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# Modelling Martin L of Type Theory in Categories

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## Abstract

We present a model of Martin-L of type theory that includes both dependent products and the identity type. It is based on the category of small categories, with cloven Grothendieck bifibrations used to model dependent types. The identity type is modelled by a path functor that seems to have independent interest from the point of view of homotopy theory. We briefly describe this model’s strengths and limitations.

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## 1. Introduction

The last few years have seen a flurry of activity in the semantics of Martin-L of’s identity type, based on the fruitful relationship with path objects in homotopy theory.

In this paper we present such a model of identity types in Martin-L of type theory which has both desirable features of including dependent products and having introduction-elimination operators that are stable under substitution. Moreover our presentation is very concrete, and calculations in the model are fairly easy; in particular no use whatsoever is made of factorization systems, which have been a favored technique in the semantics of identity types [1, 18].

This work was first presented at the Makkaifest in Montreal in June 2009. Among the other models that were being developped contemporaneously or semi-contemporaneously, one deserves special mention [5]. Not only does the construction of the simplicial path object described in that paper very much resemble ours—this is not a big surprise, since our model is built on small categories, and a category is a special kind of simplicial set—but also one of its important ingredients is what we have called a triangulator in the present paper. One additional interesting aspect of [5] is that an axiomatic framework is presented for path objects, and our own model almost certainly fits that framework, although we have not had the time to check everything. This axiomatic framework also suggests a way to obtain the present model (or something very close to it) by the means of a factorization system.

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Independently of its type-theoretical interest, our model seems to have significant simplifying power for homotopy calculations in categories, a subject we intend to investigate.

## 2. What we are looking for

What follows is the structure we will require on a category  $\mathcal{C}$  get a model of dependent type theory with an identity predicate. There is nothing original here, except that the presentation is optimized for our purposes.

The first categorical models of dependent types [2] relied on a class of maps  $\mathcal{F}$  of  $\mathcal{C}$  with the following properties.

- $\mathcal{C}$  has a terminal object,
- All isos in  $\mathcal{C}$  are in  $\mathcal{F}$ ,
- $\mathcal{F}$  is closed under pullbacks by arbitrary morphisms of  $\mathcal{C}$ .

We will try to use the following notation consistently. We try as much as we can to denote a map  $X$  of  $\mathcal{F}$  as something like

$$X: \overline{X} \longrightarrow A,$$

where the name of its domain is obtained by overlining the map's name, but the codomain can look arbitrary. Given the above along with an arbitrary  $f: B \rightarrow A$  the pullback operation is denoted by

$$\begin{array}{ccc} \overline{f^*X} & \longrightarrow & \overline{X} \\ f^*X \downarrow & & \downarrow X \\ B & \xrightarrow{f} & A \end{array} \tag{1}$$

The members of  $\mathcal{F}$  are most often called *display maps* but we prefer to call them *abstract fibrations*, and we will often just say *fibrations* when the context is clear.

The intuition should be clear and has been used in geometry since the early fifties: given  $X$  as above, it is thought of as dependent family  $(X_a)_{a \in A}$ , and in a concrete category of sets-with-structure the  $X_a$  are just the fibers  $X^{-1}(a)$ . The pullback operation corresponds to substitution: given  $f$  as above, then  $f^*X$  models the family  $(\Phi_{f(b)})_{b \in B}$ .

The maps of  $\mathcal{F}$  that have the terminal object  $\mathbf{1}$  as codomains correspond to ordinary, non-dependent types. In many models all maps to the terminal are in  $\mathcal{F}$ , but this does not have to be the case in general. Notice that since syntactic entities are built by an inductive process that starts with non-dependent types, the only objects  $A$  of  $\mathcal{C}$  that appear in the interpretation of a syntactical system are those for which there is a chain  $A \rightarrow \cdot \rightarrow \cdots \rightarrow \mathbf{1}$  of display maps to the terminal object. Abstract fibrations whose codomain is a terminal will be just

denoted by the source of the domain, since they just correspond to objects of  $\mathcal{C}$  and the overline notation become cumbersome.

When we write  $\mathcal{F}$  we usually think of this class of maps as a category, where a morphism between two abstract fibrations is a commuting square. Naturally the requirement of stability under pullbacks means that the composite

$$\mathcal{F} \longrightarrow \mathcal{C}^{\mathbf{2}} \xrightarrow{\text{Cod}} \mathcal{C}$$

of the inclusion functor with the codomain functor is a “large” Grothendieck fibration. In this context it is natural to call the maps of abstract fibrations that are pullback squares *Cartesian* maps or squares. In the ordinary world of independent type, the categorical version of a unary type constructor is just an endofunctor on the class of objects of the modelling category. In the world of dependent types, a unary type constructor is an endofunctor on the category of fibrations and cartesian maps. A type constructor which is a covariant functor is an endofunctor on the category of fibrations and all squares which also sends cartesian maps to cartesian maps. A contravariant type constructor is something a little more elaborate.

To get a completely formalized interpretation of type theory it should be required that the pullback operation be functorial, instead of pseudo-functorial as is the case for ordinary pullbacks in a category. But this requirement is completely independent of the rest and can always be obtained by massaging the target category properly [6], and no further mention of this condition will be made in this paper.

The pair  $(\mathcal{C}, \mathcal{F})$  is said to have *dependent products* when the following further condition is obeyed.

- For any  $X \in \mathcal{F}$  the pullback functor  $X^*$  has a right adjoint, which we denote  $\Pi_X$ , with the Beck-Chevalley condition holding.

Let us recall the Beck-Chevalley condition: take an arbitrary pullback square of fibrations, as in Equation (1), denoting the upper horizontal arrow by  $h$ , and let  $Y: \bar{Y} \rightarrow \bar{X}$  be another fibration. Beck-Chevalley means that the natural morphism

$$f^*(\Pi_X Y) \longrightarrow \Pi_{f^*X}(h^*Y)$$

obtained by

$$\frac{\frac{h^*X^*(\Pi_X Y) \xrightarrow{h^*\epsilon} h^*Y}{(f^*X)^* f^*(\Pi_X Y) \longrightarrow h^*Y}}{f^*(\Pi_X Y) \longrightarrow \Pi_{f^*X}(h^*Y)}$$

has to be an isomorphism. This allows us to model the operator  $\Pi$  from Martin-Löf type theory.

The Beck-Chevalley condition is something which is completely invisible in syntax, but has to be made explicit in a categorical context if we want a real correspondence between the two.

The structure above is very useful, but not all models of dependent type theory fit into it. We need to add a little more. In general we will model types by *pairs*  $(X, \Phi)$ , where  $X$  is an abstract fibration just as above, and  $\Phi$  a little extra something which we can call a *structure*. Since for us the base category  $\mathcal{C}$  will always be  $Cat$ , and the abstract fibrations will be some kinds of Grothendieck fibrations between small categories (actually Grothendieck *bifibrations*) we can think of a structure  $\Phi$  in this context as a *cleavage* for the fibration  $X$ , an algebraic structure which gives “constructive witness” to the property of being a fibration.

By defining a map between two pairs  $(Y, \Psi) \rightarrow (X, \Phi)$  as just a commutative square, (i.e., ignoring the structures) we get a category  $\mathcal{S}$  and a forgetful functor  $\mathcal{S} \rightarrow \mathcal{F}$  which is obviously an equivalence. We require that  $\mathcal{S}$  contain a specific subcategory whose maps we call *cartesian*, with the following important property:

- Given a pullback square as below and a structure  $\Phi$  for  $X$

$$\begin{array}{ccc}
 \bar{Y} & \xrightarrow{g} & \bar{X} \\
 Y \downarrow & & \downarrow X \\
 B & \xrightarrow{f} & A
 \end{array} \tag{2}$$

then there is a unique structure  $f^*\Phi$  on  $Y$  that makes the pair  $(f, g)$  a cartesian map  $(Y, f^*\Phi) \rightarrow (X, \Phi)$ .

An immediate consequence of this property is that the composite  $\mathcal{S} \rightarrow \mathcal{F} \rightarrow \mathcal{C}$  is a “large” Grothendieck fibration too, and that  $\mathcal{S} \rightarrow \mathcal{F}$  is a cartesian functor of fibrations, which is a discrete fibration when restricted to cartesian maps, because of the uniqueness condition on pullbacks of structures. Any reader who is familiar with the standard semantical technology of dependent types should be able to convert the above prescription into a *full comprehension category* [8] or a *type category* [14]. Our main departure from these standard approaches has to do with viewpoint and notation: we consider types primarily as *maps*, but equipped with *extra structure*, which we have written so far as something like  $(X, \Phi)$ . But since it will happen very seldom that we have to consider more than two structures on the same map  $X$ , we will tend to write the structure as a decorated version  $\underline{X}$  of the map  $X$ . This is a common form of abuse of notation.

The more standard approach in the literature is to consider the objects of  $\mathcal{S}$  as the primary objects of interest, and name their associated underlying maps by applying to them additional notation for the projection functor  $\mathcal{S} \rightarrow \mathcal{C}^2$ . Given  $(X, \Phi): \bar{X} \rightarrow A$  we will say that  $(X, \Phi)$  is a *type in context (or base) A*.

Thus, we can say that we are dealing with three kinds of maps, that more or less live in  $\mathcal{C}$ . There are ordinary maps, fibrations, and fibrations-with-a-chosen-structure, i.e., types. All three can coexist in the same diagram. The notion of commutation for a diagram involving types just coincides with that of

their underlying fibrations. The only additional concept which has to be added for dealing with types is that of a “pullback square” for which two parallel maps are types.

To reduce clutter, in some diagrams we will notate a type of the form  $(X, \underline{X}): \bar{X} \rightarrow A$  with just a thick arrow:  $X: \bar{X} \twoheadrightarrow A$ .

Now, the definition of dependent products has to be adapted to this more general context. It just amounts to requiring that, given

$$\bar{Y} \xrightarrow{(Y, \Psi)} \bar{X} \xrightarrow{(X, \Phi)} A$$

the adjoint on fibrations  $\Pi_X Y$  defined above exists, and that there is a structure  $\Pi_\Phi \Psi$  obtained from  $\Phi, sst$  that makes  $(\Pi_X Y, \Pi_\Phi \Psi)$  a type.

### 2.1. Path objects and identity types

Martin-Löf’s syntactic rules for identity type follows the standard pattern of natural deduction: one type constructor with the term corresponding to its introduction rule, and one term constructor for the elimination rule of the type. These requirement, formulated in the language presented above, are as follows. We want a type constructor that will map every type  $(X, \underline{X})$  of  $\mathcal{S}$ , with  $X: \bar{X} \rightarrow A$  to a type  $(\mathbf{P}X, \underline{\mathbf{P}X})$  where

$$\overline{\mathbf{P}X} \xrightarrow{\mathbf{P}X} X \times_A X$$

is also denoted sometimes

$$\overline{\mathbf{P}X} \xrightarrow{\langle \partial_0 X, \partial_1 X \rangle} X \times_A X$$

which is equipped with a map  $\iota X$  for the introduction rule

$$\begin{array}{ccc} & \overline{\mathbf{P}X} & \\ \iota X \nearrow & & \downarrow \mathbf{P}X \\ X & \xrightarrow{\Delta} & X \times_A X \end{array}$$

making the triangle above commute. This constructor is required to be stable under pullbacks in the following sense: let  $f: B \rightarrow A$  in  $\mathcal{C}$  be any map, and  $g: \bar{f}^* \bar{X} \rightarrow \bar{X}$  the top map of the pullback square, calling  $g \times_X g: \bar{f}^* X \times_B \bar{f}^* X \rightarrow X \times_A X$  (for lack of a better name) the obvious “doubled” version of that map. Then the triangle associated to the type  $\mathbf{P}f^* X$  (i.e.,  $(\mathbf{P}f^* X, \underline{\mathbf{P}f^* X})$ , along with  $\Delta_{f^* X}, r_{f^* X}$ ) is the pullback of the triangle pictured above by  $g \times_X g$  (it is easy to see that the diagonal is always stable by such pullbacks.)

It will turn out that in our model, given a fibration with structure  $(X, \Phi)$  the pair  $(\mathbf{P}X, \underline{\mathbf{P}X})$  does not actually depend on  $\Phi$ , only on  $X$ . This simplifies notation considerably and makes our abuses of notation not that abusive.

The type constructor  $\mathbf{P}$  is actually covariant functorial, but the additional structure can be obtained from the rest of the axioms we are presenting here.

The other ingredient is a term constructor  $J$  for elimination. Given a type  $(X, \underline{X}): \overline{X} \rightarrow A$ , the map  $J_X(Z, t)$  is defined for every type  $(Z, \underline{Z})$  where  $Z: \overline{Z} \rightarrow \overline{\mathbf{P}X}$  and  $t: X \rightarrow Z$  makes the left triangle below commute.

$$\begin{array}{ccc}
 & \overline{Z} & \\
 & \nearrow t & \downarrow z \\
 \overline{X} & \xrightarrow{\quad r_X \quad} & \overline{\mathbf{P}X}
 \end{array}
 \quad J_X(Z, t)
 \tag{3}$$

Moreover  $J_X(Z, t)$  is an extension of  $t$ , i.e.,  $J_X(Z, t) \circ r_X = t$ , and it is a section of  $Z$ , i.e.,  $Z \circ J_X(Z, t)$  is the identity. This constructor is also required to be stable under change of base: let  $f: B \rightarrow A$  be any map and from it construct  $g$  and  $g \times_X g$  as just above. The latter defines another map  $h: \overline{\mathbf{P}f^*X} \rightarrow \overline{\mathbf{P}X}$  by pullback. Then given  $Z, t$  also as just above,

$$\begin{array}{ccccc}
 \overline{h^*Z} & \xrightarrow{\quad} & \overline{Z} & & \\
 \uparrow & \swarrow t' & \nearrow t & & \\
 \overline{h^*Z} & \xrightarrow{\quad} & \overline{f^*X} & \xrightarrow{\quad g \quad} & \overline{X} \\
 \downarrow J' & \searrow r^{f^*X} & \downarrow r_X & & \downarrow z \\
 \overline{\mathbf{P}f^*X} & \xrightarrow{\quad} & \overline{\mathbf{P}X} & \xrightarrow{\quad h \quad} & \overline{\mathbf{P}X}
 \end{array}
 \quad J_X(Z, t)$$

we can construct the pullback of  $Z$  by  $h$  and define the maps  $t'$  and  $J'$  as the result of “pulling back”  $t$  and  $J_X(Z, t)$  along that square. The stability requirement is that

$$J_{f^*X}(h^*Z, t') = J' .$$

### 3. Our Fibrations

As we have said before, our base category  $\mathcal{C}$  will be the category  $Cat$  of small categories. The abstract fibrations will be Grothendieck bifibrations, i.e., functors that are both Grothendieck fibrations and opfibrations (fibrations on the dual of the categories involved). A choice of structure of a fibration will be a *cleavage* i.e., a choice of cartesian arrows and co-cartesian arrows for all maps involved. Let us say a little more about what we mean and fix the notation. Let  $X: \overline{X} \rightarrow A$  be a bifibration. Given objects  $x \in \overline{X}$  and  $a \in A$  as is customary we say  $x$  is above  $a$  to mean  $X(x) = a$ , and the same goes with maps. To have a chosen bifibration structure  $\Phi$  on  $X$  is to have both a fibration half:

- for every pair  $m, x$  where  $m: b \rightarrow a$  is a map in  $A$  and  $x$  an object of  $\overline{X}$  above  $a$ , a map

$$m^*x \xrightarrow{\Phi_m^*(x)} x$$

above  $m$  that has the property that

- for every  $n: c \rightarrow b$  in  $A$  and every  $r: z \rightarrow x$  which is above  $mn$ , there is a unique

$$z \xrightarrow{\Phi_m^*(n;r)} m^*x$$

above  $n$  that makes the triangle in  $\overline{X}$  commute.

and an opfibration half

- for every pair  $m, x$  where  $m: a \rightarrow b$  is a map in  $A$  and  $x$  an object of  $\overline{X}$  above  $a$ , a map

$$x \xrightarrow{\Phi_*^m(x)} m_*x$$

such that

- for every  $n: b \rightarrow c$  in  $A$  and every  $r: x \rightarrow z$  which is above  $m \circ n$ , there is a unique

$$m_*x \xrightarrow{\Phi_*^m(n;r)} z$$

above  $n$  making the triangle in  $\overline{X}$  commute.

The reader should take note that if  $m$  above is an identity, the maps  $\Phi_m^*(x)$  and  $\Phi_*^m(x)$  are not required to be identity maps, although they are necessarily isos. This is actually a desirable feature for the semantics. Obviously, the (bi)fibration structure  $\Phi$  is entirely determined by the choices  $\Phi_{(-)}^*(-, -)$ ,  $\Phi_*^{(-)}(-, -)$  of cartesian and cocartesian arrows, and the “filler maps”  $\Phi_{(-)}^{(-)}(-; -)$ ,  $\Phi_*^{(-)}(-; -)$  are not necessary to the definition. But the notation we present for these will be needed; moreover there exists a weaker notion of “semifibration” where they have to be included in the structure data, because the weaker cartesian and cocartesian maps involved do not obey a universal property.

It should be clear on how fibration structures are pulled back: given a type  $(X, \underline{X}): \overline{X} \rightarrow A$  and an arbitrary map  $f: B \rightarrow A$ , we know that an object in the pullback  $f^*\overline{X}$  is a pair  $(b, y)$  where  $b$  is an object of  $B$  and  $y$  is above  $fb$  in  $\overline{X}$ . Then given  $m: a \rightarrow b$  and  $n: b \rightarrow c$ , calling  $\Psi$  the structure obtained by pulling back  $\underline{X}$  by  $f$ , we take

$$m^*(b, y) \xrightarrow{\Psi_m^*(b,y)} (b, y) \quad \text{to be} \quad (a, (fm)^*y) \xrightarrow{(fm, \underline{X}_{fm}^*(y))} (b, y) \quad (4)$$

$$(b, y) \xrightarrow{\Psi_*^m(b,y)} n_*(b, y) \quad \text{to be} \quad (b, y) \xrightarrow{(fn, \underline{X}_{*}^{fn}(y))} (c, (fm)_*y), \quad (5)$$

the same going for the rest of the structure.

The following result was noticed more than twenty years ago by the author, as a corollary of his thesis work [9]. It can also be deduced in the same manner from [16].



**Theorem 1.** *The class  $\mathcal{S}$  of types as defined above has dependent products.*

PROOF. We will only give the main features of the proof. Let  $(X, \underline{X}): \overline{X} \rightarrow A$  and  $(Y, \underline{Y}): \overline{Y} \rightarrow \overline{X}$  be two fibrations along with their defining structures, and let  $(\Pi_X Y, \Phi)$  be the fibration and structure we want to describe. The Beck-Chevalley condition actually forces the definition of the functor  $\Pi_X Y$ . This is because, given a map  $m$  in  $A$ , and also calling  $a: \mathbf{2} \rightarrow A$  the unique map from the arrow category that names it, the Beck-Chevalley condition forces the maps in  $\overline{\Pi_X Y}$  above  $m$  to be in bijective correspondence with the “sections”  $\phi$

$$\begin{array}{ccc}
 & & \overline{Y} \\
 & \nearrow \phi & \downarrow Y \\
 \overline{m^*Y} & \longrightarrow & \overline{X} \\
 \downarrow m^*Y & & \downarrow X \\
 \mathbf{2} & \xrightarrow{m} & A
 \end{array}$$

that make the triangle above commute (the unnamed horizontal map is not necessarily an injection, so the term “section” is not entirely appropriate). The same being true, naturally, for objects, replacing  $\mathbf{2}$  by  $\mathbf{1}$  in that diagram. Thus, we define the objects and maps of  $\overline{\Pi_X Y}$  as such “sections”. We have defined a graph, but to make it a category, we have to use the (bi)fibration structure. Let  $\phi$  and  $\psi$  be composable maps in  $\overline{\Pi_X Y}$ , with  $\phi$  above  $m: a \rightarrow b$  and  $\psi$  above  $n: b \rightarrow c$ . We know that  $\psi\phi$  is a “section”  $(nm)^*\overline{X} \rightarrow \overline{Y}$ , so let  $t$  be a map in  $(nm)^*\overline{X}$ . There are three possible cases that have to be examined, but let us assume that  $t: x \rightarrow z$  is above  $nm$  (the other two are when  $t$  is above an identity, but it is rather obvious how to treat these). We take

$$(\psi\phi)(t) = \psi(\underline{X}_*^m(n; t)) \phi(\underline{X}_*^m(x)) = \psi(\underline{X}_n^*(z)) \phi(\underline{X}_n^*(m; t))$$

the equality of these two values being guaranteed by the presence of the dotted arrow  $\overline{X}_n^*(1_b; \underline{X}_*^m(n; t))$  in the diagram below.

$$\begin{array}{ccccc}
 & & m_*x & & \\
 \underline{X}_*^m(x) \nearrow & & \searrow \underline{X}_*^m(n; t) & & \\
 x & \xrightarrow{t} & z & & \\
 \searrow \underline{X}_n^*(m; t) & & \nearrow \underline{X}_n^*(z) & & \\
 & & n^*z & & 
 \end{array}$$

and the fact that  $\underline{X}_*^m(x) \circ \overline{X}_n^*(1_b; \underline{X}_*^m(n; t)) = \underline{X}_n^*(m; t)$ .

There is still a lot of things that need to be proved in order to show that  $\overline{\Pi_X Y}$  is a category and  $\Pi_X Y$  a functor, but it is rather mechanical. This category structure does not depend on the actual structures  $\underline{X}, \underline{Y}$ , but the structure  $\Phi$

does. In order to show what the latter looks like, let us show how the choice of cocartesian map is done. So let  $m: a \rightarrow b$  be a map in  $A$  and  $\alpha: \overline{a^*X} \rightarrow \overline{Y}$  be a “section” above  $a$ . We want to define  $\Phi_*^m(\alpha): \alpha \rightarrow m_*\alpha$ . First let us describe the codomain  $m_*\alpha$  as a “section”: given an object  $y$  in  $\overline{X}$  above  $y$  we take  $(m_*\alpha)y = (\underline{X}_m^*(y))_*(\alpha(m^*y))$ .

$$\begin{array}{ccc}
 \alpha(\underline{X}_m^*(1_a; s)) & \xrightarrow{\underline{Y}_{\Phi_*^m(\alpha)}(\alpha(m^*y))} & (\underline{X}_m^*(y))_*(\alpha(m^*y)) \\
 \alpha(x) & \searrow^{(\Phi_*^m(\alpha))s} & \\
 & & 
 \end{array}$$
  

$$\begin{array}{ccc}
 \underline{X}_m^*(1_a; s) & \xrightarrow{m^*y} & y \\
 x & \searrow^s & \\
 & & 
 \end{array}
 \tag{6}$$
  

$$\begin{array}{ccc}
 & a & \xrightarrow{m} & b \\
 & \parallel & & \parallel \\
 & a & \xrightarrow{m} & b
 \end{array}$$

The three horizontal arrows of the diagram above shows what is going on. It should be clear how this definition can be extended to compute  $(m_*\alpha)r$  when  $r$  is a map in the fiber above  $b$ . In order to define  $\Phi_*^m(\alpha)$  as a “section” let now  $s$  be a map above  $m$ , which is the missing one of three cases. The diagram shows the value of  $(\Phi_*^m(\alpha))s$  as the composite  $\underline{Y}_{\Phi_*^m(\alpha)}(\alpha(m^*y)) \circ \alpha(\overline{X}_m^*(1_a; s))$ . There is still a lot to do to get a complete proof, in particular making the filler maps explicit, but it is all quite mechanical.

#### 4. The Path Functor

We will first restrict ourselves to non-dependent types, by defining a path endofunctor  $\mathbf{P}$  on  $Cat$ . The additional work required for the general case is comparatively a formality, as we will see. To avoid notational clashes, given  $f: X \rightarrow Z$  in  $Cat$  will denote the action of  $\mathbf{P}$  with parentheses:  $\mathbf{P}(f): \mathbf{P}(X) \rightarrow \mathbf{P}(Z)$ . The functor  $\mathbf{P}$  will actually be obtained by quotienting another endofunctor  $\mathbf{R}$  on  $Cat$ .

So let  $X$  be a small category.

**Definition 1.** An elementary path  $\mathbf{p}$  in  $X$  is a quadruple

$$\mathbf{p} = (\mathbb{1}, \leq, \sqsubseteq, (\mathbf{p}_{x,x'})_{x,x'})$$

where

- $(\mathbb{I}, \leq)$  is a nonempty finite totally ordered set, the before-after order. Thus when  $x \leq y$  we say “ $x$  is before  $y$ ,  $y$  after  $x$ ”, etc. We denote its first element by  $\mathbf{b}$  (the beginning) and its last one by  $\mathbf{e}$  (the end of the path).
- $\sqsubseteq$  is another order structure on  $\mathbb{I}$ , the diagrammatic order, that obey the following condition, in which  $<_{\leq}, <_{\sqsubseteq}$  mean the predecessor relation on  $\leq, \sqsubseteq$  respectively:

if  $x <_{\leq} y$  then either  $x <_{\sqsubseteq} y$  or  $y <_{\sqsubseteq} x$ .

- $(\mathbf{p}_{x,x'})_{x,x' \in \mathbb{I}}$  is a diagram  $(\mathbb{I}, \sqsubseteq) \rightarrow X$ . That is, for every  $x \in \mathbb{I}$  there is an object  $\mathbf{p}_x \in X$  and for every  $x \sqsubseteq x'$  there is  $\mathbf{p}_{x,x'}: \mathbf{p}_x \rightarrow \mathbf{p}_{x'}$ , with the usual functorial identities. In particular  $\mathbf{p}_{x,x}$  is the identity on the object  $\mathbf{p}_x$ .

The *length* of an elementary path  $\mathbf{p}$  is  $\text{Card}(\mathbb{I}_{\mathbf{p}}) - 1$ .

When we deal with several elementary diagrams we use subscripts to distinguish what has to be distinguished, e.g.,  $\mathbb{I}_{\mathbf{p}}, \sqsubseteq_{\mathbf{p}} \dots$

Thus, if  $\leq$  is a total order, we see that  $\sqsubseteq$  has the shape of a zigzag, whose “branches” are totally ordered and coincide with segments of  $\leq$ , each branch of  $\sqsubseteq$  having the induced order from  $\leq$  or its opposite.

This definition is incomplete, because we need to identify two elementary paths  $\mathbf{p}, \mathbf{q}$  that differ only by the way the elements or the indexing sets  $\mathbb{I}_{\mathbf{p}}, \mathbb{I}_{\mathbf{q}}$  are named. Given two arbitrary elementary paths  $\mathbf{p}, \mathbf{q}$ , there is a natural definition of isomorphism between the structures (biposets) defined by the triples  $(\mathbb{I}_{\mathbf{p}}, \leq, \sqsubseteq)$  and  $(\mathbb{I}_{\mathbf{q}}, \leq, \sqsubseteq)$ : it’s just a bijection between  $\mathbb{I}_{\mathbf{p}}, \mathbb{I}_{\mathbf{q}}$  that preserves and reflects both orders. Since  $\leq$  is a total finite order, if an iso  $\alpha: \mathbb{I}_{\mathbf{p}} \rightarrow \mathbb{I}_{\mathbf{q}}$  exists it is unique, and we identify  $\mathbf{p}, \mathbf{q}$  when we have  $\mathbf{q}_{\alpha(x)} = \mathbf{p}_x$  for every  $x \in \mathbb{I}_{\mathbf{p}}$ . We could define the concept of elementary path without having to resort to quotienting by decreeing that  $\mathbb{I}$  is always of the form  $\{0, \dots, n\}$  but this is more complicated because composing paths forces renamings.

**Remark 1.** The terminology before-after reminds one of a progress in time, which is a pretty traditional way of thinking of paths in homotopy theory as in geometry. But a category-theoretical tradition also would like us to call this order the *vertical order*.

Here is an elementary path of length 6, drawn vertically.

$$\begin{array}{c}
\mathbf{p}_b = \mathbf{p}_0 \\
\mathbf{p}_{0,1} \downarrow \\
\mathbf{p}_1 \\
\mathbf{p}_{1,2} \downarrow \\
\mathbf{p}_2 \\
\mathbf{p}_{3,2} \uparrow \\
\mathbf{p}_3 \\
\mathbf{p}_{4,3} \uparrow \\
\mathbf{p}_4 \\
\mathbf{p}_{5,4} \uparrow \\
\mathbf{p}_5 \\
\mathbf{p}_{5,6} \downarrow \\
\mathbf{p}_e = \mathbf{p}_6
\end{array} \tag{7}$$

The  $\leq$  order is read from the top down, and so a down-arrow means that  $\leq, \sqsubseteq$  coincide and an up-arrow the opposite. An elementary path of length zero is just an object of  $X$ ; such a path is different from all the paths  $\mathbf{p}$  all whose  $\mathbf{p}_{x,y}$  are identities, which can have arbitrary length  $n \geq 1$

**Definition 2.** Let  $(\mathbb{I}, \sqsubseteq), (\mathbb{J}, \sqsubseteq)$  be two posets. An ordering on  $\mathbb{I} \triangleleft \mathbb{J}$  is an ordering  $\sqsubseteq$  of the disjoint sum  $\mathbb{I} + \mathbb{J}$  such that

- the restriction of  $\sqsubseteq$  on  $\mathbb{I}, \mathbb{J}$  is exactly  $\sqsubseteq_{\mathbb{I}}, \sqsubseteq_{\mathbb{J}}$ .
- if  $x \in \mathbb{I}, y \in \mathbb{J}$  are related by the  $\sqsubseteq$  order, then  $x \sqsubseteq y$ .

**Proposition 1.** Let  $\mathbb{I}, \mathbb{J}$  be as above and  $\sqsubseteq$  an ordering on  $\mathbb{I} \triangleleft \mathbb{J}$ . By restriction this order determines a relation  $R \subseteq \mathbb{I} \times \mathbb{J}$ , i.e.,  $xRy$  iff  $x \sqsubseteq y$ . This restriction operation determines a bijection between the set of orderings on  $\mathbb{I} \triangleleft \mathbb{J}$  and the set of relations  $R \subseteq \mathbb{I} \times \mathbb{J}$  that are

- left-down-closed:  $xRy, x' \sqsubseteq x$  implies  $x'Ry$ ,
- right-up-closed:  $xRy, y' \sqsupseteq y$  implies  $xRy'$ .

PROOF. Very easy.

Now it is well known that the class of posets along with left-down- and right-up-closed relations form a category, which has been dubbed by Lambek the category of posets and comparisons [11] and is a very special case of the extremely general bimodule/profunctor construction in enriched category theory [12].

In other words, given comparisons  $R \subseteq \mathbb{I} \times \mathbb{J}$  and  $S \subseteq \mathbb{J} \times \mathbb{K}$  it is easy to check that

$$S \circ R = \{ (x, z) \in \mathbb{I} \times \mathbb{K} \mid \text{there exists } y \in \mathbb{J} \text{ with } (x, y) \in R, (y, z) \in S \}$$

is a comparison too, and that for any poset  $\mathbb{I}$  the left-down- and right-up-closure of the identity relation acts as the identity for composition (but not the identity relation itself, since it is not a comparison unless  $\mathbb{I}$  is a discrete poset!).

**Definition 3.** *Given elementary paths  $\mathbf{p}, \mathbf{q}$  in  $X$  we define a premap  $\mathbf{f}: \mathbf{p} \rightarrow \mathbf{q}$  to be a diagram  $\mathbf{f}: (\mathbb{I}_{\mathbf{p}} + \mathbb{I}_{\mathbf{q}}, \sqsubseteq_{\mathbf{f}}) \rightarrow X$ , where  $\sqsubseteq_{\mathbf{f}}$  is an order on  $(\mathbb{I}_{\mathbf{p}}, \sqsubseteq) \triangleleft (\mathbb{I}_{\mathbf{q}}, \sqsubseteq)$ , that obeys the additional conditions*

1.  $\mathbf{p}_b \sqsubseteq \mathbf{q}_b$  and  $\mathbf{p}_e \sqsubseteq \mathbf{q}_e$
2.  $\mathbf{f}$  restricted to  $\mathbb{I}_{\mathbf{p}}, \mathbb{I}_{\mathbf{q}}$  is  $\mathbf{p}, \mathbf{q}$ .

The poset  $(\mathbb{I}_{\mathbf{p}} + \mathbb{I}_{\mathbf{q}}, \sqsubseteq_{\mathbf{f}})$  is called the *shape of  $\mathbf{f}$* . A premap  $\mathbf{f}$  is said to be a *map*, when it obeys in addition the following contractibility conditions:

- a) given  $x_1, x_2, x \in \mathbb{I}_{\mathbf{p}}$  where  $x$  is  $\leq$ -between  $x_1, x_2$ , along with  $y_1 \sqsupseteq x_1, y_2 \sqsupseteq x_2$  in  $\mathbb{I}_{\mathbf{q}}$ , then there exists  $y$   $\leq$ -between  $y_1, y_2$  with  $y \sqsupseteq x$ .
- b) given  $y_1, y_2, y \in \mathbb{I}_{\mathbf{q}}$  where  $y$  is  $\leq$ -between  $y_1, y_2$ , along with  $x_1 \sqsubseteq y_1, x_2 \sqsubseteq y_2$  in  $\mathbb{I}_{\mathbf{p}}$ , then there exists  $x$   $\leq$ -between  $x_1, x_2$  with  $x \sqsubseteq y$ .

The following result is logically independent from the rest of this work but is indicative of its topological contents.

**Theorem 2.** *Let  $\mathbf{f}: \mathbf{p} \rightarrow \mathbf{q}$  be a premap. Then the following are equivalent:*

- $\mathbf{f}$  is a map.
- The poset  $(\mathbb{I}_{\mathbf{p}} + \mathbb{I}_{\mathbf{q}}, \sqsubseteq_{\mathbf{f}})$  viewed as a simplicial set is contractible.

Whence the term “contractibility condition”. The posets  $\mathbb{I}_{\mathbf{p}}, \mathbb{I}_{\mathbf{q}}$  are themselves contractible simplicial sets, since they are linear strings of simplices, each one joined to its one or two neighbors by a single point.

Using Proposition 1 we can get a slightly different definition of map or premap between elementary paths: a premap  $\mathbf{f}: \mathbf{p} \rightarrow \mathbf{q}$  can be seen as a pair  $(G, \mathbf{f})$  where  $G \subseteq \mathbb{I}_{\mathbf{p}} \times \mathbb{I}_{\mathbf{q}}$  (the *graph* of  $\mathbf{f}$ ), where  $\mathbf{f}_{x,y}$  is a map  $\mathbf{p}_x \rightarrow \mathbf{q}_y$  for every  $(x, y) \in G$ , Condition 1 can be rephrased as

- $(b, b)$  and  $(e, e)$  are in  $G$ ,

and Condition 2 along with functoriality of  $\mathbf{f}$  can be rephrased as

- Given  $x, x' \in \mathbf{p}$  and  $y, y' \in \mathbf{q}$  such that  $x' \sqsubseteq x \sqsubseteq y \sqsubseteq y'$  then  $\mathbf{q}_{y,y'} \circ \mathbf{f}_{x,y} \circ \mathbf{p}_{x',x} = \mathbf{f}_{x',y'}$ .

In practice we can make things even simpler and just define a map to be the family  $(\mathbf{f}_{x,y})_{x,y}$  since  $G$  can be deduced as the set of  $(x, y)$  such that  $\mathbf{f}_{x,y}$  is defined. Then for example the first contractibility condition can be rephrased as

given  $\mathbf{f}_{x_1,y_1}, \mathbf{f}_{x_2,y_2}$  and  $x$  which is  $\leq$ -between  $x_1, x_2$  then there is  $\mathbf{f}_{x,y}$  where  $y$  is  $\leq$ -between  $y_1, y_2$ .

**Proposition 2.** *Let  $\mathbb{I}, \mathbb{J}, \mathbb{K}$  be posets and  $R: \mathbb{I} \rightarrow \mathbb{J}, S: \mathbb{J} \rightarrow \mathbb{K}$  be comparisons that both satisfy Conditions 1,2 above (as posets). Then  $S \circ R$  also obeys these conditions. This is also true if Conditions a,b hold additionally.*

PROOF. The proof is trivial.

Given maps  $\mathbf{f}: \mathbf{p} \rightarrow \mathbf{q}$  and  $\mathbf{g}: \mathbf{q} \rightarrow \mathbf{r}$  we define the *horizontal composite*, written  $\mathbf{g}\mathbf{f}$  or  $\mathbf{g} \circ \mathbf{f}$ , as the family

$$(\mathbf{g}_{yz} \circ \mathbf{f}_{xy})_{xz} \mid \text{there exists } y \text{ with both } \mathbf{g}_{yz} \text{ and } \mathbf{f}_{xy} \text{ defined.}$$

But we first have to show this is actually a real definition, i.e., ensure that we always have  $g_{yz} \circ f_{xy} = g_{y'z} \circ f_{xy'}$ . So let  $x \in \mathbb{I}_{\mathbf{p}}, y, y' \in \mathbb{I}_{\mathbf{q}}, z \in \mathbb{I}_{\mathbf{r}}$  be such that the situation above happens. Let  $y_0 = y, y_1, \dots, y_{n-1}, y_n = y'$  be a sequence of elements of  $\mathbb{I}_{\mathbf{q}}$  that are all  $\leq$ -between  $y, y'$  and such that we always have  $y_i \sqsubseteq y_{i+1}$  or  $y_i \supseteq y_{i+1}$ . In other words we take all the “peaks” and “valleys” between  $y, y'$ , respecting the  $\leq$ -order. If we choose an arbitrary  $0 \leq i \leq n$  we know because of (the equivalent of) Condition 3 that  $f_{x, y_i}$  is guaranteed to exist, and because of Condition 2 that  $g_{y_i, z}$  is guaranteed to exist. But the fact that  $y_i, y_{i+1}$  are  $\sqsubseteq$ -related guarantees that  $g_{y_i z} \circ f_{xy_i} = g_{y_{i+1} z} \circ f_{xy_{i+1}}$  and this shows  $g_{yz} \circ f_{xy} = g_{y'z} \circ f_{xy'}$  by induction.

Thus we get

**Proposition 3.** *Given a small category  $X$  the set of elementary paths and maps in  $X$ , along with horizontal composition, form a category  $\mathbf{Re}(X)$ .*

The identity on a path  $\mathbf{p}$  is the family  $(\mathbf{p}_{x, x'})_{x \sqsubseteq x'}$ .

Actually, it is more than just a category, but an order-enriched one.

**Definition 4.** *Let  $\mathbf{f}, \mathbf{g}: \mathbf{p} \rightarrow \mathbf{q}$  be maps. We write  $\mathbf{f} \sqsubseteq \mathbf{g}$  if  $\mathbf{g}_{x, y}$  is defined whenever  $\mathbf{f}_{x, y}$  is, and  $\mathbf{g}_{x, y} = \mathbf{f}_{x, y}$ .*

Trivially, this is an order relation, and the operation of horizontal composition is monotone in both variables.

We also have obvious functors  $\partial_0, \partial_1: \mathbf{Re}(X) \rightarrow X$  that take a map of paths  $\mathbf{f}: \mathbf{p} \rightarrow \mathbf{q}$  and restrict it to its endpoints, getting  $\mathbf{f}_b: \mathbf{p}_b \rightarrow \mathbf{q}_b$  and  $\mathbf{f}_e: \mathbf{p}_e \rightarrow \mathbf{q}_e$  respectively. There is also an obvious  $\mathbf{r}X: X \rightarrow \mathbf{Re}X$  which is a section to both  $\partial_0, \partial_1$ , that takes an object to the path of length zero it defines, composed of that object alone.

**Proposition 4.** *The process  $\mathbf{Re}(-)$  defines an endofunctor on  $\mathbf{Cat}$  and  $\partial_0, \partial_1, \mathbf{r}$  are natural.*

PROOF. Given a functor  $f: X \rightarrow Y$ , any morphism  $\mathbf{f}: \mathbf{p} \rightarrow \mathbf{q}$  in  $\mathbf{Re}(X)$  can be seen as a diagram  $\mathbf{f}: (\mathbb{I}_{\mathbf{p}} + \mathbb{I}_{\mathbf{q}}, \sqsubseteq_{\mathbf{f}}) \rightarrow X$  and thus  $f \circ \mathbf{f}$  will be a morphism in  $\mathbf{Re}(Y)$ . Functoriality of the assignment  $\mathbf{f} \mapsto f \circ \mathbf{f}$  is trivial to obtain, as is the naturality of  $\partial_0, \partial_1, \mathbf{r}$ .

4.1. Vertical composition

Let  $\mathbf{p}, \mathbf{p}'$  be two paths such that  $\mathbf{p}_e = \mathbf{p}'_b$ . It is quite obvious how to concatenate the two to get the vertical composite  $\mathbf{p}' * \mathbf{p}$ . This is just a pushout construction that involves the two orders  $\leq, \sqsubseteq$ . This also applies to maps of paths: given  $\mathbf{f}: \mathbf{p} \rightarrow \mathbf{q}$  and  $\mathbf{f}': \mathbf{p}' \rightarrow \mathbf{q}'$  with  $\mathbf{f}_e = \mathbf{f}'_b$  a slightly more involved pushout constructs

$$\mathbf{f}' * \mathbf{f}: \mathbf{p}' * \mathbf{p} \longrightarrow \mathbf{q}' * \mathbf{q}.$$

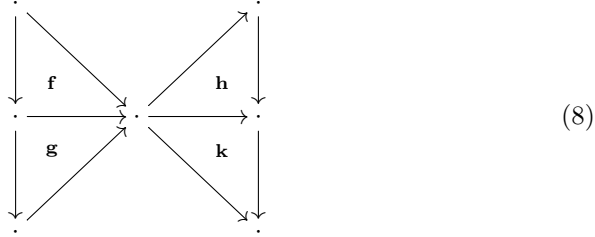
**Proposition 5.** *We always have*

$$(\mathbf{k} \circ \mathbf{h}) * (\mathbf{g} \circ \mathbf{f}) \subseteq (\mathbf{k} * \mathbf{g}) \circ (\mathbf{h} * \mathbf{f})$$

whenever  $\mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{k}$  are four maps of path for which the above is defined.

PROOF. Easy.

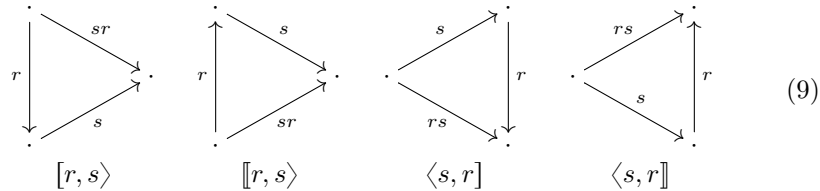
This is not an equation, and here is a counterexample:



It is easy to see that in  $(\mathbf{k} * \mathbf{g}) \circ (\mathbf{h} * \mathbf{f})$  there is an arrow from bottom left to top right of the resulting square which is not in  $(\mathbf{k} \circ \mathbf{h}) * (\mathbf{g} \circ \mathbf{f})$ .

Thus vertical composition is not a (bi)functorial operation. It is only lax-functorial when we consider the  $\subseteq$  enrichment. Nonetheless it is associative on the nose and has a real unit, and can be considered as a functorial binary operation over the category of graphs instead of  $Cat$ .

We need a notation for triangles like those we've just used. First, given a map  $s$  in a category  $X$  we write  $[s]$  for the path of length 1 that has this unique map, pointing in its natural direction (downwards) and  $\llbracket s \rrbracket$  for its reverse, where  $s$  points upwards. Then the four "elementary triangles" are as follows:



This notation should be easy to memorize. Finally given an object  $x$  we write  $\langle x \rangle$  for the one-object path it defines, of length zero, and given  $s: x \rightarrow y$  in

$X$  we take  $\langle s \rangle: \langle x \rangle \rightarrow \langle y \rangle$  to be the corresponding map of paths. Thus the operation  $\langle - \rangle$  is  $\mathfrak{r}X$  and furnishes the identities for vertical composition.

These four triangles, along with the vertical identities, generate the whole of  $\mathfrak{Re}(X)$ , in a sense which has to be made explicit and is rather nontrivial. A special and important case of this is how horizontal identities are obtained. Let  $s: x \rightarrow y$  be a map in  $X$ . Then the identity maps of  $[s], \llbracket s \rrbracket$  are

$$[s, 1_y] * \langle 1_x, s \rangle \quad \text{and} \quad \langle s, 1_x \rangle * \llbracket s, 1_y \rrbracket$$

respectively.

**Remark 2.** In what follows we will write  $[s], \llbracket s \rrbracket$  for these two identity maps. In other words we will use the objects-are-identity-maps definitional paradigm when dealing with horizontal composition, which simplifies notation considerably.

Let us show that the functor  $\langle \partial_0, \partial_1 \rangle: \mathfrak{Re}(X) \rightarrow X \times X$  has very fibration-like properties. First given  $(x, x') \in X \times X$ , let us say that  $\mathbf{p}$  is above  $(x, x')$  to mean that a path  $\mathbf{p}$  is such that  $\mathbf{p}_b = x, \mathbf{p}_e = x'$ . Given  $(s, s'): (x, x') \rightarrow (y, y')$  and  $\mathbf{p}$  above  $(x, x')$  we define

$$(s, s')_* \mathbf{p} = [s'] * \mathbf{p} * [s] \quad (10)$$

$$\underline{\mathfrak{Re}}_*^{(s, s')}(\mathbf{p}) = [1_{x'}, s'] * 1_{\mathbf{p}} * \llbracket 1_x, s \rrbracket : \mathbf{p} \longrightarrow (s, s')_* \mathbf{p} \quad (11)$$

and given  $\mathbf{q}$  above  $(y, y')$  we define

$$(s, s')^* \mathbf{q} = \llbracket s' \rrbracket * \mathbf{q} * [s] \quad (12)$$

$$\underline{\mathfrak{Re}}_{(s, s')}^*(\mathbf{q}) = \langle 1_{y'}, s' \rangle * 1_{\mathbf{q}} * \langle 1_y, s \rangle : (s, s')^* \mathbf{q} \longrightarrow \mathbf{q}. \quad (13)$$

These maps do act a lot like (co)cartesians. For instance:

**Proposition 6.** Let  $\mathbf{p}$  and  $(s, s')$  be as above, and let  $(r, r'): (y, y') \rightarrow (z, z')$  be in  $X$  and  $\mathbf{f}$  be above  $(rs, r's')$ . Then

$$[s', r'] * \mathbf{f} * \llbracket s, r \rrbracket \circ \underline{\mathfrak{Re}}_*^{(s, s')}(\mathbf{p}) = \mathbf{f}.$$

Thus we can define  $\underline{\mathfrak{Re}}_*^{(s, s')}((r, r'); \mathbf{f}) = [s', r'] * \mathbf{f} * \llbracket s, r \rrbracket$ . But this “filler” map is not uniquely defined in general. For a counterexample, suppose that  $s, s'$  are isos. Then

$$\langle s^{-1}, s \rangle * [s, s^{-1}] * 1_{\mathbf{p}} * \llbracket s, s^{-1} \rrbracket * \langle s^{-1}, s \rangle \circ \underline{\mathfrak{Re}}_*^{(s, s')}(\mathbf{p}) = \underline{\mathfrak{Re}}_*^{(s, s')}(\mathbf{p}).$$

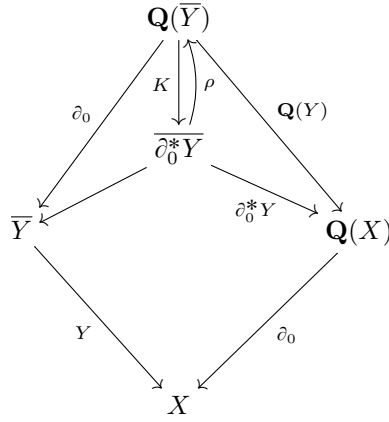
But  $\langle s^{-1}, s \rangle * [s, s^{-1}] * 1_{\mathbf{p}} * \llbracket s, s^{-1} \rrbracket * \langle s^{-1}, s \rangle$  is not the identity (which is also  $\underline{\mathfrak{Re}}_*^{(s, s')}((r, r'); \mathbf{f})$ ) because it contains two extra maps; this is very easy to see. Thus  $\underline{\mathfrak{Re}}_*^{(s, s')}(\mathbf{p})$  is not a real cocartesian in general and we do not get a standard bifibration, but something a little more general. Characterizing these generalized fibrations is a question that reaches beyond the scope of the present work.



**Definition 5.** We define  $\mathbf{P}(X)$  to be the quotient of  $\mathbf{Re}(X)$  under the symmetric closure of the order enrichment  $\subseteq$ . There is an obvious functor  $\mathbf{Re}(X) \rightarrow \mathbf{P}(X)$  and obvious projection  $\mathbf{P}(X) \rightarrow X \times X$  making the obvious triangle commute.

The objects of  $\mathbf{P}(X)$  coincide with those of  $\mathbf{Re}(X)$ , they are elementary paths. The maps are equivalence classes of maps of elementary maps. These classes can be quite complex, and there is no hope of finding normal forms in general. But notice that the class of a triangle is always a singleton, as is the class of a vertical unit. Notice also that Inequality (5) is now an equation in the quotient, and thus vertical composition in  $\mathbf{P}(X)$  is bifunctorial. Thus  $\mathbf{P}(X)$  is an object of maps for an internal category whose object of objects is  $X$ . We have constructed a natural double category structure on every small category.

**Definition 6.** Let  $\mathbf{Q}$  stand for one of  $\mathbf{Re}, \mathbf{P}$  and  $Y: \bar{Y} \rightarrow X$  be in  $\mathbf{Cat}$ . We define a Hurewicz  $\mathbf{Q}$ -action on  $Y$  to be a splitting  $\rho$  of the map  $K: \mathbf{Q}(\bar{Y}) \rightarrow \overline{\partial_0^* Y}$  determined by the universal property of pullback.



Thus a map in  $\overline{\partial_0^* Y}$  is a pair  $(\mathbf{f}, s)$  where  $\mathbf{f}$  is a map in  $\mathbf{Q}(X)$ , (either an elementary path in  $X$  or a set of such parallel paths) and  $s \in \bar{Y}$  is above  $\partial_0 X(\mathbf{f})$ . Let us write  $\rho(\mathbf{f}, s)$  as  $\mathbf{f} \star s$ . In addition to being a functor,  $\rho$  is required to respect the vertical operations in the following sense: for every  $m$  in  $X$  and  $s \in \bar{Y}$  above  $m$

$$\langle m \rangle \star s = \langle s \rangle \quad (14)$$

and for every three maps  $\mathbf{f}, \mathbf{g}$  in  $\mathbf{Q}(X)$ ,  $s$  in  $\bar{Y}$  such that  $\mathbf{g} * \mathbf{f}$  and  $\mathbf{f} \star s$  are defined, we have

$$(\mathbf{g} * \mathbf{f}) \star s = (\mathbf{g} \star \partial_1(\mathbf{f} \star s)) * (\mathbf{f} \star s). \quad (15)$$

Let us for a moment assume that  $\mathbf{Q} = \mathbf{Re}$ . Given  $\mathbf{f}$  and  $s$  such that  $\mathbf{f} \star s$  is defined, we know that  $\mathbf{f} = \mathbf{Q}(Y)(\mathbf{f} \star s)$ , and since  $\mathbf{Q}(Y)$  preserves shapes,  $\mathbf{f} \star s$  has exactly the same shape as  $s$ . Also, since horizontal identity maps in  $\overline{\partial_0^* Y}$

are pairs of the form  $([m], 1_x)$ , the path  $[m] \star 1_x$  is a horizontal identity, i.e., it is of the form  $[r]$  for a map  $r$  in  $\bar{Y}$ . Let us denote this map by

$$\Phi\rho_*^m(x): x \longrightarrow m_*x \quad \text{i.e.,} \quad [m] \star 1_x = [\Phi\rho_*^m(x)] . \quad (16)$$

Assuming that  $m: a \rightarrow b$ , we know that

$$[m] = [m, 1_b] \star \langle m, 1_a \rangle \quad \text{and} \quad [\Phi\rho_*^m(x)] = [\Phi\rho_*^m(x), 1_{m_*x}] \star \langle \Phi\rho_*^m(x), 1_x \rangle$$

and plugging these back in Equation (16)

$$([m, 1_b] \star \langle m, 1_a \rangle) \star 1_x = [\Phi\rho_*^m(x), 1_{m_*x}] \star \langle \Phi\rho_*^m(x), 1_x \rangle \quad (17)$$

but Equation (15) also tells us

$$([m, 1_b] \star \langle m, 1_a \rangle) \star 1_x = ([m, 1_b] \star \partial_1(\langle m, 1_a \rangle \star 1_x)) \star (\langle m, 1_a \rangle \star 1_x) . \quad (18)$$

Due to the unique way of decomposing the shape of such paths into triangles, we can deduce from the equality between the right sides of Equations (17,18) that

$$\langle m, 1_a \rangle \star 1_x = \langle \Phi\rho_*^m(x), 1_x \rangle \quad \text{and} \quad (19)$$

$$[m, 1_b] \star \partial_1(\langle m, 1_a \rangle \star 1_x) = [\Phi\rho_*^m(x), 1_{m_*x}] \quad (20)$$

and since  $\partial_1(\langle m, 1_a \rangle \star 1_x) = \partial_1\langle \Phi\rho_*^m(x), 1_x \rangle = \Phi\rho_*^m(x)$  we can rewrite Equation (20) as

$$[m, 1_b] \star \Phi\rho_*^m(x) = [\Phi\rho_*^m(x), 1_{m_*x}] . \quad (21)$$

Let now  $n: b \rightarrow c$  be in  $X$  and  $r: x \rightarrow z$  be above  $nm$ . The elementary path  $[m, n] \star r$  is a triangle, so let  $t: x \rightarrow y$  and  $\Phi\rho_*^m(n; r): y \rightarrow z$  be the maps that define this triangle, i.e.,

$$[m, n] \star r = [t, \Phi\rho_*^m(n; r)] .$$

Since  $[m, n]$  is a map  $[m] \rightarrow \langle c \rangle$  in  $\mathbf{Re}(X)$  and  $\rho$  is functorial, we have

$$[m] \star 1_x = [\Phi\rho_*^m(x)] \xrightarrow{[m, n] \star r} \langle z \rangle$$

and this shows

$$t = \Phi\rho_*^m(x) \quad \text{and} \quad \Phi\rho_*^m(n; r) \circ \Phi\rho_*^m(x) = r .$$

Now take  $u: m_*x \rightarrow z$  any map such that  $u \circ \Phi\rho_*^m(x) = r$ . Obviously  $\langle n \rangle \circ [m, 1_b] = [m, n]$ . By functoriality of the action  $\rho$ , we also have

$$\begin{array}{ccccc} & x & \xrightarrow{\Phi\rho_*^m(x)} & m_*x & \xrightarrow{u} & z \\ \Phi\rho_*^m(x) \swarrow & & \searrow [m, 1_b] \star \Phi\rho_*^m(x) & & \searrow \langle n \rangle \star u & \\ m_*x & & & & & \\ & & & & & \\ & a & \xrightarrow{m} & b & \xrightarrow{n} & c \\ m \swarrow & & \searrow [m, 1_b] & & \searrow \langle n \rangle & \\ b & & & & & \end{array}$$

$$(\langle n \rangle \star u) \circ ([m, 1_b] \star \Phi \rho_*^m(x)) = (\langle n \rangle \circ [m, 1_b]) \star (u \circ \Phi \rho_*^m(x)) = [m, n] \star r$$

But by Equation (21) the left side of this is  $\langle u \rangle \circ [\Phi \rho_*^m(x), 1_{m_*x}]$  which is  $[\Phi \rho_*^m(x), u]$  trivially and we have seen above that the right side is  $[\Phi \rho_*^m(x), \Phi \rho_*^m(n; r)]$ . Thus  $u = \Phi \rho_*^m(n; r)$  and we have proved that  $\Phi \rho_*^m(x)$  is a cocartesian arrow.

This can be done dually; that is, if  $m: a \rightarrow b$  is in  $X$  and  $y$  in  $\bar{Y}$  above  $b$ , then defining  $\Phi \rho_m^*(y): m^*y \rightarrow y$  as the unique  $s$  such that  $\llbracket s \rrbracket = \llbracket m \rrbracket \star 1_y$ , we can show that  $\Phi \rho_m^*(y): m^*y \rightarrow y$  is a Cartesian arrow.

Let now  $\mathbf{Q} = \mathbf{P}$ . The argument above only needs slight modifications to be applicable. First, given an arbitrary small category  $Z$ , recall that the equivalence class in  $\mathbf{P}(Z)$  of a triangle or a path of length zero is a singleton, and notice that the only elementary paths in this argument that do not belong to the above are those of the form  $[u]$  or  $\llbracket u \rrbracket$  for a morphism  $u$  of  $Z$ . But it is easy to see that such a horizontal unit in  $\mathbf{P}(Z)$  is either the singleton  $\{[u]\}$ ,  $\{\llbracket u \rrbracket\}$  or a doubleton, the latter case happening only if  $u$  is an isomorphism. Thus it is easy to see everything we have said also holds in that context.

So we conclude

**Proposition 7.** *Let  $Y: \bar{Y} \rightarrow X$  be a functor and  $\mathbf{Q}$  be either  $\mathbf{R}$  or  $\mathbf{P}$ . Every Hurewicz  $\mathbf{Q}$ -action  $\rho$  on  $Y$  determines a bifibration structure  $\Phi \rho$  on  $Y$ .*

This has a converse

**Theorem 3.** *The correspondence  $\rho \mapsto \Phi \rho$  is bijective.*

An immediate consequence is that there is also a bijective correspondence between Hurewicz  $\mathbf{R}$ -actions and Hurewicz  $\mathbf{P}$ -actions on  $Y$ .

**Theorem 4.** *Given a small category  $X$  then  $\mathbf{P}(X) \rightarrow X \times X$  is a bifibration.*

Thus, the “almost cartesian” and “almost cocartesian” maps in  $\mathbf{R}(X)$  become truly cartesian and cocartesian in the quotient  $\mathbf{P}(X)$ .

The proofs of these results [10] is rather more technical and depends on a close analysis of the order enrichment  $\subseteq$  on  $\mathbf{R}$ .

**Remark 3.** Our functor  $\mathbf{P}$  almost certainly fits in the axiomatic framework of a *path object category* presented in [5]. But the definition of fibration structure used there gives rise to a category of types which is equivalent, but not isomorphic to the one we present. The reason is that we insist on  $\mathbf{P}$ -actions to respect the vertical structure on  $\mathbf{P}$ , which is not the case in [5]. It is not hard to see, plugging our  $\mathbf{P}$  in that framework, that the abstract fibrations that will result, which are called (cloven)  $\mathcal{R}$ -maps, will be Grothendieck bifibrations, just as expected. But a fibration *structure* in that framework (a cleavage) involves a new, arbitrary choice of a cartesian or cocartesian arrow for a map every time it appears in a path, while our concept of Hurewicz action is an ordinary cleavage, a single choice of (co)cartesian which is used consistently and repeatedly with paths to define the action of  $\mathbf{P}$ . Thus there are many more cloven  $\mathcal{R}$ -maps than there are cloven bifibrations for us.

#### 4.2. The fibered path functor

An elementary path is said to be *constant* if it has length zero or all its component maps are identities. This obviously defines a full subfunctor  $\mathbf{Ce}$  for which the inclusion map  $\mathbf{Ce} \rightarrow \mathbf{Re}$  obviously equalizes the two projections  $\partial_0, \partial_1$ , and we call the composite  $\partial$ . By definition the transformation  $r$  factors through that inclusion:

$$\begin{array}{ccc} \mathbf{Ce}(X) & \longrightarrow & \mathbf{Re}(X) \\ & \searrow \partial & \swarrow \partial_0 \\ & & X \\ & & \swarrow \partial_1 \\ & & X \end{array} \qquad \begin{array}{ccc} \mathbf{Ce}(X) & \longrightarrow & \mathbf{Re}(X) \\ & \searrow r & \swarrow r \\ & & X \end{array}$$

Notice that given a category  $X$  and constant paths  $\mathbf{p}, \mathbf{q}$ , then given a map  $\mathbf{f}: \mathbf{p} \rightarrow \mathbf{q}$  we see that every component  $\mathbf{f}_{x,y}$  is the same map of  $X$ . Moreover every such  $\mathbf{f}$  has an  $\subseteq$ -maximal element above it, whose shape is the full product  $\mathbb{1}_{\mathbf{p}} \times \mathbb{1}_{\mathbf{q}}$ . Thus when we take the associated quotient  $\mathbf{Ce}X \rightarrow \mathbf{C}X$  not only do we also get a subfunctor of  $\mathbf{P}$ , whose inclusion equalizes  $\partial_0, \partial_1$ , but in addition the composite  $\partial$  is an equivalence between  $\mathbf{C}X$  and  $X$ .

$$\begin{array}{ccc} \mathbf{Ce}(X) & \longrightarrow & \mathbf{Re}(X) \\ \downarrow & & \downarrow \\ \mathbf{C}(X) & \longrightarrow & \mathbf{P}(X) \\ & \searrow \partial & \swarrow \partial_0 \\ & & X \\ & & \swarrow \partial_1 \end{array}$$

Let now  $X: \overline{X} \rightarrow A$  be a bifibration. In order to define  $\mathbf{Re}X: \overline{\mathbf{Re}X} \rightarrow X \times_A X$ , we take  $\overline{\mathbf{Re}X} \subseteq \mathbf{Re}(\overline{X})$  as the full subcategory of all elementary paths in  $\overline{X}$  that are mapped by  $X$  to a constant path in  $A$ . This ensures that the inclusion functor followed by the projection  $\langle \partial_0, \partial_1 \rangle: \mathbf{Re}(\overline{X}) \rightarrow \overline{X} \times \overline{X}$  will factor through  $\overline{X} \times_A \overline{X}$ , and this defines the bifibration  $\mathbf{Re}X$ .

$$\begin{array}{ccccc} \overline{X} \times_A \overline{X} & \xrightarrow{\quad} & \overline{X} \times \overline{X} & & \\ \downarrow X \times_A X & \swarrow \mathbf{Re}X & \searrow \langle \partial_0, \partial_1 \rangle & & \\ & \overline{\mathbf{Re}X} & \longrightarrow & \mathbf{Re}(\overline{X}) & \\ & \downarrow i^* \mathbf{Re}(X) & & \downarrow \mathbf{Re}(X) & \\ & \mathbf{Ce}(A) & \xrightarrow{i} & \mathbf{Re}(A) & \\ & \downarrow \partial & & \downarrow \langle \partial_0, \partial_1 \rangle & \\ A & \xrightarrow{\quad} & A \times A & & \\ & \Delta & & & \end{array}$$

This definition says that  $\mathbf{Re}X$  is really “the original  $\mathbf{Re}(-)$  on  $Cat$  applied fiberwise”. In particular it is easy to see that the order enrichment  $\subseteq$  is fiberwise,

i.e., if two parallel maps in  $\overline{\mathbf{Re}X}$  are  $\subseteq$ -related then they are above the same map in  $A$ . Thus, if we define the category  $\overline{\mathbf{P}X}$  by quotienting with that order,  $\mathbf{Re}X$  will factor through that quotient, making the expected triangle commute.

$$\begin{array}{ccc}
\overline{\mathbf{Re}X} & \xrightarrow{\quad} & \overline{\mathbf{P}X} \\
\downarrow \mathbf{Re}X & & \downarrow \mathbf{P}X \\
& X \times_A X &
\end{array}$$

Naturally we could also define  $\mathbf{P}X$  by using the construction above, replacing  $\mathbf{Re}(\overline{X})$  and  $\mathbf{Ce}(\overline{X})$  with  $\mathbf{P}(\overline{X})$  and  $\mathbf{C}(\overline{X})$ , and this shows that  $\mathbf{P}X$  is a fibration.

We are left to show that these two path constructions on maps are stable under change of base/context. This follows trivially from their fiberwise nature.

#### 4.3. The Identity Type.

We now have all machinery necessary to model the Martin-Löf identity type rules. Let  $X$  be a small category. Since  $\langle \partial_0, \partial_1 \rangle: \mathbf{P}(X) \rightarrow X \times X$  is a bifibration, obviously  $\partial_1$  is one too. We already know a structure  $\Psi$  for it: it is “one half” of Equations (10–13). That is, given a path  $\mathbf{p} \in \mathbf{P}(X)$ , whose endpoint  $\mathbf{p}_e = \partial_1 \mathbf{p}$  we call  $y$ , along with  $s: x \rightarrow y$  and  $r: y \rightarrow z$ , we have

$$\begin{array}{ll}
r_* \mathbf{p} = [r] * \mathbf{p} & s^* \mathbf{p} = \llbracket s \rrbracket * \mathbf{p} \\
\Psi_*^r(y) = \langle 1_y, r \rangle * \mathbf{p} & \Psi_*^s(y) = \llbracket s, 1_y \rrbracket * \mathbf{p} .
\end{array} \tag{22}$$

(for simplicity we are using the same notation for the denizens of  $\mathbf{P}$  as we did for those of  $\mathbf{Re}$ ).

**Definition 7.** Let  $X$  be a category and  $Y \subseteq \mathbf{P}(X)$  a full subcategory of its category of paths. A triangulator is a functor  $T: Y \rightarrow \mathbf{P}(\mathbf{P}(X))$  which satisfies the two equations

$$\partial_1(T\mathbf{f}) = \mathbf{f}, \quad \partial_0(T\mathbf{f}) = \langle \partial_0 \mathbf{f} \rangle \tag{23}$$

for any map  $\mathbf{f}$  in  $Y$ .

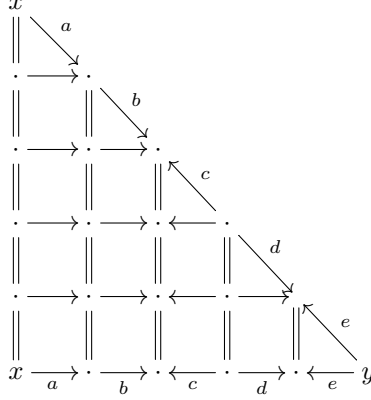
As an example of a triangulator we take

$$T\mathbf{f} = \mathbf{f} \star_{\Psi} \langle \partial_0 \mathbf{f} \rangle, \text{ which is a map in } \mathbf{P}(\mathbf{P}(X)). \tag{24}$$

Let us show what this looks like in practice. if  $\mathbf{f}$  is the path

$$x \xrightarrow{a} \cdot \xrightarrow{b} \cdot \xleftarrow{c} \cdot \xrightarrow{d} \cdot \xleftarrow{e} y$$

then  $T\mathbf{f}$  looks like



where every horizontal combination of squares + one triangle is a map of paths.

Let now  $X: \bar{X} \rightarrow A$  be a fibration,  $(Z, \underline{Z}): \bar{Z} \rightarrow \mathbf{P}\bar{X}$  an arbitrary type, and  $t: \bar{X} \rightarrow \bar{Z}$  be just as in Equation (3), i.e.  $Z(t(s)) = \langle s \rangle$  for every map  $s \in X$ . Let also  $T$  be a triangulator defined on  $\mathbf{P}\bar{X} \subseteq \mathbf{P}(\bar{X})$ . Given a map  $\mathbf{f}$  in  $\mathbf{P}X$  we define

$$(J_X(Z, t)) \mathbf{f} = \partial_1(T\mathbf{f} \star_Z t(\partial_0\mathbf{f})) . \quad (25)$$

This is well defined because  $Z\langle t(\partial_0\mathbf{f}) \rangle = \langle \partial_0\mathbf{f} \rangle = \partial_0(T\mathbf{f})$ . We need to show this is a section of  $Z$ , but

$$\begin{aligned} Z(\partial_1(T\mathbf{f} \star_Z t(\partial_0\mathbf{f}))) &= \partial_1(\mathbf{P}(Z)(T\mathbf{f} \star_Z t(\partial_0\mathbf{f}))) && \text{by naturality of } \partial_1 \\ &= \partial_1(T\mathbf{f}) && \text{by definition of Hur. Action} \\ &= \mathbf{f} && \text{by definition of triangulator.} \end{aligned}$$

We also have to show this is an extension of  $t$ . Let  $\mathbf{f} = \langle s \rangle$  for  $s$  in  $X$ . Notice that

$$T\langle s \rangle = \langle s \rangle \star_{\Psi} \langle \partial_0\langle s \rangle \rangle = \langle s \rangle \star_{\Psi} \langle s \rangle = \langle \langle s \rangle \rangle ,$$

the latter equation because of (14). Then

$$(J_X(Z, t))\langle s \rangle = \partial_1(T\langle s \rangle \star_Z t(\partial_0\langle s \rangle)) = \partial_1(\langle \langle s \rangle \rangle \star_Z t(s)) = \partial_1\langle t(s) \rangle = t(s) .$$

We are left to show that there are triangulators that are stable under change of base/context. But this is quite trivial in the case of the one given by Equation (24).

The action of  $J_X(Z, t)$  is easy to describe on objects. First, start with paths of length one: let us denote the source and target of a map  $s$  in  $\bar{X}$  by  $\mathbf{d}_0s, \mathbf{d}_1s$  respectively. Then it is easy to see that

$$(J_X(Z, t)) [s] = \langle 1, s \rangle_* (t\langle \mathbf{d}_0s \rangle) \quad \text{and} \quad (J_X(Z, t)) \llbracket s \rrbracket = \llbracket s, 1 \rrbracket^* (t\langle \mathbf{d}_1s \rangle) .$$

For longer paths is just use induction. The calculations for maps are a little more involved, naturally.

## 5. Some calculations

Iterating the operator  $\mathbf{P}$  on a fibration  $X: \top X \rightarrow A$  naturally produces a globular object:

$$\dots \quad \overline{\mathbf{P}\mathbf{P}\mathbf{P}X} \xrightarrow[\partial_1 \mathbf{P}\mathbf{P}X]{\partial_0 \mathbf{P}\mathbf{P}X} \overline{\mathbf{P}\mathbf{P}X} \xrightarrow[\partial_1 \mathbf{P}X]{\partial_0 \mathbf{P}X} \overline{\mathbf{P}X} \xrightarrow[\partial_1 X]{\partial_0 X} \overline{X}$$

which is equipped with an “augmentation”  $X$ . Since  $\mathbf{P}$  is equipped with its own internal category structure, and since everything happens in  $Cat$ , this globular structure in our specific model is actually a higher-dimensional cubical category.

In general, given a model of the identity type, a certain measure of “how degenerate” it is given by observing how large  $n$  has to be for the internal graph

$$\overline{\mathbf{P}^n X} \xrightarrow{\mathbf{P}^n X} \overline{\mathbf{P}^{n-1} X} \times_{\overline{\mathbf{P}^{n-2} X}} \overline{\mathbf{P}^{n-1} X} \quad (26)$$

to be degenerate in some sense. Here  $\mathbf{P}^0 X = \overline{X}$ ,  $\mathbf{P}^{-1} X = A$ . For example in the first ever model which was not completely degenerate [7], the equivalent to  $\mathbf{P}^2 X$  is a discrete graph (i.e., isomorphic to the diagonal). Another form of degeneracy is when  $\mathbf{P}^n X$  a relation, i.e., (26) is a monomorphism. A comprehensive taxonomy of these situations is given in [18]. This could be called an “absolute” test of degeneracy; under this test our model comes out with no degeneracy in any dimension.

Another way to measure degeneracy, which could be called “relative”, is to consider the internal higher categorical structure which is deduced from the identity type axioms, and which in general is a kind of weak  $\omega$ -groupoid. There have been several descriptions of the specific kind of weak  $\omega$ -groupoid structure which is obtained in general from identity types [4, 3, 13]. For example, although the inverse operation is always strictly involutive, composition in general is not strictly associative and strict associativity in a model is a form of degeneracy. The only completely nondegenerated model has been constructed so far is the Kan complex model [17, 15].

We want to give an impression of where our model stands in these hierarchies. Intuitively, since it is based on Grothendieck fibrations, which are defined through a universal property (i.e., not *uniquely*, but up to *unique*) isomorphism, relative degeneracy should begin at dimension 3, and this is what indeed happens. In what follows we will show the rudiments of the computations that are necessary to assert this.

In the presence of dependent products, the set of rules we have given for the identity type  $\mathbf{P}X$  associated to a dependent type  $X$  can be generalized to a “parametrized” version as follows: Let  $(\underline{W}\underline{W}): \overline{W} \rightarrow \overline{\mathbf{P}X}$  be a type,  $(\underline{Z}, \underline{Z}): \overline{Z} \rightarrow \overline{W}$  another one, and let now  $t: \overline{\mathbf{P}^*W} \rightarrow \overline{Z}$  be such that  $Z \circ t = h$ ,

where  $h$  is the upper part of the pullback:

$$\begin{array}{ccc}
& & \overline{Z} \\
& & \downarrow Z \\
& & \overline{W} \\
& \nearrow t & \downarrow W \\
\overline{r^*W} & \xrightarrow{h} & \overline{W} \\
\downarrow r^*W & & \downarrow W \\
\overline{X} & \xrightarrow{r} & \overline{\mathbf{P}X} \\
& \searrow \Delta & \downarrow \mathbf{P}X \\
& & \overline{X \times_A X}
\end{array}$$

Then the parametrized version of the identity rules requires that there is a section  $J_X(W; Z, t)$  that extends  $t$ . This generalization is easy to prove using  $\Pi$ ; it is also the way to define an identity type in the absence of  $\Pi$  (the true general definition replaces the single type  $W$  by an arbitrary sequence  $\overline{W}_n \rightarrow \overline{W}_{n-1} \rightarrow \dots \rightarrow \overline{W}_1 \rightarrow \overline{\mathbf{P}X}$  but our seemingly more restricted version is enough for the purposes of this illustrative section, and is equivalent anyway because our model has the sum operator  $\Sigma$ , as the interested reader can show easily).

Let now  $w$  be an object of  $\overline{W}$ , which is above the path  $\mathbf{p}$ . The discussion right above allows us to assume that  $\mathbf{p}$  has length one, since we can use induction to deduce the general case. We know that objects of  $r^*W$  are pairs  $(x, y)$ , where  $x$  is an object of  $\overline{X}$  and  $y$  an object of  $W$  which is above  $\langle x \rangle$ .

It is not hard to see, using the calculations displayed in Equation (6) that

$$J_X(W; Z, t)[s] = \underline{Z}_*^{W_{\langle 1, s \rangle}(z)}(t(\mathbf{d}_0 s, \langle 1, s \rangle^* w))$$

in more detail:

$$\begin{array}{ccc}
t(\mathbf{d}_0 s, \langle 1, s \rangle^* w) & \longrightarrow & \underline{Z}_*^{W_{\langle 1, s \rangle}(z)}(t(\mathbf{d}_0 s, \langle 1, s \rangle^* w)) \\
\langle 1, s \rangle^* w & \xrightarrow{W_{\langle 1, s \rangle}(z)} & w \\
\langle \mathbf{d}_0 s \rangle & \xrightarrow{\langle 1, s \rangle} & [s]
\end{array} \tag{27}$$

and thus symmetrically

$$J_X(W; Z, t)[s] = \underline{Z}_*^{W_{[s, 1]}(z)}(t(\mathbf{d}_1 s, [s, 1]^* w)) .$$

These calculations simplify considerably when  $Z, W$  are types that depend only on  $\overline{X \times_A X}$  (i.e. obtained by pulling back with  $\mathbf{P}X$ ). One important case is the substitution rule for equality. Let  $(Y, \underline{Y}): \overline{Y} \rightarrow \overline{X}$  be a type, and denote by  $Y_0$  the pullback of  $Y$  by the first projection  $\overline{X \times_A X} \rightarrow \overline{X}$  and by  $Y_1$  the pullback of  $Y$  by the second one. Let now  $W = (\mathbf{P}X)^* Y_0$  and  $Z = (\mathbf{P}X \circ W)^* Y_1$ . An



object of  $\overline{W}$  can be presented as a pair  $(\mathbf{q}, y)$  where  $y \in \overline{Y}$  is above  $\partial_0 \mathbf{q}$ , while an object of  $\overline{Z}$  as a triple  $(\mathbf{q}, y, z)$ , where  $(\mathbf{q}, y)$  is just as above and  $z \in Y$  is above  $\partial_1 \mathbf{q}$ . Given maps of paths  $\mathbf{f}: \mathbf{p} \rightarrow \mathbf{q}$  and  $\mathbf{g}: \mathbf{q} \rightarrow \mathbf{r}$  then according to (4) and (5) the cartesian and cocartesian maps associated to  $(\mathbf{q}, y)$  are

$$(\mathbf{p}, (\partial_0 \mathbf{f})^* y) \xrightarrow{(\mathbf{f}, \underline{Y}_{\partial_0 \mathbf{f}}^*(y))} (\mathbf{q}, y) \quad \text{and} \quad (\mathbf{q}, y) \xrightarrow{(\mathbf{g}, \underline{Y}_{\mathbf{g}}^{\partial_1 \mathbf{g}}(y))} (\mathbf{r}, (\mathbf{g} * \mathbf{q}), y)$$

respectively. Much the same goes with the fibration  $Z$ : let now  $(\mathbf{f}, u): (\mathbf{p}, x) \rightarrow (\mathbf{q}, y)$  and  $(\mathbf{g}, r): (\mathbf{q}, y) \rightarrow (\mathbf{r}, w)$  be two maps in  $\overline{W}$ , and  $(\mathbf{q}, y, z)$  an object in  $\overline{Z}$ . The cartesian  $\underline{Z}_{(\mathbf{f}, u)}^*(\mathbf{q}, y, z)$  and cocartesian  $\underline{Z}_{\mathbf{g}}^{(\mathbf{g}, r)}(\mathbf{q}, y, z)$  are

$$(\mathbf{p}, x, (\partial_1 \mathbf{f})^* z) \xrightarrow{(\mathbf{f}, u, \underline{Y}_{\partial_1 \mathbf{f}}^*(z))} (\mathbf{q}, y, z) \quad \text{and} \quad (\mathbf{q}, y, z) \xrightarrow{(\mathbf{g}, r, \underline{Y}_{\mathbf{g}}^{\partial_1 \mathbf{g}}(z))} (\mathbf{r}, w, (\partial_1 \mathbf{g})^* z)$$

respectively.

Now, given the above data, and seeing an object of  $\overline{\mathbf{r}^* \overline{W}}$  as a pair  $(x, y)$  where  $x \in \overline{X}$  and  $y \in \overline{Y}$  is above  $x = \partial_0 \langle x \rangle$ , the substitution rule for equality on the type  $Y$  is obtained directly from  $J_X(W; Z, t)$  when

$$t(x, y) = (\langle x \rangle, y, y) . \tag{28}$$

Choosing an arbitrary map  $s$  in  $\overline{X}$  and object  $([s], y)$  in  $\overline{W}$ , and taking into account the fact that  $\partial_0 \langle 1, s \rangle = 1_{\mathbf{d}_0 s}$ ,  $\partial_1 \langle 1, s \rangle = s$ , we get

$$J_X(W; Z, t)([s], y) = ([s], y, s_*(1^* y))$$

where, in more details, Diagram (27) becomes

$$\begin{array}{ccc} (\langle \mathbf{d}_0 s \rangle, 1^* y, 1^* y) & \xrightarrow{(\langle 1, s \rangle, \underline{Y}_1^*(y), \underline{Y}_{\mathbf{g}}^s(1^* y))} & ([s], y, s_*(1^* y)) \\ (\langle \mathbf{d}_0 s \rangle, 1_{\mathbf{d}_0 s}^* y) & \xrightarrow{(\langle 1, s \rangle, \underline{Y}_1^*(y))} & ([s], y) \\ \langle \mathbf{d}_0 s \rangle & \xrightarrow{\langle 1, s \rangle} & [s] \end{array}$$

and symmetrically

$$J_X(W; Z, t)([s], y) = ([s], y, s^*(1_* y)) .$$

Now standard calculations tell us that the internal 1-composition of the weak  $\omega$ -groupoid  $\mathbf{P}X$  is obtained by substituting for  $(Y, \underline{Y})$  in the above the type  $(\partial_1, \Psi)$  that was described at the beginning of 4.3. In other words, given paths  $\mathbf{p}, \mathbf{q}$  such that  $\partial_1 \mathbf{p} = \partial_0 \mathbf{q}$ , denoting that internal composition operator by  $\mathbf{q} \cdot \mathbf{p}$ , those standard calculations say that

$$J_X(W; Z, t)(\mathbf{p}, \mathbf{q}) = (\mathbf{p}, \mathbf{q}, \mathbf{q} \cdot \mathbf{p})$$

when  $(Y, \underline{Y}) = (\partial_0 X, \Psi)$  and  $t$  is just as in (28). But, using Equation (22) we see that

$$[s] \cdot \mathbf{p} = [s] * [1] * \mathbf{p} \quad \text{and} \quad \llbracket s \rrbracket \cdot \mathbf{p} = \llbracket 1 \rrbracket * \llbracket s \rrbracket * \mathbf{p}$$

Let  $\mathbf{q} = \mathbf{q}_1 * \mathbf{q}_2 * \cdots * \mathbf{q}_n$  be the uniquely defined decomposition of path  $\mathbf{q}$  as a vertical composite of paths of length one. Applying induction on the length of  $\mathbf{q}$ , we see that  $\mathbf{q} \cdot \mathbf{p} = \tilde{\mathbf{q}}_1 * \tilde{\mathbf{q}}_2 * \cdots * \tilde{\mathbf{q}}_n * \mathbf{p}$  where

$$\tilde{\mathbf{q}}_i = \begin{cases} [s_i] * [1] & \text{if } \mathbf{q}_i = [s_i] \\ \llbracket 1 \rrbracket * \llbracket s_i \rrbracket & \text{if } \mathbf{q}_i = \llbracket s_i \rrbracket \end{cases}$$

Thus we immediately see that  $\mathbf{r} \cdot (\mathbf{q} \cdot \mathbf{p}) = (\mathbf{r} \cdot \mathbf{q}) \cdot \mathbf{p}$ . But *this does not mean that 1-composition is strict*. This is because the real test of strictness is the path between  $\mathbf{r} \cdot (\mathbf{q} \cdot \mathbf{p})$  and  $(\mathbf{r} \cdot \mathbf{q}) \cdot \mathbf{p}$  in  $\mathbf{PPX}$ , and further computations will show that this path is not the identity path, but a constant path of nonzero length. Thus the 2-cells of the weak  $\omega$ -groupoid are not degenerate, but being constant paths, they are isomorphisms, and the cell structure in dimension 3 becomes truly degenerate.

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