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An algorithmic characterization of P-matrixity

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An algorithmic characterization of P-matricity

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Abstract: It is shown that a matrix M is a **P**-matrix if and only if, whatever is the vector q , the Newton-min algorithm does not cycle between two points when it is used to solve the linear complementarity problem $0 \leq x \perp (Mx + q) \geq 0$.

Key-words: Linear complementarity problem, semismooth Newton method, Newton-min algorithm, NM-matrix, **P**-matricity characterization, **P**-matrix.

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Une caractérisation algorithmique de la **P**-matricité

Résumé : Nous montrons dans cet article qu'une matrice M est une **P**-matrice si, et seulement si, quel que soit le vecteur q , l'algorithme de Newton-min ne fait pas de cycle de deux points lorsqu'il est utilisé pour résoudre le problème de complémentarité linéaire $0 \leq x \perp (Mx + q) \geq 0$.

Mots-clés : Algorithme de Newton-min, caractérisation de la **P**-matricité, méthode de Newton semi-lisse, **NM**-matrice, **P**-matrice, problème de complémentarité linéaire.

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1 Introduction

Being given a positive integer n , a matrix $M \in \mathbb{R}^{n \times n}$, and a vector $q \in \mathbb{R}^n$, the *linear complementarity problem* consists in determining a vector $x \in \mathbb{R}^n$ such that $x \geq 0$ (inequalities have to be understood componentwise; hence $x \geq 0$ means $x_i \geq 0$ for all $i \in \llbracket 1, n \rrbracket := \{1, \dots, n\}$), $Mx + q \geq 0$, and $x^\top(Mx + q) = 0$ (perpendicularity of the two vectors x and $Mx + q$ for the Euclidean scalar product). This problem is sometimes written in compact form as follows

$$\text{LC}(M, q) : \quad 0 \leq x \perp (Mx + q) \geq 0.$$

Let M_{II} denote the square principal submatrix of a matrix M formed of its rows and columns with indices in I . An $n \times n$ real matrix M is a \mathbf{P} -matrix if its principal minors are positive: for all $I \subseteq \llbracket 1, n \rrbracket$, $\det M_{II} > 0$ (by convention $\det M_{\emptyset\emptyset} = 1$). The class of \mathbf{P} -matrices is denoted by \mathbf{P} (the order n of the matrices is implicit and assumed fixed in that notation). These matrices have an eminent role in linear complementarity problems since it can be shown that $M \in \mathbf{P}$ if and only if $\text{LC}(M, q)$ has one and only one solution, whatever is q [16, 4; 1958]. Another characterization of \mathbf{P} -matricity, which will be useful below, is the following [6, 4; 1962]:

$$M \in \mathbf{P} \quad \Longleftrightarrow \quad \text{any } x \text{ verifying } x \cdot (Mx) \leq 0 \text{ vanishes,} \quad (2.1)$$

where we have denoted by $u \cdot v$ the *Hadamard product* of the vectors u and v , which is a vector whose i th component is $u_i v_i$.

There are many other equivalent conditions for a matrix to be in \mathbf{P} , than the three given above [1, 9, 15, 4, 14]. This paper gives still another characterization of \mathbf{P} -matricity in terms of a property of the algorithm for solving $\text{LC}(M, q)$ that is described in section 2, the Newton-min algorithm. We show in this paper that $M \in \mathbf{P}$ if and only if the Newton-min algorithm does not cycle between two distinct points, whatever is q .

2 The Newton-min algorithm

Let I be a subset of $\llbracket 1, n \rrbracket$. We denote by $I^c := \llbracket 1, n \rrbracket \setminus I$ the complementary set of I in $\llbracket 1, n \rrbracket$ and by $|I|$ the cardinality of I . For a vector $v \in \mathbb{R}^n$, v_I is the vector in $\mathbb{R}^{|I|}$ whose components are the components v_i 's of v with index $i \in I$.

The *Newton-min algorithm* is a short name for the semismooth Newton method for solving the nonsmooth piecewise linear equation $\min(x, Mx + q) = 0$, which is equivalent to $\text{LC}(M, q)$ [11, 13; 1976-1977]. More specifically, at the current iterate $x \in \mathbb{R}^n$, the algorithm determines a set of indices

$$I := \{i \in \llbracket 1, n \rrbracket : x_i > (Mx + q)_i\} \quad (2.1)$$

and computes the next iterate x^+ as the unique solution to the system

$$x_{I^c}^+ = 0 \quad \text{and} \quad (Mx^+ + q)_I = 0. \quad (2.2)$$

The uniqueness of the solution of the system (2.2) is certainly ensured if M is *nondegenerate*, meaning that the principal minors of M do not vanish (we denote by \mathbf{D}^c the set of nondegenerate matrices), so that this wellposedness condition of the algorithm will always be assumed. In that case,

$$x_I^+ = -M_{II}^{-1}q_I.$$

A more general form of the algorithm puts also in I some of the indices i such that $x_i = (Mx + q)_i$ (see section 2 in [3] for instance), but we do not consider this version of the algorithm below; numerically, the difference is not essential and the present version is more natural (the linear systems to solve have a smaller size $|I|$). The Newton-min algorithm can be traced back at least to Aganagić in [1; 1984]; see paragraph 7 of the introduction of [3] for more details on its origin and its developments.

By definition (see (2.2)), except possibly for the initial iterate, the Newton-min algorithm only visits points x satisfying $x \cdot (Mx + q) = 0$ or equivalently

$$x_{I^c} = 0 \quad \text{and} \quad (Mx + q)_I = 0, \quad (2.3)$$

for some (possibly empty) index set $I \subseteq \llbracket 1, n \rrbracket$. The point x satisfying (2.3) is denoted by

$$x^{(I)}$$

and is called a *node*. Clearly $x^{(\emptyset)} = 0$. Since there are 2^n different index sets I , there are *at most* 2^n nodes (two different index sets may yield the same node; for example, there is a single node, zero, if and only if $q = 0$).

Since the Newton-min algorithm only generates nodes, either it converges or it cycles by visiting a finite number of nodes repetitively. The identification of the conditions of convergence of the Newton-min algorithm may, therefore, go through the analysis of the conditions that prevent cycles from occurring. The case of the 2-cycles is considered in the next section (for $k \geq 2$, a *k-cycle* is a cycle made of k distinct nodes). Let us formalize this a bit more.

For $k \geq 2$, we denote by \mathbf{NM}_k the set of $n \times n$ matrices M such that the Newton-min algorithm does not make k -cycles when it is used to solve $\text{LC}(M, q)$, whatever is q (and the initial point). Therefore

$$\mathbf{NM} := \bigcap_{k \geq 2} \mathbf{NM}_k$$

is the class of $n \times n$ matrices M such that the Newton-min algorithm converges, whatever is q (and the initial point). In this paper, we prove that

$$\mathbf{NM}_2 = \mathbf{P}.$$

Since \mathbf{NM} is included in \mathbf{NM}_2 , this identity implies in particular that

$$\mathbf{NM} \subseteq \mathbf{P}, \quad (2.4)$$

i.e., the set of matrices that ensure the convergence of the Newton-min algorithm, whatever is q (and the initial point), is contained in \mathbf{P} . It has been shown in [2, 3; 2009] that $\mathbf{P} \subseteq \mathbf{NM}$ if $n = 1$ or $n = 2$ (hence $\mathbf{P} = \mathbf{NM}$ in that case) and that $\mathbf{P} \not\subseteq \mathbf{NM}$ if $n \geq 3$ (hence the inclusion (2.4) is strict in that case).

We recall that an **M-matrix** is a **P-matrix** with nonpositive off-diagonal elements ($M_{ij} \leq 0$ when $i \neq j$). It has been shown in [1; 1984, theorem 6.2] that

$$\mathbf{M} \subseteq \mathbf{NM}.$$

It is also known that $M \in \mathbf{NM}$ if M is sufficiently close to an **M-matrix** [8; 2003].

3 A dual characterization of the absence of 2-cycle

The Newton-min algorithm makes a 2-cycle if it generates successively the iterates

$$x^{(I)} \rightarrow x^{(J)} \rightarrow x^{(I)} \rightarrow x^{(J)} \rightarrow \dots,$$

for distinct nodes $x^{(I)} \neq x^{(J)}$ associated with index sets I and $J \subseteq \llbracket 1, n \rrbracket$. The dual characterization of the absence of 2-cycle of the Newton-min algorithm given in this section is a first step in the derivation of our main result (theorem 4.2).

We start by recalling Motzkin's theorem of the alternative (see [7; 2010, theorem 3.15 or 7.17] for instance), on which the dual characterization of proposition 3.2 rests.

Lemma 3.1 (Motzkin) *Let $A \in \mathbb{R}^{m_A \times n}$ and $B \in \mathbb{R}^{m_B \times n}$ be two matrices with the same number of columns. Then, there is a vector $x \in \mathbb{R}^n$ satisfying*

$$Ax < 0 \quad \text{and} \quad Bx \leq 0 \quad (3.1)$$

if and only if there is no vector $\alpha \in \mathbb{R}_+^{m_A} \setminus \{0\}$ and $\beta \in \mathbb{R}_+^{m_B}$ such that

$$A^\top \alpha + B^\top \beta = 0. \quad (3.2)$$

Strict inequalities on vectors must also be understood componentwise; hence $Ax < 0$ in (3.1) means $(Ax)_i < 0$ for all $i \in \llbracket 1, m_A \rrbracket$. The (components of the) vectors α and β in the lemma will be called *Motzkin multipliers* below. There is one such scalar multiplier for each inequality in (3.1).

The next proposition gives a dual condition on the matrix M such that the Newton-min algorithm does not cycle between two given nodes when it is used to solve $\text{LC}(M, q)$, whatever is the vector q . The dual aspect of condition (3.3) comes from lemma 3.1 and therefore involves, like in (3.2), the transpose of (modified) submatrices of M . The *symmetric difference* of two index sets I and $J \subseteq \llbracket 1, n \rrbracket$ is denoted by

$$I \triangle J := (I \cap J^c) \cup (I^c \cap J) = (I \cup J) \setminus (I \cap J).$$

Proposition 3.2 (no cycle $x^{(I)} \rightarrow x^{(J)} \rightarrow x^{(I)}$) *Suppose that $M \in \mathbf{D}^c$ and let be given two different subsets I and $J \subseteq \llbracket 1, n \rrbracket$. Then the following conditions are equivalent:*

(i) *there is an $\alpha \in \mathbb{R}_+^{I \triangle J} \setminus \{0\}$ such that*

$$\begin{aligned} & \begin{pmatrix} M_{(I \cap J^c)(I \cap J^c)} & -M_{(I \cap J^c)(I^c \cap J)} \\ -M_{(I^c \cap J)(I \cap J^c)} & M_{(I^c \cap J)(I^c \cap J)} \end{pmatrix}^\top \alpha \\ & \geq \begin{pmatrix} -M_{(I \cap J)(I \cap J^c)} & M_{(I \cap J)(I^c \cap J)} \end{pmatrix}^\top M_{(I \cap J)(I \cap J)}^{-\top} \begin{pmatrix} -M_{(I \cap J^c)(I \cap J)} \\ M_{(I^c \cap J)(I \cap J)} \end{pmatrix}^\top \alpha, \end{aligned} \quad (3.3)$$

(ii) *whatever is q , the Newton-min algorithm does not make the cycle $x^{(I)} \rightarrow x^{(J)} \rightarrow x^{(I)}$ when it is used to solve $\text{LC}(M, q)$.*

PROOF. [(i) \Leftrightarrow (ii)] Let us first specify the conditions on q such that the Newton-min algorithm makes the cycle $x^{(I)} \rightarrow x^{(J)} \rightarrow x^{(I)}$ when it is used to solve $\text{LC}(M, q)$. Let $x^1 = x^{(I)}$, so that by (2.3),

$$\begin{cases} x_I^1 = -M_{II}^{-1} q_I \\ x_{I^c}^1 = 0 \end{cases} \quad \text{and} \quad \begin{cases} (Mx^1 + q)_I = 0 \\ (Mx^1 + q)_{I^c} = q_{I^c} - M_{I^c I} M_{II}^{-1} q_I. \end{cases}$$

We have used the nonsingularity of M_{II} , which comes from the nondegeneracy of M . Now, by definition of the Newton-min algorithm (2.1)-(2.2), the next iterate $x^2 = x^{(J)}$ if and only if

$$\begin{aligned} & \underbrace{-(M_{II}^{-1}q_I)_{J^c} \leq 0}_{\beta_{I \cap J^c}}, \quad \underbrace{(M_{II}^{-1}q_I)_J < 0}_{\alpha'_{I \cap J}}, \\ & \underbrace{q_{I^c \cap J} < M_{(I^c \cap J)I} M_{II}^{-1} q_I}_{\alpha_{I^c \cap J}}, \quad \text{and} \quad M_{(I \cup J)^c I} M_{II}^{-1} q_I \leq q_{(I \cup J)^c}. \end{aligned} \quad (3.4)$$

The vectors under the braces are Motzkin multipliers, which will be used below. In that case

$$\begin{cases} x_J^2 = -M_{JJ}^{-1} q_J \\ x_{J^c}^2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} (Mx^2 + q)_J = 0 \\ (Mx^2 + q)_{J^c} = q_{J^c} - M_{J^c J} M_{JJ}^{-1} q_J. \end{cases}$$

We have used the nonsingularity of M_{JJ} , which comes from the nondegeneracy of M . Now, $x^3 = x^{(I)}$, if and only if

$$\begin{aligned} & \underbrace{-(M_{JJ}^{-1}q_J)_{I^c} \leq 0}_{\beta_{I^c \cap J}}, \quad \underbrace{(M_{JJ}^{-1}q_J)_I < 0}_{\alpha''_{I \cap J}}, \\ & \underbrace{q_{I \cap J^c} < M_{(I \cap J^c)J} M_{JJ}^{-1} q_J}_{\alpha_{I \cap J^c}}, \quad \text{and} \quad M_{(I \cup J)^c J} M_{JJ}^{-1} q_J \leq q_{(I \cup J)^c}. \end{aligned} \quad (3.5)$$

The vectors under the braces are Motzkin multipliers, which will be used below. We have shown that the Newton-min algorithm makes the cycle $x^{(I)} \rightarrow x^{(J)} \rightarrow x^{(I)}$ if and only if q satisfies the linear inequalities in (3.4)-(3.5).

Observe now that the components of q with indices in $(I \cup J)^c$ intervene only in the last inequalities in (3.4)-(3.5) and that these inequalities can be satisfied by taking these components of q sufficiently large. Therefore, below, we do not have to consider the satisfiability of these last inequalities. This is the reason why we have not assigned Motzkin multipliers to these inequalities.

By Motzkin's theorem of the alternative (lemma 3.1), there is a q satisfying the linear inequalities in (3.4)-(3.5) if and only if one cannot find

$$(\alpha, \alpha', \alpha'', \beta) \in \mathbb{R}_+^{|I \Delta J|} \times \mathbb{R}_+^{|I \cap J|} \times \mathbb{R}_+^{|I \cap J|} \times \mathbb{R}_+^{|I \Delta J|}$$

(these are the vectors under the braces in (3.4)-(3.5)) such that $(\alpha, \alpha', \alpha'') \neq 0$ and

$$\begin{aligned} & \bullet (M_{II}^{-\top})_{I \cap J^c} \begin{pmatrix} -\beta_{I \cap J^c} \\ \alpha'_{I \cap J} \end{pmatrix} - (M_{II}^{-\top})_{I \cap J^c} M_{(I^c \cap J)I}^{\top} \alpha_{I^c \cap J} + \alpha_{I \cap J^c} = 0, \\ & \bullet (M_{II}^{-\top})_{I \cap J} \begin{pmatrix} -\beta_{I \cap J^c} \\ \alpha'_{I \cap J} \end{pmatrix} - (M_{II}^{-\top})_{I \cap J} M_{(I^c \cap J)I}^{\top} \alpha_{I^c \cap J} + (M_{JJ}^{-\top})_{I \cap J} \begin{pmatrix} \alpha''_{I \cap J} \\ -\beta_{I^c \cap J} \end{pmatrix} \\ & \quad - (M_{JJ}^{-\top})_{I \cap J} M_{(I \cap J^c)J}^{\top} \alpha_{I \cap J^c} = 0, \\ & \bullet \alpha_{I^c \cap J} + (M_{JJ}^{-\top})_{I^c \cap J} \begin{pmatrix} \alpha''_{I \cap J} \\ -\beta_{I^c \cap J} \end{pmatrix} - (M_{JJ}^{-\top})_{I^c \cap J} M_{(I \cap J^c)J}^{\top} \alpha_{I \cap J^c} = 0, \end{aligned}$$

where A_R denotes the submatrix of a matrix A formed of its rows with indices in R . With the notation

$$u := \begin{pmatrix} -\beta_{I \cap J^c} \\ \alpha'_{I \cap J} \end{pmatrix} - M_{(I^c \cap J)I}^{\top} \alpha_{I^c \cap J} \quad \text{and} \quad v := \begin{pmatrix} \alpha''_{I \cap J} \\ -\beta_{I^c \cap J} \end{pmatrix} - M_{(I \cap J^c)J}^{\top} \alpha_{I \cap J^c} \quad (3.6)$$

the system above reads equivalently

$$\begin{pmatrix} (M_{II}^{-\top})_{I \cap J^c} u \\ (M_{II}^{-\top})_{I \cap J} u \\ 0 \end{pmatrix} + \begin{pmatrix} \alpha_{I \cap J^c} \\ 0 \\ \alpha_{I^c \cap J} \end{pmatrix} + \begin{pmatrix} 0 \\ (M_{JJ}^{-\top})_{I \cap J} v \\ (M_{JJ}^{-\top})_{I^c \cap J} v \end{pmatrix} = 0. \quad (3.7)$$

If we multiply the first two equations to the left by M_{II}^\top , if we multiply the last two equations to the left by M_{JJ}^\top , and if we use the second equation, we obtain the equivalent system

$$u + M_{(I \cap J^c)I}^\top \alpha_{I \cap J^c} - M_{(I \cap J)I}^\top (M_{II}^{-\top})_{I \cap J} u = 0, \quad (3.8a)$$

$$v + M_{(I^c \cap J)J}^\top \alpha_{I^c \cap J} + M_{(I \cap J)J}^\top (M_{II}^{-\top})_{I \cap J} u = 0. \quad (3.8b)$$

Now the rows with indices in $I \cap J$ of these last two equations read

$$\alpha'_{I \cap J} - M_{(I^c \cap J)(I \cap J)}^\top \alpha_{I^c \cap J} + M_{(I \cap J^c)(I \cap J)}^\top \alpha_{I \cap J^c} - M_{(I \cap J)(I \cap J)}^\top (M_{II}^{-\top})_{I \cap J} u = 0,$$

$$\alpha''_{I \cap J} - M_{(I \cap J^c)(I \cap J)}^\top \alpha_{I \cap J^c} + M_{(I^c \cap J)(I \cap J)}^\top \alpha_{I^c \cap J} + M_{(I \cap J)(I \cap J)}^\top (M_{II}^{-\top})_{I \cap J} u = 0.$$

By adding these equations, we obtain $\alpha'_{I \cap J} + \alpha''_{I \cap J} = 0$, which implies that $\alpha'_{I \cap J} = \alpha''_{I \cap J} = 0$ since the components of these two vectors are nonnegative. One deduces then from any of these last equations that

$$(M_{II}^{-\top})_{I \cap J} u = M_{(I \cap J)(I \cap J)}^{-\top} \left(M_{(I \cap J^c)(I \cap J)}^\top \alpha_{I \cap J^c} - M_{(I^c \cap J)(I \cap J)}^\top \alpha_{I^c \cap J} \right). \quad (3.9)$$

Now the $I \cap J^c$ component of equation (3.8a) and the $I^c \cap J$ component of equation (3.8b) read

$$\begin{aligned} & -M_{(I^c \cap J)(I \cap J^c)}^\top \alpha_{I^c \cap J} + M_{(I \cap J^c)(I \cap J^c)}^\top \alpha_{I \cap J^c} \\ & - M_{(I \cap J)(I \cap J^c)}^\top M_{(I \cap J)(I \cap J)}^{-\top} \left(M_{(I \cap J^c)(I \cap J)}^\top \alpha_{I \cap J^c} - M_{(I^c \cap J)(I \cap J)}^\top \alpha_{I^c \cap J} \right) \\ & = \beta_{I \cap J^c}, \end{aligned} \quad (3.10a)$$

$$\begin{aligned} & -M_{(I \cap J^c)(I^c \cap J)}^\top \alpha_{I \cap J^c} + M_{(I^c \cap J)(I^c \cap J)}^\top \alpha_{I^c \cap J} \\ & + M_{(I \cap J)(I^c \cap J)}^\top M_{(I \cap J)(I \cap J)}^{-\top} \left(M_{(I \cap J^c)(I \cap J)}^\top \alpha_{I \cap J^c} - M_{(I^c \cap J)(I \cap J)}^\top \alpha_{I^c \cap J} \right) \\ & = \beta_{I^c \cap J}. \end{aligned} \quad (3.10b)$$

We have deduced the system (3.10) and $\alpha'_{I \cap J} = \alpha''_{I \cap J} = 0$ from (3.8). Reciprocally, to show that there has been no loss of information in that operation, let us show that one can recover the system (3.8) from (3.10) and $\alpha'_{I \cap J} = \alpha''_{I \cap J} = 0$, provided we define u and v by (3.6). Indeed, let us denote by w the right hand side of (3.9) and let us first show that the identity (3.9) is verified:

$$\begin{aligned} (M_{II}^{-\top})_{I \cap J} u &= (M_{II}^{-\top})_{I \cap J} \begin{pmatrix} -\beta_{I \cap J^c} - M_{(I^c \cap J)(I \cap J^c)}^\top \alpha_{I^c \cap J} \\ -M_{(I^c \cap J)(I \cap J)}^\top \alpha_{I^c \cap J} \end{pmatrix} \quad [(3.6), \alpha'_{I \cap J} = 0] \\ &= (M_{II}^{-\top})_{I \cap J} \begin{pmatrix} -M_{(I \cap J^c)(I \cap J^c)}^\top \alpha_{I \cap J^c} + M_{(I \cap J)(I \cap J^c)}^\top w \\ -M_{(I \cap J^c)(I \cap J)}^\top \alpha_{I \cap J^c} + M_{(I \cap J)(I \cap J)}^\top w \end{pmatrix} \quad \left[\begin{array}{l} (3.10a) \text{ and} \\ \text{definition of } w \end{array} \right] \\ &= (M_{II}^{-\top})_{I \cap J} M_{II}^\top \begin{pmatrix} -\alpha_{I \cap J^c} \\ w \end{pmatrix} \\ &= w. \end{aligned}$$

Now the $I \cap J^c$ component of (3.8a) is a consequence of (3.10a); the $I \cap J$ components of (3.8a) and (3.8b) are consequences of $\alpha'_{I \cap J} = \alpha''_{I \cap J} = 0$ and (3.9); and the $I^c \cap J$ component of (3.8b) is a consequence of (3.10b).

Therefore, getting rid of the vector $\beta \in \mathbb{R}_+^{|I \Delta J|}$ in (3.10), we see that there is a q such that the Newton-min algorithm makes the cycle $x^{(I)} \rightarrow x^{(J)} \rightarrow x^{(I)}$ when it is used to solve $\text{LC}(M, q)$ if and only if one cannot find an $\alpha \in \mathbb{R}_+^{|I \Delta J|} \setminus \{0\}$ such that

$$\begin{aligned} & \begin{pmatrix} M_{(I \cap J^c)(I \cap J^c)} & -M_{(I \cap J^c)(I^c \cap J)} \\ -M_{(I^c \cap J)(I \cap J^c)} & M_{(I^c \cap J)(I^c \cap J)} \end{pmatrix}^\top \alpha \\ & \geq \begin{pmatrix} -M_{(I \cap J)(I \cap J^c)} & M_{(I \cap J)(I^c \cap J)} \end{pmatrix}^\top M_{(I \cap J)(I \cap J)}^{-\top} \begin{pmatrix} -M_{(I \cap J^c)(I \cap J)} \\ M_{(I^c \cap J)(I \cap J)} \end{pmatrix}^\top \alpha. \end{aligned}$$

The equivalence (i) \Leftrightarrow (ii) follows. \square

Observe that, as expected, condition (3.3) is symmetric in I and J , in the sense that the permutation $I \leftrightarrow J$ does not modify the condition.

Proposition 3.2 is apparently difficult to use if the goal is to characterize the class \mathbf{NM}_2 of matrices. It requires indeed to consider all the possible pairs of distinct subsets I and $J \subseteq \llbracket 1, n \rrbracket$. In addition, for each of these pairs, each choice of $\alpha \in \mathbb{R}_+^{I \triangle J} \setminus \{0\}$ in (3.3) may yield different conditions on M that make the Newton-min algorithm avoid the 2-cycle $x^{(I)} \rightarrow x^{(J)} \rightarrow x^{(I)}$. This looks like a long-term and hazardous task. We show in theorem 4.2 below, however, that it is equivalent to avoid all the 2-cycles or to avoid the small subset of 2-cycles $x^{(I)} \rightarrow x^{(J)} \rightarrow x^{(I)}$, in which $I = J \cup \{i\}$. In other words, this small subset of 2-cycles contains all the information on M that is needed to prevent the Newton-min algorithm from making 2-cycles. A precious interest of the choice $I = J \cup \{i\}$ is that α is then a positive *scalar*, which can be eliminated from the inequality (3.3). The task of characterizing the matrices M in \mathbf{NM}_2 is then much easier. It is shown in the next section that these matrices are the \mathbf{P} -matrices.

4 A characterization of \mathbf{P} -matricity

Let us start by an elementary lemma.

Lemma 4.1 *Suppose that I and J are two index sets included in $\llbracket 1, n \rrbracket$ such that $J \setminus I \neq \emptyset$ and that $x^{(I)} = x^{(J)}$ for some q . Then the Newton-min algorithm (2.1)-(2.2) does not make the displacement $x^{(I)} \rightarrow x^{(J)}$ when it is used to solve $\text{LC}(M, q)$.*

PROOF. We argue by contradiction, assuming that the Newton-min algorithm goes from $x^{(I)}$ to $x^{(J)}$. Then, $(Mx^{(I)} + q)_{I^c \cap J} < x_{I^c \cap J}^{(I)}$ [by the algorithm rule (2.1)] and $x_{I^c \cap J}^{(I)} = 0$ [by the definition of $x^{(I)}$, see (2.3)₁], so that

$$(Mx^{(I)} + q)_{I^c \cap J} < 0.$$

On the other hand,

$$(Mx^{(J)} + q)_{I^c \cap J} = 0,$$

by the definition of $x^{(J)}$, see (2.3)₂. Therefore $(Mx^{(I)} + q)_{I^c \cap J} \neq (Mx^{(J)} + q)_{I^c \cap J}$, contradicting the fact that $x^{(I)} = x^{(J)}$ for the given q . \square

We denote by $\text{cof}(M)$ the *cofactor matrix* of a matrix $M \in \mathbb{R}^{n \times n}$, whose element $[\text{cof}(M)]_{ij}$ is the *cofactor* $\text{cof}(M_{ij})$ of the element M_{ij} of M , that is

$$\text{cof}(M_{ij}) := (-1)^{i+j} \det M_{(\llbracket 1, n \rrbracket \setminus \{i\})(\llbracket 1, n \rrbracket \setminus \{j\})}. \quad (4.1)$$

We use the notation $\text{cof}_{II}(M_{ij})$ for the cofactor of the element M_{ij} in M_{II} . Recall [12; 1987, chapter VI] that for any index i and j :

$$\det M = \sum_{i'} M_{i'j} \text{cof}(M_{i'j}) = \sum_{j'} M_{ij'} \text{cof}(M_{ij'}) \quad (4.2)$$

and that

$$M^{-1} = (\det M)^{-1} \text{cof}(M^T). \quad (4.3)$$

Our main result is given in theorem 4.2 below. The implication (i) \Rightarrow (ii) of the theorem was already proven in [2; 2009, lemma 4.3], but this part of that paper was not selected by the refereeing process for appearing in the published version of the paper [3; 2012].

Theorem 4.2 (a characterization of \mathbf{P} -matricity) *Suppose that $M \in \mathbf{D}^c$. Then the following conditions are equivalent:*

- (i) $M \in \mathbf{P}$,
- (ii) for any q , the Newton-min algorithm does not cycle between two different nodes when it is used to solve $\text{LC}(M, q)$,
- (iii) for any q , for any subset $J \subseteq \llbracket 1, n \rrbracket$, and for any index $i \in \llbracket 1, n \rrbracket \setminus J$, the Newton-min algorithm does not cycle between the nodes $x^{(J)}$ and $x^{(J \cup \{i\})}$ when it is used to solve $\text{LC}(M, q)$,
- (iv) for any subset $J \subseteq \llbracket 1, n \rrbracket$ and any index $i \in \llbracket 1, n \rrbracket \setminus J$, there holds

$$M_{ii} \geq M_{\{i\}J} M_{JJ}^{-1} M_{J\{i\}}. \quad (4.4)$$

PROOF. [(i) \Rightarrow (ii)] We prove the contrapositive, assuming that the algorithm visits in order the following nodes $x^{(I)} \rightarrow x^{(J)} \rightarrow x^{(I)}$, for some I and $J \subseteq \llbracket 1, n \rrbracket$ and some $q \in \mathbb{R}^n$ such that $x^{(I)} \neq x^{(J)}$. We simplify the notation by setting $x^1 := x^{(I)}$ and $x^2 := x^{(J)}$. Since the Newton-min algorithm goes from x^1 to x^2 and from x^2 from x^1 , the very definition (2.1)-(2.2) of the algorithm implies that

$$x_{J^c}^1 \leq (Mx^1 + q)_{J^c} \quad \text{and} \quad x_J^1 > (Mx^1 + q)_J, \quad (4.5)$$

$$x_{I^c}^2 \leq (Mx^2 + q)_{I^c} \quad \text{and} \quad x_I^2 > (Mx^2 + q)_I. \quad (4.6)$$

After a possible rearrangement of the component order, we get

$$x^2 - x^1 = \begin{pmatrix} 0_{I \cap J^c} \\ x_{I \cap J}^2 \\ x_{I^c \cap J}^2 \\ 0_{I^c \cap J^c} \end{pmatrix} - \begin{pmatrix} x_{I \cap J^c}^1 \\ x_{I \cap J}^1 \\ 0_{I^c \cap J} \\ 0_{I^c \cap J^c} \end{pmatrix} = \begin{pmatrix} -x_{I \cap J^c}^1 \\ (x^2 - x^1)_{I \cap J} \\ x_{I^c \cap J}^2 \\ 0_{I^c \cap J^c} \end{pmatrix} \cdot \begin{matrix} [+] \\ [?] \\ [-] \\ [0] \end{matrix}$$

The extra column on the right gives the sign of each component, when appropriate: the components of $x^2 - x^1$ with indices in $I \cap J^c$ are nonnegative since $-x_{I \cap J^c}^1 \geq -(Mx^1 + q)_{I \cap J^c}$ [by (4.5)₁] = 0 [by (2.2)₂] and that the components with indices in $I^c \cap J$ are nonpositive since $x_{I^c \cap J}^2 \leq (Mx^2 + q)_{I^c \cap J}$ [by (4.6)₁] = 0 [by (2.2)₂]. On the other hand, by (2.2)₂, there holds

$$M(x^2 - x^1) = \begin{pmatrix} (Mx^2)_{I \cap J^c} \\ -q_{I \cap J} \\ -q_{I^c \cap J} \\ (Mx^2)_{I^c \cap J^c} \end{pmatrix} - \begin{pmatrix} -q_{I \cap J^c} \\ -q_{I \cap J} \\ (Mx^1)_{I^c \cap J} \\ (Mx^1)_{I^c \cap J^c} \end{pmatrix} = \begin{pmatrix} (Mx^2 + q)_{I \cap J^c} \\ 0 \\ -(Mx^1 + q)_{I^c \cap J} \\ (M(x^2 - x^1))_{I^c \cap J^c} \end{pmatrix} \cdot \begin{matrix} [-] \\ [0] \\ [+] \\ [?] \end{matrix}$$

The extra column on the right gives the sign of each component, when appropriate: the components with indices in $I \cap J^c$ are nonpositive since $(Mx^2 + q)_{I \cap J^c} \leq x_{I \cap J^c}^2$ [by (4.6)₂] = 0 [by (2.2)₁] and the components with indices in $I^c \cap J$ are nonnegative since $-(Mx^1 + q)_{I^c \cap J} \geq -x_{I^c \cap J}^1$ [by (4.5)₂] = 0 [by (2.2)₁]. Therefore

$$(x^2 - x^1) \cdot M(x^2 - x^1) \leq 0.$$

Since $x^1 \neq x^2$, M cannot be a \mathbf{P} -matrix (see (1.1)).

[(ii) \Rightarrow (iii)] Let $q \in \mathbb{R}^n$, $J \subseteq \llbracket 1, n \rrbracket$, and $i \in \llbracket 1, n \rrbracket \setminus J$. Set $I := J \cup \{i\}$. If $x^{(I)} \neq x^{(J)}$, then (iii) is a clear consequence of (ii). If $x^{(I)} = x^{(J)}$, then (iii) is a consequence of lemma 4.1, according to which the Newton min does not go from $x^{(J)}$ to $x^{(I)}$ because $I \setminus J \neq \emptyset$.

[(iii) \Rightarrow (iv)] Let J and i like in (iv), and set $I = J \cup \{i\}$. By (iii), whatever is q , the Newton-min algorithm does not cycle between the nodes $x^{(I)}$ and $x^{(J)}$ when it is used to solve $\text{LC}(M, q)$. Then, the implication (ii) \Rightarrow (i) of proposition 3.2 shows that there is an $\alpha > 0$ such that (3.2) holds. Since $I \cap J^c = \{i\}$, $I^c \cap J = \emptyset$, $I \cap J = J$, $I \triangle J = \{i\}$, and α is a positive scalar that can be eliminated from (3.2), this inequality simplifies in (4.4) (use also the fact that $M_{\{i\}J} M_{JJ}^{-1} M_{J\{i\}}$ is a scalar).

[(iv) \Rightarrow (i)] We prove by induction that $\det M_{II} > 0$ for any $I \subseteq \llbracket 1, n \rrbracket$, which is equivalent to $M \in \mathbf{P}$. By applying (iv) with $J = \emptyset$, we obtain $M_{ii} > 0$ for a nondegenerate matrix, so that $\det M_{II} > 0$ when $|I| = 1$. Now, assume that J and i are chosen like in (iv), that $I = J \cup \{i\}$, that $\det M_{JJ} > 0$ (induction assumption), and let us show that $\det M_{II} > 0$, which will conclude the proof of (iv) \Rightarrow (i).

Let us denote the indices in J by j_k , $k \in \llbracket 1, |J| \rrbracket$. Using the cofactor matrix of M_{JJ} in (4.4) and the induction assumption $\det M_{JJ} > 0$, one gets

$$\begin{aligned}
0 &\leq M_{ii} \det M_{JJ} - M_{\{i\}J} \text{cof}(M_{JJ}^T) M_{J\{i\}} && [(4.4), (4.3), \det M_{JJ} > 0] \\
&= M_{ii} \det M_{JJ} - \sum_{k=1}^{|J|} \sum_{l=1}^{|J|} M_{ij_k} \text{cof}_{JJ}([M_{JJ}]_{j_l j_k}) M_{j_l i} \\
&= M_{ii} \det M_{JJ} - \sum_{k=1}^{|J|} \sum_{l=1}^{|J|} M_{ij_k} (-1)^{l+k} \det M_{(J \setminus \{j_l\})(J \setminus \{j_k\})} M_{j_l i} && [(4.1)] \\
&= M_{ii} \det M_{JJ} + \sum_{k=1}^{|J|} (-1)^{k+|J|+1} M_{ij_k} \sum_{l=1}^{|J|} M_{j_l i} (-1)^{l+|J|} \det M_{(J \setminus \{j_l\})(J \setminus \{j_k\})} \\
&= M_{ii} \det M_{JJ} + \sum_{k=1}^{|J|} (-1)^{k+|J|+1} M_{ij_k} \det (M_{J(J \setminus \{j_k\})}, M_{J\{i\}}) && [(4.2)] \\
&= \det M_{II} && [(4.2)].
\end{aligned}$$

Therefore $\det M_{II} > 0$ by the nondegeneracy of M . \square

The implication (ii) \Rightarrow (i) of theorem 4.2, according to which only the \mathbf{P} -matrices in \mathbf{D}^c prevent the Newton-min algorithm from cycling between two nodes, is ultimately based on Motzkin's theorem of the alternative, which supports the implication (iii) \Rightarrow (iv) of theorem 4.2, while the implication (ii) \Rightarrow (iii) is straightforward and the implication (iv) \Rightarrow (i) has an algebraic nature. Motzkin's theorem is not constructive whereas, for $M \in \mathbf{D}^c \setminus \mathbf{P}$, it would certainly be interesting (and reassuring) to be able to construct a q and to select two nodes such that the Newton-min algorithm cycles between these nodes when it is used to solve $\text{LC}(M, q)$. This is the goal of the next proposition, which takes inspiration from the contrapositive of the implications (iii) \Rightarrow (iv) \Rightarrow (i) of theorem 4.2: if $M \in \mathbf{D}^c \setminus \mathbf{P}$, there are a vector $q \in \mathbb{R}^n$, a subset $J \subseteq \llbracket 1, n \rrbracket$, and an index $i \in \llbracket 1, n \rrbracket \setminus J$ such that the Newton-min algorithm cycles between the nodes $x^{(J)}$ and $x^{(J \cup \{i\})}$. According to the contrapositive of the implication (iv) \Rightarrow (i), the index sets J and $I := J \cup \{i\}$ are such that $\det M_{JJ} > 0$ and $\det M_{II} < 0$. One still has to find the vector q and this is precisely what the next proposition brings by exhibiting a whole family of vectors q such that this cycle occurs.

We adopt the notation $t^+ := \max(0, t)$ for $t \in \mathbb{R}$. The operator 'max' is supposed to act componentwise on vectors.

Proposition 4.3 (2-cycle for $M \notin \mathbf{P}$) Suppose that $M \in \mathbf{D}^c \setminus \mathbf{P}$. Then

- 1) there are two index sets I and $J \subseteq \llbracket 1, n \rrbracket$ and an index $i \in \llbracket 1, n \rrbracket$ such that $I = J \cup \{i\}$, $\det M_{II} < 0$, and $\det M_{JJ} > 0$,
- 2) for any two index sets I and $J \subseteq \llbracket 1, n \rrbracket$ and an index $i \in \llbracket 1, n \rrbracket$ having the properties given in point 1, the Newton-min algorithm cycles between $x^{(I)}$ and $x^{(J)}$ when the components of q are determined in order as follows

$$q_J = -M_{JJ}e^J, \quad (4.7)$$

$$q_i = -M_{iJ}e^J - \varepsilon, \quad \text{with } 0 < \varepsilon < \frac{|\det M_{II}|}{\max_{j \in J} [\text{cof}_{II}(M_{ij})]^+}, \quad (4.8)$$

$$q_{I^c} \geq \max(M_{I^cJ}M_{JJ}^{-1}q_J, M_{I^cI}M_{II}^{-1}q_I), \quad (4.9)$$

where $e^J \in \mathbb{R}^{|J|}$ is a vector of all ones.

PROOF. 1) Since $M \notin \mathbf{P}$, some principal minor of M is negative. Then one can choose an index set I with the smallest cardinal number $|I|$ such that $\det M_{II} < 0$. Since $|I| \geq 1$, one can choose an index $i \in I$ and set $J := I \setminus \{i\}$ (J may be empty). The properties in point 1 are verified for the selected sets I and J , and the index i (recall that $\det M_{\emptyset\emptyset} = 1$ by convention).

2) Let the index sets I and J and the index i be such that $I = J \cup \{i\}$, $\det M_{II} < 0$, and $\det M_{JJ} > 0$. Suppose that q satisfies (4.7)-(4.9). Let $x^1 := x^{(J)}$, so that

$$\begin{cases} x_J^1 = -M_{JJ}^{-1}q_J \\ x_{J^c}^1 = 0 \end{cases} \quad \text{and} \quad \begin{cases} (Mx^1 + q)_J = 0 \\ (Mx^1 + q)_{J^c} = q_{J^c} - M_{J^cJ}M_{JJ}^{-1}q_J. \end{cases}$$

For a q satisfying the assumption, there hold

$$\begin{aligned} M_{JJ}^{-1}q_J &= -e^J < 0 && [(4.7)], \\ q_i - M_{iJ}M_{JJ}^{-1}q_J &= -\varepsilon < 0 && [(4.8)], \\ q_{I^c} - M_{I^cJ}M_{JJ}^{-1}q_J &\geq 0 && [(4.9)]. \end{aligned}$$

These inequalities imply that the next iterate visited by the Newton-min algorithm (2.1)-(2.2) is $x^2 := x^{(I)}$, so that

$$\begin{cases} x_I^2 = -M_{II}^{-1}q_I \\ x_{I^c}^2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} (Mx^2 + q)_I = 0 \\ (Mx^2 + q)_{I^c} = q_{I^c} - M_{I^cI}M_{II}^{-1}q_I. \end{cases}$$

We now want to show that appropriate inequalities on q are verified that ensure that the iterate following x^2 is x^1 . Observe that by (4.7) and (4.8)

$$M_{II}^{-1}q_I = -M_{II}^{-1}M_{IJ}e^J - \varepsilon M_{II}^{-1}e^{I,i} = -(e^I - e^{I,i}) - \varepsilon (\det M_{II})^{-1} \text{cof}_{II}(M_{ii})^\top,$$

where $e^{I,i} \in \mathbb{R}^{|I|}$ is a vector whose components are all zero, except the one at the position of i in I whose value is 1. Therefore

$$\forall j \in J: \quad (M_{II}^{-1}q_I)_j = -1 - \varepsilon (\det M_{II})^{-1} \text{cof}_{II}(M_{ij}) < 0, \quad (4.10)$$

since $-\varepsilon (\det M_{II})^{-1} \text{cof}_{II}(M_{ij}) \leq \varepsilon |\det M_{II}|^{-1} \max_{j \in J} [\text{cof}_{II}(M_{ij})]^+ < 1$, by the choice of ε in (4.8). On the other hand,

$$(M_{II}^{-1}q_I)_i = -\varepsilon (\det M_{II})^{-1} \text{cof}_{II}(M_{ii}) = -\varepsilon (\det M_{II})^{-1} (\det M_{JJ}) > 0, \quad (4.11)$$

since $(\det M_{II}) < 0$ and $(\det M_{JJ}) > 0$ by assumption. Finally, by (4.9), there holds

$$q_{I^c} - M_{I^c I} M_{II}^{-1} q_I \geq 0. \quad (4.12)$$

The inequalities (4.10), (4.11), and (4.12) imply that the iterate following x^2 is indeed x^1 . Hence the Newton-min algorithm cycles between the nodes $x^{(J)}$ and $x^{(I)}$. \square

Thanks to proposition 4.3, the equivalence (i) \Leftrightarrow (ii) of proposition 4.2 can now be proven without proposition 3.2 and Motzkin's theorem of the alternative, so that one can wonder whether section 3 is still useful. We have maintained it in the paper for two reasons. First, proposition 3.2 has brought a clear indication on the way of choosing the index sets I and J in proposition 4.3, whose origin could be obscure otherwise. Next, the dual characterization of the absence of 2-cycle that it provides may be useful in the extension of this work.

5 Discussion and perspectives

A natural extension of the present work would consist in looking at the possibility to give a *simple* algebraic description of the classes \mathbf{NM}_k , for $k \geq 3$, and \mathbf{NM} . An intriguing feature of the approach followed in this paper to characterize \mathbf{NM}_2 is its intrinsic conjunctive-disjunctive nature. It is conjunctive in the sense that it provides conditions to satisfy for each possible 2-cycle that the matrix must prevent. Its disjunctive aspect comes from the use of Motzkin's theorem of the alternative (lemma 3.1), which provides a possibility to avoid a given cycle for each acceptable choice of multipliers α in proposition 3.2, and there may be many. For example, when $M \in \mathbf{P}$, the same approach shows that the Newton-min algorithm does not make the 3-cycle $x^{\{i\}} \rightarrow x^{\{j\}} \rightarrow x^{\{k\}} \rightarrow x^{\{i\}}$, for three different indices i, j , and $k \in \llbracket 1, n \rrbracket$, when it is used to solve $\text{LC}(M, q)$, whatever is $q \in \mathbb{R}^n$, if and only if *one* of the following eight conditions holds

$$\begin{aligned} M_{ik} M_{jj} &\leq M_{ij}^+ M_{jk}, & M_{ji} M_{kk} &\leq M_{jk}^+ M_{ki}, & M_{kj} M_{ii} &\leq M_{ki}^+ M_{ij}, \\ M_{ji}^+ M_{ik} &\leq M_{jk} M_{ii}, & M_{kj}^+ M_{ji} &\leq M_{ki} M_{jj}, & M_{ik}^+ M_{kj} &\leq M_{ij} M_{kk}, \\ M_{ik}^+ M_{kj}^+ M_{ji}^+ &\leq M_{ii} M_{jj} M_{kk}, & \text{or} & & M_{ij}^+ M_{jk}^+ M_{ki}^+ &> M_{ii} M_{jj} M_{kk}. \end{aligned}$$

Note that conditions 1, 2, 3, and 7 are satisfied by an \mathbf{M} -matrix. We don't know whether this disjunctive form of the conditions disappears for the whole classes \mathbf{NM}_k or \mathbf{NM} , i.e., when all the possible cycles must be avoided, as it does for $\mathbf{NM}_2 = \mathbf{P}$.

Another natural question is whether the membership to \mathbf{NM} can be determined in polynomial time; we recall that recognizing a \mathbf{P} -matrix is a co-NP-complete problem [5, 17; 1994-2000]. It would also be important to know whether the Newton-min algorithm solves the linear complementarity problem $\text{LC}(M, q)$ in polynomial time when $M \in \mathbf{NM}$; recall that it does when $M \in \mathbf{M}$ [10; 2004].

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