positive probability, so that G is ergodic. Conversely, if $\mathsf{E}\tau = \infty$ then $X(t)$ is transient and so is G.

For an arbitrary positive function f, denote by f^* its ordinary Laplace transform

$$f^*(s) \stackrel{\mathrm{def}}{=} \int_0^\infty e^{-st} f(t)dt, \quad \Re(s) \geq 0.$$

Later on we will also need the associated inversion formula (see e.g. [7])

$$f(t) = \frac{1}{2i\pi} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{st} f^*(s)ds, \quad \Re(\sigma) > 0. \qquad\qquad (2.7)$$

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To show the necessity of condition (2.5), we suppose G is ergodic. In this case, by (2.1), the quantity $\lim_{t\to\infty} \beta(t)$ does exist and

$$\mu e^{-m} \leq \beta(t) \leq \mu,$$

so that we can apply a limiting relation of Abelian type (see e.g. [7]). Hence equations (2.1) and (2.2)—the latter belonging to the Volterra class—yield respectively

$$\begin{cases} \lim_{t\to\infty} \beta(t) = \lim_{s\to 0} s\beta^*(s) = \lim_{s\to 0} \dfrac{s^2 p^*(s)}{1 - sp^*(s)} = \dfrac{1}{m}, \\ \lim_{t\to\infty} \beta(t) = \mu e^{-\lambda m}, \end{cases} \tag{2.8}$$

whence the equality

$$\rho = \lambda m e^{-\lambda m}.$$

As the function xe^{-x} reaches its maximum e^{-1} at $x = 1$, we conclude that necessarily $\rho \leq e^{-1}$.

In order to prove the sufficiency of (2.5), we have to get a deeper insight into system (S). There will be done along two main steps.

(a) Although (S) reduces to a second order nonlinear integro-differential equation, this does not help much. What is more useful is that all derivatives $p^{(n)}(0)$, $\beta^{(n)}(0)$, taken at the the origin in the complex t-plane, can be recursively computed for all n. This can be checked at once, rewriting (2.1) in the differential form

$$\frac{d\beta(t)}{dt} + \lambda(1 - p(t))\beta(t) = 0. \tag{2.9}$$

Noticing the derivatives $p^{(n)}(0)$ and $\beta^{(n)}(0)$ have alternate signs when n varies, it is direct to verify that β and p are analytic functions around the origin, and that their respective power series have a non-zero radius of convergence. The first singularities of p and β are on the negative real axis, but not easy to locate precisely. Thus (S) has a solution, which is unique, remarking also that uniqueness is a mere consequence of the Lipschitz character of $dp(t)/dt$ with respect to β in the Volterra integral equation (2.2) (see e.g. [2]). En passant, it is worth noting that the solution in the whole complex plane—which is not really needed for our purpose—could be obtained by analytic continuation directly on system (S).

(b) When (2.5) holds, the next stage consists in exhibiting a *non-defective* probabilistic solution $p(t)$ [necessarily unique by step (a)], with a finite mean $m < \infty$. This is more intricate and will be achieved by constructing a converging iterative scheme.

Consider the system

$$
\begin{cases}
\beta_0(t) & = \mu, \quad t \geq 0, \\
\beta_k(t) & = \dfrac{dp_k(t)}{dt} + \displaystyle\int_0^t \beta_k(t-y)dp_k(y), \\
\beta_{k+1}(t) & = \mu \exp\left\{-\lambda \displaystyle\int_0^t \left(1 - p_k(y)\right)dy\right\}, \\
p_k(0) & = 0, \ \forall k \geq 0.
\end{cases}
\tag{2.10}
$$

The second equation in (2.10) is equivalent to

$$
sp_k^*(s) = \frac{\beta_k^*(s)}{1 + \beta_k^*(s)},
\tag{2.11}
$$

allowing to derive p_k from β_k by means of (2.7) (see also [6] for various inversion formulas in the real plane). Then the computational algorithm becomes simple:

1. $p_0(t) = 1 - e^{-\mu t}$.

2. Compute $\beta_1(t) = \mu \exp\left[-\rho(1 - e^{-\mu t})\right]$.

3. Compute $p_1(t)$, then $\beta_2(t), p_2(t)$, etc.

In the scheme (2.10), the initial condition $\beta_0(t) = \mu$ is tantamount to take an implicit fictitious function, say p_{-1}, satisfying $p_{-1}(t) = 1, \ \forall t \geq 0$.

At each step, the successive p_k's are non-defective probability distributions, with finite means denoted by m_k. The scheme (2.10) enjoys two nice properties.

(i) It is monotone *decreasing*. Suppose $p_k(t) \leq p_{k-1}(t)$, which is in particular true for $k = 1$. In the third equation of (2.10), $\beta_{k+1}(t)/\mu$ is simply the probability of being empty for an M/G/∞ queue with arrival rate λ and service time distribution function p_k. It is therefore possible, by a coupling argument, to build two M/G/∞ queues corresponding to $\beta_k(t)$ and $\beta_{k+1}(t)$ such that the μ-event pertaining to level $k + 1$ will always occur later than the one for level k. Thus $p_{k+1}(t) \leq p_k(t)$.

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So, the positive sequences $\{p_k(t), \beta_k(t), k \geq 0\}$ are uniformly bounded and non-increasing for each fixed t. Consequently,

$$p(t) = \lim_{k \to \infty} \searrow p_k(t) \quad \text{and} \quad \beta(t) = \lim_{k \to \infty} \searrow \beta_k(t)$$

form the unique solutions of (S).

(ii) Letting $r_k \stackrel{\text{def}}{=} \lambda m_k$ and combining the two main equations of (2.10), we get

$$r_{k+1} = \rho e^{r_k}, \ \forall k \geq 0, \quad \text{with } r_0 = \rho.$$

When $\rho \leq e^{-1}$, the r_k's form an increasing sequence of positive real numbers, with a finite positive limit r satisfying equation (2.6). Since $1 - p_k(t)$ is also an increasing sequence of positive functions, the theorem of Beppo Levi ensures the equality

$$\int_0^\infty (1 - p(t))dt = \lim_{k \to \infty} \int_0^\infty (1 - p_k(t))dt = \lim_{k \to \infty} m_k = \frac{r}{\lambda}. \qquad (2.12)$$

It is worth to point out that (2.10) is equivalent to the construction of a sequence of trees $\{G_k, k \geq 0\}$, such that, for any finite k, G_k is ergodic and has a height not greater than k.

This completes the proof of points (A) and (B) of the theorem. $\blacksquare$

2.2 Transience

It turns out that the classification of the process for $\rho > e^{-1}$ can be obtained from analytic arguments.

Recalling that $\ell = \lim_{t \to \infty} p(t)$, we define

$$\varepsilon(t) \stackrel{\text{def}}{=} \lambda \int_0^t (\ell - p(x))dx, \quad \lim_{t \to \infty} \uparrow \varepsilon(t) \stackrel{\text{def}}{=} \bar{\varepsilon}, \qquad (2.13)$$

and it will be convenient to write $\varepsilon'(t) \stackrel{\text{def}}{=} \frac{d\varepsilon(t)}{dt}$.

The quantity $\bar{\varepsilon}$ exists, but a priori is not necessarily finite. It represents the mean number of descendants of an arbitrary vertex v, conditioning on the fact that v is the root of an almost surely finite tree. In fact we will show that if $\rho > e^{-1}$ then $\ell < 1$, in which case the system is transient. Thus there is no null-recurrence region in the parameter space. However, $\bar{\varepsilon}$ given by (2.13) will appear to be *finite for all $\rho > 0$*.

Unless otherwise stated, s in this subsection will stand for a positive real variable. Then by a direct computation we get

$$\beta^*(s) = \mu \int_0^\infty \exp\big[-\big(\varepsilon(t) + (\lambda(1 - \ell) + s)t\big)\big]dt,$$

together with the functional equation

$$\frac{s^2\varepsilon^*(s)}{\lambda} = \frac{\ell + (\ell - 1)\beta^*(s)}{1 + \beta^*(s)}. \tag{2.14}$$

(i) *Suppose now for a while $\ell = 1$. Then* (2.14) *reduces to*

$$s^2\varepsilon^*(s)\big[1 + \beta^*(s)\big] = \lambda, \tag{2.15}$$

and the idea is to show that, for $\rho > e^{-1}$, (2.15) *has no admissible solution, i.e. a solution such that* $\lim\searrow_{t\to\infty} \varepsilon'(t) = 0$.

By well known theorems for Laplace transforms (see e.g. [7]), we have the relations

$$\lim_{s\to 0} s\varepsilon^*(s) = \bar{\varepsilon}, \quad \lim_{s\to 0} s^2\varepsilon^*(s) = 0, \quad \lim_{s\to 0} s\beta^*(s) = \mu\exp(-\bar{\varepsilon}).$$

When the system is ergodic, $\bar{\varepsilon} < \infty$ and the above limit equations give at once

$$\lim_{s\to 0} s^2\varepsilon^*(s)\beta^*(s) = \mu\bar{\varepsilon}\exp(-\bar{\varepsilon}). \tag{2.16}$$

In the case $\bar{\varepsilon} = \infty$, the question is more difficult and (2.16) does not hold without additional conditions on $\varepsilon(t)$, as for instance slow variation (see tauberian theorems in [6]). The only cheap by-product of (2.15) is the existence of the decreasing limit

$$\lim_{s\to 0} \uparrow s^2\varepsilon^*(s)\beta^*(s) = \lambda - \lim\searrow_{s\to 0} s^2\varepsilon^*(s).$$

To get deeper insight into (2.15), we remark that the quantity $s\varepsilon^*(s)\beta^*(s)$ can be viewed as the Laplace transform of a convolution measure with density

$$\int_0^t \varepsilon'(z)\exp[-\varepsilon(t - z)]dz,$$

so that (2.15) is equivalent to the integro-differential equation

$$\rho = \frac{1}{\mu}\varepsilon'(t) + \int_0^t \varepsilon'(z)\exp[-\varepsilon(t - z)]dz. \tag{2.17}$$

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It is worth remembering that we are searching for solutions of (2.17) in the class of functions $\varepsilon(t)$ which have positive monotone decreasing derivatives [this property can in fact be established by taking derivatives of higher order in (2.17)], and satisfy the intial condition $\varepsilon(0) = 0$. Hence, as $t \to \infty$, the limit of the integral in (2.17) exists and

$$\lim_{t \to \infty} \int_0^t \varepsilon'(z) \exp[-\varepsilon(t-z)]dz = \rho - \lim_{t \to \infty} \searrow \frac{\varepsilon'(t)}{\mu} > 0.$$

Choose $T, 0 < t < T$. Then the decomposition

$$\rho = \frac{1}{\mu}\varepsilon'(T) + \int_0^t \varepsilon'(z) \exp[-\varepsilon(T-z)]dz + \int_t^T \varepsilon'(z) \exp[-\varepsilon(T-z)]dz$$

yields by monotonicity

$$\rho \le \frac{1}{\mu}\varepsilon'(T) + \varepsilon(t) \exp[-\varepsilon(T-t)] + \int_t^T \varepsilon'(z) \exp[-\varepsilon(T-z)]dz. \qquad (2.18)$$

Putting $T = 2t$ in (2.18) and using $\varepsilon'(2t) \le \varepsilon'(t)$, we obtain the main inequality

$$\rho - \varepsilon(t) \exp[-\varepsilon(t)] \le \varepsilon'(t)\left[\frac{1}{\mu} + \int_0^t \exp[-\varepsilon(z)]dz\right]. \qquad (2.19)$$

On the other hand, (2.17) shows immediately that the right-hand side member of (2.19) is bounded by ρ. Finally, any solution of (2.17) must satisfy

$$\rho - \varepsilon(t) \exp[-\varepsilon(t)] \le \varepsilon'(t)\left[\frac{1}{\mu} + \int_0^t \exp[-\varepsilon(z)]dz\right] \le \rho. \qquad (2.20)$$

Assume also now $\rho > e^{-1}$. Then

$$\sup_{t \ge 0}\left[\rho - \varepsilon(t) \exp[-\varepsilon(t)]\right] \overset{\text{def}}{=} \xi > 0,$$

so that (2.20) implies

$$\xi \le \varepsilon'(t)\left[\frac{1}{\mu} + \int_0^t \exp[-\varepsilon(z)]dz\right] \le \rho. \qquad (2.21)$$

Let $a \in [\xi, \rho]$ be a real parameter and consider the differential equation

$$f'(t)\left[\frac{1}{\mu} + \int_0^t \exp[-f(z)]dz\right] = a. \qquad (2.22)$$

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The next step is to show that all solutions of (2.21) are located in the *strip* delimited by the maximal and minimal solutions of (2.22). Setting $g(t) = \exp[-f(t)]$, a simple calculation gives

$$
\begin{cases}
g'(t) = -a\log\big(1 + \mu g(t)\big) + 1 \geq 0, \\
\displaystyle\int_0^{g(t)} \frac{dy}{1 - a\log(1 + \mu y)} = t.
\end{cases}
\tag{2.23}
$$

The first equation of (2.23) yields $1 + \mu g(t) \leq e^{\frac{1}{a}}$, whence $\sup_{t\geq 0} g(t) < \infty$. On the other hand, the second equation of (2.23) implies, for any fixed t, that $g(t)$ (resp. $f'(t)$) is decreasing (resp. increasing) with respect to a. Thus the maximal and minimal solutions of (2.22) take place respectively for $a = \rho$ and $a = \xi$, and we have

$$
\inf_{t\geq 0} f'(t) \geq \xi \exp[\xi^{-1}], \quad \forall a \in [\xi, \rho].
$$

Consequently, by (2.21), $\lim\limits_{t\to\infty} \searrow \varepsilon'(t) > 0$, which contradicts null-recurrence, and shows finally that if $\rho > e^{-1}$ then necessarily $\ell < 1$ (the announced transience).

(ii) Consider now the case $\ell < 1$. Letting $s \to 0$ in (2.14) yields immediately

$$
\beta^*(0) = \frac{\ell}{1 - \ell}.
$$

Then rewriting (2.14) in the form

$$
\frac{s\varepsilon^*(s)}{\lambda} = \frac{(1 - \ell)[\beta^*(0) - \beta^*(s)]}{s + s\beta^*(s)},
\tag{2.24}
$$

letting again $s \to 0$ in (2.24) and using l'Hôpital's rule, we obtain

$$
\bar{\varepsilon} = \lambda(1 - \ell)^2 \int_0^\infty t\beta(t)dt < \infty.
$$

The exact computation of ℓ proves to be a difficult project. Actually, since one can hardly expect more than approximate formulas, we shall present various results, both formal and concrete, some of them yielding bounds for ℓ.

2.2.1 Formal approach

Using the definition (2.13), it appears that the right-hand side member of (2.14) can be analytically continued to the region $\Re(s) < -\lambda(1-\ell)$. Thus an analysis of singularities becomes theoretically possible, which should hopefully allow to compute ℓ. We roughly sketch the method, without giving an exhaustive presentation of all technicalities.

Owing to the inversion formula (2.7), we can rewrite (2.14) in the functional form

$$\frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}\beta^*(s)ds = \mu \exp\left[\frac{-\lambda}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{st}ds}{s^2\big(1+\beta^*(s)\big)}\right], \quad \Re(\sigma) > 0. \tag{2.25}$$

Arguing by analytic continuation in (2.25), it is possible to prove that $\beta^*(s)$ is a meromorphic function with real negative poles. Hence, $\beta(t)$ can be represented by the Dirichlet series

$$\beta(t) = C\exp\left[\frac{-\lambda t}{1+\beta^*(0)}\right] + \sum_{i \geq 0} u_i e^{-\sigma_i t}, \tag{2.26}$$

where C is a constant, the σ_i's form a sequence of positive increasing numbers satisfying

$$\sigma_i > \frac{\lambda}{1+\beta^*(0)}, \quad \forall i \geq 0,$$

and the u_i's are *ad hoc* residues. In the ergodic case $\beta^*(0) = \infty$ and the first term in (2.26) reduces to the constant C. Then, $\varepsilon(t)$ could be obtained by formal inversion of $\beta^*(s)$. Alas, the computation becomes formidable and we did not get an exact tractable form (if any at all !) for ℓ, since this is equivalent to compute $u_i, \sigma_i, i \geq 0$.

2.2.2 Bounds and tail distribution

Beforehand, it is worth quoting some simple facts. First, the value of ℓ does solely depend on ρ, as can be seen by scaling in system (2.1–2.2), with the new functions

$$\widetilde{\beta}(t) = \frac{1}{\mu}\beta\Big(\frac{t}{\mu}\Big), \quad \widetilde{p}(t) = p\Big(\frac{t}{\mu}\Big).$$

Secondly, combining (2.1) and (2.2) leads to the inequality

$$\beta(t) \leq \mu - \lambda p(t),$$

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which yields

$$\ell \le \min\left(1, \frac{1}{\rho}\right), \quad \forall \rho < \infty. \tag{2.27}$$

In Section 2.1, we also could have considered the scheme

$$\begin{cases} \gamma_0(t) &= \mu e^{-\lambda t}, \quad t \ge 0, \\[2mm] \gamma_k(t) &= \dfrac{dq_k(t)}{dt} + \displaystyle\int_0^t \gamma_k(t-y)dq_k(y), \\[2mm] \gamma_{k+1}(t) &= \mu \exp\left\{-\lambda \displaystyle\int_0^t \big(1 - q_k(y)\big)dy\right\}, \\[2mm] q_k(0) &= 0, \ \forall k \ge 0, \end{cases} \tag{2.28}$$

which differs from (2.10) only by its first equation, but this is a crucial difference, corresponding in some sense to a fictitious function $q_{-1}(t) = 0$, $\forall t \ge 0$. Actually, this scheme produces a sequence of trees $\{L_k, k \ge 0\}$, with the property that the leaves of L_k at level k never die. Its basic properties are the following:

- the q_k's form an *increasing* sequence of *defective* distributions;

- for all $k \ge 0$, the tail distribution of q_k dominates a defective exponential distribution with density of the form $a_k b_k e^{-b_k t}$. Moreover, under condition (2.5), we have

$$\lim_{k \to \infty} a_k = 1, \quad \lim_{k \to \infty} b_k = \frac{\lambda}{r}$$

and q_k converges in L_1 to the proper distribution p.

The iterative scheme (2.28) is convergent *for all* ρ, but the distributions $q_k(t)$, $k \ge 0$, are *defective*, their limit being proper if and only if $\rho \le e^{-1}$. When $\rho > e^{-1}$, the limiting function $p(t)$ remains defective and

$$\lim_{t \to \infty} p(t) = \lim_{k \to \infty} \lim_{t \to \infty} q_k(t) = \ell < 1.$$

We shall derive bounds on ℓ, in showing by induction that $q_k(t)$, for t sufficiently large, dominates an exponential distribution. The idea of proof will appear from the very first step $k = 1$. Actually, we have

$$\begin{cases} q_0(t) = \ell_0(1 - e^{-\theta_0 t}), \\[2mm] \ell_0 = \dfrac{\mu}{\lambda + \mu}, \quad \theta_0 = \lambda + \mu, \\[2mm] \gamma_1(t) = \mu \exp\left[-\lambda\big(1 - \ell_0\big)t + \dfrac{\lambda \ell_0}{\theta_0}\big(e^{-\theta_0 t} - 1\big)\right], \end{cases}$$

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and the Laplace transform $\gamma_1^*(s)$ has an explicit form, based on the formula (which involves the incomplete gamma function, see e.g. [9])

$$\mathcal{I}(x,y) = \int_0^\infty \exp\left[-xt + ye^{-t}\right] dt = \sum_{n=0}^\infty \frac{y^n}{n!(x+n)}, \quad \Re(x) > 0. \quad (2.29)$$

By scaling, for any constant $c > 0$, we have

$$\frac{1}{c}\mathcal{I}\left(\frac{x}{c}, y\right) = \int_0^\infty \exp\left[-xt + ye^{-ct}\right] dt, \quad (2.30)$$

so that

$$\gamma_1^*(s) = \frac{\mu}{\theta_0}\exp\left(\frac{-\lambda\ell_0}{\theta_0}\right)\mathcal{I}\left[\frac{s+\lambda(1-\ell_0)}{\theta_0}, \frac{\lambda\ell_0}{\theta_0}\right], \quad \Re\big(s+\lambda(1-\ell_0)\big) > 0.$$

The series in equation (2.29) shows that $\gamma_1^*(s)$ can be analytically continued as a meromorphic function of s, with simple poles $s_n = -\lambda(1-\ell_0)-n, n \geq 0$.

Similarly, one checks easily the roots in s of $\gamma_1^*(s) + 1 = 0$ are simple, real and negative. Denoting them by $-z_n, n \geq 0$, we have the following

Lemma 2.2.

$$q_1^*(s) = \frac{\gamma_1^*(s)}{s(1+\gamma_1^*(s))}$$

is a meromorphic function of s, with poles at $0, -z_0, -z_1, \ldots$, where

$$\lambda(1-\ell_0) + n\theta_0 < z_n < \lambda(1-\ell_0) + (n+1)\theta_0, \quad n \geq 0,$$

with the more precise bounds

$$\frac{\mu\theta_0}{\mu+\theta_0}\exp\left(\frac{-\lambda\ell_0}{\theta_0}\right) \leq z_0 - \lambda(1-\ell_0) \leq \min\left[\theta_0, \mu\exp\left(\frac{-\lambda\ell_0}{\theta_0}\right)\right]. \quad (2.31)$$

Hence

$$q_1(t) = \ell_1 - \sum_{n\geq 0} r_n e^{-z_n t}, \quad (2.32)$$

where the residue r_n of $q_1^*(s)e^{st}$ at the pole $z_n, n \geq 0$, is positive and given by the linear relation

$$r_n z_n \frac{d\gamma_1^*}{ds}\bigg|_{s=-z_n} + 1 = 0,$$

and

$$\ell_1 = \sum_{n\geq 0} r_n = \frac{\gamma_1^*(0)}{1+\gamma_1^*(0)}.$$

Moreover, (2.32) yields

$$\ell_1(1 - e^{-z_0 t}) \leq q_1(t) \leq \ell_1. \quad (2.33)$$

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Proof. Only the first part of (2.31) needs some explanation. It is obtained by checking that, for all $y \geq 0$, the first negative root in x of the equation

$$\mu x \exp(-y)\mathcal{I}(x, y) + \theta x = 0, \ y \geq 0,$$

satisfies

$$\begin{cases} x\theta_0 \geq -\mu e^{-y} \\ x^2 + \left(1 + \dfrac{\mu}{\theta_0}\right) + \dfrac{\mu}{\theta_0}e^{-y} \leq 0. \end{cases}$$

$\blacksquare$

We shall prove by induction that $p(t)$ dominates a *reasonable* exponential distribution. To this end, assume

$$\ell_k(1 - e^{-\theta_k t}) \leq q_k(t),$$

which is in particular true for $k = 0, 1$, as shown in lemma 2.2. Then the calculus which led to (2.33) yields also

$$\ell_{k+1}(1 - e^{-\theta_{k+1} t}) \leq q_{k+1}(t),$$

where $(\ell_{k+1}, \theta_{k+1})$ can be derived from (ℓ_k, θ_k) by the formulas

$$\begin{cases} \alpha_k = \dfrac{\mu}{\theta_k} \exp\left(\dfrac{-\lambda\ell_k}{\theta_k}\right)\mathcal{I}\left[\dfrac{\lambda(1 - \ell_k)}{\theta_k}, \dfrac{\lambda\ell_k}{\theta_k}\right], \\ \ell_{k+1} = \dfrac{\alpha_k}{1 + \alpha_k}. \end{cases} \tag{2.34}$$

and θ_{k+1} is the first positive root of the equation

$$\frac{\mu}{\theta_k} \exp\left(\frac{-\lambda\ell_k}{\theta_k}\right)\mathcal{I}\left[\frac{-\theta_{k+1} + \lambda(1 - \ell_k)}{\theta_k}, \frac{\lambda\ell_k}{\theta_k}\right] + 1 = 0. \tag{2.35}$$

Replacing θ_0 and z_0 in (2.31) by θ_k and θ_{k+1} respectively, one can prove the existence of

$$\lim_{k \to \infty} (\ell_k, \theta_k) = (\ell_d, \theta), \quad \theta < \infty, \ \ell_d \leq \ell \leq 1,$$

where $0 < \theta < \infty$ when $\ell < 1$.

The numerical computation of ℓ_d is freakish in the ergodicity region (where the determination of the θ_k's is a source of numerical instability), but proves very satisfactory for $\rho \gg e^{-1}$.

Next, instead of providing as in lemma 2.2 a stochastic ordering for all t, we get a tail-ordering, which has the advantage of achieving the *exact* value $\ell = 1$ for $\rho \leq e^{-1}$.

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Lemma 2.3.

$$q_k(t) \geq a_k(1 - e^{-b_k t}) + o(e^{-b_k t}), \quad \forall\, k \geq 0, \tag{2.36}$$

where the sequence (a_k, b_k) satisfies the recursive scheme

$$\begin{cases} a_{k+1} b_{k+1} = \mu \exp\left(\dfrac{-\lambda a_k}{b_k}\right), \\[2mm] b_{k+1}(1 - a_{k+1}) = \lambda(1 - a_k), \end{cases} \tag{2.37}$$

with $a_0 = \mu/(\lambda + \mu)$ and $b_0 = \lambda + \mu$.

Setting $a \stackrel{def}{=} \lim\limits_{k \to \infty} a_k$ and $b \stackrel{def}{=} \lim\limits_{k \to \infty} b_k$ in (2.37), one has the limits

$$\begin{cases} a = 1, \ b = \dfrac{\lambda}{r}, \ \text{if} \quad \rho \leq e^{-1}, \\[2mm] a = x, \ b = \lambda, \ \text{if} \quad \rho \geq e^{-1}, \end{cases} \tag{2.38}$$

where $x \leq 1$ is the root of the equation

$$xe^x = \frac{1}{\rho}. \tag{2.39}$$

In the course of the proof of lemma 2.3, we will have to characterize positive measures when then are defined from a Laplace transform of the form $\frac{f^*(s)}{1+f^*(s)}$, as for instance in (2.11), which a priori does not correspond to a *completely monotone* function, according to the classical definition of [6]. The following lemma does address this question and might be of intrinsic interest.

Lemma 2.4. *Let Q be a measure concentrated on $[0, \infty[$, and its corresponding Laplace transform $Q^*(s) = \int_0^\infty e^{-st} dQ(t)$, for any complex s with $\Re(s) \geq 0$. Define*

$$\psi(s) \stackrel{def}{=} \int_0^\infty \mu e^{-(\lambda Q(t) + st)} dt,$$

$$\omega(s) \stackrel{def}{=} \frac{\psi(s)}{1 + \psi(s)}.$$

Then $\omega(s + \mu)$ is the Laplace transform of a positive measure Δ_Q $[0, \infty[$. In addition Δ_Q is a decreasing functional of Q, in the sense that, for all $R \geq Q$, R being Q-continuous,

$$\Delta_R \leq \Delta_Q.$$

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Proof. For any complex number s with $\Re(s) \geq 0$, $\widetilde{\psi}(s) \stackrel{\text{def}}{=} \psi(s + \mu)$ can be viewed as the the Laplace transform of a positive measure U having the density $\mu e^{-(\lambda Q(t) + \mu t)}$. Thus

$$\mathsf{P}\{U \leq t\} = \int_0^t \mu e^{-(\lambda Q(t) + \mu t)} dt \leq 1 - e^{-\mu t} \leq 1, \qquad (2.40)$$

and the following expansion holds

$$\omega(s + \mu) = \sum_{k=0}^{\infty} (-1)^k \widetilde{\psi}^{k+1}(s), \qquad (2.41)$$

where $\widetilde{\psi}^k(s)$ stands for the transform of the k-fold convolution of U defined in (2.40). A function being uniquely determined—up to values in a set of measure zero—by the values of its Laplace transform in the region $\Re(s) \geq \mu$, the first part of the lemma is proved. As for the monotony, one can differentiate the inverse of (2.41) term by term (with respect to Q): since each term is multiplied by $(-\lambda)^k$, the resulting series is negative and the conclusion follows. ∎

Proof of lemma 2.3. Most of the ingredients reside in the integral representation of $\gamma_1^*(s)$ by means of formula (2.29), and we shall present the main lines of argument.

Fix a number D, $b_0 < D < \infty$. Then (2.30) yields the inequality

$$\gamma_1^*(s) \geq \psi_1(s) \stackrel{\text{def}}{=} \frac{\mu}{D} \exp\left(\frac{-\lambda a_0}{b_0}\right) \mathcal{I}\left[\frac{s + \lambda(1 - a_0)}{D}, \frac{\lambda a_0}{b_0}\right], \qquad (2.42)$$

whence

$$sq_1^*(s) = \frac{\gamma_1^*(s)}{1 + \gamma_1^*(s)} \geq \frac{\psi_1(s)}{1 + \psi_1(s)}, \qquad \forall s > 0.$$

Setting

$$u(s) = \mu \exp\left(\frac{-\lambda a_0}{b_0}\right) \frac{1}{\lambda(1 - a_0) + s}, \qquad (2.43)$$

and isolating the first term is the power series expansion of (2.42) by means of (2.29), we obtain after a routine algebra

$$\frac{\psi_1(s)}{1 + \psi_1(s)} = \frac{u(s)}{u(s) + 1} + \frac{w(s, D)}{D}. \qquad (2.44)$$

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Remarking the first pole of $w(s, D)$ is the root of $u(s) + 1 = 0$, we can use lemma 2.4 and a term by term inversion of (2.44) to obtain

$$q_1(t) \geq a_1(1 - e^{-b_1 t}) + \frac{\mathcal{O}(e^{-b_1 t})}{D},$$

where the couples $(a_1, b_1), (a_0, b_0)$ satisfy system (2.37).

At step $k = 2$, one would introduce a constant, say D_1, and repeat the same procedure to obtain (2.36). It might be useful to note that it is not possible to take $D = \infty$, since this would create an atom at $t = 0$, in which case lemma 2.4 does not work in general.

At last, it is a simple exercise (therefore omitted) to verify the existence of $(a, b) = \lim_{k \to \infty} (a_k, b_k)$, given by (2.38). The proof of the lemma is terminated. ∎

Now, to conclude the proof of theorem 2.1, it merely suffices to note that, by (2.39),

$$\rho^{-1} - \rho^{-2} \leq x \leq \ell \leq \rho^{-1}, \quad \forall \rho \geq e^{-1}.$$

∎

Subsidiary comments   The method of schemes to analyze nonlinear operators in a probabilistic context is extremely powerful (see e.g. [3] for problems related to systems in thermodynamical limit), and in some sense deeply related to the construction of Lyapounov functions. Up to sharp technicalities, the schemes (2.10) and (2.28) can be exploited to derive precise information about the speed of convergence as $t \to \infty$, for any ρ, $0 < \rho < \infty$, and when pushing exact computations slightly farther, one perceives underlying relationships with intricate continued fractions. Finally, we note that the question of transience could be studied from a large deviation point of view, by considering $\varepsilon(t)$ as the member of a family indexed by the parameter $(\rho - e^{-1})$—see in this respect section 4.

3   Some stationary distributions

In this section, we derive the stationary laws of some performance measures of interest when the system is ergodic, i.e. $\rho \leq e^{-1}$. Incidentally, note that the only process studied so far, that is the number X of vertices attached

to the root, behaves like the number of customers in a M/G/∞ queue with arrival rate λ and service time distribution p, so that

$$\lim_{t \to \infty} \mathsf{P}\{X(t) = k\} = e^{-r} \frac{r^k}{k!}, \ \forall k \geq 0.$$

Another point worth mentioning is that, as for the model in [16], the Markov process $G(t)$ is reversible and hence has an explicit invariant measure. To see this, notice that at each vertex v, leaves are added at rate λ, and removed at rate $\mu \eta(v)$, where $\eta(v)$ stands for the number of leaves attached to v. Therefore, the stationary probability of some configuration G is

$$\pi(G) \stackrel{\text{def}}{=} K \frac{\rho^{N_G}}{\prod_{v \in G} \eta(v)!},$$

where K is a normalization constant. The ergodicity of the Markov process $G(t)$ is then equivalent to the convergence of the series $\sum \pi(G)$, where the sum is taken over all admissible trees G. However, while counting Catalan trees as in [16] is not that difficult, the combinatorics is more involved in the present setting, and this direction will be pursued no further.

3.1 Volume of the tree

Let $N \stackrel{\text{def}}{=} \lim_{t \to \infty} N(t)$, where $N(t)$ introduced in section 1 stands for the volume of $G(t)$ and the limit is taken in distribution.

Theorem 3.1. *When $\rho \leq e^{-1}$, the distribution of the stationary volume N is given by*

$$\mathsf{P}\{N = k\} = \frac{1}{r} \frac{k^{k-1}}{k!} \rho^k, \tag{3.1}$$

where r is given by (2.6). Moreover, the mean value of N is given by

$$\mathsf{E}N = \frac{1}{1 - r}. \tag{3.2}$$

Proof. We proceed as in lemma 2.1, saying that the number of vertices in the tree at time t is equal to 1 plus the numbers of vertices in all the descendants that have appeared in $[0, t]$ and are not yet dead. The construction mimics the former proposed for the process $X_v(t)$: the volume of a subtree rooted at some vertex v is distributed as $N(t)$ for $t \leq \tau_v$.

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For any complex number z, $|z| < 1$, we have therefore

$$\mathsf{E}z^{N(t)} = z\sum_{k=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^k}{k!}\left\{\int_0^t\frac{dx}{t}\mathsf{E}\Big[z^{N(x)}\mathbb{1}_{\{x\leq\tau\}}\Big]\right\}^k$$

$$= z\exp\left\{\lambda\mathsf{E}\Big[\int_0^t\big(z^{N(x)}-1\big)\mathbb{1}_{\{x\leq\tau\}}dx\Big]\right\}, \qquad (3.3)$$

and, letting $t\to\infty$,

$$\mathsf{E}z^{N} = z\exp\left\{\lambda\mathsf{E}\Big[\int_0^{\tau}\big(z^{N(x)}-1\big)dx\Big]\right\} = \frac{\rho z}{r}\exp\left\{\lambda\mathsf{E}\Big[\int_0^{\tau}z^{N(x)}dx\Big]\right\}. \quad (3.4)$$

It is easy to write a renewal equation similar to (2.2), namely

$$\mathsf{E}z^{N(t)} = \mathsf{E}[z^{N(t)}\mathbb{1}_{\{t\leq\tau\}}] + \int_0^t\mathsf{E}z^{N(t-y)}dp(y),$$

which, after setting $\phi(z,t)\stackrel{\text{def}}{=}\mathsf{E}z^{N(t)}$ and $\widetilde{\phi}(z,t)\stackrel{\text{def}}{=}\mathsf{E}[z^{N(t)}\mathbb{1}_{\{t\leq\tau\}}]$, and taking Laplace transforms with respect to t, yields the equation

$$\phi^*(z,s) = \widetilde{\phi}^*(z,s) + \phi^*(z,s)sp^*(s).$$

Then, as in the case of (2.8), using the boundedness of $z^{N(t)}$, we get

$$\phi(z)\stackrel{\text{def}}{=}\lim_{t\to\infty}\phi(z,t) = \frac{1}{m}\int_0^{\infty}\widetilde{\phi}(z,t)dt = \frac{1}{m}\mathsf{E}\Big[\int_0^{\tau}z^{N(x)}dx\Big].$$

Then (3.4) can be rewritten as

$$r\phi(z) = \rho z\exp\{r\phi(z)\}, \qquad (3.5)$$

and hence $r\phi(z) = C(\rho z)$, where C stands for the classical Cayley tree generating function (see e.g. [17]). Using the well-known series expansion for C (which follows from Lagrange's inversion formula), we get (3.1), since

$$\phi(z) = \frac{1}{r}\sum_{k=0}^{\infty}\frac{k^{k-1}}{k!}(\rho z)^k.$$

The mean (3.2) is obtained by differentiating (3.5) with respect to z and taking $z = 1$. ∎

The analysis of the asymptotics of (3.1) with respect to k confirms an interesting change of behavior when $\rho = e^{-1}$. Indeed, for k sufficiently large, Stirling's formula yields

$$\mathsf{P}\{N = k\} = \frac{1}{r}\frac{k^{k-1}}{k!}\rho^k \approx \frac{1}{r}\frac{k^{k-1}}{\sqrt{2\pi k}\,k^k\,e^{-k}}\rho^k = \frac{1}{r}\frac{(\rho e)^k}{\sqrt{2\pi}\,k^{\frac{3}{2}}}.$$

Moreover, a straightforward Taylor expansion of (2.6) gives the following estimate of $\mathsf{E}N$, as $\rho \to e^{-1}$:

$$\mathsf{E}N = \frac{1}{1-r} \approx \frac{1}{\sqrt{2(1-\rho e)}}.$$

Thus, while all moments of N exist for $\rho < e^{-1}$, there is no finite mean as soon as $\rho = e^{-1}$. We note in passing that this phenomenon appears sometimes in branching processes and can be viewed as a phase transition inside the parameter region, as already remarked in [5].

3.2 Height of the tree

Let $H \stackrel{\text{def}}{=} \lim_{t\to\infty} H(t)$, where $H(t)$ introduced in section 1 stands for the height of $G(t)$. The distribution of H is given by the following theorem.
Theorem 3.2.

1. For $\rho \leq e^{-1}$, the following relations hold:

$$\mathsf{P}\{H = 0\} = e^{-r}, \tag{3.6}$$
$$\mathsf{P}\{H > h+1\} = 1 - \exp\left[-r\mathsf{P}\{H > h\}\right], \quad \forall h \geq 0. \tag{3.7}$$

2. If $\rho < e^{-1}$, then there exists a positive constant $\theta(r,1)$, such that

$$\mathsf{P}\{H > h\} = \theta(r,1)r^{h+1} + O\left(\frac{r^{2h}}{1-r}\right) \tag{3.8}$$

where the function $\theta(r,x)$ is the locally analytic w.r.t. x solution of the functional Schröder equation

$$\theta(r, 1 - e^{-rx}) = r\theta(r,x),$$

subject to the boundary condition

$$\frac{\partial\theta}{\partial x}(r,0) = 1. \tag{3.9}$$

3. When $\rho = e^{-1}$,

$$P\{H > h\} = \frac{2}{h} + O\left(\frac{\log h}{h^2}\right). \tag{3.10}$$

Proof. As in the previous proof, one writes the height of the tree at time t is less than $h + 1$ if, and only if, all the descendants that have appeared in $[0, t]$ are either dead or have a height smaller than h:

$$
\begin{aligned}
P\{H(t) \leq h + 1\} &= \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \left\{\int_0^t \frac{dx}{t}\left[1 - P\{H(x) > h, x \leq \tau\}\right]\right\}^k \\
&= \exp\left\{-\lambda \int_0^t P\{H(x) > h, x \leq \tau\} dx\right\}.
\end{aligned}
$$

Letting $t \to \infty$ and arguing as in theorem 3.1, we can write

$$P\{H \leq h + 1\} = \exp\left[-rP\{H > h\}\right],$$

which proves (3.7). On the other hand, equation (3.6) is immediate, since it is in fact a plain rewriting of (2.3).

To prove the remainder of the theorem, let $d_0 \overset{\text{def}}{=} x$, where x is a positive real number, and consider the sequence

$$d_{h+1} = 1 - e^{-rd_h}, \quad h = 0, 1, \dots \tag{3.11}$$

When $x = 1$, note that we have exactly $d_{h+1} = P\{H > h\}$. The question that faces us now is to compute and to estimate the iterates of an analytic function, in the circumstances $1 - e^{-rx}$. This subject concerns a wide branch of mathematics (including functional equations, automorphic functions, boundary value problems), and it has received considerable attention since the nineteen twenties. We shall employ classical arguments without further comment, referring the interested reader to e.g. [12] and [1] for a more extensive treatment.

A Taylor expansion up to second order in (3.11) gives

$$rd_h \geq d_{h+1} = 1 - e^{-rd_h} \geq rd_h - \frac{r^2 d_h^2}{2}, \tag{3.12}$$

which implies that $r^{-h} d_h$ is a decreasing sequence with $\lim_{h \to \infty} \downarrow d_h = 0$ (that we already knew!) and

$$d_h \leq xr^h. \tag{3.13}$$

As $h \to \infty$, the asymptotic behavior of d_h has a twofold nature, depending on whether $r = 1$ or $r < 1$.

Case $r = 1$. This is the easy part. Writing

$$\frac{1}{d_{h+1}} = \frac{1}{1 - e^{-d_h}} = \frac{1}{d_h} + \frac{1}{2} + O(d_h),$$

we get immediately $d_h = O\left(\frac{1}{h}\right)$, and hence

$$\frac{1}{d_h} = \frac{h}{2} + O\left(\sum_{k=0}^{h-1} d_k\right) = \frac{h}{2} + O(\log h),$$

which leads to (3.10).

Case $r < 1$. The analysis is less direct. From (3.12) and (3.13), we infer that, when $h \to \infty$, $r^{-h} d_h$ has a limit denoted by $\theta(r, x)$, with

$$0 \leq r^{-h} d_h - \theta(r, x) \leq \frac{rx^2}{2} \frac{r^h}{1 - r}.$$

First let us show that $\theta(r, x)$ is strictly positive. Indeed,

$$\frac{d_{h+1}}{r^{h+1}} = x \prod_{m=0}^{h} \left(1 - \varphi_m(r, x)\right), \tag{3.14}$$

where, $\forall m \geq 0$, the quantity $\varphi_m(r, x) = O(r^{m+1})$ is an analytic function of the pair of real variables (r, x) in the region $[0, 1[\times [0, A]$, with $0 \leq A < \infty$. Hence, as $h \to \infty$, the infinite product in (3.14) converges uniformly to a strictly positive value, $\forall x > 0$, so that $\theta(r, x)$ is also analytic of (r, x) in the aforementioned region. To summarize,

$$\lim_{h \to \infty} r^{-h} d_h \overset{\text{def}}{=} \theta(r, x) > 0.$$

The pleasant fact is that θ, taken as a function of x, satisfies the so-called Schröder equation

$$\theta(r, 1 - e^{-rx}) = r\theta(r, x). \tag{3.15}$$

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While it is clear that $\theta(r,0) = 0$, (3.15) does not impose any constraint on $\dfrac{\partial \theta}{\partial x}(r,0)$. However, it is easy to show by induction that $d_h(x)$ (where the dependency on x is for a while explicitly written) satisfies

$$\frac{\partial d_h}{\partial x}_{\,|x=0} = r^h, \ \forall h > 0,$$

and thus condition (3.9) also holds for θ. To conclude the proof of (3.8), it suffices to choose $x = 1$. ∎

Remark We have taken the variable x on the positive real half-line to get sharper bounds, e.g. (3.13). Actually, arguing as above, it is immediate to check that θ has an analytic continuation in the complex x-plane in a a neighborhood of the origin. In this respect, without going into a full discussion, we mention the relationships with automorphic functions and boundary value problems, which would allow integral representations. For our purpose, simply writing

$$\theta(r,x) = \sum_{i \geq 0} \theta_i x^i, \quad \theta_0 = 0, \ \theta_1 = 1,$$

we see that all the θ_i's can be computed recursively. Furthermore the iteration of (3.15) yields

$$\theta(r, d_h) = r^h \theta(r, x).$$

from which we obtain

$$d_h = \omega\big(r, r^h \theta(r, x)\big),$$

where $\omega(r, x)$ denotes the inverse function of θ with respect to the variable x and satisfies the functional relation

$$1 - \exp\{-r\omega(r, y)\} = \omega(r, ry). \tag{3.16}$$

We have

$$\omega(r, y) = \sum_{i \geq 0} \omega_i y^i, \quad \omega_0 = 0, \ \omega_1 = 1,$$

and again the ω_i's are obtained recursively.

4 Scaling and limit laws in the transient case

In this section, we present some limit laws for $N(t)$ and $H(t)$, which are especially of interest when the system is transient. Beforehand, for every integer k and all $t > 0$, we define the quantities

$$\begin{cases} X_k(t) \stackrel{\text{def}}{=} \#\{v \in G(t) : h(v) = k\}, \\[2mm] Y_k(t) \stackrel{\text{def}}{=} \sum_{j=k}^{\infty} X_j(t) \mathbb{1}_{\{t \leq \tau\}}. \end{cases}$$

So, $X_k(t)$ stands for the number of vertices at level k in the whole tree at time t.

4.1 Scaling for $N(t)$ in the pure birth case $\mu = 0$

Lemma 4.1. *When $\mu = 0$, $\mathsf{E}X_n(t)$ has the explicit form*

$$\mathsf{E}X_n(t) = \frac{(\lambda t)^n}{n!}. \tag{4.1}$$

Proof. Since

$$\mathsf{P}\{X_n(t+dt) = X_n(t) + 1 | X_n(t), X_{n-1}(t)\} = \lambda X_{n-1}(t)dt + o(dt),$$

we obtain

$$\begin{cases} \dfrac{d}{dt}\mathsf{E}X_n(t) = \lambda\mathsf{E}X_{n-1}(t), \quad n \geq 1, \\[2mm] \mathsf{E}X_0(t) = 1, \end{cases}$$

and the result is immediate by induction. ∎

Theorem 4.1. *When $\mu = 0$, the expected volume at time t is given by*

$$\mathsf{E}N(t) = e^{\lambda t}, \tag{4.2}$$

and

$$\lim_{t \to \infty} \frac{N(t)}{\mathsf{E}N(t)} = \mathrm{Exp}(1), \tag{4.3}$$

where the limit is taken in distribution and $\mathrm{Exp}(1)$ denotes an exponentially distributed variable with parameter 1.

Proof. Equation (4.2) is a mere consequence of lemma 4.1, since

$$\mathsf{E}N(t) = \sum_{n=0}^{\infty} \mathsf{E}X_n(t) = e^{\lambda t}.$$

Let now $\phi(z,t) \stackrel{\text{def}}{=} \mathsf{E}z^{N(t)}$, for z complex with $|z| < 1$. We start from equation (3.3), in which we take $\tau = \infty$. Then the following relation holds:

$$\phi(z,t) = z \exp\left\{\lambda\left[\int_0^t (\phi(z,x) - 1)dx\right]\right\}.$$

Differentiating with respect to t yields

$$\frac{\partial}{\partial t}\phi(z,t) = \lambda\big[\phi(z,t) - 1\big]\phi(z,t),$$

whence

$$\frac{1 - \phi(z,t)}{\phi(z,t)} = Ke^{\lambda t},$$

where K does not depend on t. Since $\phi(z,0) = z$, we deduce $K = z^{-1} - 1$, and finally

$$\phi(z,t) = \frac{1}{1 + [z^{-1} - 1]e^{\lambda t}}.$$

The Laplace transform of $e^{-\lambda t}N(t)$ is, for $\Re(s) \geq 0$,

$$\mathsf{E}\exp\{-se^{-\lambda t}N(t)\} = \phi\left(\exp\{-se^{-\lambda t}\}, t\right)$$

$$= \frac{1}{1 + \big[\exp\{se^{-\lambda t}\} - 1\big]e^{\lambda t}},$$

so that, letting $t \to \infty$,

$$\lim_{t\to\infty} \mathsf{E}\exp\{-se^{-\lambda t}N(t)\} = \frac{1}{1+s}.$$

Now (4.3) follows directly from Feller's continuity theorem (see [6]). ∎

4.2   An ergodic theorem for $H(t)$

The key result of this section concerns the height of the tree and is formulated in the next theorem.

Let

$$b(s,c) \stackrel{\text{def}}{=} \frac{s}{c} + \log\left[\frac{\lambda(1 - sp^*(s))}{s}\right], \quad \Re(s) \geq 0. \tag{4.4}$$

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Theorem 4.2. *With probability 1,*

$$\lim_{t \to \infty} \frac{H(t)}{t} = \delta,$$

where $\delta \geq 0$ is uniquely defined from the system of equations

$$b(s, \delta) = \frac{\partial b(s, \delta)}{\partial s} = 0.$$

In the ergodic case, $\delta = 0$.

The proof is constructed around the three forthcoming lemmas.

Lemma 4.2. *Define the events*

$$A_c = \left\{ \liminf_{t \to \infty} \frac{H(t)}{t} \geq c \right\}, \qquad B_c = \left\{ \limsup_{t \to \infty} \frac{H(t)}{t} \leq c \right\}.$$

Then $\mathsf{P}\{A_c\} = 0$ or 1 and $\mathsf{P}\{B_c\} = 0$ or 1. In other words, A_c and B_c satisfy a zero-one law and can only be trivial events (i.e. sure or impossible).

Proof. Fixing an arbitrary t_0, with $G(t_0) = G_0$, we want to show that A_c does not depend on G_0. For this purpose, consider the random process $G'(t) \subset G(t)$ constructed as follows: for $t \leq t_0$, it consists only of the root, and for $t > t_0$ it contains exactly that part of G grown from the root after time t_0. Then the probability that A_c holds for G' is clearly equal to the probability that A_c holds for G without conditioning. In other words, since $H_G(t) \geq H_{G'}(t)$, we have

$$\mathsf{P}\{A_c \mid G(t_0) = G_0\} \geq \mathsf{P}\{A_c\}.$$

Then basic properties of the conditional expectation yield

$$\mathsf{E}[\mathsf{P}\{A_c \mid G(t_0)\}] = \mathsf{P}\{A_c\},$$

so that

$$\mathsf{P}\{A_c \mid G(t_0) = G_0\} = \mathsf{P}\{A_c\} \tag{4.5}$$

for any G_0. On conditioning with respect to $G(t_0), G(t_1), \ldots, G(t_k)$, for any arbitrary increasing sequence of times t_k, we see (4.5) still holds. Hence, the assertion for A_c is a direct consequence of the zero-one law for martingales (see e.g. [10]).

Quite similarly, if the event B_c holds for G, then it is also in force for any subtree rooted at a vertex of G_0, which reads

$$\mathsf{P}\{B_c \mid G(t_0) = G_0\} \leq \mathsf{P}\{B_c\}.$$

The lemma is proved. ∎

Lemma 4.3.

(i) If, for some integer n and real number $c > 0$, $\mathsf{E}[Y_n(n/c)] > 1$, then

$$\mathsf{P}\{A_c\} = 1.$$

(ii) If, for some n and real number $c > 0$, $\sum_{k=0}^{\infty} \mathsf{E}[X_{kn}(kn/c)] < \infty$, then

$$\mathsf{P}\{B_c\} = 1.$$

Proof.
For the sake of brevity, let $J_n(k)$ denote the time interval $[kn/c, (k+1)n/c]$.

(i) Consider a standard branching process $\xi_k, k \geq 0$, endowed with an offspring distribution equal to that of $Y_n(n/c)$. From the condition in 4.3(i), this process has a probability of non extinction which is strictly positive and will be denoted by $y(n, c)$. The key point is that ξ_k can be viewed as defining a subtree $G' \subset G$ such that $H_{G'}(kn/c) \geq kn$. Then $\mathsf{P}\{A_c\} \geq y(n, c) > 0$, and we have $\mathsf{P}\{A_c\} = 1$ by lemma 4.2.

Indeed, to build such a subtree G', we associate with each generation of ξ_k a set of vertices $s_k \subset G(k\tau)$, such that $\xi_k = |s_k|$.

Let $s_0 = \{v_0\}$ and, for each $v \in s_k, k \geq 0$, let G_v be a subtree rooted at v and born during $J_n(k)$ (by convention G_v is empty if v dies). We put

$$s_{k+1} = \bigcup_{v \in s_k} \{v' \in G_v : d(v', v) \geq n\}.$$

This construction produces the desired tree, since the volume of each set belonging to the above union is exactly distributed as $Y_n(n/c)$, and because s_k consists of vertices located at a distance at least kn from the root.

(ii) Let $a_k = o(k)$ be a sequence of non-decreasing positive integers. Then for any fixed integer k_0, we have the inequality

$$\mathsf{P}(B_c) \geq \mathsf{P}\left\{ \sup_{t \in J_n(k)} H(t) < (k+1)n + a_k, \forall k \geq k_0 \right\}.$$

We observe the height of the tree decreases at a rate not faster than μ, so that, given the event $\{H((k+1)n/c) < (k+1)n\}$, the supremum of $H(.)$ on the interval $J_n(k)$ is bounded by

$$\sup_{t \in J_n(k)} H(t) \leq (k+1)n + \pi(\mu n/c),$$

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where $\pi(x)$ denotes a Poisson random variable with rate x. Thus we have

$$\mathsf{P}(B_c) \geq \mathsf{P}\{H(kn/c) < kn, \forall k \geq k_0\} \prod_{k \geq k_0} \mathsf{P}\{\pi(\mu n/c) < a_k\}, \qquad (4.6)$$

and we will show that the right-hand side of (4.6) can be rendered positive. First, we remark that

$$
\begin{aligned}
\mathsf{P}\{H(kn/c) < kn, \forall k \geq k_0\} &\geq 1 - \sum_{k \geq k_0} \mathsf{P}\{H(kn/c) \geq kn\} \\
&\geq 1 - \sum_{k \geq k_0} \mathsf{E}[X_{kn}(kn/c)] \to 1, \text{ as } k_0 \to \infty.
\end{aligned}
$$

Secondly, we choose the sequence

$$a_k = j, \ \forall k \in [j(j-1)/2 + 1, \ j(j+1)/2], \forall j \geq 1,$$

which consists of blocks of repeated integers satisfying $a_k = \mathcal{O}(k^{1/2})$.

Setting $\nu \stackrel{\text{def}}{=} \mu n/c$, the product in (4.6) will be positive, provided that the following sum is finite

$$
\begin{aligned}
\sum_{k \geq k_0} \mathsf{P}\{\pi(\nu) \geq a_k\} &\leq \sum_{k \geq k_0} e^{-\nu} \frac{\nu^{a_k}}{a_k!} \left(1 - \frac{\nu}{a_k}\right)^{-1} \\
&\leq \sum_{j \geq \sqrt{k_0}} e^{-\nu} \frac{\nu^j}{(j-1)!} \left(1 - \frac{\nu}{j}\right)^{-1} \leq \nu \left(1 - \frac{\nu}{\sqrt{k_0}}\right)^{-1} < \infty,
\end{aligned}
$$

and hence *(ii)* follows from the zero-one property of B_c.

The proof of the lemma is concluded. ∎

Lemma 4.4. *For any $\Re(s) > 0$, let $\varphi_k(s)$ and $\widetilde{\varphi}_k(s)$ be the Laplace transforms*

$$
\begin{aligned}
\varphi_k(s) &\stackrel{\text{def}}{=} \int_0^\infty \mathsf{E}X_k(t)\, e^{-st} dt, \\
\widetilde{\varphi}_k(s) &\stackrel{\text{def}}{=} \int_0^\infty \mathsf{E}\big[X_k(t)\mathbb{1}_{\{t \leq \tau\}}\big]\, e^{-st} dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
\varphi_k(s) &= \frac{\lambda^k (1 - sp^*(s))^k}{s^{k+1}}, \\
\widetilde{\varphi}_k(s) &= \frac{\lambda^k (1 - sp^*(s))^{k+1}}{s^{k+1}}.
\end{aligned}
$$

Proof. It is not difficult to check the following relations

$$\mathsf{E}X_k(t) = \int_0^t \mathsf{E}\big[X_{k-1}(y)\mathbb{1}_{\{y\leq\tau\}}\big]\lambda dy, \quad k \geq 1$$

$$\mathsf{E}X_k(t) = \mathsf{E}\big[X_k(t)\mathbb{1}_{\{t\leq\tau\}}\big] + \int_0^t \mathsf{E}X_k(t-y)dp(y),$$

with the initial condition $\mathsf{E}X_0(t) = 1$. Actually, the first equation follows from an argument already employed before. Namely, the number of vertices at level k are the direct descendants of vertices at level $(k-1)$ still alive at time t, remarking that each such descendant on $[0,t]$ appears independently at rate λ. The second equation is a straight regeneration relation. Therefore,

$$\varphi_k(s) = \frac{\lambda\widetilde{\varphi}_{k-1}(s)}{s} \quad k \geq 1,$$

$$\varphi_k(s) = \widetilde{\varphi}_k(s) + \varphi_k(s)sp^*(s),$$

whence, since $\varphi_0(s) = 1/s$,

$$\varphi_k(s) = \frac{\lambda\big(1 - sp^*(s)\big)\varphi_{k-1}(s)}{s} = \frac{\lambda^k\big(1 - sp^*(s)\big)^k}{s^{k+1}},$$

and the result follows. ∎

We are now in a position to prove theorem 4.2.

Proof of theorem 4.2. The proof is split into two parts, each one corresponding respectively to criteria *(i)* and *(ii)* of lemma 4.3.

First we shall find the *largest* c, denoted by $c_{\inf}$, ensuring criterion *(i)* of lemma 4.3 is fulfilled. Applying the results of lemma 4.4 and the inversion formula (2.7), we have

$$\mathsf{E}[Y_n(n/c)] = \sum_{j=n}^{\infty} \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \widetilde{\varphi}_j(s)e^{sn/c}ds$$

$$= \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \left[\frac{\lambda(1 - sp^*(s))}{s}e^{s/c}\right]^n \frac{1 - sp^*(s)}{s - \lambda(1 - sp^*(s))}ds,$$

$$= \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{nb(s,c)}\left[1 - sp^*(s)\right]}{s - \lambda(1 - sp^*(s))}ds, \tag{4.7}$$

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in the region $\mathcal{U} \overset{\text{def}}{=} \left\{ \sigma > 0, \; \sigma > \lambda(1 - \sigma p^*(\sigma)) \right\}$, where $b(s, c)$ has been defined in (4.4). When the system is ergodic, it is immediate to check the region $\mathcal{U}$ coincides with the complex half-plane $\Re(s) > 0$. On the other hand, in the transient case, the equation

$$s = \lambda(1 - sp^*(s))$$

has exactly one root, which is real and belongs to the open interval $]0, \lambda[$. Computing the residues of the integral (4.7) (by shifting the line of integration to the left, after analytic continuation of $p^*(s)$ to the region $\sigma = -\epsilon$, for some $\epsilon > 0$) is a tedious task, in particular due to the pole of order n at $s = 0$. We will rather proceed by a kind of *saddle-point* approach (see e.g. [7]).

The form of the integrand in (4.7) shows that, as $n \to \infty$, the boundedness of $\mathsf{E}[Y_n(n/c)]$, depends primarily on the value of the modulus of $b(s, c)$. In fact one can see precisely that $\mathsf{E}[Y_n(n/c)]$, for each fixed c, does not tend to zero iff the minimum of $b(s, c)$ is non-negative at any possible real saddle-point $s \in \mathcal{U}$, where

$$\frac{\partial b(s, c)}{\partial s} = 0, \quad s \in \mathcal{U}.$$

It follows that c_{inf} is the unique real solution of the system

$$b(s, c_{\text{inf}}) = \frac{\partial b(s, c_{\text{inf}})}{\partial s} = 0, \quad s \in \mathcal{U}. \tag{4.8}$$

Without presenting a detailed discussion, we shall simply stress that in the real plane (s, y) the curves

$$y = s/c \quad \text{and} \quad y = -\log\left[\frac{\lambda(1 - sp^*(s))}{s}\right]$$

are tangent (resp. intersecting, non-intersecting) for $c = c_{\text{inf}}$ (resp. $c > c_{\text{inf}}$, $c < c_{\text{inf}}$).

As for the second part of the theorem, the question is to find the value c_{sup}, equal to the smallest positive number c satisfying criterion *(ii)* of lemma 4.3,

which implies the finiteness of the quantity

$$
\begin{aligned}
\sum_{k=0}^{\infty} \mathsf{E}[X_{kn}(kn/c)] &= \sum_{k=0}^{\infty} \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \varphi_{kn}(s) e^{skn/c} ds \\
&= \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{ds}{s} \left[1 - \left(\frac{\lambda(1-sp^*(s))}{s} e^{s/c} \right)^n \right]^{-1} \\
&= \frac{1}{2i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{ds}{s} \left[1 - e^{nb(s,c)} \right]^{-1}, \qquad (4.9)
\end{aligned}
$$

where (4.9) holds in the region $\mathcal{V} \overset{\text{def}}{=} \left\{ \sigma > 0, \ \sigma > \lambda(1 - \sigma p^*(\sigma)) e^{\sigma/c} \right\}$.

Clearly, the existence of the last integral in (4.9), as $n \to \infty$, amounts again to find the sign of $\Re(b(s,c))$, for $s \in \mathcal{V}$. Arguing exactly as above, one can find at once the equality

$$
c_{\sup} = c_{\inf} \overset{\text{def}}{=} \delta.
$$

When the system is ergodic, $\lim_{s \to 0} b(s,c) = \log \lambda m = \log r \leq 0$, which yields $\delta = 0$ as might be expected.

The proof of the theorem is concluded. ∎

As a by-product, we state the following corollary, of which the almost sure convergence part has been derived in [4, 15] through different and less terse methods.

Corollary 4.1. *In the pure birth case $\mu = 0$, almost surely and in L_1,*

$$
\lim_{t \to \infty} \frac{H(t)}{t} = \lambda e. \qquad (4.10)
$$

Proof. Instantiating equation (4.1) in criteria *(i)* and *(ii)* of lemma 4.3 yields directly the first part of (4.10). On the other hand, it is immediate to check that the function $\mathsf{E}H(t)$ is superadditive (this would be not true for $\mu \neq 0$), namely

$$
\mathsf{E}H(s+t) \geq \mathsf{E}H(s) + \mathsf{E}H(t),
$$

so that, by a variant of Kingman's theorem (see [10]), the limit $\lim_{t \to \infty} \frac{\mathsf{E}H(t)}{t}$ does exist. Then the convergence in L_1 will follow if one can show

$$
\mathsf{E}H(t) \leq At, \quad \forall t > 0,
$$

for some positive constant A. Using the obvious inequality

$$\mathsf{P}\{H(t) \geq k\} = \mathsf{P}\{X_k(t) > 0\} \leq \mathsf{E}X_k(t),$$

we can write

$$\mathsf{E}H(t) = \sum_{k=1}^{\infty} \mathsf{P}\{H(t) \geq k\} \leq \sum_{k=1}^{k_0} 1 + \sum_{k=k_0+1}^{\infty} \frac{(\lambda t)^k}{k!}.$$

Then, taking $k_0 = \lceil \lambda e t \rceil$ and using Stirling's formula, we obtain

$$
\begin{aligned}
\mathsf{E}H(t) &\leq k_0 + \frac{(\lambda t)^{k_0+1}}{(k_0+1)!} \sum_{k=0}^{\infty} \Big(\frac{\lambda t}{k_0+1}\Big)^k \leq \lambda e t + \frac{(\lambda t)^{\lambda e t+1}}{(\lambda t)^{\lambda e t}\sqrt{2\pi\lambda e t}} \frac{e}{e-1} \\
&\leq \lambda e t + \frac{\sqrt{\lambda e t}}{\sqrt{2\pi}(e-1)}.
\end{aligned}
$$

∎

5   Extension to the multiclass case

The extension of the results of section 2 to models encompassing several classes of vertices is very tempting, although not quite evident. We solve hereafter a case where the birth and death parameters depend on classes in a reasonably general way.

Let $\mathcal{C}$ be a finite set of classes. Then the multiclass Markov chain $G_{\mathcal{C}}$ has the following evolution.

- At any given vertex of class c, a new edge of class $c' \in \mathcal{C}$ can be added at the epochs of a Poisson process with parameter $\lambda_{cc'} \geq 0$.

- Any leaf attached to an edge of class c' and having an ancestor of class c can be deleted at rate $\mu_{cc'} > 0$.

- The root v_0 of the tree belongs to class $c \in \mathcal{C}$, say with probability π_c, with $\sum_{c \in \mathcal{C}} \pi_c = 1$, albeit these probabilities will not really matter in our analysis.

Let $p_{cc'}$ be the lifetime distribution of a vertex of class c' which descend from a vertex of class c. Also, $X_c(t)$ will denote the distribution of the number of direct descendants of a vertex of class c. The following lemma is the analogous of lemma 2.1.

Lemma 5.1. *The lifetime distributions $p_{cc'}$, $c, c' \in \mathcal{C}$ satisfy the following set of equations.*

$$\mathsf{P}\{X_c(t) = 0\} = \exp\Big\{-\sum_{c' \in \mathcal{C}} \lambda_{cc'} \int_0^t (1 - p_{cc'}(x))dx\Big\}, \qquad (5.1)$$

$$\mathsf{P}\{X_c(t) = 0\} = \frac{1}{\mu_{bc}} \frac{dp_{bc}(t)}{dt} + \int_0^t \mathsf{P}\{X_c(t-y) = 0\}dp_{bc}(y), \quad \forall b \in \mathcal{C}, \qquad (5.2)$$

with the initial conditions $p_{cc'}(0) = 0, \forall c, c' \in \mathcal{C}$.

Proof. Details are omitted, as it suffices to mimic the proof of lemma 2.1. Note however that the dependency with respect to c' disappears surprisingly enough in (5.2). Indeed, while the lifetime of a vertex depends on the class of its direct ascendant, the distribution of the number of its descendants merely depends on its own class. ∎

In the setting of this section, it is actually not easy to come up with a natural explicit extension of theorem 2.1. However, the following theorem provides a necessary and sufficient condition for ergodicity.

The following notation will be useful in the theorem:

$$\rho_{cc'} \overset{\text{def}}{=} \frac{\lambda_{cc'}}{\mu_{cc'}}, \ \forall c, c' \in \mathcal{C}$$

$$\rho_c \overset{\text{def}}{=} \sum_{c' \in \mathcal{C}} \rho_{cc'}, \ \forall c \in \mathcal{C}.$$

We will also denote by $\rho \geq 0$ the Perron-Frobenius eigenvalue (see [8]) of the non-negative matrix $\big(\rho_{cc'}\big)_{c, c' \in \mathcal{C}}$.

Theorem 5.1.

1. *The Markov chain $G_{\mathcal{C}}$ is ergodic if, and only if, the nonlinear system*

$$y_c = \sum_{d \in \mathcal{C}} \rho_{cd} \exp\{y_d\}, \ \forall c \in \mathcal{C}, \qquad (5.3)$$

   has at least one real-valued (and obviously non-negative) solution. In this case, the mean lifetime of a vertex of class c with an ascendant of class b can be written as

$$m_{bc} = \frac{1}{\mu_{bc}} \exp\{r_c\}, \qquad (5.4)$$

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where the r_c form the smallest solution of (5.3), that is $r_c \leq y_c$, $\forall c \in \mathcal{C}$. Note that (5.3) implies that $r_c \geq \rho_c$.

2. *A simple sufficient condition for ergodicity is*

$$\rho_c \leq \frac{1}{e}, \ \forall c \in \mathcal{C} \tag{5.5}$$

in which case $r_c \leq \rho_c e$.

3. *A simple necessary condition for ergodicity is*

$$\rho \leq \frac{1}{e}. \tag{5.6}$$

Remark Before stating the proof of the theorem, it is worth pointing out that equation (5.3) may in general have several real solutions (as in dimension 1). Therefore, there is no guarantee that the solution y_c is the correct one. However, its sole existence proves ergodicity and (5.4).

Proof. Assume first that $G_{\mathcal{C}}$ is ergodic. Then, as in theorem 2.1, we let $t \to \infty$ in (5.1)–(5.2) to obtain the relation

$$\frac{1}{\mu_{bc} m_{bc}} = \exp\Big\{ - \sum_{c' \in \mathcal{C}} \lambda_{cc'} m_{cc'} \Big\},$$

which in its turn yields (5.3) and (5.4), just choosing

$$y_c = r_c \overset{\text{def}}{=} \sum_{c' \in \mathcal{C}} \lambda_{cc'} m_{cc'}.$$

As for the proof of sufficiency in item 1, we introduce the following modified version of scheme (2.10):

$$\begin{cases} p_{cc';0}(t) & = 1, \ t \geq 0, \\[2mm] \alpha_{c;k+1}(t) & = \exp\Big\{ - \sum_{c' \in \mathcal{C}} \lambda_{cc'} \int_0^t \big(1 - p_{cc';k}(y)\big) dy \Big\}, \ k \geq 0, \\[2mm] \alpha_{c;k}(t) & = \frac{1}{\mu_{bc}} \frac{dp_{bc;k}(t)}{dt} + \int_0^t \alpha_{c;k}(t - y) dp_{bc;k}(y), \ k \geq 1, \\[2mm] p_{cc';k}(0) & = 0, \ k \geq 1. \end{cases} \tag{5.7}$$

Then, for any $b, c \in \mathcal{C}$, we have

$$\alpha_{c;1}(t) = 1 \quad \text{and} \quad p_{bc;1}(t) = 1 - e^{-\mu_{bc}t} \ \leq \ p_{bc;0}(t).$$

Here again, the positive sequences $\{p_{cc';k}(t); k \geq 0\}$ and $\{\alpha_{c;k}(t); k \geq 0\}$ are uniformly bounded and non-increasing, for each fixed $t > 0$. Consequently,

$$p_{cc'}(t) = \lim_{k \to \infty} \searrow p_{cc';k}(t) \quad \text{and} \quad \alpha_c(t) = \lim_{k \to \infty} \searrow \alpha_{c;k}(t)$$

form the unique solution of (5.1)–(5.2), uniqueness resulting from the Lipschitz character of equation (5.2).

Letting $m_{cc';k}$ denote the finite mean associated with each distribution $p_{cc';k}$ and

$$r_{c;k} \stackrel{\text{def}}{=} \sum_{c' \in \mathcal{C}} \lambda_{cc'} m_{cc';k},$$

we can write the following recursive equation

$$r_{c;k+1} = \sum_{c' \in \mathcal{C}} \rho_{cc'} \exp\{r_{c';k}\}, \ \forall c \in \mathcal{C}.$$

The $p_{cc';k}$'s are decreasing sequences, and hence the $r_{c;k}$'s are non-decreasing, with $r_{c;0} = 0, \forall c \in \mathcal{C}$. If (5.3) has a solution, then the relation

$$r_{c;k+1} - y_c = \sum_{c' \in \mathcal{C}} \rho_{cc'} \big[\exp\{r_{c';k}\} - \exp\{y_{c'}\}\big]$$

yields $r_{c;k} \leq y_c$, for all $c \in \mathcal{C}$. Therefore, each sequence $r_{c;k}$ converges as $k \to \infty$ to a finite value $r_c \leq y_c$, and $G_{\mathcal{C}}$ is ergodic since, by (5.4), the $m_{cc'}$'s are also finite. When (5.5) holds, the same line of argument shows that the sequences $r_{c;k}$ are non-decreasing and bounded from above by $\rho_c e$.

Finally, to prove (5.6), we use the following inequality (see [8]), valid for any $x_c > 0$, $c \in \mathcal{C}$:

$$\rho \leq \max_{c \in \mathcal{C}} \sum_{c' \in \mathcal{C}} \frac{\rho_{cc'} x_{c'}}{x_c}.$$

When the r_c's satisfy (5.3), the choice $x_c = \exp\{r_c\}$ implies

$$\rho \leq \max_{c \in \mathcal{C}} \left[\sum_{c' \in \mathcal{C}} \rho_{cc'} \exp\{r_{c'}\} \exp\{-r_c\}\right] = \max_{c \in \mathcal{C}} \left[r_c \exp\{-r_c\}\right] \leq \frac{1}{e},$$

which concludes the proof of the theorem. ∎

It is possible to extend the results of section 3 to the multiclass case. We will only sketch the proofs in what follows, since they are very similar to their single class counterparts. At time $t > 0$, let $N_{cd}(t)$ be the number of vertices of class d inside a tree, the root of which is of class c. Then, as in proof of theorem 3.1, if z_c is a complex number such that $|z_c| < 1$, $\forall c \in \mathcal{C}$,

$$\mathsf{E}\Big[\prod_{d\in\mathcal{C}} z_d^{N_{cd}(t)}\Big] = z_c \exp\Big\{\sum_{c'\in\mathcal{C}} \lambda_{cc'}\mathsf{E}\Big[\int_0^t \Big(\prod_{d\in\mathcal{C}} z_d^{N_{c'd}(x)} - 1\Big)\mathbb{1}_{\{x\le\tau_{cc'}\}}dx\Big]\Big\}$$

Assume the system is ergodic, let $N_{cd} \overset{\text{def}}{=} \lim_{t\to\infty} N_{cd}(t)$ and

$$\phi_c(\vec{z}) \overset{\text{def}}{=} \mathsf{E}\Big[\prod_{d\in\mathcal{C}} z_d^{N_{cd}}\Big].$$

Then computations similar to the ones in theorem 3.1 yield

$$\phi_c(\vec{z}) = z_c \exp\Big\{\sum_{c'\in\mathcal{C}} \lambda_{cc'}m_{cc'}(\phi_{c'}(\vec{z}) - 1)\Big\}, \quad c \in \mathcal{C}. \tag{5.8}$$

Unfortunately, no closed form solution is known for ϕ_c from this equation. It is however possible, as for (3.2), to write down a system of equations for the expectations of the N_{cd}'s.

$$\mathsf{E}\big[N_{cd}\big] = \mathbb{1}_{\{c=d\}} + \sum_{c'\in\mathcal{C}} \rho_{cc'} \exp\{r_{c'}\}\mathsf{E}\big[N_{c'd}\big].$$

This system admits of a non-negative matrix solution if, and only if, the Perron-Frobenius eigenvalue of the matrix

$$M \overset{\text{def}}{=} \Big(\rho_{cc'} \exp\{r_{c'}\}\Big)_{c,c'\in\mathcal{C}}$$

is smaller than 1. A simple necessary condition for this to hold is (5.5).

Finally, the same line of argument allows to extend (3.7). If H_c is the height in stationary regime of a tree which root is of class $c \in \mathcal{C}$, then

$$
\begin{aligned}
\mathsf{P}\{H_c = 0\} &= e^{-r_c}, \\
\mathsf{P}\{H_c > h + 1\} &= 1 - \exp\Big[-\sum_{c'\in\mathcal{C}} \rho_{cc'} \exp\{r_{c'}\}\mathsf{P}\{H_{c'} > h\}\Big], \quad \forall h \ge 0.
\end{aligned}
$$

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