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Sensitivity analysis for relaxed optimal control problems with final-state constraints

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Abstract: In this article, we compute a second-order expansion of the value function of a family of relaxed optimal control problems with final-state constraints, parameterized by a perturbation variable. The sensitivity analysis is performed for controls that we call R -strong solutions. They are optimal solutions with respect to the set of feasible controls with a uniform norm smaller than a given R and having an associated trajectory in a small neighborhood for the uniform norm. In this framework, relaxation enables us to consider a wide class of perturbations and therefore to derive sharp estimates of the value function.

Key-words: optimal control, sensitivity analysis, relaxation, Young measures, Pontryagin's principle, strong solutions.

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Analyse de sensibilité pour des problèmes de contrôle optimal relaxés avec contraintes sur l'état final

Résumé : Dans cet article, nous calculons un développement au second ordre de la fonction valeur d'une famille de problèmes de contrôle optimal avec contraintes sur l'état final, paramétrée par une variable de perturbation. L'analyse de sensibilité est réalisée pour des contrôles nommés *R-strong solutions*. Ce sont des solutions optimales par rapport à l'ensemble des contrôles admissibles de norme infinie inférieure à R ayant une trajectoire associée dans un petit voisinage pour la norme infinie. Dans ce cadre, la relaxation nous permet de considérer une large classe de perturbations et ainsi d'obtenir des estimations précises de la fonction valeur.

Mots-clés : contrôle optimal, analyse de sensibilité, relaxation, mesures de Young, principe de Pontryagin, solutions fortes.

1 Introduction

Let us consider a family of relaxed optimal control problems with final-state constraints, parameterized by a perturbation variable θ . The variable θ can perturb the initial state, the dynamic of the system, the cost function and the final-state constraints. The aim of the article is to compute a second-order expansion of the value $V(\theta)$ of the problems, in the neighborhood of a reference value of θ , say $\bar{\theta}$.

This second-order expansion is obtained by applying the methodology described in [5] and originally in [3]. The approach is the following: we begin by linearizing the family of optimization problems in the neighborhood of an optimal solution of the reference problem. The first-order and the second-order linearizations provide a second-order upper estimate of the value function. A first lower estimate is obtained by expanding the Lagrangian up to the second order. Considering a strong sufficient second-order condition, we show that the distance between the reference solution and solutions to the perturbed problems is of order $|\theta - \bar{\theta}|$. Then, it follows directly that the lower estimate corresponds to the upper estimate previously obtained.

In this study, we use the notion of R -strong optimal controls. We say that a control is an R -strong optimal solution if it is optimal with respect to the controls with a uniform norm smaller than a given $R > 0$ and having a trajectory sufficiently close for the uniform norm. This notion is related to the one of bounded strong solutions [18], since by definition, a bounded strong solution is an R -strong solution for all $R > 0$. In order to obtain a sharp upper estimate of V , we must derive a linearized problem from a wide class of perturbations of the control. Typically, we must be able to perturb the reference optimal control with close controls for the L^1 -distance, but not necessarily close for the L^∞ -distance. For such perturbations of the control, we use a special linearization of the dynamic of the system, the Pontryagin linearization [18].

We perform the sensitivity analysis in the framework of relaxed optimal controls. Roughly speaking, at each time, the control variable is not anymore a vector in a space U , but a probability measure on U , like if we were able to use several controls simultaneously. The new control variable is now a Young measure, in reference to the pioneering work of L.C. Young. Relaxation of optimal control problems with Young measures has been much studied, in particular in [6, 7, 12, 17, 23, 24, 25]. It is expected that a classical optimal control problem and its relaxed version have the same value, since any Young measure is the weak-* limit of a sequence of classical controls. This question is studied in [2, 13].

Three aspects motivate the use of the relaxation. First, by considering convex combinations of controls in the sense of measures, we manage to describe in a convenient way a large class of perturbations of the optimal control. This idea was used in [9] to prove Pontryagin's principle in the case of mixed constraints and more recently in [16]. Moreover, in this framework, we derive directly from abstract results [11] a regularity metric theorem for the L^1 -distance. The associated qualification condition uses the Pontryagin linearization. This theorem not only justifies the linearization of the problem but it also permits to exhibit some sufficient conditions to ensure the equality of classical and relaxed problems. Finally, the existence of relaxed solutions to the perturbed problem is guaranteed, which is not the case in a classical framework.

We obtain a lower estimate of the value function by assuming a sufficient second-order condition of the same kind as the one in [4]. We assume that a certain quadratic form is positive and that the Hamiltonian satisfies a quadratic growth condition. In order to expand the Lagrangian up to the second-order, we split the controls into two parts, one accounting for the small of the control in L^∞ -distance, the other accounting for the large variations. We obtain the decomposition principle described in [4] and a lower estimate which corresponds to the upper estimate previously obtained.

The outline of the paper is as follows. In section 2, we use the Pontryagin linearization to

derive Pontryagin's principle. We also prove our metric regularity theorem and study conditions ensuring the equality of classical and relaxed problems in the context of R -strong solutions. In section 3, we obtain a first-order upper estimate of V and in section 4 a second-order upper estimate. In section 5, we prove the decomposition principle and we obtain the lower estimate. Two examples are discussed in section 6. All the theoretical material related to Young measures is recalled in the appendix, with precise references from [1, 8, 21, 22].

2 Necessary conditions in a relaxed framework

2.1 Setting

In this section, we study the first-order necessary optimality conditions of a bounded strong solution to an unperturbed optimal control problem with final-state constraints, in a relaxed framework. Let us begin by defining the problem in a classical problem. The control and state spaces are respectively

$$\mathcal{U} := L^\infty(0, T; \mathbb{R}^m), \quad \mathcal{Y} := W^{1, \infty}(0, T; \mathbb{R}^n). \quad (2.1)$$

The state equation is

$$\begin{cases} \dot{y}_t = f(u_t, y_t), & \text{for a.a. } t \in [0, T], \\ y_0 = y^0. \end{cases} \quad (2.2)$$

For a control u in \mathcal{U} , we denote by $y[u]$ the trajectory satisfying the differential system (2.2). We consider the following final state constraint:

$$\Phi(y_T) \in K, \quad (2.3)$$

where K stands for a finite number n_C of equalities and inequalities, as follows:

$$K := \{0_{n_E}\} \times \mathbb{R}_{-}^{n_I} \subset \mathbb{R}^{n_C}, \quad (2.4)$$

with $n_C = n_E + n_I$. A control u is *feasible* if $\Phi(y_T[u]) \in K$. The classical optimal control problem that we consider is

$$\text{Min}_{u \in \mathcal{U}} \phi(y_T[u]), \quad \text{s.t. } \Phi(y_T[u]) \in K. \quad (\mathcal{P})$$

All the functions introduced (f , ϕ , and Φ) are supposed to be $C^{2,1}$ (twice differentiable with a Lipschitz second-order derivative).

Let us consider a feasible control \bar{u} and its associated trajectory \bar{y} . Let $R > 0$ and $\eta > 0$, we define the *localized problem* as follows:

$$\text{Min}_{u \in \mathcal{U}, \|u\|_\infty \leq R} \phi(y_T[u]), \quad \text{s.t. } \Phi(y_T[u]) \in K, \quad \|y[u] - \bar{y}\|_\infty \leq \eta. \quad (\mathcal{P}^{\eta, R})$$

Definition 1. Let $R > 0$, the control \bar{u} is said to be an R -strong optimal solution if there exists $\eta > 0$ such that \bar{u} is solution to $(\mathcal{P}^{\eta, R})$.

Note that the control \bar{u} is a *bounded strong* solution if for all $R > \|\bar{u}\|_\infty$, it is an R -strong optimal solution [18, page 291].

From now, we fix $\bar{u} \in \mathcal{U}$ and $R > \|\bar{u}\|_\infty$, and we suppose that \bar{u} is an R -strong optimal solution.

2.2 Relaxed problem

Let us denote by U_R the closed ball of radius R and centre 0 in \mathbb{R}^m . We consider a relaxed formulation of the localized problem by taking controls in the space of Young measures on $[0, T] \times U_R$, \mathcal{M}_R^Y . The basic definitions related to Young measures are recalled in the appendix.

The dynamic associated with a Young measure μ in \mathcal{M}_R^Y is the following:

$$\begin{cases} \dot{y}_t = \int_{U_R} f(u, y_t) d\mu_t(u), & \text{for a.a. } t \in [0, T], \\ y_0 = y^0. \end{cases} \quad (2.5)$$

This definition is compatible with (2.2) for controls in \mathcal{U} . We extend the mapping $y[\mu]$ to Young measures. In our study, we will call elements of \mathcal{M}_R^Y *relaxed controls*, by contrast with elements of \mathcal{U} which will be called *classical controls*.

Given $\eta > 0$, we relax the localized problem $(\mathcal{P}^{\eta, R})$ as follows:

$$\text{Min}_{\mu \in \mathcal{M}_R^Y} \phi(y_T[\mu]), \quad \text{s.t. } \Phi(y_T[\mu]) \in K, \quad \|y[\mu] - \bar{y}\|_\infty \leq \eta. \quad (\mathcal{P}^{Y, \eta, R})$$

Denoting by $\bar{\mu}$ the Young measure associated to \bar{u} , we say that $\bar{\mu}$ is a *relaxed R -strong optimal solution* if it is a solution to problem $(\mathcal{P}^{Y, \eta, R})$ for some $\eta > 0$ sufficiently small. The following assumption is motivated by the fact that any relaxed control is the weak-* limit of a sequence of classical controls.

Hypothesis 2. *The relaxed control $\bar{\mu}$ is a relaxed R -strong solution.*

From now, we focus on the notion of relaxed R -strong solution. In this section, we study the optimality condition of problem $(\mathcal{P}^{Y, \eta, R})$. In section 3, we will introduce a perturbation parameter θ and start then the sensitivity analysis.

In theorem 20, we provide some sufficient conditions to ensure hypothesis 2.

2.3 Metric regularity for the L^1 -distance

In our study, the addition of Young measures must be understood as the addition of measures on $[0, T] \times U_R$. It is clear that with this definition of the addition, a convex combination of Young measures is still a Young measure. In the sequel, we use the notation $g[t] := g(\bar{u}_t, \bar{y}_t)$ for every function g of (u, y) .

The following definition of the Pontryagin linearization is a non-standard linearization of the state equation. Indeed, we only linearize the dynamic with respect to the state variable. We extend the definition of [18, page 40] to Young measures.

Definition 3. *For a given control μ , we define the Pontryagin linearization $\xi[\mu]$ in \mathcal{Y} as the solution of*

$$\begin{cases} \dot{\xi}_t[\mu] = f_y[t]\xi_t[\mu] + \int_{U_R} f(u, \bar{y}_t) d\mu_t(u) - f[t], & \text{for a.a. } t \in [0, T], \\ \xi_0[\mu] = 0. \end{cases}$$

Lemma 4. *The following estimates hold:*

$$\|y[\mu] - \bar{y}\|_\infty = O(d_1(\mu, \bar{\mu})), \quad (2.6)$$

$$\|y[\mu] - (\bar{y} + \xi[\mu])\|_\infty = O(d_1(\mu, \bar{\mu})^2). \quad (2.7)$$

The distance d_1 is defined by the Wasserstein distance (A.1).

Proof. See lemma 21, which extends this result. \square

Let $q \in \mathbb{N}^*$, we denote by Δ the following polytope of \mathbb{R}^q :

$$\Delta = \left\{ \gamma \in \mathbb{R}_+^q, \sum_{i=1}^q \gamma_i \leq 1 \right\}.$$

Let $\mu^1, \dots, \mu^q \in \mathcal{M}_R^Y$, let us denote by S the following mapping:

$$\begin{cases} S : (\mathcal{M}_R^Y \times \Delta) & \rightarrow & \mathcal{M}_R^Y \\ (\mu^0, \gamma) & \mapsto & (1 - \sum_{i=1}^q \gamma_i) \mu^0 + \sum_{i=1}^q \gamma_i \mu^i, \end{cases} \quad (2.8)$$

where the addition is the addition of measures.

Lemma 5. *Let $\gamma, \gamma' \in \Delta$, let $\mu^0 \in \mathcal{M}_R^Y$. Then,*

$$d_1(S(\mu^0, \gamma), S(\mu^0, \gamma')) \leq \sum_{i=1}^q |\gamma'_i - \gamma_i| d_1(\mu^i, \mu^0) \leq 2RT \left(\sum_{i=1}^q |\gamma'_i - \gamma_i| \right).$$

Proof. \triangleright *First case:* $\gamma' \geq \gamma$.

Let us suppose that for all i in $\{1, \dots, q\}$, $\gamma'_i \geq \gamma_i$. Using the definitions of section A.1, we denote for all i in $\{1, \dots, q\}$ by $\tilde{\pi}^i$ the transportation plan from μ^0 to μ^i which is optimal for the L^1 -distance. For i in $\{0, \dots, q\}$, we denote by π^i the transportation plan which is the image of μ^i induced by the mapping $(t, u) \in [0, T] \times U_R \mapsto (t, u, u) \in [0, T] \times U_R \times U_R$. We set

$$\pi = \sum_{i=1}^q (\gamma'_i - \gamma_i) \tilde{\pi}^i + \left(1 - \sum_{i=1}^q \gamma'_i \right) \pi^0 + \sum_{i=1}^q \gamma_i \pi^i.$$

It is clear that $\pi \in \Pi(S(\mu^0, \gamma_i), S(\mu^0, \gamma'_i))$. Moreover,

$$d_1(S(\mu^0, \gamma_i), S(\mu^0, \gamma'_i)) \leq \int_0^T \int_{U_R \times U_R} |u - v| d\pi_t(u, v) dt = \sum_{i=1}^q (\gamma'_i - \gamma_i) d_1(\mu^0, \mu^i).$$

This proves the first inequality in this particular case.

\triangleright *General case.*

Let us define $\tilde{\gamma}$ by $\tilde{\gamma}_i = \min\{\gamma_i, \gamma'_i\}$. Then, $\tilde{\gamma} \in \Delta$, $\tilde{\gamma} \leq \gamma$, $\tilde{\gamma} \leq \gamma'$, and therefore,

$$\begin{aligned} d_1(S(\mu^0, \gamma), S(\mu^0, \gamma')) &\leq d_1(S(\mu^0, \gamma), S(\mu^0, \tilde{\gamma})) + d_1(S(\mu^0, \tilde{\gamma}), S(\mu^0, \gamma')) \\ &\leq \sum_{i=1}^q (\gamma_i - \tilde{\gamma}_i) d_1(\mu^0, \mu^i) + \sum_{i=1}^q (\gamma'_i - \tilde{\gamma}_i) d_1(\mu^0, \mu^i) \\ &= \sum_{i=1}^q |\gamma'_i - \gamma_i| d_1(\mu^0, \mu^i), \end{aligned}$$

which proves the first inequality. Finally, for all i in $\{1, \dots, q\}$, for all π in $\Pi(\mu^0, \mu^i)$,

$$\int_0^T \int_{U_R} |u - v| d\pi_t(u, v) dt \leq 2RT,$$

hence, for all i in $\{1, \dots, q\}$, $d_1(\mu^0, \mu^i) \leq 2RT$ and the lemma follows. \square

Corollary 6. Let μ^0 and μ^1 in \mathcal{M}_R^Y , let σ in $[0, 1]$. Then,

$$d_1(\mu^0, (1 - \sigma)\mu^0 + \sigma\mu^1) \leq \sigma d_1(\mu^0, \mu^1) \leq 2RT\sigma.$$

We introduce the following set:

$$\mathcal{R}_T := \{\xi_T[\mu], \mu \in \mathcal{M}_R^Y\}.$$

The Pontryagin linearization being affine with respect to μ , \mathcal{R}_T is clearly convex. We denote by $\mathcal{C}(\mathcal{R}_T)$ the smallest closed cone containing \mathcal{R}_T . Since \mathcal{R}_T is convex, $\mathcal{C}(\mathcal{R}_T)$ is also convex.

Definition 7. The control $\bar{\mu}$ is qualified if there exists $\varepsilon > 0$ such that

$$\varepsilon B \subset \Phi(\bar{y}_T) + \Phi'(\bar{y}_T)\mathcal{C}(\mathcal{R}_T) - K, \quad (2.9)$$

where B is the unit ball of \mathbb{R}^{n_c} .

In the sequel, we will always assume that $\bar{\mu}$ is qualified. Note that in remark 31, we justify why this assumption is weaker than the standard assumed qualification condition. The following theorem establishes a property of metric regularity for the relaxed problem.

Theorem 8. If $\bar{\mu}$ is qualified, then there exist $\delta > 0$ and $C > 0$ such that for all μ satisfying $d_1(\mu, \bar{\mu}) \leq \delta$, there exists a control μ' satisfying

$$\Phi(y_T[\mu']) \in K \quad \text{and} \quad d_1(\mu', \mu) \leq C \text{dist}(\Phi(y_T[\mu]), K). \quad (2.10)$$

Proof. We prove this theorem in three steps, by using the theory of multifunctions. First, we prove a metric regularity result for an affine subspace of \mathcal{M}_R^Y by showing that a certain multifunction $\Psi_{\bar{\mu}}(\gamma)$ is metric regular. Then, by perturbing $\bar{\mu}$, we extend this result to a family of multifunctions. Finally, we obtain the result.

▷ *First step: metric regularity of $\Psi_{\bar{\mu}}(\gamma)$.*

Let us consider a family $(\xi^i)_i$, $i = 1, \dots, n_A$ (with $n_A \leq n_C + 1$) in $\mathcal{C}(\mathcal{R}_T)$ such that for some $\varepsilon_1 > 0$,

$$\varepsilon_1 B \subset \Phi(\bar{y}_T) + \Phi'(\bar{y}_T)(\text{conv}\{\xi^1, \dots, \xi^{n_A}\}) - K.$$

We can easily show the existence of a family $(\alpha_i, \mu^i)_i$, $i = 1, \dots, n_A$ in $\mathbb{R}_+ \times \mathcal{M}_R^Y$ which is such that

$$\frac{\varepsilon_1 B}{2} \subset \Phi(\bar{y}_T) + \Phi'(\bar{y}_T)(\text{conv}\{\alpha_1 \xi_T[\mu^1], \dots, \alpha_{n_A} \xi_T[\mu^{n_A}]\}) - K.$$

Let us set $\hat{\alpha} = \max\{\max\{\alpha_i\}, 1\}$. Then,

$$\frac{\varepsilon_1}{2\hat{\alpha}} B \subset \frac{1}{\hat{\alpha}} [\Phi(\bar{y}_T) - K] + \Phi'(\bar{y}_T)\left(\text{conv}\left\{\frac{\alpha_1}{\hat{\alpha}} \xi_T[\mu^1], \dots, \frac{\alpha_{n_A}}{\hat{\alpha}} \xi_T[\mu^{n_A}]\right\}\right).$$

Since $\hat{\alpha} \geq 1$,

$$\frac{1}{\hat{\alpha}} [\Phi(\bar{y}_T) - K] \subset \Phi(\bar{y}_T) - K$$

and then, for all i in $\{1, \dots, n_A\}$, we can replace μ^i by $(1 - \frac{\alpha_i}{\hat{\alpha}})\bar{\mu} + \frac{\alpha_i}{\hat{\alpha}}\mu^i$ since $0 \leq \alpha_i/\hat{\alpha} \leq 1$. We obtain that

$$\frac{\varepsilon_1}{2\hat{\alpha}} B \subset \Phi(\bar{y}_T) + \Phi'(\bar{y}_T)(\text{conv}\{\xi_T[\mu^1], \dots, \xi_T[\mu^{n_A}]\}) - K.$$

Using the mapping S defined by (2.8), with $q = n_A$, we consider the mapping G_μ defined by

$$G_\mu : \gamma \in \Delta \mapsto \Phi(y_T[S(\mu, \gamma)]) \in \mathbb{R}^{n_c}, \quad (2.11)$$

for all μ in \mathcal{M}_R^Y . Note that $G_\mu(0_{n_A}) = \Phi(y_T[\mu])$.

Let us fix μ , let us study the differentiability of G_μ (with respect to γ). Let γ in Δ , let us set $\tilde{\mu} = S(\mu, \gamma)$, $\tilde{y} = y[\tilde{\mu}]$. For all i in $\{1, \dots, n_A\}$, we denote by $\zeta^i[\mu, \gamma]$ the solution ζ^i of the following differential system:

$$\begin{cases} \dot{\zeta}_t^i = \left[\int_{U_R} f_y(u, \tilde{y}_t) d\tilde{\mu}_t(u) \right] \zeta_t^i \\ \quad + \int_{U_R} f(u, \tilde{y}_t) [d\mu_t^i(u) - d\mu_t(u)], \quad \text{for a.a. } t \in [0, T], \\ \zeta_0^i = 0. \end{cases} \quad (2.12)$$

Let us denote by $G'_\mu(\gamma)$ the following mapping:

$$G'_\mu(\gamma) = \Phi'(\tilde{y}_T)(\zeta_T^1[\mu, \gamma], \dots, \zeta_T^{n_A}[\mu, \gamma]). \quad (2.13)$$

The notation $(\zeta_T^1[\mu, \gamma], \dots, \zeta_T^{n_A}[\mu, \gamma])$ stands for the matrix with n_A columns, $\zeta_T^1[\mu, \gamma], \dots, \zeta_T^{n_A}[\mu, \gamma]$. We prove in lemma 9 that for all γ' in Δ ,

$$G_\mu(\gamma') = G_\mu(\gamma) + G'_\mu(\gamma)(\gamma' - \gamma) + o(|\gamma' - \gamma|).$$

Note that

$$G'_{\bar{\mu}}(0_{n_A}) = \Phi'(\bar{y}_T)(\xi_T[\mu^1], \dots, \xi_T[\mu^{n_A}]). \quad (2.14)$$

Now, let us consider the family of multifunctions:

$$\Psi_\mu : \gamma \in \Delta \mapsto G_\mu(\gamma) - K.$$

Robinson's qualification condition for $\Psi_{\bar{\mu}}$ at the point $(0_{n_A}, 0_{n_C})$ holds if there exists $\varepsilon_2 > 0$ such that

$$\varepsilon_2 B \subset G_{\bar{\mu}}(0_{n_A}) + G'_{\bar{\mu}}(0_{n_A})\Delta - K, \quad (2.15)$$

see [5, condition 2.194]. This condition is satisfied for $\varepsilon_2 = \varepsilon_1/(2\hat{\alpha})$ since by (2.14),

$$G_{\bar{\mu}}(0_{n_A}) + G'_{\bar{\mu}}(0_{n_A})\Delta = \Phi(\bar{y}_T) + \Phi'(\bar{y}_T)(\text{conv}\{\xi_T[\mu^1], \dots, \xi_T[\mu^{n_A}]\}).$$

By the Robinson-Ursescu stability theorem (see e.g. [5, theorem 2.87] or [19] and [20] for early references), $\Psi_{\bar{\mu}}$ is metric regular at $(0_{n_A}, 0_{n_C})$, i.e., there exists $C_1 > 0$ and two neighborhood \mathcal{O}^γ and \mathcal{O}^ρ of 0_{n_A} and 0_{n_C} such that for all (γ, ρ) in $(\mathcal{O}^\gamma \cap \Delta) \times \mathcal{O}^\rho$,

$$\text{dist}(\gamma, \Psi_{\bar{\mu}}^{-1}(\rho)) \leq C_1 \text{dist}(\Psi_{\bar{\mu}}(\gamma), \rho).$$

▷ *Second step: metric regularity of Ψ_μ .*

We prove in lemma 9 that the mapping $(\mu, \gamma) \in (\mathcal{M}_R^Y, \Delta) \mapsto G'_\mu(\gamma)$ is continuous at $(\bar{\mu}, 0_{n_A})$, when \mathcal{M}_R^Y is equipped with the L^1 -distance. Then, restricting if necessary \mathcal{O}^γ , there exists a neighborhood \mathcal{O}^μ of $\bar{\mu}$ such that for all (μ, γ) in $\mathcal{O}^\mu \times (\mathcal{O}^\gamma \cap \Delta)$,

$$|G'_\mu(\gamma) - G'_{\bar{\mu}}(\gamma)| \leq \frac{C_1}{2},$$

and thus, $G_\mu(\cdot) - G_{\bar{\mu}}(\cdot)$ is $C_1/2$ -Lipschitz on $\mathcal{O}^\gamma \cap \Delta$. By [5, theorem 2.84], for μ in N^μ , Ψ_μ is metric regular at $(0_{n_A}, G_\mu(0_{n_A}) - G_{\bar{\mu}}(0_{n_A}))$. More precisely, restricting if necessary \mathcal{O}^μ , \mathcal{O}^γ , \mathcal{O}^ρ , there exists $C_2 > 0$ such that for all μ in \mathcal{O}^μ , for all $\gamma \in \Delta \cap \mathcal{O}^\gamma$, for all ρ in \mathcal{O}^ρ ,

$$\text{dist}(\gamma, \Psi_\mu^{-1}(\rho)) \leq C_2 \text{dist}(\Psi_\mu(\gamma), \rho). \quad (2.16)$$

▷ *Third step: proof of the theorem.* Let μ in \mathcal{O}^μ , since $G_\mu(0_{n_A}) = \Phi(y_T[\mu])$,

$$\text{dist}(\Phi(y_T[\mu]), K) = \text{dist}(0_{n_C}, \Psi_\mu(0_{n_A})). \quad (2.17)$$

Specializing (2.16) to $\gamma = 0_{n_A}$ and $\rho = 0_{n_C}$ and using (2.17), we obtain the existence of $\tilde{\gamma}$ in $\Psi_\mu^{-1}(0_{n_C})$ such that

$$|\tilde{\gamma} - 0_{n_A}| \leq C_2 \text{dist}(\Phi(y_T[\mu]), K).$$

Finally, we set $\mu' = S(\mu, \tilde{\gamma})$. This control is feasible and by lemma 5, $d_1(\mu', \mu) \leq 2RT|\tilde{\gamma}|$. Restricting \mathcal{O}^μ to a ball (for the L^1 -distance) of radius δ and center $\bar{\mu}$, we obtain the theorem with δ , μ' , and $C = 2RTC_2$. □

Lemma 9. *Consider the mapping $\gamma \mapsto G_\mu(\gamma)$ defined by (2.11) in the proof of theorem 8 and its derivative $G'_\mu(\gamma)$, defined by (2.12) and (2.13). Then, for all μ in \mathcal{M}_R^Y , for all γ and γ' in Δ ,*

$$y_T[S(\mu, \gamma')] = y_T[S(\mu, \gamma)] + \zeta_T(\gamma' - \gamma) + o(|\gamma' - \gamma|), \quad (2.18)$$

$$G'_\mu(\gamma') = G'_\mu(\gamma) + G''_\mu(\gamma)(\gamma' - \gamma) + o(|\gamma' - \gamma|). \quad (2.19)$$

Moreover, the mapping $(\mu, \gamma) \in (\mathcal{M}_R^Y \times \Delta) \mapsto G'_\mu(\gamma)$ is continuous when \mathcal{M}_R^Y is equipped with the L^1 -distance and \mathcal{Y} with the L^∞ -norm.

Proof. Let $\mu \in \mathcal{M}_R^Y$, let $\gamma, \gamma' \in \Delta$. Let us set

$$\tilde{\mu} = S(\mu, \gamma), \quad \tilde{y} = y[\tilde{\mu}], \quad \tilde{\mu}' = S(\mu, \gamma'), \quad \tilde{y}' = y[\tilde{\mu}'].$$

Note that $\tilde{\mu}' - \tilde{\mu} = \sum_{i=1}^{n_A} (\gamma'_i - \gamma_i)(\mu^i - \mu)$. By lemma 5, we know that

$$d_1(\tilde{\mu}, \tilde{\mu}') = O(|\gamma' - \gamma|),$$

and by lemma (56), we obtain that $\|\tilde{y}' - \tilde{y}\|_\infty = O(|\gamma' - \gamma|)$.

Let us set $r_t = \tilde{y}'_t - (\tilde{y}_t + \zeta_t(\gamma' - \gamma))$, with the notation $\zeta_t = (\zeta_t^1[\mu, \gamma], \dots, \zeta_t^{n_A}[\mu, \gamma])$. Then,

$$\begin{aligned} \dot{r}_t &= \sum_{i=1}^{n_A} \int_{U_R} (f(u, \tilde{y}'_t) - f(u, \tilde{y}_t))(\gamma'_i - \gamma_i)(d\mu_t^i(u) - d\mu_t(u)) \\ &\quad + \int_{U_R} (f(u, \tilde{y}'_t) - f(u, \tilde{y}_t)) d\tilde{\mu}_t(u) - \left(\int_{U_R} f_y(u, \tilde{y}_t) d\tilde{\mu}_t(u) \right) \zeta_t(\gamma' - \gamma). \end{aligned}$$

The first term of the r.h.s. is of order $|\gamma' - \gamma|^2$, in the second term, we can replace $f(u, \tilde{y}'_t) - f(u, \tilde{y}_t)$ by $f_y(u, \tilde{y}_t)(\tilde{y}'_t - \tilde{y}_t)$, since the error realized is then of order $|\gamma' - \gamma|^2$. We obtain that

$$\dot{r}_t = \left(\int_{U_R} f_y(u, \tilde{y}_t) d\tilde{\mu}_t(u) \right) r_t + O(|\gamma' - \gamma|^2)$$

thus, by Gronwall's lemma,

$$y_T[S(\mu, \gamma')] = y_T[S(\mu, \gamma)] + \zeta_T(\gamma' - \gamma) + O(|\gamma' - \gamma|^2).$$

This proves (2.18). Expansion (2.19) follows directly, since Φ is continuously differentiable.

Now, let us study the continuity of $(\mu, \gamma) \in (\mathcal{M}_R^Y \times \Delta) \mapsto G'_\mu(\gamma)$. Recall that $G'_\mu(\gamma)$ is the product of

$$(\mu, \gamma) \in (\mathcal{M}_R^Y, \Delta) \mapsto (\zeta_T^1[\mu, \gamma], \dots, \zeta_T^{n_A}[\mu, \gamma]) \quad (2.20)$$

and

$$(\mu, \gamma) \in (\mathcal{M}_R^Y, \Delta) \mapsto \Phi'(y_T[S(\mu, \gamma)]). \quad (2.21)$$

It can be proved that the mapping defined by (2.20) is continuous by extending lemma 56. Moreover, by lemma 5, for all μ and μ' in \mathcal{M}_R^Y , for all γ and γ' in Δ ,

$$\begin{aligned} d_1(S(\mu, \gamma), S(\mu', \gamma')) &\leq d_1(S(\mu, \gamma), S(\mu, \gamma')) + d_1(S(\mu, \gamma'), S(\mu', \gamma')) \\ &= O(|\gamma' - \gamma| + d_1(\mu, \mu')), \end{aligned}$$

thus S is continuous and by extending lemma 56, we obtain the continuity of the mapping defined by (2.21). Finally, $(\mu, \gamma) \mapsto G'_\mu(\gamma)$ is continuous with respect to (μ, γ) . \square

2.4 Pontryagin's principle

We denote respectively by E , I and I' the sets of equality, active and inactive inequality constraints:

$$\begin{aligned} E &= \{1, \dots, n_E\}, \\ I &= \{i, n_E + 1 \leq i \leq n_C, \Phi_i(\bar{y}_T) = 0\}, \\ I' &= \{i, n_E + 1 \leq i \leq n_C, \Phi_i(\bar{y}_T) < 0\}. \end{aligned}$$

Let us denote by Y_T^η the set of feasible reachable final states of the localized problem $(\mathcal{P}^{Y, \eta, R})$:

$$Y_T^\eta = \{y_T \in \mathbb{R}^n, \text{ s.t. } \exists \mu \in \mathcal{M}_R^Y, y_T = y_T[\mu], \Phi(y_T) \in K, \|y[\mu] - \bar{y}\|_\infty \leq \eta\}.$$

In the following theorem, the notation T refers to the tangent cone. In particular, note that for all ρ in \mathbb{R}^{n_C} , $\rho \in T_K(\Phi(\bar{y}_T))$ if and only if for all i in E , $\rho_i = 0$ and for all i in I , $\rho_i \geq 0$.

Theorem 10. *For any $\eta > 0$, the following inclusion holds:*

$$\{\xi \in \mathcal{C}(\mathcal{R}_T), \text{ s.t. } \Phi'(\bar{y}_T)\xi \in T_K(\Phi(\bar{y}_T))\} \subset T_{Y_T^\eta}(\bar{y}_T).$$

Proof. Let ξ in $\mathcal{C}(\mathcal{R}_T)$, by definition, there exists a sequence $(\alpha_k, \nu^k, \xi^k)_k$ in $\mathbb{R}_+ \times \mathcal{M}_R^Y \times \mathcal{R}_T$ such that

$$\begin{cases} \xi = \lim \alpha_k \xi^k, \\ \xi^k = \xi_T[\nu^k], \quad \forall k. \end{cases} \quad (2.22)$$

Note that it may happen that $\alpha_k \rightarrow +\infty$. We set $\sigma_k = \min\{\frac{1}{\alpha_k}, \frac{1}{k}, \frac{1}{k\alpha_k^2}\}$ and

$$\mu^k = (1 - \sigma_k \alpha_k) \bar{\mu} + \sigma_k \alpha_k \nu^k.$$

The sequence $(\mu^k)_k$ is a sequence of Young measures since $\sigma_k \alpha_k \leq 1$ and we also have that $\sigma_k \rightarrow 0$. By lemma 4,

$$\|y[\mu^k] - (\bar{y} + \xi[\mu^k])\|_\infty = O(d_1(\bar{\mu}, \mu^k)^2).$$

Then, by corollary 6,

$$d_1(\bar{\mu}, \mu^k)^2 = O(\sigma_k^2 \alpha_k^2) = O\left(\frac{\alpha_k^2 \sigma_k}{k \alpha_k^2}\right) = O\left(\frac{\sigma_k}{k}\right) = o(\sigma_k), \quad (2.23)$$

thus,

$$\|y[\mu^k] - (\bar{y} + \sigma_k \alpha_k \xi^k)\|_\infty = o(\sigma_k)$$

and finally, by (2.22), $\|y[\mu^k] - (\bar{y} + \sigma_k \xi)\|_\infty = o(\sigma_k)$. We obtain the expansion

$$\Phi(y_T[\mu^k]) = \Phi(\bar{y}_T) + \sigma_k \Phi'(\bar{y}_T)\xi + o(\sigma_k). \quad (2.24)$$

Since $\Phi'(\bar{y}_T)\xi \in T_K(\Phi(\bar{y}_T))$,

$$\text{dist}(\Phi(\bar{y}_T) + \sigma_k \Phi'(\bar{y}_T)\xi, K) = o(\sigma_k). \quad (2.25)$$

Combining (2.24) and (2.25), we obtain that

$$\text{dist}(\Phi(y_T[\mu^k]), K) = o(\sigma_k),$$

and by the metric regularity theorem (theorem 8), we obtain the existence of a feasible sequence $\tilde{\mu}^k$ such that $d_1(\tilde{\mu}^k, \mu^k) = o(\sigma_k)$. Moreover, by lemma 56, we know that the mapping $\mu \in \mathcal{M}_R^Y \mapsto y_T[\mu]$ is Lipschitz for the L^1 -distance, therefore,

$$y_T[\tilde{\mu}^k] = \bar{y}_T + \sigma_k \xi + o(\sigma_k).$$

By estimate (2.23), we also have the estimate

$$\|y[\tilde{\mu}^k] - \bar{y}\|_\infty = O(d_1(\bar{\mu}, \tilde{\mu}^k)) = O(d_1(\bar{\mu}, \mu^k) + d_1(\mu^k, \tilde{\mu}^k)) = o(\sqrt{\sigma_k}) = o(1),$$

thus, for any $\eta > 0$, for k sufficiently large, $\tilde{\mu}^k$ is a feasible control of the localized problem $(\mathcal{P}^{\eta, R})$. The theorem follows. \square

The first-order necessary optimality conditions follow in their primal form.

Corollary 11. *If \bar{u} is a relaxed R -strong solution, then the value of problem (PL)*

$$\text{Min}_{\xi \in \mathcal{C}(\mathcal{R}_T)} \phi'(\bar{y}_T)\xi, \quad \text{s.t. } \Phi'(\bar{y}_T)\xi \in T_K(\Phi(\bar{y}_T)) \quad (PL)$$

is equal to 0.

Proof. Let $\eta > 0$ be such that $(\bar{\mu}, \bar{y})$ is a solution to $(\mathcal{P}^{Y, \eta, R})$. Let ξ be in $\mathcal{C}(\mathcal{R}_T)$ such that $\Phi'(\bar{y}_T)\xi \in T_K(\Phi(\bar{y}_T))$. By theorem 10, there exist a sequence $(\sigma_k)_k \downarrow 0$ and a sequence $(\mu^k)_k$ of feasible controls of problem $(\mathcal{P}^{Y, \eta, R})$ such that

$$y[\mu^k]_T = \bar{y}_T + \sigma_k \xi + o(\sigma_k).$$

As a consequence, $0 \leq \phi(y_T[\mu^k]) - \phi(\bar{y}_T) = \sigma_k \phi'(\bar{y}_T)\xi + o(\sigma_k)$, and then, $\phi'(\bar{y}_T)\xi \geq 0$. The conclusion follows. \square

We introduce now the *Hamiltonian* function $H : \mathbb{R}^{n^*} \times \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ defined by

$$H[p](u, y) := pf(u, y). \quad (2.26)$$

We also define the *end-point Lagrangian* $\Phi : \mathbb{R}^{n^*} \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$\Phi[\lambda](y_T) := \phi(y_T) + \lambda \Phi(y_T). \quad (2.27)$$

Definition 12. *Let $\lambda \in \mathbb{R}^{n^*}$. We say that p^λ in \mathcal{Y} is the costate associated with λ if it satisfies the following differential equation:*

$$\begin{cases} -\dot{p}_t^\lambda = H_y[p_t^\lambda](\bar{u}_t, \bar{y}_t), & \text{for a.a. } t \in [0, T], \\ p_T^\lambda = \Phi'[\lambda](\bar{y}_T). \end{cases} \quad (2.28)$$

Lemma 13. *Given $v \in L^\infty(0, T; \mathbb{R}^n)$ and $z^0 \in \mathbb{R}^n$, let $z \in \mathcal{Y}$ be the solution of*

$$\begin{cases} \dot{z}_t = f_y[t]z_t + v_t \\ z_0 = z^0. \end{cases}$$

*Then, for all λ in $\mathbb{R}^{n \times *}$,*

$$\Phi'[\lambda](\bar{y}_T)z_T = p_0^\lambda z^0 + \int_0^T p_t^\lambda v_t dt.$$

Proof. The lemma is obtained with an integration by parts:

$$\begin{aligned} \Phi'[\lambda](\bar{y}_T)z_T &= p_T^\lambda z_T - p_0^\lambda z^0 + p_0^\lambda z^0 \\ &= \int_0^T (\dot{p}_t^\lambda z_t + p_t^\lambda \dot{z}_t) dt + p_0^\lambda z^0 \\ &= \int_0^T (-p_t^\lambda f_y[t]z_t + p_t^\lambda f_y[t]z_t + p_t^\lambda v_t) dt + p_0^\lambda z^0 \\ &= \int_0^T p_t^\lambda v_t dt + p_0^\lambda z^0, \end{aligned}$$

as was to be proved. \square

In the following definition, the notation N refers to the normal cone.

Definition 14. *Let λ in $N_K(\Phi(\bar{y}_T))$, we say that λ is a Pontryagin multiplier if for almost all t in $[0, T]$, for all u in U_R ,*

$$H[p_t^\lambda](\bar{u}_t, \bar{y}_t) \leq H[p_t^\lambda](u, \bar{y}_t). \quad (2.29)$$

We denote by Λ^P the set of Pontryagin multipliers.

Remark 15. *Note that for our problem, $\lambda \in N_K(\Phi(\bar{y}_T))$ if and only if for all i in I , $\lambda_i \geq 0$ and for all i in I' , $\lambda_i = 0$. Note also that (2.29) is equivalent to: for all μ in \mathcal{M}_R^Y ,*

$$\int_0^T \int_{U_R} H[p_t^\lambda](u, \bar{y}_t) - H[p_t^\lambda](\bar{u}_t, \bar{y}_t) d\mu_t(u) dt \geq 0. \quad (2.30)$$

Proof. The implication is obvious. Suppose that (2.30) holds. By the Lebesgue differentiation theorem, we know that for almost all t in $[0, T]$,

$$H[p_t^\lambda](\bar{u}_t, \bar{y}_t) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} H[p_s^\lambda](\bar{u}_s, \bar{y}_s) ds.$$

Let t be such a point. Let u in U_R , for all $\varepsilon > 0$, let μ^ε be the Young measure associated with $s \mapsto u \mathbf{1}_{[t-\varepsilon, t+\varepsilon]}(s) + \bar{u}_s \mathbf{1}_{[0, T] \setminus [t-\varepsilon, t+\varepsilon]}(s)$. Applying (2.30) to μ^ε , we obtain that for all $\varepsilon > 0$,

$$\int_{t-\varepsilon}^{t+\varepsilon} H[p_s^\lambda](\bar{u}_s, \bar{y}_s) ds \leq \int_{t-\varepsilon}^{t+\varepsilon} H[p_s^\lambda](u, \bar{y}_s) ds.$$

Since p^λ and \bar{y} are continuous in time, we obtain to the limit when $\varepsilon \downarrow 0$ that

$$H[p_t^\lambda](\bar{u}_t, \bar{y}_t) \leq H[p_t^\lambda](u, \bar{y}_t),$$

which proves (2.29). \square

Now, we state the first-order necessary optimality conditions in their dual form. The theorem that we obtain is nothing but Pontryagin's principle.

Theorem 16. *If (\bar{u}, \bar{y}) is a relaxed R -strong solution and if the qualification condition (2.9) holds, then the set of Pontryagin multipliers is non-empty, convex and compact.*

Proof. Problem (PL) can be reformulated as follows:

$$\text{Min}_{\xi \in \mathcal{C}(\mathcal{R}_T)} \sup_{\lambda \in N_K(\Phi(\bar{y}_T))} \Phi'[\lambda](\bar{y}_T)\xi. \quad (PL)$$

The theorem is obtained by studying its dual, which is the following:

$$\text{Max}_{\lambda \in N_K(\Phi(\bar{y}_T))} \inf_{\xi \in \mathcal{C}(\mathcal{R}_T)} \Phi'[\lambda](\bar{y}_T)\xi, \quad (DL)$$

see [5, problem (2.308)]. Problem (DL) has its value equal to 0 and has a non-empty, convex and compact set of solution $S(DL)$, as a consequence of [5, theorem 2.165]. Indeed, the primal problem is convex and qualified. The qualification condition [5, condition 2.312] is the following:

$$\varepsilon B \in \Phi(\bar{y}_T) + \Phi'(\bar{y}_T)\mathcal{C}(\mathcal{R}_T) - T_K(\Phi(\bar{y}_T)), \quad (2.31)$$

which is satisfied since (2.9) holds and $K \subset T_K(\Phi(\bar{y}_T))$, K being a convex closed cone.

Now, we claim that for $\lambda \in N_K(\Phi(\bar{y}_T))$,

$$\left[\inf_{\xi \in \mathcal{C}(\mathcal{R}_T)} \Phi'[\lambda](\bar{y}_T)\xi \right] \in \{0, -\infty\} \quad (2.32)$$

and

$$\left[\inf_{\xi \in \mathcal{C}(\mathcal{R}_T)} \Phi'[\lambda](\bar{y}_T)\xi = 0 \right] \iff [\lambda \in \Lambda^P]. \quad (2.33)$$

Claim (2.32) is obvious since $\Phi'[\lambda](\bar{y}_T)\xi$ is linear with respect to ξ and $\mathcal{C}(\mathcal{R}_T)$ is a cone. Let us show (2.33), let λ be a Pontryagin multiplier. By lemma 13, for ξ in \mathcal{R}_T with associated control μ ,

$$\begin{aligned} & \Phi'[\lambda](\bar{y}_T)\xi \\ &= \int_0^T \left(-p_t^\lambda f_y[t]\xi_t[\mu] + p_t^\lambda f_y[t]\xi_t[\mu] + \int_{U_R} p_t^\lambda [f(u, \bar{y}_t) - f(\bar{u}_t, \bar{y}_t)] d\mu_t(u) \right) dt \\ &= \int_0^T \int_{U_R} (H[p_t^\lambda](u, \bar{y}_t) - H[p_t^\lambda](\bar{u}_t, \bar{y}_t)) d\mu_t(u) dt \\ &\geq 0. \end{aligned} \quad (2.34)$$

Let $\xi \in \mathcal{C}(\mathcal{R}_T)$, there exists a sequence $(\alpha_k, \xi^k)_k$ in $(\mathbb{R}_+ \times \mathcal{R}_T)$ such that $\xi = \lim \alpha_k \xi^k$. With (2.34), we obtain that

$$\Phi'[\lambda](\bar{y}_T)\xi = \lim_k \alpha_k \Phi'[\lambda](\bar{y}_T)\xi^k \geq 0.$$

Since $0 \in \mathcal{C}(\mathcal{R}_T)$,

$$\inf_{\xi \in \mathcal{C}(\mathcal{R}_T)} \Phi'[\lambda](\bar{y}_T)\xi = 0.$$

Conversely, if λ is not a Pontryagin multiplier, by remark 15, there exists a control μ such that

$$\Phi'[\lambda](\bar{y}_T)\xi_T[\mu] = \int_0^T \int_{U_R} H[p_t^\lambda](u, \bar{y}_t) - H[p_t^\lambda](\bar{u}_t, \bar{y}_t) d\mu_t(u) dt < 0.$$

Consequently,

$$\inf_{\xi \in \mathcal{C}(\mathcal{R}_T)} \Phi'[\lambda](\bar{y}_T)\xi < 0.$$

Combining claims (2.32) and (2.33), we obtain that the dual linearized problem is equivalent to the following one:

$$\text{Max}_{\lambda \in \Lambda^P} 0.$$

Since its value is equal to 0, Λ^P is equal to the set of dual solutions $S(DL)$, which is non-empty, convex and compact. \square

2.5 Equality of the values of the classical and the relaxed problems

In this subsection, we investigate sufficient conditions for the equality of the values of the classical and the relaxed localized problems. While any relaxed control is the limit for the weak-* topology of a sequence of classical controls, we have no guarantee that any feasible relaxed control is the limit for the weak-* topology of a sequence of feasible classical controls. However, in lemma 17, we prove a restoration result for classical controls under the qualification condition (2.9). This result enables us to obtain sufficient conditions to ensure hypothesis 2.

Lemma 17. *If $\bar{\mu}$ is qualified, then there exist δ_1 and C_1 such that for all classical control u with $\|u - \bar{u}\|_1 \leq \delta_1$, there exists a classical control u' such that*

$$\Phi(y_T[u']) \in K \quad \text{and} \quad \|u' - u\|_1 \leq C_1 \text{dist}(\Phi(y_T[u]), K).$$

Proof. Let δ and C be the constants given by the metric regularity theorem (theorem 8). Let us set

$$\delta_1 = \frac{\delta}{2C + 3}.$$

Let u be a classical control such that $\|u - \bar{u}\|_1 \leq \delta_1$. We set $d = \text{dist}(\Phi(y_T[u]), K)$. Let us build by induction a sequence $(u^k)_k$ of classical controls such that for all k ,

$$\|u^{k+1} - u^k\|_1 \leq \frac{(C+1)d}{2^k} \quad \text{and} \quad \Phi(y_T[u^k]) \leq \frac{d}{2^k}. \quad (2.35)$$

Let us set $u^0 = u$, by definition of d , $\text{dist}(\Phi(y_T[u^0]), K) \leq d/2^0$. Let k in \mathbb{N} , let us suppose that we have built u^0, \dots, u^k such that $\text{dist}(\Phi(y_T[u^k]), K) \leq d/2^k$ and $\|u^{j+1} - u^j\|_1 \leq (C+1)d/2^j$ for all j in $\{0, \dots, k-1\}$. Therefore,

$$\begin{aligned} d_1(u^k, \bar{\mu}) &\leq \|u^k - u^0\|_1 + d_1(u^0, \bar{\mu}) \leq \sum_{j=0}^{k-1} \|u^{j+1} - u^j\|_1 + \delta_1 \\ &\leq \sum_{j=0}^{k-1} \frac{(C+1)d}{2^j} + \delta_1 \leq 2(C+1)\delta_1 + \delta_1 \leq \delta. \end{aligned}$$

Thus, we can apply the metric regularity theorem and we obtain the existence a feasible relaxed control μ such that $d_1(u^k, \mu) \leq Cd/2^k$. Let $(v^j)_j$ be a sequence of classical controls converging to μ for the weak-* star topology. By lemma 56,

$$\text{dist}(\Phi(y_T[v^j]), K) \rightarrow \text{dist}(\Phi(y_T[\mu]), K) = 0$$

and by expression (A.2), $d_1(u^k, \cdot)$ is weakly-* continuous, thus

$$\|v^j - u^k\|_1 \xrightarrow{j \rightarrow \infty} d_1(u^k, \mu) \leq \frac{Cd}{2^k}.$$

Therefore, there exists j such that

$$\|v^j - u^k\|_1 \leq \frac{(C+1)d}{2^k} \quad \text{and} \quad \Phi(y_T[v^j]) \leq \frac{d}{2^{k+1}}.$$

We set $u^{k+1} = v^j$. This justifies the existence of a sequence satisfying (2.35). Finally, we have built a sequence $(u^k)_k$ of classical controls which converges for the L^1 -norm. Let us denote by u' its limit, it follows by lemma 56 that

$$\text{dist}(\Phi(y_T[u']), K) \leq \lim_k \text{dist}(\Phi(y_T[u^k]), K) = 0$$

and

$$\|u' - u\|_1 \leq \sum_{k=0}^{\infty} \|u^{k+1} - u^k\|_1 \leq \sum_{k=0}^{\infty} \frac{(C+1)d}{2^k} = 2(C+1)d.$$

The lemma holds with δ_1 , u' and $C_1 = 2(C+1)$. □

Lemma 18. *Let us consider the classical and relaxed localized problems associated respectively with the notion of Pontryagin extremal:*

$$\underset{u \in \mathcal{U}, \|u\|_{\infty} \leq R}{\text{Min}} \phi(y_T[u]), \quad \text{s.t. } y_T[u] \in K, \|u - \bar{u}\|_1 \leq \beta, \quad (2.36)$$

$$\underset{\mu \in \mathcal{M}_R^Y}{\text{Min}} \phi(y_T[\mu]), \quad \text{s.t. } y_T[\mu] \in K, \|\mu - \bar{\mu}\|_1 \leq \tilde{\beta}. \quad (2.37)$$

Let $\beta \in (0, \delta_1)$, where δ_1 is the constant given by lemma 17. If \bar{u} is a solution to (2.36) for the value β , then $\bar{\mu}$ is a solution to (2.37) for any $\tilde{\beta} \in (0, \beta)$.

Proof. Let $0 < \tilde{\beta} < \beta < \delta_1$ and let $\mu \in \mathcal{M}_R^Y$ be a feasible relaxed control such that $d_1(\bar{\mu}, \mu) \leq \tilde{\beta}$. Let $(u^k)_k$ be a sequence of classical controls, not necessarily feasible, converging to μ for the weak-* topology. By lemma 56, we obtain that $\text{dist}(\Phi(y_T[u^k]), K) \rightarrow 0$. Moreover, by expression (A.2), $d_1(u^k, \cdot)$ is weakly-* continuous, thus

$$d_1(\bar{\mu}, u^k) \longrightarrow d_1(\bar{\mu}, \mu).$$

For k sufficiently large, $d_1(\bar{\mu}, u^k) < \delta_1$. By lemma 17, there exists a sequence $(v^k)_k$ of feasible classical controls such that:

$$\|v^k - u^k\|_1 = O(\text{dist}(\Phi(y_T[u^k]), K)) = o(1), \quad (2.38)$$

thus, $(v^k)_k$ converges to μ for the weak-* topology and for k sufficiently large,

$$\|v^k - \bar{u}\|_1 \leq \beta$$

and necessarily, $\phi(y_T[v^k]) \geq \phi(\bar{y}_T)$. By lemma 56, we obtain that $\phi(y_T[\mu]) \geq \phi(\bar{y}_T)$, which proves that $\bar{\mu}$ is a solution to (2.37). □

Remark 19. *The qualification condition (2.9) can be defined for any feasible relaxed control $\tilde{\mu}$. Indeed, it suffices to define the Pontryagin linearization for $\tilde{\mu}$. The metric regularity theorem is then satisfied for all μ in a neighborhood of $\tilde{\mu}$ for the L^1 -distance. Moreover, this theorem is also satisfied for all μ in a weak-* neighborhood of $\tilde{\mu}$. Finally, lemma 17 also holds for all classical control u in a weak-* neighborhood of $\tilde{\mu}$.*

Proof. Let $\tilde{\mu}$ in \mathcal{M}_R^Y . The Pontryagin linearization $\xi^{\tilde{\mu}}$ can be defined for all μ by

$$\begin{cases} \dot{\xi}_t^{\tilde{\mu}}[\mu] = \int_{U_R} f_y(u, y_t[\tilde{\mu}]) d\tilde{\mu}_t(u) \xi_t^{\tilde{\mu}}[\mu] + \int_{U_R} f(u, y_t[\tilde{\mu}]) (d\mu_t(u) - d\tilde{\mu}_t(u)), \\ \xi_0^{\tilde{\mu}}[\mu] = 0. \end{cases}$$

We set then $\mathcal{R}_T^{\tilde{\mu}} = \{\xi_T^{\tilde{\mu}}[\mu], \mu \in \mathcal{M}_R^Y\}$ and denote by $\mathcal{C}(\mathcal{R}_T^{\tilde{\mu}})$ the smallest closed cone containing $\mathcal{R}_T^{\tilde{\mu}}$. The relaxed control $\tilde{\mu}$ is said to be qualified if there exists $\varepsilon > 0$ such that

$$\varepsilon B \subset \Phi(y_T[\tilde{\mu}]) + \Phi'(y_T[\tilde{\mu}])\mathcal{C}(\mathcal{R}_T^{\tilde{\mu}}) - K. \quad (2.39)$$

Now, remember that the metric regularity theorem follows directly from the fact that a family of multifunctions Ψ_μ is metric regular for all μ sufficiently close to $\tilde{\mu}$ for the L^1 -distance. We can prove that Ψ_μ is also metric regular for all μ in some neighborhood of $\tilde{\mu}$ for the weak-* topology by applying [5, theorem 2.84]. The only difference in the proof is that we have to show that $(\mu, \gamma) \in \mathcal{M}_R^Y \times \Delta \mapsto G'_\mu(\gamma)$ is continuous when \mathcal{M}_R^Y is equipped with the weak-* topology. Since the weak-* topology is metrizable, all the continuity properties can still be proved "sequentially". Lemma 17 follows equally, as a consequence of the metric regularity theorem. \square

The next theorem gives three different conditions which ensure hypothesis (2).

Theorem 20. *Suppose that the qualification condition (2.9) holds. Consider the three following hypotheses.*

- (i) *All the relaxed controls μ in \mathcal{M}_R^Y such that $y[\mu] = \bar{y}$ are qualified (in the sense of condition (2.39)).*
- (ii) *There is only one relaxed control μ such that $y[\mu] = \bar{y}$, which is the solution $\bar{\mu}$.*
- (iii) *For some Pontryagin multiplier λ , for almost all t in $[0, T]$, the function*

$$u \in U_R \mapsto H[p_t^\lambda](u, \bar{y}_t)$$

has a unique minimizer, which is \bar{u}_t .

Let one of these assumptions be satisfied. If \bar{u} is a classical R -strong solution for some $\eta > 0$, then $\bar{\mu}$ is an R -strong solution for some $\tilde{\eta} \in (0, \eta)$.

Proof. Note first that (iii) \implies (ii) \implies (i). Indeed, suppose that (iii) holds for some Pontryagin multiplier λ . Let $\mu \in \mathcal{M}_R^Y$ be such that $y[\mu] = \bar{y}$. Then, for almost all t in $[0, T]$,

$$\int_{U_R} H[p_t^\lambda](u, \bar{y}_t) d\mu_t(u) = p_t^\lambda \dot{y}_t[\mu] = p_t^\lambda \dot{\bar{y}}_t = H[p_t^\lambda](\bar{u}_t, \bar{y}_t).$$

As a consequence, $\mu_t = \delta_{\bar{u}_t}$ and thus $\mu = \bar{\mu}$. The implication (iii) \implies (ii) follows. The implication (ii) \implies (i) is obvious.

Now assume that (i) holds. Let us prove the theorem by contradiction. We suppose that \bar{u} is a classical R -strong solution for the value η and we suppose that there exist a sequence $(\eta_k)_k$ of

positive real numbers converging to 0 and a sequence (μ^k) of feasible relaxed controls such that for all k ,

$$\|y[\mu^k] - \bar{y}\|_\infty \leq \eta^k, \quad \text{and} \quad \phi(y_T[\mu^k]) < \phi(\bar{y}_T).$$

Up to a subsequence, $(\mu^k)_k$ converges to some $\tilde{\mu}$, for the weak-* topology, and by lemma 56, this control $\tilde{\mu}$ is such that $y[\tilde{\mu}] = \bar{y}$. As a consequence, $\tilde{\mu}$ is qualified and by remark 19, there exists an open neighborhood \mathcal{O} of μ for the weak-* topology such that for all μ in \mathcal{O} , lemma 17 holds and

$$\|y[\mu] - \bar{y}\|_\infty \leq \frac{\eta}{2}.$$

Let j be sufficiently large so that μ^j belongs to \mathcal{O} . Then, let $(u^k)_k$ be a sequence of classical controls converging to μ^j for the weak-* topology, for k sufficiently large, lemma 17 can be applied to u^k . We obtain a sequence $(v^k)_k$ of feasible classical controls such that for k sufficiently large, $\|y[v^k] - y[\mu^j]\|_\infty \leq \eta$ and such that

$$\lim \phi(y_T[v^k]) = \phi(y_T[\tilde{\mu}]) < \phi(\bar{y}_T),$$

contradicting the fact that \bar{u} is an R -strong solution to the classical problem, with the value η . \square

3 First-order upper estimate of the value function

3.1 Framework

We introduce now a perturbation variable θ in the relaxed localized problem $(\mathcal{P}^{Y,\eta,R})$. We consider a reference value $\bar{\theta}$ for θ . The state equation is

$$\begin{cases} \dot{y}_t = \int_{U_R} f(u, y_t, \theta) d\mu_t(u), & \text{for a.a. } t \in [0, T], \\ y_0 = y^0(\theta). \end{cases} \quad (3.1)$$

We denote by $y[\mu, \theta]$ its solution and consider the following final state constraint:

$$\Phi(y_T, \theta) \in K, \quad (3.2)$$

where K is as in (2.4). The family of relaxed optimal control problems that we consider is

$$V(\theta) = \min_{\mu \in \mathcal{M}_R^Y} \phi(y_T[\mu, \theta], \theta), \quad \text{s.t. } \Phi(y_T[\mu, \theta], \theta) \in K, \quad \|y[\mu, \theta] - \bar{y}\|_\infty \leq \eta. \quad (\mathcal{P}_\theta^{Y,R,\eta})$$

The functions f , y^0 , ϕ , and Φ are always supposed to be $C^{2,1}$.

Our goal is to obtain a second-order expansion of $V(\theta)$ with respect to θ . To that purpose, we suppose without loss of generality that θ is of dimension 1 and that $\bar{\theta} = 0$ and we consider only perturbations such that $\theta \geq \bar{\theta} = 0$.

Note that for all $\theta \geq 0$ and $\eta > 0$, the relaxed problem has a solution. Indeed, consider a minimizing sequence, by compactness of \mathcal{M}_R^Y for the weak-* topology, it has a limit point which is an optimal solution by lemma 56.

The notations of section 2 extend to this new framework by adding the variable θ when necessary. We always consider a classical solution (\bar{u}, \bar{y}) to the relaxed problem for the value $\theta = 0$. We still use the Pontryagin linearization $\xi[\mu]$, which is taken for $\theta = 0$. Finally, we denote by $g[t] = g(\bar{u}_t, \bar{y}_t, 0)$.

As before, we suppose that the qualification condition (2.9) holds for $\theta = 0$. If one of the assumptions of theorem 20 is satisfied for the reference problem, it can be shown, using theorem

22 that for $\eta > 0$ and $\theta > 0$ sufficiently small, the problems $(\mathcal{P}_\theta^{Y,R,\eta})$ have the same value as their classical version.

In this section, we prove a first-order upper estimate of the value function V . The approach is very close to the one that we used to derive Pontryagin's principle.

3.2 First-order upper estimate

Denote by ξ^θ the solution of the following differential system:

$$\begin{cases} \dot{\xi}_t^\theta = f_y[t]\xi_t^\theta + f_\theta[t], & \text{for a.a. } t \in [0, T], \\ \xi_0^\theta = y_\theta^0(0). \end{cases}$$

Lemma 21. *The following estimates hold:*

$$\|y[\mu, \theta] - \bar{y}\|_\infty = O(d_1(\mu, \bar{\mu}) + \theta), \quad (3.3)$$

$$\|y[\mu, \theta] - (\bar{y} + \xi[\mu] + \theta\xi^\theta)\|_\infty = O(d_1(\mu, \bar{\mu})^2 + \theta^2). \quad (3.4)$$

Proof. For all t in $[0, T]$,

$$\begin{aligned} & |y[\mu, \theta]_t - \bar{y}_t| \\ &= \left| \int_0^t \int_{U_R} (f(u, y_s[\mu, \theta], \theta) - f(\bar{u}_s, \bar{y}_s, 0)) \, d\mu_s(u) \, ds \right| + |y^0(\theta) - y^0(0)| \\ &= \int_0^t \int_{U_R} O(|u - \bar{u}_t|) + O(|y_s[\mu, \theta] - \bar{y}_s| + \theta) \, d\mu_s(u) \, ds + O(\theta) \\ &= O(d_1(\mu, \bar{\mu})) + O(\theta) + \int_0^t O(|y[\mu, \theta]_s - \bar{y}_s|) \, ds, \end{aligned}$$

whence estimate (3.3) by Gronwall's lemma. Now, set $r = y[\mu, \theta] - (\bar{y} + \xi[\mu] + \theta\xi^\theta)$, then, for all t in $[0, T]$,

$$\begin{aligned} |r_t| &= \left| \int_0^t \left[-f_y[s]\xi_s[\mu] - f_y[s]\theta\xi_s^\theta - f_\theta[s]\theta \right. \right. \\ &\quad \left. \left. + \int_{U_R} (f(u, y_s[\mu, \theta], \theta) - f(u, \bar{y}_s, 0)) \, d\mu_s(u) \right] \, ds \right| \\ &\quad + |y^0(\theta) - [y^0(0) + \theta y_\theta^0(0)]| \\ &\leq \left| \int_0^t -f_y[s]\xi_s[\mu] - f_y[s]\theta\xi_s^\theta - f_\theta[s]\theta \right. \\ &\quad \left. + f(\bar{u}_s, y_s[\mu, \theta], \theta) - f(\bar{u}_s, \bar{y}_s, 0) \, ds \right| \\ &\quad + \int_0^t \int_{U_R} |f(u, y_s[\mu, \theta], \theta) - f(u, \bar{y}_s, 0) \\ &\quad - f(\bar{u}_s, y_s[\mu, \theta], \theta) + f(\bar{u}_s, \bar{y}_s, 0)| \, d\mu_s(u) \, ds + O(\theta^2) \\ &= \int_0^t |f_y[s](y_s[\mu, \theta] - (\bar{y}_s + \xi_s[\mu] + \theta\xi_s^\theta))| \, ds \\ &\quad + O(\|y[\mu, \theta] - \bar{y}\|_\infty^2) + d_1(\mu, \bar{\mu})(\|y[\mu] - \bar{y}\|_\infty + \theta) \\ &\quad + O(\theta^2) \\ &= \int_0^t O(|r_s|) \, ds + O(d_1(\mu, \bar{\mu})^2 + \theta^2). \end{aligned}$$

Estimate (3.4) follows with Gronwall's lemma. \square

Theorem 22. *If $\bar{\mu}$ is qualified, then there exist $\theta_2 > 0$, $\delta_2 > 0$ and $C_2 > 0$ such that for all θ in $[0, \theta_2]$ and for all μ in \mathcal{M}_R^Y satisfying $d_1(\mu, \bar{\mu}) \leq \delta_2$, there exists a control μ' such that*

$$\Phi(y_T[\mu', \theta], \theta) \in K \quad \text{and} \quad d_1(\mu, \mu') \leq C_2 \text{dist}(\Phi(y_T[\mu, \theta], \theta), K). \quad (3.5)$$

Proof. This theorem is a simple extension of theorem 8. We define $G_{\mu, \theta}$ and $\Psi_{\mu, \theta}$ by

$$G_{\mu, \theta} : \gamma \in \mathbb{R}^{n_A} \mapsto \Phi(y[S(\mu, \gamma), \theta]_T, \theta)$$

and

$$\Psi_{\mu, \theta} : \gamma \in \Delta \mapsto G_{\mu, \theta}(\gamma) - K.$$

Like previously, we can show by [5, theorem 2.84] that if (μ, θ) is sufficiently close to $(\bar{\mu}, 0)$, $\Psi_{\mu, \theta}$ is metric regular, and the theorem follows. \square

Consider the Pontryagin linearized problem

$$\begin{cases} \text{Min}_{\xi \in \mathcal{C}(\mathcal{R}_T)} \phi'(\bar{y}_T, 0)(\xi + \xi_T^\theta, 1), \\ \text{s.t. } \Phi'(\bar{y}_T, 0)(\xi + \xi_T^\theta, 1) \in T_K(\Phi(\bar{y}_T, 0)) \end{cases} \quad (PL_\theta)$$

Lemma 23. *The following upper estimate on the value function holds:*

$$V(\theta) \leq V(0) + \theta \text{Val}(PL_\theta) + o(\theta). \quad (3.6)$$

Proof. Let $(\theta_k)_k \downarrow 0$ be such that

$$\lim_{k \rightarrow \infty} \frac{V(\theta_k) - V(0)}{\theta_k} = \limsup_{\theta \downarrow 0} \frac{V(\theta) - V(0)}{\theta} \quad (3.7)$$

Let $\xi \in F(PL_\theta)$. There exists a sequence $(\alpha_k, \nu^k, \xi^k)_k$ in $\mathbb{R}_+ \times \mathcal{M}_R^Y \times \mathcal{R}_T$ such that

$$\xi = \lim \alpha_k \xi^k, \quad \text{and} \quad \xi^k = \xi_T[\nu^k], \quad \forall k.$$

Note that it may happen that $\alpha_k \rightarrow +\infty$. Extracting if necessary a subsequence of $(\theta_k)_k$, we can suppose that

$$\theta_k \alpha_k \leq 1, \quad \alpha_k^2 \leq \frac{1}{k \theta_k}.$$

We set

$$\mu^k = (1 - \theta_k \alpha_k) \bar{\mu} + \theta_k \alpha_k \nu^k.$$

Then $(\mu^k)_k$ is a sequence of Young measures and lemma 21 implies that

$$\|y[\mu^k, \theta_k] - (\bar{y} + \xi[\mu^k] + \theta_k \xi^\theta)\|_\infty = O(d_1(\mu^k, \bar{\mu})^2 + \theta_k^2).$$

Since

$$d_1(\mu^k, \bar{\mu})^2 = O(\theta_k^2 \alpha_k^2) = O\left(\frac{\theta_k}{k}\right) = o(\theta_k),$$

we obtain that

$$\|y[\mu^k, \theta_k] - [\bar{y} + \theta_k(\alpha_k \xi^k + \xi^\theta)]\|_\infty = o(\theta_k)$$

and since $\theta_k |\xi - \alpha_k \xi^k| = o(\theta_k)$,

$$\|y[\mu^k, \theta_k] - [\bar{y} + \theta_k(\xi + \xi^\theta)]\|_\infty = o(\theta_k).$$

We obtain the two following expansions:

$$\phi(y[\mu^k, \theta_k]_T, \theta_k) = \phi(\bar{y}_T, 0) + \theta_k \phi'(\bar{y}_T, 0)(\xi + \xi_T^\theta, 1) + o(\theta_k), \quad (3.8)$$

$$\Phi(y[\mu^k, \theta_k]_T, \theta_k) = \Phi(\bar{y}_T, 0) + \theta_k \Phi'(\bar{y}_T, 0)(\xi + \xi_T^\theta, 1) + o(\theta_k). \quad (3.9)$$

Since

$$\Phi'(\bar{y}_T, 0)(\xi + \xi_T^\theta, 1) \in T_K(\Phi(\bar{y}_T, 0)),$$

we obtain that

$$\text{dist}(\Phi(\bar{y}_T, 0) + \theta_k \Phi'(\bar{y}_T, 0)(\xi + \xi_T^\theta, 1), K) = o(\theta_k). \quad (3.10)$$

Combining (3.9) and (3.10), we obtain that $\text{dist}(\Phi(y_T[\mu^k], \theta_k), K) = o(\theta_k)$ and by the metric regularity theorem for the perturbed problem (theorem 22), we obtain the existence of a feasible sequence $\tilde{\mu}^k$ such that $d_1(\tilde{\mu}^k, \mu^k) = o(\theta_k)$. By lemma 56, estimate (3.8) holds for $\tilde{\mu}^k$ and therefore,

$$\begin{aligned} V(\theta_k) - V(\bar{\theta}) &\leq \phi(y_T[\tilde{\mu}^k, \theta_k], \theta_k) - \phi(\bar{y}_T, 0) \\ &= \theta_k \phi'(\bar{y}_T, 0)(\xi + \xi_T^\theta, 1) + o(\theta_k). \end{aligned}$$

Finally,

$$\limsup_{\theta \downarrow 0} \frac{V(\theta) - V(0)}{\theta} \leq \text{Val}(PL_\theta)$$

and the lemma follows. \square

Let us define (formally) the Lagrangian of the problem by

$$\mathcal{L}(u, y, \lambda, \theta) := p_0^\lambda y^0(\theta) + \int_0^T H[p_t^\lambda](u_t, y_t, \theta) dt + \Phi[\lambda](y_T, \theta)$$

and the dual linearized problem (DL_θ) by

$$\text{Max}_{\lambda \in \Lambda^P} \mathcal{L}_\theta(\bar{u}, \bar{y}, \lambda, \bar{\theta}), \quad (DL_\theta)$$

with

$$\mathcal{L}_\theta(\bar{u}, \bar{y}, \lambda, \bar{\theta}) := p_0^\lambda y_\theta^0(0) + \int_0^T H_\theta[p_t^\lambda][t] dt + \Phi_\theta[\lambda](\bar{y}_T, 0). \quad (3.11)$$

Theorem 24. *Under the qualification condition (2.9), problem (PL_θ) has the same value as its dual, problem (DL_θ) .*

Proof. Let us begin by checking the qualification condition. By (2.31),

$$\varepsilon B \subset \Phi(\bar{y}_T, 0) + \Phi_y(\bar{y}_T, 0)\mathcal{C}(\mathcal{R}_T) - T_K(\Phi(\bar{y}_T, 0)).$$

The r.h.s. contains necessarily the whole space \mathbb{R}^{n_C} , since it is a cone. Thus,

$$\varepsilon B \subset \mathbb{R}^{n_C} = \Phi(\bar{y}_T, 0) + \Phi'(\bar{y}_T, 0)(\xi_T^\theta + \mathcal{C}(\mathcal{R}_T), 1) - T_K(\Phi(\bar{y}_T, 0)),$$

which is the qualification condition for the linearized problem.

Now, let us study the dual problem, which is:

$$\text{Max}_{\lambda \in N_K(\bar{\Phi}(\bar{y}_T, 0))} \inf_{\xi \in \mathcal{C}(\mathcal{R}_T)} \Phi'[\lambda](\bar{y}_T, 0)(\xi_T^\theta + \xi, 1). \quad (3.12)$$

Following the proof of theorem 16, we obtain directly that this problem is equivalent to the following one:

$$\text{Max}_{\lambda \in \Lambda^P} \Phi'[\lambda](\bar{y}_T, 0)(\xi_T^\theta, 1). \quad (DL_\theta)$$

By lemma 13, we obtain that

$$\Phi'[\lambda](\bar{y}_T, 0)(\xi_T^\theta, 1) = \int_0^T H_\theta[p_t^\lambda][t] dt + p_0^\lambda y_\theta^0(0) + \Phi_\theta[\lambda](\bar{y}_T, 0).$$

Thus, the dual problem is equivalent to problem (DL_θ) and has the same value as problem (PL_θ) as a consequence of [5, theorem 2.165]. \square

4 Second-order upper estimate of the value function

In this section, we obtain a second-order upper estimate of the value function by using a “standard” linearization to the first order and a “Pontryagin” linearization to the second order. Indeed, to obtain a second-order estimate, we need to have a solution to some linearized first-order problem. Unfortunately, problem (PL_θ) is a conic linear problem, thus, it does not have necessarily a solution. This is why we consider now a different kind of linearization, which is such that the associated linearized problem has a solution.

In this section and in the sequel, we use properties of Young measures detailed in subsection A.3.

4.1 Basic tools

In the sequel, for p in $[1, +\infty]$, we will write L^p instead of $L^p(0, T; \mathbb{R}^m)$.

Definition 25. For a given ν in \mathcal{M}_2^Y , we define its mean value $\rho[\nu]$ in L^2 by

$$\rho_t[\nu] := \int_{\mathbb{R}^m} u d\nu_t(u), \quad \text{for a.a. } t \in [0, T].$$

This mapping is well-defined, affine, and Lipschitz continuous with modulus 1.

Proof. Let us check the Lipschitz continuity. Let $\nu^1, \nu^2 \in \mathcal{M}_2^Y$, let π a transportation plan from ν^1 to ν^2 be optimal for the L^2 -distance. Then,

$$\begin{aligned} \|\rho[\nu^2] - \rho[\nu^1]\|_2^2 &= \int_0^T \left[\int_{\mathbb{R}^m} u d\nu_t^1(u) - \int_{\mathbb{R}^m} v d\nu_t^2(v) \right]^2 dt \\ &= \int_0^T \left[\int_{\mathbb{R}^m \times \mathbb{R}^m} (u - v) d\pi_t(u, v) \right]^2 dt \\ &\leq \int_0^T \left[\int_{\mathbb{R}^m} |u - v|^2 d\pi_t(u, v) \int_{\mathbb{R}^m} 1 d\pi_t(u, v) \right] dt \\ &= d_2(\nu^1, \nu^2)^2, \end{aligned}$$

as was to be proved. \square

Definition 26. Let $\nu \in \mathcal{M}^Y$, $w \in L^\infty(0, T; \mathbb{R}^m)$, and $\theta \in \mathbb{R}$. We denote by

$$w \oplus \theta\nu$$

the unique Young measure μ in \mathcal{M}^Y such that for all g in $C^0([0, T] \times \mathbb{R}^m)$,

$$\int_0^T \int_{\mathbb{R}^m} g(t, u) d\mu_t(u) dt = \int_0^T \int_{\mathbb{R}^m} g(t, w_t + \theta u) d\nu_t(u).$$

If $\theta \neq 0$, we denote by

$$\frac{\nu \ominus w}{\theta}$$

the unique Young measure μ in \mathcal{M}^Y which is such that for all g in $C^0([0, T] \times \mathbb{R}^m)$,

$$\int_0^T \int_{\mathbb{R}^m} g(t, u) d\mu_t(u) dt = \int_0^T \int_{\mathbb{R}^m} g\left(t, \frac{u - w_t}{\theta}\right) d\nu_t(u).$$

We also denote: $\nu \ominus w = \frac{\nu \ominus w}{1}$.

The addition \oplus (resp. the subtraction \ominus) must be viewed as translations on \mathbb{R}^m of vector w_t (resp. $-w_t$) at each time t . The multiplication (resp. the division) by θ must be viewed as an homothety of ratio θ (resp. $\frac{1}{\theta}$) on \mathbb{R}^m , at each time t . Note that it will always be clear from the context if the multiplication (by constants), or the division, is the operation described in the previous definition or if it the multiplication of measures by constants, which we used up to now.

Lemma 27. Let ν be in \mathcal{M}_2^Y , let us set $v = \rho[\nu]$. Then, for all $\varepsilon > 0$, for all v' in L^2 , there exists ν' in \mathcal{M}_2^Y such that

$$\rho[\nu'] = v' \quad \text{and} \quad d_2(\nu, \nu') \leq \|v' - v\|_2 + \varepsilon,$$

and such that if v' is bounded, then ν' has a bounded support.

Proof. Let ν be in \mathcal{M}_2^Y , v' be in L^2 , and $\varepsilon > 0$. We set $v = \rho[\nu]$. Let us consider a measurable subset A of $[0, T]$ such that

$$\int_A \int_{\mathbb{R}^m} |u|^2 d\nu_t(u) dt \leq \frac{\varepsilon^2}{8}, \quad (4.1)$$

and such that v is bounded on $[0, T] \setminus A$. We observe first that by the dominated convergence theorem, we can fix C sufficiently large so that

$$\int_{[0, T] \setminus A} \int_{|u| > C} u^2 d\nu_t(u) dt \leq \frac{\varepsilon^2}{8}.$$

Let us denote by $\tilde{\nu}$ the unique Young measure which is such that, for all g in $C^0([0, T] \times \mathbb{R}^m)$,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^m} g(t, u) d\tilde{\nu}_t(u) dt \\ &= \int_{[0, T] \setminus A} \left[\int_{|u| \leq C} g(t, u) d\nu_t(u) + g(t, 0) \int_{|u| > C} 1 d\nu_t(u) \right] dt + \int_A g(t, 0) dt. \end{aligned}$$

By construction, $\tilde{\nu}$ is bounded (by C). This Young measure is such that

$$d_2(\nu, \tilde{\nu})^2 \leq \int_A \int_{\mathbb{R}^m} |u|^2 d\nu_t(u) dt + \int_{[0, T] \setminus A} \int_{|u| > C} |u|^2 d\nu_t(u) dt \leq \frac{\varepsilon^2}{4}. \quad (4.2)$$

Now, let us set $\tilde{v} = \rho[\tilde{\nu}]$. Then,

$$\|v - \tilde{v}\|_2 \leq d_2(\nu, \tilde{\nu}) \leq \frac{\varepsilon}{2}. \quad (4.3)$$

Finally, let us set $\nu' = (v' - \tilde{v}) \oplus \tilde{\nu}$. Then, $\rho[\nu'] = \rho[v'] - \rho[\tilde{v}] + \rho[\tilde{\nu}] = \rho[v'] = v'$ and

$$d_2(\nu', \tilde{\nu}) \leq \|v' - \tilde{v}\|_2. \quad (4.4)$$

Combining (4.2), (4.3), and (4.4), we obtain

$$\begin{aligned} d_2(\nu', \nu) &\leq d_2(\nu', \tilde{\nu}) + d_2(\tilde{\nu}, \nu) \\ &\leq \|v' - \tilde{v}\|_2 + \varepsilon/2 \\ &\leq \|v' - v\|_2 + \|v - \tilde{v}\|_2 + \varepsilon/2 \\ &\leq \|v' - v\|_2 + \varepsilon. \end{aligned}$$

Since $\tilde{\nu}$ is bounded, if v' is bounded, then ν' is also bounded. \square

4.2 Standard linearizations and estimates

Definition 28. For a given ν in \mathcal{M}_2^Y , we define the standard linearization $z[\nu]$ by

$$\begin{cases} \dot{z}_t[\nu] = f_y[t]z_t[\nu] + f_u[t]\rho_t[\nu], & \text{for a.a. } t \in [0, T], \\ z_0[\nu] = 0. \end{cases}$$

We also set $z^1[\nu] = z[\nu] + \xi^\theta$, which is the solution of the following system:

$$\begin{cases} \dot{z}_t^1[\nu] = f'[t](z_t^1[\nu], \rho_t[\nu], 1), & \text{for a.a. } t \in [0, T], \\ z_0^1[\nu] = y_\theta^0(0). \end{cases}$$

Lemma 29. For μ in \mathcal{M}_R^Y , the following estimates hold:

$$\|z[\mu] - \xi[\mu]\|_\infty = O(d_1(\bar{\mu}, \mu)^2), \quad (4.5)$$

$$\|y[\mu, \theta] - \bar{y}\|_\infty = O(\|\mu \ominus \bar{\mu}\|_1 + \theta), \quad (4.6)$$

$$\|y[\mu, \theta] - (\bar{y} + z[\mu \ominus \bar{\mu}] + \theta\xi^\theta)\|_\infty = O(\|\mu \ominus \bar{\mu}\|_1^2 + \theta^2). \quad (4.7)$$

Proof. The dynamic of $z = z[\mu \ominus \bar{\mu}]$ is the following:

$$\begin{cases} \dot{z}_t = f_y[t]z_t + f_u[t] \left(\int_{U_R} (u - \bar{u}_t) d\mu_t(u) \right), \\ z_0 = 0. \end{cases} \quad (4.8)$$

Note also that $\|\mu \ominus \bar{\mu}\|_1 = d_1(\bar{\mu}, \mu)$. Setting $r = \xi[\mu] - z[\mu \ominus \bar{\mu}]$, we obtain that for almost all t in $[0, T]$,

$$\begin{aligned} \dot{r}_t &= f_y[t]r_t + \int_{U_R} [f(\bar{y}_t, u) - (f[t] + f_u[t](u - \bar{u}_t))] d\mu_t(u) \\ &= O(|r_t|) + \int_{U_R} O(|u - \bar{u}_t|^2) d\mu_t(u), \end{aligned}$$

thus, by Gronwall's lemma, $\|r\|_\infty = O(\|\mu \ominus \bar{\mu}\|_1^2)$, which proves estimate (4.5). Replacing $\xi[\mu]$ by $z[\mu \ominus \bar{\mu}]$ in estimates (3.3) and (3.4) of lemma 21, we obtain estimates (4.6) and (4.7). \square

Corollary 30. For all ν in \mathcal{M}_∞^Y ,

$$z[\nu] = \lim_{\theta \downarrow 0} \frac{\xi[\bar{u} \oplus \theta\nu]}{\theta}.$$

Proof. By estimate (4.5), for $\theta > 0$ sufficiently small,

$$\left\| z[\nu] - \frac{\xi[\bar{u} \oplus \theta\nu]}{\theta} \right\|_\infty = \frac{1}{\theta} \|z[\theta\nu] - \xi[\bar{u} \oplus \theta\nu]\|_\infty = \frac{O(\theta^2)}{\theta} = O(\theta).$$

The corollary follows. \square

Remark 31. Denoting by \mathcal{C} the smallest closed convex cone containing $\{z[\nu]_T, \nu \in L^\infty\}$, we obtain by corollary 30 that $\mathcal{C} \subset \mathcal{C}(\mathcal{R}_T)$. A standard qualification condition for the problem would have been to assume that for some $\varepsilon' > 0$,

$$\varepsilon' B \subset \Phi(\bar{y}_T, 0) + \Phi'_{y_T}(\bar{y}_T, 0)\mathcal{C} - K.$$

This assumption is stronger than the qualification condition that we assumed.

4.3 Standard first-order upper estimate

Consider the standard linearized problem

$$\begin{cases} \text{Min}_{v \in L^2} \phi'(\bar{y}_T, 0)(z_T^1[v], 1), \\ \text{s.t. } \Phi'(\bar{y}_T, 0)(z_T^1[v], 1) \in T_K(\Phi(\bar{y}_T, 0)). \end{cases} \quad (SPL_\theta)$$

Definition 32. Let λ in $N_K(\Phi(\bar{y}_T, 0))$, we say that it is a Lagrange multiplier if for almost all t in $[0, T]$, $H_u[p_t^\lambda](\bar{u}_t, \bar{y}_t) = 0$. We denote by Λ^L the set of Lagrange multipliers.

Note that the inclusion $\Lambda^P \subset \Lambda^L$ holds.

Lemma 33. The dual of problem (SPL_θ) is the following problem:

$$\text{Max}_{\lambda \in \Lambda^L} \mathcal{L}_\theta(\bar{u}, \bar{y}, \lambda, 0), \quad (SDL_\theta)$$

it has the same value as the primal problem. Moreover, $\text{Val}(PL_\theta) \leq \text{Val}(SPL_\theta)$.

Proof. Remember the definition of the derivative of the Lagrangian, given by (3.11). The dual of problem (SPL_θ) is the following:

$$\text{Max}_{\lambda \in N_K(\Phi(\bar{y}_T, 0))} \text{Min}_{\nu \in L^2} \Phi'[\lambda](z_T^1[\nu], 1).$$

By lemma 13, we obtain directly that for all v in L^2 ,

$$\Phi'[\lambda](z_T^1[v], 1) = \mathcal{L}_\theta(\bar{u}, \bar{y}, \lambda, 0) + \int_0^T H_u[p_t^\lambda][t](v_t - \bar{u}_t) dt.$$

Let λ be in $N_K(\Phi(\bar{y}_T, 0))$. It can be easily checked that if λ is a Lagrange multiplier, then

$$\text{Min}_{v \in L^2} \int_0^T H_u[p_t^\lambda][t](v_t - \bar{u}_t) dt = 0,$$

while if λ is not a Lagrange multiplier, then

$$\text{Min}_{v \in L^2} \int_0^T H_u[p_t^\lambda][t](v_t - \bar{u}_t) dt = -\infty.$$

This proves that problem (SDL_θ) is the dual of problem (SPL_θ) . Moreover, it follows directly from the inclusion $\Lambda^P \subset \Lambda^L$ that

$$-\infty < \text{Val}(PL_\theta) = \text{Val}(DL_\theta) \leq \text{Val}(SDL_\theta) \leq \text{Val}(SPL_\theta),$$

We also obtain from the inclusion that problem (SDL_θ) is feasible. Since (SPL_θ) is linear and since the value of its dual is not $-\infty$, it follows by [5, theorem 2.204] that the two problems have the same value. \square

Remark 34. *In the previous lemma, we have obtained the inequality $\text{Val}(PL_\theta) \leq \text{Val}(SPL_\theta)$ by working with the associated dual problems. It would have been possible to show this inequality by working with the primal problems, and by using the inclusion*

$$\{z[\nu]_T, \nu \in \mathcal{M}_\infty^Y\} \subset \mathcal{C}(\mathcal{R}_T),$$

which derives from corollary 30.

From now, we suppose that the following restrictive assumption holds.

Hypothesis 35. *The Pontryagin and the classical linearized problems have the same value: $\text{Val}(SPL_\theta) = \text{Val}(PL_\theta)$.*

This hypothesis is satisfied in particular if the set of Lagrange multipliers is a singleton. This hypothesis is also satisfied if the Hamiltonian is convex with respect to u , since then the definitions of Lagrange and Pontryagin multipliers are equivalent.

Lemma 36. *The set $S(SPL_\theta)$ of solutions to problem (SPL_θ) is non-empty. Moreover, the intersection $S(SPL_\theta) \cap L^\infty$ is dense in $S(SPL_\theta)$ for the L^2 -distance.*

Proof. By hypothesis 35, problem (SPL_θ) has a finite value, thus, it has solutions (see [5, theorem 2.202]). Moreover, since L^∞ is dense in L^2 , we obtain by Dmitruk's density lemma (see [10]) that $S(SPL_\theta) \cap L^\infty$ is dense in $S(SPL_\theta) \cap L^2$ (for the L^2 -norm). \square

Now, let us consider the standard linearized problem in the space of Young measures \mathcal{M}_2^Y ,

$$\begin{cases} \text{Min}_{\nu \in \mathcal{M}_2^Y} \phi'(\bar{y}_T, 0)(z_T^1[\nu], 1), \\ \text{s.t. } \Phi'(\bar{y}_T, 0)(z_T^1[\nu], 1) \in T_K(\Phi(\bar{y}_T, 0)). \end{cases} \quad (SYPL_\theta)$$

Note first that for all ν in \mathcal{M}_2^Y , $\rho[\nu]$ belongs to L^2 and $z^1[\nu] = z^1[\rho[\nu]]$. As a consequence, problems (SPL_θ) and $(SYPL_\theta)$ have the same value.

Corollary 37. *Problem $(SYPL_\theta)$ has a non-empty set of solutions and $S(SYPL_\theta) \cap \mathcal{M}_\infty^Y$ is dense in $S(SYPL_\theta)$ for the L^2 -distance (on \mathcal{M}_2^Y).*

Proof. Since problems (SPL_θ) and $(SYPL_\theta)$ have the same value, we obtain the inclusion $S(SPL_\theta) \subset S(SYPL_\theta)$, which proves that $S(SYPL_\theta)$ is non-empty, by lemma 36. Let ν in

\mathcal{M}_2^Y be a solution to problem $(SYPL_\theta)$, then $\rho[\nu]$ is a solution to (SPL_θ) . Let $(v^k)_k$ be a sequence of solutions in L^∞ converging to $\rho[\nu]$. By lemma 27, we obtain the existence of a sequence $(\nu^k)_k$ of Young measures in \mathcal{M}_∞^Y such that

$$d_2(\nu, \nu^k) \leq \|v^k - \rho[\nu]\|_2 + \frac{1}{k} \rightarrow 0,$$

and such that for all k , $\rho[\nu^k] = v^k$, therefore for all k , ν^k is a solution to problem (SPL_θ) . This proves the corollary. \square

4.4 Second-order upper estimate

Definition 38. For a given ν in \mathcal{M}_2^Y , we define the second-order linearization $z^2[\nu]$ by

$$\begin{cases} z_t^2[\nu] = f_y[t]z_t^2[\nu] + \frac{1}{2} \int_{\mathbb{R}^m} f''[t](u, z_t^1[\nu], 1)^2 d\nu_t(u), \\ z_0^2[\nu] = \frac{1}{2} y_{\theta\theta}^0(0). \end{cases}$$

In the following problem, the notation T^2 refers to the second-order tangent set [5, definition 3.28]. Given a solution ν to problem $(SYPL_\theta)$, consider the following associated linearized problem:

$$\begin{cases} \text{Min}_{\xi \in \mathcal{C}(\mathcal{R}_T)} \frac{1}{2} \phi''(\bar{y}_T, 0)(z_T^1[\nu], 1)^2 + \phi_{y_T}(\bar{y}_T, 0)(z_T^2[\nu] + \xi), \\ \text{s.t. } \frac{1}{2} \Phi''(\bar{y}_T, 0)(z_T^1[\nu], 1)^2 + \Phi_{y_T}(\bar{y}_T, 0)(z_T^2[\nu] + \xi) \\ \qquad \qquad \qquad \in T_K^2(\Phi(\bar{y}_T, 0), \Phi'(\bar{y}_T, 0)(z_T^1[\nu], 1)). \end{cases} \quad (PQ_\theta(\nu))$$

Let us define the mapping Ω^θ on $\mathbb{R}^{n_E^*} \times \mathcal{M}_2^Y$ as follows:

$$\begin{aligned} \Omega^\theta[\lambda](\nu) := & \frac{1}{2} \left[\int_0^T \int_{\mathbb{R}^m} H''[p_t^\lambda][t](u, z_t^1[\nu], 1)^2 d\nu_t(u) dt \right. \\ & \left. + p_0^\lambda y_{\theta\theta}^0(0) + \Phi''[\lambda](\bar{y}_T, 0)(z_T^1[\nu], 1)^2 \right]. \end{aligned} \quad (4.9)$$

Theorem 39. For all ν in $S(SYPL_\theta) \cap \mathcal{M}_\infty^Y$, the following second-order upper estimate of the value function holds:

$$V(\theta) \leq V(0) + \theta \text{Val}(PL_\theta) + \theta^2 \text{Val}(PQ_\theta(\nu)) + o(\theta^2). \quad (4.10)$$

Proof. We follow the proof of lemma 23. Let $\nu \in S(SYPL_\theta) \cap \mathcal{M}_\infty^Y$, $\xi \in F(PQ_\theta(\nu))$, and $(\theta_k)_k \downarrow 0$ be such that

$$\lim_{k \rightarrow \infty} \frac{V(\theta_k^2) - [V(0) + \theta_k \text{Val}(PL_\theta)]}{\theta_k^2} = \limsup_{\theta \downarrow 0} \frac{V(\theta^2) - [V(0) + \theta \text{Val}(PL_\theta)]}{\theta^2}.$$

Let $(\tilde{\mu}_k, \alpha_k)_k$ be a sequence in $\mathcal{M}_R^Y \times \mathbb{R}_+$ such that $\xi = \lim \alpha_k \xi_T[\tilde{\mu}^k]$. Extracting a subsequence of $(\theta_k)_k$ if necessary, we can suppose that

$$\theta_k \alpha_k = o(1) \quad \text{and} \quad \alpha_k \theta_k^2 \leq 1.$$

We define

$$\mu^k = (1 - \alpha_k \theta_k^2)(\bar{u} \oplus \theta_k \nu) + (\alpha_k \theta_k^2) \tilde{\mu}^k.$$

Since $\alpha_k \theta_k^2 \leq 1$, μ^k is a Young measure. We set $y^k = y[\mu^k, \theta_k]$. Let us show the expansion

$$\|y^k - (\bar{y} + \theta_k z_1[\nu] + \theta_k^2(z^2[\nu] + \xi))\|_\infty = o(\theta_k^2). \quad (4.11)$$

We know that $d_1(\bar{\mu}, \mu^k) = O(\theta_k)$. Moreover,

$$\theta z[\nu] - z[\mu^k \ominus \bar{\mu}] = \alpha_k \theta_k^3 z[\nu] - \alpha_k \theta_k^2 z[\tilde{\mu}^k] = o(\theta_k),$$

thus, using lemma 29, we obtain that

$$\|y^k - (\bar{y} + \theta_k z^1[\nu])\|_\infty = o(\theta_k).$$

Let us set $r^k = y^k - (\bar{y} + \theta_k z^1[\nu] + \theta_k^2(z^2[\nu] + \alpha_k \xi[\tilde{\mu}^k]))$. Then,

$$\begin{aligned} r_t^k &= (1 - \alpha_k \theta_k^2) \int_0^t \int_B f(\bar{u}_s + \theta_k u, y_s^k, \theta_k) d\nu_t(u) ds \\ &\quad - \int_0^t \left(f[s] + \theta_k \int_B f'[s](u, z_s^1[\nu], 1) d\nu_t(u) \right) ds \\ &\quad - \theta_k^2 \int_B \frac{1}{2} f''[s](u, z_s^1[\nu], 1)^2 d\nu_t(u) ds \\ &\quad - \theta_k^2 \int_0^t f_y[s](z_s^2[\nu] + \alpha_k \xi[\tilde{\mu}^k]_s) ds \\ &\quad + \alpha_k \theta_k^2 \int_0^t \int_{U_R} f(u, y_s^k, \theta_k) d\tilde{\mu}_s^k(u) ds \\ &\quad - \alpha_k \theta_k^2 \int_0^t \int_{U_R} (f(u, \bar{y}_s, 0) - f[s]) d\tilde{\mu}_s^k(u) ds \\ &= (1 - \alpha_k \theta_k^2) \int_0^t \int_B \left(f(\bar{u}_s + \theta_k u, y_s^k, \theta_k) - f[s] - f'[s](\theta_k u, \theta_k z_s^1[\nu], \theta_k) \right) d\nu_t(u) ds \\ &\quad - \int_0^t \int_B \frac{1}{2} f''[s](\theta_k u, \theta_k z_s^1[\nu], \theta_k)^2 d\nu_t(u) ds \\ &\quad - \theta_k^2 \int_0^t f_y[s](z_s^2[\nu] + \alpha_k \xi[\tilde{\mu}^k]_s) ds \\ &\quad + \alpha_k \theta_k^2 \int_0^t \int_B (f(u, y_s^k, \theta_k) - f(u, \bar{y}_s, \theta_k)) d\tilde{\mu}_s^k(u) ds + o(\theta_k^2) \\ &= \int_0^t \int_B \left(f'[s](\theta_k u, \theta_k (y_s^k - \bar{y}_s), \theta_k) + \frac{1}{2} f''[s](\theta_k u, y_s^k - \bar{y}_s, \theta_k)^2 \right) d\nu_s(t) dt \\ &\quad - \int_0^t \int_B f'[s](\theta_k u, \theta_k z_s^1[\nu], \theta_k) d\nu_t(u) ds \\ &\quad - \int_0^t \int_B \frac{1}{2} f''[s](\theta_k u, \theta_k z_s^1[\nu], \theta_k)^2 d\nu_t(u) ds \\ &\quad - \theta_k^2 \int_0^t f_y[s](z_s^2[\nu] + \alpha_k \xi_s[\tilde{\mu}^k]) ds + o(\theta_k^2) \\ &= \int_0^t f'[s] r_s^k ds + o(\theta_k^2). \end{aligned}$$

By Gronwall's lemma, $\|r^k\|_\infty = o(\theta_k^2)$ and since $\alpha_k \xi_T[\mu^k] \rightarrow \xi$, expansion (4.11) follows. As a consequence, the two following second-order expansions hold:

$$\begin{aligned}\phi(y_T[\mu^k, \theta_k]) &= \phi(\bar{y}_T, 0) + \theta_k \phi'(\bar{y}_T, 0)(z_T^1[\nu], 1) \\ &\quad + \theta_k^2 \left[\frac{1}{2} \phi''(\bar{y}_T, 0)(z_T^1[\nu], 1)^2 + \phi_{y_T}(\bar{y}_T, 0)(z_T^2[\nu] + \xi) \right] + o(\theta_k^2), \\ \Phi(y_T[\mu^k, \theta_k]) &= \Phi(\bar{y}_T, 0) + \theta_k \Phi'(\bar{y}_T, 0)(z_T^1[\nu], 1) \\ &\quad + \theta_k^2 \left[\frac{1}{2} \Phi''(\bar{y}_T, 0)(z_T^1[\nu], 1)^2 + \Phi_{y_T}(\bar{y}_T, 0)(z_T^2[\nu] + \xi) \right] + o(\theta_k^2).\end{aligned}$$

We obtain that $\text{dist}(\Phi(y_T^k), K) = o(\theta_k^2)$. By the metric regularity theorem (theorem 22) and by lemma 56, there exists a sequence $\hat{\mu}^k$ of feasible controls such that $d_1(\mu^k, \hat{\mu}^k) = o(\theta_k^2)$ and such that

$$\begin{aligned}\phi(y[\hat{\mu}^k, \theta_k]_T) &= \phi(\bar{y}_T, 0) + \theta_k \phi'(\bar{y}_T, 0)(z_T^1[\nu], 1) \\ &\quad + \theta_k^2 \left[\frac{1}{2} \phi''(\bar{y}_T, 0)(z_T^1[\nu], 1)^2 + \phi_{y_T}(\bar{y}_T, 0)(z_T^2[\nu] + \xi) \right] + o(\theta_k^2).\end{aligned}$$

Minimizing with respect to ξ , we obtain that

$$\limsup_{\theta \downarrow 0} \frac{V(\theta^2) - [V(0) + \theta \text{Val}(PL_\theta)]}{\theta^2} \leq \text{Val}(PQ_\theta(\nu)).$$

The theorem follows. \square

Lemma 40. *If \mathcal{M}_2^Y is equipped with the L^2 -distance, then $(\lambda, \nu) \mapsto \Omega^\theta[\lambda](\nu)$ is continuous.*

Proof. Let $\lambda \in \mathbb{R}^{nE^*}$ and $\nu \in \mathcal{M}_2^Y$. Let $(\lambda_k)_k \rightarrow \lambda$ be a sequence in \mathbb{R}^{nE^*} and $(\nu^k)_k \rightarrow \nu$ a sequence in \mathcal{M}_2^Y . Let us show that $\Omega^\theta[\lambda_k](\nu^k) \rightarrow \Omega^\theta[\lambda](\nu)$. By lemma 58, we know that $(z^1[\nu^k])_k$ and $(p^{\lambda_k})_k$ uniformly converge to resp. $z^1[\nu]$ and p^λ . Then, it suffices to study the convergence of

$$\begin{aligned}& |\Omega^\theta[\lambda^k](\nu^k) - \Omega^\theta[\lambda](\nu)| \\ & \leq \left| \int_0^T \int_{\mathbb{R}^m} \left(p_t^{\lambda_k} f''[t](u, z_t^1[\nu^k], 1)^2 - p_t^\lambda f''[t](u, z_t^1[\nu], 1)^2 \right) d\nu_t^k(u) dt \right| \\ & \quad + \left| \int_0^T \int_{\mathbb{R}^m} H''[p_t^\lambda][t](u, z_t^1[\nu], 1)^2 [d\nu_t^k(u) - d\nu_t(u)] dt \right| + o(1).\end{aligned}$$

The convergence to 0 of the first term is easily checked. For the second one, let us consider for all k the optimal transportation plan π^k between ν and ν^k for the L^2 -distance. The second term is equal to

$$\begin{aligned}& \int_0^T \int_{\mathbb{R}^m \times \mathbb{R}^m} H''[p_t^\lambda][t](u, z_t^1[\nu], 1)^2 - H''[p_t^\lambda][t](v, z_t^1[\nu], 1)^2 d\pi_t^k(u, v) dt \\ & = \int_0^T \int_{\mathbb{R}^m \times \mathbb{R}^m} H''[p_t^\lambda][t](u - v, 0, 0)(u + v, 2z_t^1[\nu], 2) d\pi_t^k(u, v) dt \\ & = \int_0^T \int_{\mathbb{R}^m \times \mathbb{R}^m} O(|u - v| \cdot |1 + u + v|) d\pi_t^k(u, v) dt \\ & = \left(\int_0^T \int_{\mathbb{R}^m \times \mathbb{R}^m} O(|u - v|^2) d\pi_t^k(u, v) dt \right)^{1/2}\end{aligned}$$

$$\begin{aligned} & \cdot \left(\int_0^T \int_{\mathbb{R}^m \times \mathbb{R}^m} O(|1+u+v|^2) d\pi_t^k(u,v) dt \right)^{1/2} \\ & = O(d_2(\nu, \nu^k)(1 + \|\nu\|_2 + \|\nu^k\|_2)), \end{aligned}$$

thus, it converges to 0, which proves the lemma. \square

Lemma 41. *The dual of problem $(PQ_\theta(\nu))$ is the following problem,*

$$\text{Max}_{\lambda \in S(DL_\theta)} \frac{1}{2} \Omega^\theta[\lambda](\nu), \quad (DQ_\theta(\nu))$$

and it has the same value as $(PQ_\theta(\nu))$.

Proof. It is proved in [5, proposition 3.34, equality 3.64] that since K is polyhedic,

$$T_K^2(\Phi(\bar{y}_T, 0), \Phi'(\bar{y}_T, 0)(z_T^1[\nu], 1)) = T_K(\Phi(\bar{y}_T, 0) + \Phi'(\bar{y}_T, 0)(z_T^1[\nu], 1)\mathbb{R},$$

where the addition $+$ is the Minkowski sum. Since the second-order tangent set is included into the tangent cone, we obtain, like in the proof of theorem 24 that

$$\begin{aligned} \varepsilon B \subset \mathbb{R}^{n_c} & = \frac{1}{2} \Phi''(\bar{y}_T, 0)(z_T^1[\nu], 1)^2 + \Phi_{y_T}(\bar{y}_T, 0)(z_T^2[\nu] + \mathcal{C}(\mathcal{R}_T)) \\ & \quad - T_K^2(\Phi(\bar{y}_T, 0), \Phi'(\bar{y}_T, 0)(z_T^1[\nu], 1)), \end{aligned}$$

which is the qualification condition. By [5, theorem 2.165], problem $(PQ_\theta(\nu))$ has the same value as its dual.

Let us denote by N the polar cone of the second-order tangent set. For all λ in $\mathbb{R}^{n_c^*}$,

$$\lambda \in N \iff \{ \lambda \in N_K(\Phi(\bar{y}_T, 0)) \quad \text{and} \quad \lambda \Phi'(\bar{y}_T, 0)(z_T^1[\nu], 1) = 0 \}.$$

Following the proofs of theorems 16 and 24, we obtain that the dual of problem $(PQ_\theta(\nu))$ is the following problem:

$$\begin{aligned} & \text{Max}_{\substack{\lambda \in \Lambda^P, \\ \lambda \Phi'(\bar{y}_T, 0)(z_T^1[\nu], 1) = 0}} \Phi''[\lambda](\bar{y}_T, 0)(z_T^1[\nu], 1)^2 + \Phi_{y_T}[\lambda](z_T^2[\nu]). \end{aligned}$$

Using lemma 13, we obtain that

$$\Phi_{y_T}[\lambda]z_T^2[\nu] = \frac{1}{2} \left[p_0^\lambda y_{\theta\theta}^0(0) + \int_0^T \int_{\mathbb{R}^m} H''[p_t^\lambda][t](u, z_t^1[\nu], 1)^2 d\nu_t(u) \right].$$

Moreover, by lemma 13 and hypothesis (35), for all λ in Λ^P ,

$$\begin{aligned} & \lambda \Phi'(\bar{y}_T, 0)(z_T^1[\nu], 1) = 0 \\ & \iff \Phi'[\lambda](\bar{y}_T, 0)(z_T^1[\nu], 1) = \phi'(\bar{y}_T, 0)(z_T^1[\nu], 1) \\ & \iff p_0^\lambda y_{\theta\theta}^0(0) + \int_0^T H_\theta[p_t^\lambda][t] dt + \Phi_\theta[\lambda](\bar{y}_T, 0) = \text{Val}(SPL_\theta) \\ & \iff L_\theta(\bar{u}, \bar{y}, \lambda, 0) = \text{Val}(PL_\theta) \\ & \iff \lambda \in S(DL_\theta). \end{aligned}$$

The lemma follows. \square

Remark 42. If problems (PL_θ) and (SPL_θ) do not have the same value, then the feasible set of problem $(DQ_\theta(\nu))$ is $S(SDL_\theta) \cap \Lambda^P$, which is then empty, and thus, $\text{Val}(DQ_\theta) = -\infty$.

Consider the problem (PQ_θ) defined by

$$\text{Min}_{\nu \in S(SYPL_\theta)} \text{Val}(PQ_\theta(\nu)). \quad (PQ_\theta)$$

Corollary 43. The following second-order upper estimate holds:

$$V(\theta) \leq V(0) + \theta \text{Val}(PL_\theta) + \theta^2 \text{Val}(PQ_\theta) + o(\theta^2). \quad (4.12)$$

Proof. Let us set

$$A = \limsup_{\theta \downarrow 0} \frac{V(\theta) - [V(0) + \theta \text{Val}(PL_\theta)]}{\theta^2}.$$

By theorem 39 and lemma 41, we already know that for all ν in $S(SYPL_\theta) \cap \mathcal{M}_\infty^Y$,

$$A \leq \text{Val}(PQ_\theta(\nu)) = \text{Max}_{\lambda \in S(DL_\theta)} \Omega^\theta[\lambda](\nu). \quad (4.13)$$

Now, let ν in $S(SYPL_\theta)$, let $(\nu^k)_k$ be a sequence of solutions to $S(SYPL_\theta)$ in \mathcal{M}_∞ converging to ν for the L^2 -distance. The existence of this sequence is given by corollary 37. For all k , let us denote by λ_k a solution to

$$\text{Max}_{\lambda \in S(DL_\theta)} \Omega^\theta[\lambda](\nu^k).$$

This solution exists, since $S(DL_\theta)$ is compact (being closed and bounded, by theorem 16) and since Ω^θ is continuous (by lemma 40). Extracting if necessary a subsequence, we can suppose that λ^k converges to some $\tilde{\lambda}$ in $S(DL_\theta)$. By continuity of Ω^θ , we obtain that

$$\limsup_{k \rightarrow \infty} \left(\text{Max}_{\lambda \in S(DL_\theta)} \Omega^\theta[\lambda](\nu^k) \right) \leq \Omega^\theta[\tilde{\lambda}](\nu) \leq \text{Max}_{\lambda \in S(DL_\theta)} \Omega^\theta[\lambda](\nu). \quad (4.14)$$

Combining (4.13) and (4.14), we obtain that

$$A \leq \inf_{\nu \in S(SYPL_\theta)} \text{Max}_{\lambda \in S(DL_\theta)} \Omega^\theta[\lambda](\nu)$$

and the result follows. \square

5 Lower estimates of the value function

5.1 A decomposition principle

In the family of optimization problems that we consider, the expression $\Phi[\lambda](y_T, \theta)$ plays the role of a Lagrangian. The basic idea to obtain a lower estimate for the value function is to use a second-order expansion of the right-hand-side of the following inequality:

$$\phi(y_T, \theta) - \phi(\bar{y}_T, 0) \geq \Phi[\lambda](y_T, \theta) - \Phi[\lambda](\bar{y}_T, 0), \quad (5.1)$$

for a feasible trajectory y (for the perturbed problem $(\mathcal{P}_\theta^{Y,R,\eta})$). This inequality holds since

$$\Phi(y_T, \theta) - \Phi(\bar{y}_T, 0) \in T_K(\Phi(\bar{y}_T, 0)) \quad \text{and} \quad \lambda \in N_K(\Phi(\bar{y}_T, 0)).$$

The main difficulty in computing an expansion of the difference of Lagrangians is that we cannot perform Taylor expansions with respect to the control variable, since we are interested by perturbations of the control which are not small for the L^∞ -norm. The idea to deal with this difficulty is to split the control into two intermediate controls, one accounting for the small perturbations and one accounting for the large perturbations (both for the L^∞ -norm). The decomposition principle that we obtain is an extension of [4, theorem 2.13].

In this part, we fix a sequence $(\theta_k)_k \downarrow 0$ and a sequence $(\mu^k, y^k)_k$ of feasible trajectories for the perturbed problems with $\theta = \theta_k$. We set $\delta y^k = y^k - \bar{y}$. We also fix $\lambda \in S(DL_\theta)$. In the proofs of lemma 44, corollary 45, lemma 46, theorem 47, and corollary 48, we omit to mention the dependence of the Hamiltonian with respect to p_t^λ (since the multiplier λ is fixed). For example, we will write $H(u, \bar{y}_t, \theta)$ instead of $H[p_t^\lambda](u, \bar{y}_t, \theta)$. Moreover, we set $R_{1,k} = d_1(\bar{\mu}, \mu^k)$. Note that by lemma 21, $\|\delta y_k\|_\infty = O(R_{1,k} + \theta_k)$.

Lemma 44. *The following expansion holds:*

$$\begin{aligned} & \Phi[\lambda](y_T^k, \theta_k) - \Phi[\lambda](\bar{y}_T, 0) \\ &= \text{Val}(PL_\theta)\theta_k \\ &+ \int_0^T \int_{U_R} (H[p_t^\lambda](u, \bar{y}_t, 0) - H[p_t^\lambda][t]) d\mu_t(u) dt \\ &+ \int_0^T \int_{U_R} (H_y[p_t^\lambda](u, \bar{y}_t, 0) - H_y[p_t^\lambda][t]) \delta y_t^k d\mu_t^k(u) dt \end{aligned} \quad (5.2a)$$

$$+ \int_0^T \int_{U_R} (H_\theta[p_t^\lambda](u, \bar{y}_t, 0) - H_\theta[p_t^\lambda][t]) \theta_k d\mu_t^k(u) dt \quad (5.2b)$$

$$\begin{aligned} &+ \frac{1}{2} \left[p_{\theta\theta}^0(0) \theta_k^2 + \int_0^T H_{(y,\theta)^2}[p_t^\lambda][t] (\delta y_t^k, \theta_k)^2 dt + \Phi''[\lambda](\delta y_T^k, \theta_k)^2 \right] \\ &+ o(\theta_k^2 + R_{1,k}^2). \end{aligned} \quad (5.2c)$$

Proof. Expanding the difference of Lagrangians up to the second order, we obtain

$$\begin{aligned} & \Phi[\lambda](y_T^k, \theta_k) - \Phi[\lambda](\bar{y}_T, 0) \\ &= \Phi'[\lambda](\bar{y}_T, 0)(\delta y_T^k, \theta_k) + \frac{1}{2} \Phi''[\lambda](\delta y_T^k, \theta_k)^2 + o(\theta_k^2 + |\delta y_T^k|^2). \end{aligned} \quad (5.3)$$

We also have

$$\begin{aligned} & \Phi_{y_T}[\lambda](\bar{y}_T, 0) \delta y_T^k \\ &= p_T^\lambda \delta y_T^k \\ &= [p_t^\lambda \delta y_t^k]_0^T + p_0^\lambda \delta y_0^k \\ &= \int_0^T (p_t^\lambda \delta y_t^k + \dot{p}_t^\lambda \delta y_t^k) dt + p_0^\lambda [y^0(\theta_k) - y^0(0)] \\ &= \int_0^T \left(\int_{U_R} (H(u, y_t^k, \theta_k) - H[t]) d\mu_t^k(u) - H_y[t] \delta y_t^k \right) dt \\ &+ p_0^\lambda \left[y_\theta^0(0) \theta_k + \frac{1}{2} y_{\theta\theta}^0(0) \theta_k^2 + o(\theta_k^2) \right]. \end{aligned} \quad (5.4)$$

Moreover,

$$\begin{aligned}
& \int_0^T \int_{U_R} (H(u, y_t^k, \theta_k) - H[t]) \, d\mu_t^k(u) \, dt \\
&= \int_0^T \int_{U_R} (H(u, y_t^k, \theta_k) - H(u, \bar{y}_t, 0)) \, d\mu_t^k(u) \, dt \\
&\quad + \int_0^T \int_{U_R} (H(u, \bar{y}_t, 0) - H[t]) \, d\mu_t^k(u) \, dt \\
&= \int_0^T \int_{U_R} H_{(y,\theta)}(u, \bar{y}_t, 0)(\delta y_t^k, \theta_k) \, d\mu_t^k(u) \, dt \\
&\quad + \int_0^T \int_{U_R} \frac{1}{2} H_{(y,\theta)^2}(u, \bar{y}_t, 0)(\delta y_t^k, \theta_k)^2 \, d\mu_t^k(u) \, dt \\
&\quad + \int_0^T \int_{U_R} (H(u, \bar{y}_t, 0) - H[t]) \, d\mu_t^k(u) \, dt \\
&\quad + o(\theta_k^2 + R_{1,k}^2) \\
&= \int_0^T \int_{U_R} H_{(y,\theta)}(u, \bar{y}_t, 0)(\delta y_t^k, \theta_k) \, d\mu_t^k(u) \, dt \\
&\quad + \int_0^T \int_{U_R} \frac{1}{2} H_{(y,\theta)^2}(u, \bar{y}_t, 0)(\delta y_t^k, \theta_k)^2 \, d\mu_t^k(u) \, dt \\
&\quad + \int_0^T \int_{U_R} (H(u, \bar{y}_t, 0) - H[t]) \, d\mu_t^k(u) \, dt \\
&\quad + o(\theta_k^2 + R_{1,k}^2). \tag{5.5}
\end{aligned}$$

Remember that

$$\text{Val}(PL_\theta) = p_0^\lambda y_\theta^0(0) + \int_0^T H_\theta[t] \, dt + \Phi_\theta[\lambda](\bar{y}_T, 0). \tag{5.6}$$

Finally,

$$\begin{aligned}
& \int_0^T \int_{U_R} \frac{1}{2} H_{(y,\theta)^2}(u, \bar{y}_t, 0)(\delta y_t^k, \theta_k)^2 \, d\mu_t^k(u) \, dt \\
&= \int_0^T \int_{U_R} \frac{1}{2} H_{(y,\theta)^2}[t](\delta y_t^k, \theta_k)^2 \, d\mu_t^k(u) \, dt + O(R_{1,k}(R_{1,k}^2 + \theta_k^2)). \tag{5.7}
\end{aligned}$$

and

$$\begin{aligned}
R_{1,k}(R_{1,k}^2 + \theta_k^2) &= R_{1,k}^3 + R_{1,k}\theta_k^{1/2} \cdot \theta_k^{3/2} \\
&= o(R_{1,k}^2) + O(R_{1,k}^2\theta_k + \theta_k^3) \\
&= o(R_{1,k}^2 + \theta_k^2). \tag{5.8}
\end{aligned}$$

Combining expansions (5.3-5.8), we obtain the lemma. \square

Corollary 45. *The following expansion holds:*

$$\begin{aligned} & \Phi[\lambda](y_T^k, \theta_k) - \Phi[\lambda](\bar{y}_T, 0) \\ &= \text{Val}(PL_\theta)\theta_k + \int_0^T \int_{U_R} H[p_t^\lambda](u, \bar{y}_t, 0) - H[p_t^\lambda][t] d\mu_t(u) dt \\ & \quad + O(R_{1,k} \|\delta y^k\|_\infty) + O(R_{1,k}\theta_k) + O(\theta_k^2) + o(R_{1,k}^2). \end{aligned} \quad (5.9)$$

Proof. This corollary follows directly from the expansion given in lemma 44. We replace respectively terms (5.2a), (5.2b), and (5.2c) by the following estimates: $O(R_{1,k} \|\delta y^k\|_\infty)$, $O(R_{1,k}\theta_k)$, $O(\theta_k^2 + \|\delta y^k\|^2) = O(\theta_k^2 + R_{1,k}\theta_k)$, and the corollary follows. \square

From now, we set $z^k := z[\mu^k \ominus \bar{u}]$. Note that the dynamic of z^k is given by equation (4.8).

Lemma 46. *The following expansion holds:*

$$\begin{aligned} & \Phi[\lambda](y_T^k, \theta_k) - \Phi[\lambda](\bar{y}_T, 0) \\ &= \text{Val}(PL_\theta)\theta_k \\ & \quad + \int_0^T \int_{U_R} (H[p_t^\lambda](u, \bar{y}_t, 0) - H[p_t^\lambda][t]) d\mu_t(u) dt \end{aligned} \quad (5.10a)$$

$$+ \int_0^T \int_{U_R} (H_y[p_t^\lambda](u, \bar{y}_t, 0) - H_y[p_t^\lambda][t])(z_t^k + \theta_k \xi_t^\theta) d\mu_t^k(u) dt \quad (5.10b)$$

$$+ \int_0^T \int_{U_R} (H_\theta[p_t^\lambda](u, \bar{y}_t, 0) - H_\theta[p_t^\lambda][t])\theta_k d\mu_t^k(u) dt \quad (5.10c)$$

$$+ \frac{1}{2} \left[p_{\theta\theta}^0(0)\theta_k^2 + \int_0^T H_{(y,\theta)^2}[p_t^\lambda][t](z_t^k + \xi_t^\theta \theta_k, \theta_k)^2 dt \right] \quad (5.10d)$$

$$+ \Phi''[\lambda](z_T^k + \xi_T^\theta \theta_k, \theta_k)^2 \quad (5.10e)$$

$$+ o(\theta_k^2 + R_{1,k}^2).$$

Proof. We have already proved in lemma 29 the following estimate:

$$\|\delta y^k - (z^k + \theta_k \xi^\theta)\|_\infty = O(R_{1,k}^2 + \theta_k^2).$$

Therefore, we replace δy^k by its standard expansion $z^k + \theta_k \xi^\theta$ in terms (5.2a) and (5.2c) of lemma 44. The errors that we make are respectively of order $R_{1,k}(R_{1,k}^2 + \theta_k^2)$ and $R_{1,k}^4 + \theta_k^4$. We have already proved (see estimate (5.3)) that

$$R_{1,k}(R_{1,k}^2 + \theta_k^2) = o(R_{1,k}^2 + \theta_k^2).$$

The lemma follows. \square

Now, in order to go further in the expansions, we need to split the control μ^k into two controls. To that purpose, we consider a sequence $(A^k, B^k)_k$ of measurable subsets of $[0, T] \times U_R$ such that for all k , (A^k, B^k) is a measurable partition of $[0, T] \times U_R$. We consider the Young measures $\mu^{A,k}$ and $\mu^{B,k}$ which are the unique Young measures such that for all g in $C^0([0, T] \times U_R)$,

$$\begin{cases} \int_0^T \int_{U_R} g(t, u) d\mu_t^{A,k}(u) dt = \int_{A^k} g(t, u) d\mu^k(t, u) + \int_{B^k} g(t, \bar{u}_t) d\mu^k(t, u), \\ \int_0^T \int_{U_R} g(t, u) d\mu_t^{B,k}(u) dt = \int_{B^k} g(t, u) d\mu^k(t, u) + \int_{A^k} g(t, \bar{u}_t) d\mu^k(t, u). \end{cases}$$

Note that if g is such that for almost all t in $[0, T]$, $g(t, \bar{u}_t) = 0$, then

$$\int_0^T \int_{U_R} g(t, u) d\mu_t^k(u) dt = \int_0^T \int_{U_R} g(t, u) d\mu_t^{A,k}(u) dt + \int_0^T \int_{U_R} g(t, u) d\mu_t^{B,k}(u).$$

For $i = 1, 2$, we set $R_{i,A,k} := d_i(\bar{\mu}, \mu^{A,k})$ and $R_{i,B,k} := d_i(\bar{\mu}, \mu^{B,k})$. We also set $z^{A,k} := z[\mu^{A,k} \ominus \bar{u}]$, and $z^{B,k} := z[\mu^{B,k} \ominus \bar{u}]$. Remember the definition of Ω^θ given by (4.9).

Theorem 47 (Decomposition principle). *Assume that*

$$\mu^k(B^k) \longrightarrow 0 \quad \text{and} \quad \text{ess sup}_{k \rightarrow \infty} \{|u - \bar{u}_t|, (t, u) \in A^k\} \rightarrow 0. \quad (5.11)$$

Then,

$$z^k = z^{A,k} + o(R_{2,A,k}) \quad (5.12)$$

and the following expansion holds:

$$\begin{aligned} & \Phi[\lambda](y_T^k, \theta_k) - \Phi[\lambda](\bar{y}_T, 0) \\ &= \text{Val}(PL_\theta)\theta_k + \frac{1}{2}\Omega^\theta[\lambda](\mu^{A,k} \ominus \bar{u}) \\ & \quad + \int_0^T \int_{U_R} H[p_t^\lambda](u, \bar{y}_t, 0) - H[p_t^\lambda][t] d\mu_t^{B,k}(u) dt \\ & \quad + o(R_{2,A,k}^2 + R_{2,B,k}^2 + \theta_k^2). \end{aligned} \quad (5.13)$$

Proof. With the Cauchy-Schwartz inequality, we get $R_{1,A,k} = O(R_{2,A,k})$ and since $\mu^k(B^k) \rightarrow 0$,

$$\begin{aligned} R_{1,B,k} &= \int_{B^k} |u - \bar{u}_t| d\mu_t^k(t, u) dt \\ &\leq (\mu^k(B^k))^{1/2} \left[\int_{B^k} |u - \bar{u}_t|^2 d\mu^k(t, u) \right]^{1/2} = o(R_{2,B,k}). \end{aligned} \quad (5.14)$$

Estimate (5.12) follows from (5.14) and $z^k = z^{A,k} + z^{B,k}$. In order to obtain expansion (5.13), we work with the terms of the expansion of lemma 46.

With term (5.10a), we obtain

$$\begin{aligned} & \int_0^T \int_{U_R} (H(u, \bar{y}_t, 0) - H[t]) d\mu_t^k(u) dt \\ &= \int_0^T \int_{U_R} (H(u, \bar{y}_t, 0) - H[t]) d\mu_t^{A,k}(u) dt \\ & \quad + \int_0^T \int_{U_R} (H(u, \bar{y}_t, 0) - H[t]) d\mu_t^{B,k}(u) dt \\ &= \int_0^T \int_{U_R} H_{uu}[t](u - \bar{u}_t)^2 d\mu_t^{A,k}(u) dt \\ & \quad + \int_0^T \int_{U_R} (H(u, \bar{y}_t, 0) - H[t]) d\mu_t^{B,k}(u) dt \\ & \quad + o(R_{2,A,k}^2). \end{aligned} \quad (5.15)$$

With term (5.10b), we obtain

$$\begin{aligned}
& \int_0^T \int_{U_R} (H_y(u, \bar{y}_t, 0) - H_y[t])(z_t^k + \theta_k \xi_t^\theta) d\mu_t^k(u) dt \\
&= \int_0^T \int_{U_R} (H_y(u, \bar{y}_t, 0) - H_y[t])(z_t^{A,k} + \theta_k \xi_t^\theta) d\mu_t^{A,k}(u) dt \\
&\quad + \int_0^T \int_{U_R} (H_y(u, \bar{y}_t, 0) - H_y[t]) z_t^{B,k} d\mu_t^{A,k}(u) dt \\
&\quad + \int_0^T \int_{U_R} (H_y(u, \bar{y}_t, 0) - H_y[t])(z_t^{B,k} + \xi_t^\theta \theta_k) d\mu_t^{B,k}(u) dt \\
&= \int_0^T \int_{U_R} H_{u,y}[t](u - \bar{u}_t, z_t^{A,k} + \theta_k \xi_t^\theta) d\mu_t^{A,k}(u) dt \\
&\quad + O(R_{2,A,k}^2(R_{1,A,k} + \theta_k)) + O(R_{1,B,k}(R_{1,A,k} + R_{1,B,k} + \theta_k)). \tag{5.16}
\end{aligned}$$

We also have that

$$R_{2,A,k}^2(R_{1,A,k} + \theta_k) = o(R_{2,A,k}^2) \tag{5.17}$$

and

$$\begin{aligned}
& R_{1,B,k}(R_{1,A,k} + R_{1,B,k} + \theta_k) \\
&= o(R_{2,B,k}(R_{2,A,k} + R_{2,B,k} + \theta_k)) \\
&= o(R_{2,A,k}^2 + R_{2,B,k}^2 + \theta_k^2). \tag{5.18}
\end{aligned}$$

With term (5.10c), we obtain similarly that

$$\begin{aligned}
& \int_0^T \int_{U_R} (H_\theta(u, \bar{y}_t, 0) - H_\theta[t]) \theta_k d\mu_t^k(u) dt \\
&= \int_0^T \int_{U_R} H_{(u,\theta)}[t](u - \bar{u}_t, \theta_k) d\mu_t^{A,k}(u) dt + O(R_{2,A,k}^2 \theta_k) + O(R_{1,B,k} \theta_k). \tag{5.19}
\end{aligned}$$

We also have that

$$R_{2,A,k}^2 \theta_k = o(R_{2,A,k}^2) \quad \text{and} \quad R_{1,B,k} \theta_k = o(R_{2,B,k} \theta_k) = o(R_{2,B,k}^2 + \theta_k^2). \tag{5.20}$$

With terms (5.10d-5.10e), we obtain

$$\begin{aligned}
& \int_0^T \int_{U_R} H_{(y,\theta)^2}[t](z_t^k + \theta_k \xi_t^\theta, \theta_k)^2 dt + \Phi''[\lambda](z_T^k + \theta_k \xi_T^\theta, \theta_k)^2 \\
&= \int_0^T \int_{U_R} H_{(y,\theta)^2}[t](z_t^{A,k} + \theta_k \xi_t^\theta, \theta_k)^2 d\mu_t^{A,k}(u) dt \\
&\quad + \Phi''[\lambda](z_T^{A,k} + \theta_k \xi_T^\theta, \theta_k)^2 + o(R_{2,A,k}^2 + \theta_k^2). \tag{5.21}
\end{aligned}$$

Finally, combining lemma 46 and estimates (5.15-5.21), we obtain the result. \square

Let us denote by $\Omega[\lambda] : \mathcal{M}_2^Y \rightarrow \mathbb{R}$ the following mapping:

$$\Omega[\lambda](\nu) = \int_0^T \int_{U_R} H_{(u,y)^2}[p_t^\lambda][t](u, z[\nu])^2 d\mu_t(u) dt + \Phi_{(y_T)^2}[\lambda](\bar{y}_T, 0)(z[\nu]_T)^2. \tag{5.22}$$

With a similar proof to the proof of lemma 40, we can show that $\Omega[\lambda]$ is a continuous with respect to ν , when \mathcal{M}_2^Y is equipped with the L^2 -distance.

Corollary 48. *Let us suppose that the assumptions of theorem 47 are satisfied, then the following estimate holds:*

$$\begin{aligned}
& \Phi[\lambda](y_T^k, \theta_k) - \Phi[\lambda](\bar{y}_T, 0) \\
&= \text{Val}(PL_\theta)\theta_k + \frac{1}{2}\Omega[\lambda](\mu^{A,k} \ominus \bar{u}) \\
&\quad + \int_0^T \int_{U_R} (H[p_t^\lambda](u, \bar{y}_t, 0) - H[p_t^\lambda](t)) d\mu_t^{B,k}(u) dt \\
&\quad + o(R_{2,A,k}^2 + R_{2,B,k}^2) + O(\theta_k^2) + O(\theta_k R_{2,A,k}). \tag{5.23}
\end{aligned}$$

Proof. This expansion derives directly from the expansion of theorem 47 in which we have replaced the second-order terms involving θ_k by the estimate $O(R_{2,A,k}\theta_k)$. \square

5.2 Study of the rate of convergence of perturbed solutions

In this part, we give estimates of the L^2 -distance between a solution to the perturbed problem $(\mathcal{P}_\theta^{Y,R,\eta})$ and $\bar{\mu}$ under a strong second-order sufficient condition. We will also suppose that the parameter $\eta > 0$ (which a uniform bound between the reference trajectory and a perturbed trajectory) is sufficiently small.

Definition 49. *We call critical cone C_2 and relaxed critical cone C_2^Y the following sets:*

$$C_2 := \{\nu \in L^2, \phi_{y_T}(\bar{y}_T, 0)z[\nu]_T = 0, \Phi_{y_T}(\bar{y}_T, 0)z[\nu]_T \in T_K(\Phi(\bar{y}_T, 0))\}, \tag{5.24}$$

$$C_2^Y := \{\nu \in \mathcal{M}_2^Y, \phi_{y_T}(\bar{y}_T, 0)z[\nu]_T = 0, \Phi_{y_T}(\bar{y}_T, 0)z[\nu]_T \in T_K(\Phi(\bar{y}_T, 0))\}. \tag{5.25}$$

In the following assumption, we denote by $\text{ri}(S(DL_\theta))$ the relative interior of $S(DL_\theta)$, which is the interior of $S(DL_\theta)$ for the topology induced by its affine hull.

Hypothesis 50 (Second-order sufficient conditions). *There exists $\alpha > 0$ such that*

1. *for some $\lambda \in \text{ri}(S(DL_\theta))$, for almost all t in $[0, T]$, for all v in U_R ,*

$$H[p_t^\lambda](v, \bar{y}_t, 0) - H[p_t^\lambda](\bar{u}_t, \bar{y}_t, 0) \geq \alpha|v - \bar{u}_t|^2,$$

2. *for all ν in C_2^Y ,*

$$\sup_{\lambda \in S(DL_\theta)} \Omega[\lambda](\nu) \geq \alpha\|\nu\|_2^2.$$

Remark 51. *It is shown in [4, lemma 2.3] that since $S(DL_\theta)$ is compact, then for all $\bar{\lambda} \in \text{ri}(S(DL_\theta))$, there exists $\beta > 0$ such that for almost all t , for all v in U_R ,*

$$H[p_t^{\bar{\lambda}}](v, \bar{y}_t, 0) - H[p_t^{\bar{\lambda}}](\bar{u}_t, \bar{y}_t, 0) \geq \beta \left(\sup_{\lambda \in S(DL_\theta)} \{H[p_t^\lambda](v, \bar{y}_t, 0) - H[p_t^\lambda](\bar{u}_t, \bar{y}_t, 0)\} \right).$$

It follows from this result that condition (50.1) is equivalent to: there exists $\alpha' > 0$ such that for all μ in \mathcal{M}_R^Y ,

$$\sup_{\lambda \in S(DL_\theta)} \left\{ \int_0^T \int_{U_R} (H[p_t^\lambda](u, \bar{y}_t, 0) - H[p_t^\lambda](\bar{u}_t, \bar{y}_t, 0)) d\mu_t(u) dt \right\} \geq \alpha' \|\mu \ominus \bar{u}\|_2^2.$$

In the sequel, we use this form of the second-order sufficient condition.

Lemma 52. *If $\eta > 0$ is sufficiently small, then for any sequence $(\theta_k)_k \downarrow 0$, for any sequence of solutions $(\mu^k, y^k)_k$ to $(\mathcal{P}_\theta^{Y,R,\eta})$, with $\theta = \theta_k$,*

$$R_{2,k} = d_2(\bar{\mu}, \mu^k) \rightarrow 0. \quad (5.26)$$

Proof. We prove this lemma by contradiction. We suppose that there exists two sequences $(\eta_k)_k \downarrow 0$ and $(\theta_k)_k \downarrow 0$ and a sequence of solutions (μ^k, y^k) to $(\mathcal{P}_\theta^{Y,R,\eta})$ with $\eta = \eta_k$ and $\theta = \theta_k$ such that

$$\liminf_{k \rightarrow \infty} R_{2,k} = \liminf_{k \rightarrow \infty} d_2(\bar{\mu}, \mu^k) > 0. \quad (5.27)$$

It follows from the boundedness of $S(DL_\theta)$, inequality (5.1), corollary 45, and assumption (50.1) that

$$\begin{aligned} & \phi(y_T^k, \theta_k) - \phi(\bar{y}_T, 0) \\ & \geq \text{Val}(PL_\theta)\theta_k + \sup_{\lambda \in S(DL_\theta)} \left\{ \int_0^T \int_{U_R} (H[p_t^\lambda](u, \bar{y}_t, 0) - H[p_t^\lambda][t]) d\mu_t^k(u) dt \right\} \\ & \quad + O(R_{1,k}\theta_k) + O(R_{1,k}\|\delta y^k\|_\infty) + O(\theta_k^2) + o(R_{1,k}^2) \\ & \geq \text{Val}(PL_\theta)\theta_k + \alpha R_{2,k}^2 + O(R_{1,k}\theta_k) + O(R_{1,k}\eta_k) + O(\theta_k^2) + o(R_{1,k}^2). \end{aligned}$$

Using the first-order estimate (lemma 23), we obtain that

$$\begin{aligned} \alpha R_{2,k}^2 & \leq o(\theta_k) + O(R_{1,k}\theta_k) + O(R_{1,k}\eta_k) + O(\theta_k^2) + o(R_{1,k}^2) \\ & \leq o(\theta_k) + O(R_{1,k}\theta_k) + O(R_{1,k}\eta_k) + O(\theta_k^2) + o(R_{2,k}^2) \end{aligned}$$

thus, since the sequence $(R_{1,k})_k$ is bounded,

$$R_{2,k}^2 = o(\theta_k) + O(R_{1,k}\theta_k) + O(R_{1,k}\eta_k) + O(\theta_k^2) = o(1),$$

in contradiction with (5.27). \square

From now, we fix a parameter $\eta > 0$ sufficiently small so that lemma 52 is satisfied. We are now able to build a sequence $(A^k, B^k)_k$ which can be used in the decomposition principle. Let us set

$$\begin{aligned} A^k & = \{ (t, u) \in [0, T] \times U_R, |u - \bar{u}_t| < \sqrt{R_{1,k}} \}, \\ B^k & = \{ (t, u) \in [0, T] \times U_R, |u - \bar{u}_t| \geq \sqrt{R_{1,k}} \}. \end{aligned}$$

We consider the sequences $(\mu^{A,k})_k$ and $(\mu^{B,k})_k$ associated with $(\mu^k)_k$ and the sequence of partitions $(A^k, B^k)_k$. We still use the notations $z^{A,k}$ and $z^{B,k}$. Then,

$$\begin{aligned} R_{1,k} & = \int_0^T \int_{U_R} |u - \bar{u}_t| d\mu_t^k(u) dt \\ & \geq \int_0^T \int_{U_R} \sqrt{R_{1,k}} \mathbf{1}_{B^k}(t, u) d\mu_t^k(u) dt \\ & \geq \sqrt{R_{1,k}} \cdot \mu^k(B^k). \end{aligned}$$

Thus, $\mu^k(B^k) \leq \sqrt{R_{1,k}} = O(\sqrt{R_{2,k}}) \rightarrow 0$, by lemma 52. Moreover,

$$\text{ess sup}_{k \rightarrow \infty} \{|u - \bar{u}_t|, (t, u) \in A^k\} \leq \sqrt{R_{1,k}} = O(\sqrt{R_{2,k}}) \rightarrow 0.$$

As a consequence, we can apply the decomposition principle to the partition.

Theorem 53. *The following estimates on the rate of convergence of perturbed solutions holds:*

$$R_{2,k} = d_2(\bar{\mu}, \mu^k) = O(\theta_k), \quad \|y^k - \bar{y}\|_\infty = O(\theta_k). \quad (5.28)$$

Proof. \triangleright *First step:* $R_{2,B,k} = O(R_{2,A,k} + \theta_k)$.

With corollary 48 and the second-order upper estimate (4.12), we obtain that for all $\lambda \in S(DL_\theta)$,

$$\begin{aligned} & \frac{1}{2} \Omega[\lambda](\mu^{A,k} \ominus \bar{\mu}) + \int_0^T \int_{U_R} H[p_t^\lambda](u, \bar{y}_t, 0) - H[p_t^\lambda][t] d\mu_t^{B,k}(u) dt \\ & \leq o(R_{2,A,k}^2 + R_{2,B,k}^2) + O(\theta_k R_{2,A,k}) + O(\theta_k^2). \end{aligned} \quad (5.29)$$

The estimates are uniform in λ , since $S(DL_\theta)$ is bounded. Since $\Omega[\lambda](\mu^{A,k} \ominus \bar{\mu}) = O(R_{2,A,k}^2)$, we obtain by the second-order sufficient condition (hypothesis 50.1) that

$$\alpha R_{2,B,k}^2 = O(R_{2,A,k}^2 + \theta_k^2),$$

thus, $R_{2,B,k} = O(R_{2,A,k} + \theta_k)$.

\triangleright *Second step:* $R_{2,A,k} = O(\theta_k)$.

Let us prove by contradiction that $R_{2,A,k} = O(\theta_k)$. Extracting if necessary a subsequence, we may assume that $\theta_k = o(R_{2,A,k})$. It follows directly that $R_{2,B,k} = O(R_{2,A,k})$. For all $\lambda \in S(DL_\theta)$,

$$\int_0^T \int_{U_R} H(u, \bar{y}_t, 0) - H[t] d\mu_t^{B,k}(u) dt \geq 0,$$

thus, by (5.29),

$$\sup_{\lambda \in S(DL_\theta)} \Omega[\lambda](\mu^{A,k} \ominus \bar{\mu}) \leq O(\theta_k^2) + O(\theta_k R_{2,A,k}) + o(R_{2,A,k}^2) = o(R_{2,A,k}^2). \quad (5.30)$$

Using definition 26, we set

$$\nu^k = \frac{\mu^{A,k} \ominus \bar{\mu}}{R_{2,A,k}} \quad \text{and} \quad z^k = z[\nu^k] = \frac{z[\mu^{A,k} \ominus \bar{\mu}]}{R_{2,A,k}}.$$

Let us prove that

$$\phi_{y_T}(\bar{y}_T, 0) z_T[\nu^k] = o(1), \quad (5.31)$$

$$\text{dist}(\Phi_{y_T}(\bar{y}_T, 0) z_T[\nu^k], T_K(\Phi(\bar{y}_T, 0))) = o(1). \quad (5.32)$$

By lemma 29, we obtain that

$$\begin{aligned} \delta y_T^k &= z_T^{A,k} + z_T^{B,k} + \theta_k \xi_\theta^T + O(\theta_k^2) + O(R_{1,A,k}^2 + R_{1,B,k}^2) \\ &= z_T^{A,k} + o(R_{2,A,k}). \end{aligned}$$

As a consequence,

$$\begin{aligned} \phi(y_T^k, \theta^k) - \phi(\bar{y}_T, 0) &= \phi_{y_T}(\bar{y}_T, 0) z_T^{A,k} + o(R_{2,A,k}) \\ &= R_{2,A,k} [\phi_{y_T}(\bar{y}_T, 0) z_T[\nu^k] + o(1)], \end{aligned} \quad (5.33)$$

$$\begin{aligned} \Phi(y_T^k, \theta^k) - \Phi(\bar{y}_T, 0) &= \Phi_{y_T}(\bar{y}_T, 0) z_T^{A,k} + o(R_{2,A,k}) \\ &= R_{2,A,k} [\Phi_{y_T}(\bar{y}_T, 0) z_T[\nu^k] + o(1)]. \end{aligned} \quad (5.34)$$

Estimate (5.31) follows from (5.33) and from the following first-order upper estimate:

$$\phi(y_T^k, \theta_k) - \phi(\bar{y}_T, 0) \leq O(\theta_k) = o(R_{2,A,k}).$$

Estimate (5.32) follows from (5.34) and from the following inclusion:

$$\frac{\Phi(y_T^k, \theta_k) - \Phi(\bar{y}_T, 0)}{R_{2,A,k}} \in T_K(\Phi(\bar{y}_T, 0)).$$

We can apply Hoffman's lemma (see [5, theorem 2.200] or [15] for a historical reference) to $\rho[\nu^k]$ and to the critical cone, since $\rho[\nu^k]$ satisfies also expansions (5.31) and (5.32). We obtain the existence of a sequence $(\tilde{\nu}^k)_k$ in C_2 which is such that $\|\rho[\nu^k] - \tilde{\nu}^k\|_2 \rightarrow 0$. By lemma 27, we obtain the existence of a sequence $(\tilde{\nu}^k)_k$ in C_2^Y such that $d_2(\nu^k, \tilde{\nu}^k) \rightarrow 0$. It follows from (5.30) that

$$\sup_{\lambda \in S(DL_\theta)} \Omega[\lambda](\nu^k) = \sup_{\lambda \in S(DL_\theta)} \frac{\Omega[\lambda](\mu^{A,k} \ominus \bar{u})}{R_{2,A,k}^2} = o(1). \quad (5.35)$$

By the strong sufficient second-order condition and the continuity of $\Omega[\lambda]$, we obtain that

$$\begin{aligned} \sup_{\lambda \in S(DL_\theta)} \Omega[\lambda](\nu^k) &= \sup_{\lambda \in S(DL_\theta)} \Omega[\lambda](\tilde{\nu}^k) + o(1) \\ &\geq \alpha \|\tilde{\nu}^k\|_2^2 + o(1) \\ &= \alpha \|\nu^k\|_2^2 + o(1) \\ &= \alpha + o(1), \end{aligned}$$

in contradiction with (5.35). It follows that $R_{2,A,k} = O(\theta_k)$, thus $R_{2,k} = O(R_{2,A,k} + R_{2,B,k}) = \theta_k$ and finally that $\|y^k - \bar{y}\|_\infty = O(\theta_k)$, by lemma (56). \square

5.3 First- and second-order lower estimates

In this section, we compute a first and a second-order lower estimate for the value function. The first-order lower estimate derives directly from inequality (5.1), corollary 45, and theorem 53:

$$V(\theta_k) - V(0) \geq \text{Val}(PL_\theta)\theta_k + O(\theta_k^2).$$

Theorem 54. *The following second-order estimate holds:*

$$V(\theta) = V(0) + \theta \text{Val}(PL_\theta) + \theta^2 \text{Val}(PQ_\theta) + o(\theta^2). \quad (5.36)$$

Moreover, for any $\theta_k \downarrow 0$, we can extract a subsequence of solutions μ^k to $(\mathcal{P}_\theta^{Y,R,\eta})$ such that $\frac{\mu^k \ominus \bar{u}}{\theta_k}$ converges narrowly to some $\bar{\nu}$ solution of (PQ_θ) .

Proof. Let $(\theta_k)_k \downarrow 0$. We set

$$\nu^{A,k} = \frac{\mu^{A,k} \ominus \bar{u}}{\theta_k}, \quad \nu^k = \frac{\mu^k \ominus \bar{u}}{\theta_k}.$$

By theorem 53, $R_{2,A,k}^2 = O(\theta_k^2)$. Therefore, $(\nu^{A,k})_k$ is bounded for the L^2 -norm and we can extract a subsequence such that $(\nu^{A,k})_k$ converges for the narrow topology to some $\bar{\nu}$ in \mathcal{M}_2^Y . Moreover, we can show that

$$d_1(\nu^k, \nu^{A,k}) \leq \frac{\|\mu^{B,k} \ominus \bar{u}\|_1}{\theta_k} = o(1),$$

thus, ν^k equally converges to $\bar{\nu}$ for the narrow topology. For all $\lambda \in S(DL_\theta)$,

$$\int_0^T \int_{U_R} H[p_t^\lambda](u, \bar{y}_t, 0) - H[p_t^\lambda][t] d\mu_t^{B,k}(u) dt \geq 0,$$

thus, by inequality (5.1) and by the decomposition principle (theorem 47),

$$V(\theta_k) - V(0) \geq \theta_k \text{Val}(PL_\theta) + \frac{\theta_k^2}{2} \Omega^\theta[\lambda](\nu^{A,k}) + o(\theta_k^2).$$

Let us prove that $\nu \mapsto \Omega^\theta[\lambda](\nu)$ is lower semi-continuous for the narrow topology at $\bar{\nu}$. We already know that $(z[\nu^k])_k$ converges uniformly to $z[\bar{\nu}]$. Then, all the terms involving a second-order derivative different from H_{uu} have a linear growth, thus lemma 59 apply and we obtain that, for example,

$$\int_0^T \int_{\mathbb{R}^m} H_{u,y}[p_t^\lambda][t](u - \bar{u}_t, z_t[\nu^k])(d\nu_t^k(u) - d\bar{\nu}_t(u)) dt \rightarrow 0.$$

Moreover,

$$\liminf_{k \rightarrow \infty} \int_0^T \int_{\mathbb{R}^m} H_{uu}[p_t^\lambda][t](u - \bar{u}_t)^2 (d\nu_t^k(u) - d\bar{\nu}_t(u)) dt \geq 0$$

since the integrand $H_{uu}[p_t^\lambda][t](u - \bar{u}_t)^2$ is non-negative. Finally, we obtain that

$$V(\theta_k) - V(0) \geq \theta_k \text{Val}(PL_\theta) + \frac{\theta_k^2}{2} \Omega^\theta[\lambda](\bar{\nu}) + o(\theta_k^2).$$

Let us prove that $\bar{\nu}$ is a solution to problem $(SYPL_\theta)$. Following the proof of theorem (53), we obtain that

$$\delta y_T^k = \theta_k z_T^{A,k} + o(\theta_k)$$

and therefore that

$$\begin{aligned} \phi(y_T^k, \theta_k) - \phi(\bar{y}_T, 0) &= \theta_k \phi_{y_T}(\bar{y}_T, 0) z_T^{A,k} + o(\theta_k), \\ \Phi(y_T^k, \theta_k) - \Phi(\bar{y}_T, 0) &= \theta_k \Phi_{y_T}(\bar{y}_T, 0) z_T^{A,k} + o(\theta_k). \end{aligned}$$

Since

$$\phi(y_T^k, \theta_k) - \phi(\bar{y}_T, 0) = \text{Val}(PL_\theta) \theta_k + o(\theta_k),$$

we obtain that $\phi_{y_T}(\bar{y}_T, 0) z_T^{A,k} = \text{Val}(PL_\theta)$. Since

$$\Phi(y_T^k, \theta_k) - \Phi(\bar{y}_T, 0) \in T_K(\Phi(\bar{y}_T, 0)),$$

we obtain that $\Phi_{y_T}(\bar{y}_T, 0) z_T^{A,k} \in T_K(\Phi(\bar{y}_T, 0))$. This proves that $\bar{\nu}$ is a solution to $(SYPL_\theta)$. By lemma 41 and corollary 43, we obtain that

$$\text{Val}(PQ_\theta(\bar{\nu})) \leq \inf_{\nu \in S(SYPL_\theta)} \text{Val}(PQ_\theta(\nu)),$$

thus, $\bar{\nu}$ is a solution to problem (PQ_θ) and the theorem follows. \square

Remark 55. *The second-order expansion can be simplified as follows:*

$$V(\theta) = V(0) + \theta \text{Val}(PL_\theta) + \theta^2 \left(\underset{\nu \in S(SYPL_\theta)}{\text{Min}} \text{Val}(PQ_\theta(\nu)) \right) + o(\theta^2).$$

Proof. Let $\theta_k \downarrow 0$. By lemma 52, we know that $d_2(\bar{\mu}, \mu^k) \rightarrow 0$, thus, by lemma 17, we obtain the existence of a sequence $(u^k)_k$ of $o(\theta_k^2)$ -optimal controls such that

$$\|u^k - \bar{u}\|_2 = O(\theta_k).$$

Therefore, we can apply the decomposition principle to $(u^k)_k$ and we obtain the existence of sequences $(u^{A,k})_k$ and $(u^{B,k})_k$ which satisfy (5.13). Following the proof of theorem 54, we prove that $(u^{A,k} - \bar{u})/\theta_k$ converges to some \bar{v} in L^2 for the weak topology of L^2 , and that \bar{v} is a solution to problem (SPL_θ) . Finally, since $v \mapsto \Omega^\theta[\lambda](v)$ is lower semi-continuous for the weak topology of L^2 (see [14, lemma 21] for the idea of a proof), we obtain that the r.h.s. of (55) is a lower estimate of $V(\theta)$. It is also an upper estimate since

$$\text{Val}(PQ_\theta) \leq \text{Min}_{\nu \in S(SPL_\theta)} \text{Val}(PQ_\theta(\nu)).$$

Expansion (55) follows. □

6 Two examples

6.1 A different value for the Pontryagin and the standard linearized problem

Let us consider the following dynamic in \mathbb{R}^2 :

$$\begin{cases} \dot{y}_t = (u_t^3, u_t^2)^T, \text{ for a.a. } t \in [0, T], \\ y_0 = (0, 0)^T. \end{cases}$$

The control u is such that $\|u\|_\infty \leq 1$ and we minimize $y_{2,T}[u]$ under the constraint $y_{1,T}[u] = \theta$, with $\theta \geq 0$ and $\bar{\theta} = 0$. The coordinate y_2 correspond to the integral which would have been used in a Bolza formulation of the problem. For $\bar{\theta} = 0$, the problem has a unique solution $\bar{u} = 0$, $\bar{y} = (0, 0)^T$. This solution is qualified in the sense of definition 7, since for $v = 1$, $\xi_1[v] = T$ and for $v = -1$, $\xi_1[v] = -T$. However, the solution is not qualified in the sense of the standard definition, since the standard linearized dynamic z is equal to 0.

For $\theta \leq T$, the problem has infinitely many solutions, one of them being:

$$u_t^\theta = \begin{cases} 1, & \text{if } t \in (0, \theta), \\ 0, & \text{if } t \in (\theta, T). \end{cases}$$

Indeed, $y_{1,T}[u^\theta] = \theta$, $y_{2,T}[u^\theta] = \theta$ and if v^θ is feasible, then

$$\theta = y_{1,T}[v^\theta] = \int_0^T (v_t^\theta)^3 dt \leq \int_0^T (v_t^\theta)^2 dt = y_{2,T}[v],$$

which proves that u^θ is optimal. Moreover, if v^θ is optimal, then the previous inequality is an equality and thus, for almost all t , $(v_t^\theta)^3 = (v_t^\theta)^2$, that is to say, $v_t^\theta \in \{0, 1\}$. We also obtain that $\|v^\theta - \bar{u}\|_2 = \sqrt{\theta}$ and $\|v^\theta - \bar{u}\|_\infty = 1$.

Now, let us compute the sets of multiplier Λ^L and Λ^P (for the reference problem). Since the dynamic does not depend on y , denoting by $\lambda \in \mathbb{R}$ the dual variable associated with the constraint $y_{1,T}[u] - \theta = 0$, the costate p^λ is constant and given by $p_t = (\lambda, 1)$. The Hamiltonian is given by

$$H[\lambda](u) = u^2 + \lambda u^3.$$

As a consequence, we obtain that $\Lambda^L = \mathbb{R} \times \{1\}$ and $\Lambda^P = [-1, 1] \times \{1\}$. The Lagrangian associated with our family of problem is given by

$$\mathcal{L}(u, y, \lambda, \theta) = \int_0^T (u_t^2 + \lambda u_t^3) dt + \lambda(y_{1,\theta} - \theta),$$

therefore, $\mathcal{L}_\theta(\bar{u}, \bar{y}, \lambda, \bar{\theta}) = -\lambda$, $\text{Val}(PL_\theta) = 1$, and $\text{Val}(SPL_\theta) = +\infty$. In this example, the Pontryagin linearized problem enables a more accurate estimation of the value function. Since the solution \bar{u} is not qualified in a standard definition, it is not surprising that the associated linearized problem has a value equal to $+\infty$.

Note that the second-order theory developed in the article cannot be used to study this example, since we do not have the equality of $\text{Val}(PL_\theta)$ and $\text{Val}(SPL_\theta)$. Moreover, observe that for the solution $\lambda = -1$ of (DL_θ) , the Hamiltonian $H[\lambda](u) = u^2 - u^3$ has two minimizers: 0 and 1. The set of minimizers contains the support of the solutions to the perturbed problems.

6.2 No classical solutions for the perturbed problems

This second example shows a family of problems for which the perturbed problems do not have a classical solution. This example does not fit to the framework of the study since we consider control constraints. However, we believe it is interesting since in this case, the ratio $(\mu^\theta \ominus \bar{u})/\theta$ converges to a purely relaxed element of \mathcal{M}_2^Y for the narrow topology. This confirms us in the idea to use relaxation to perform a sensitivity analysis of optimal control problems.

Let us consider the following dynamic in \mathbb{R}^2 :

$$\begin{cases} (\dot{y}_{1,t}, \dot{y}_{2,t})^T = (u_t, y_1^2 + 2(v_t - \theta)^2 - u_t^2)^T, \text{ for a.a. } t \in [0, T], \\ (y_{1,0}, y_{2,0})^T = (0, 0)^T, \end{cases}$$

where for almost all t in $[0, T]$, $v_t \geq u_t$ and $v_t \geq -u_t$. The perturbation parameter θ is nonnegative and $\bar{\theta} = 0$. We minimize $y_{2,T}$. For $\theta = 0$, the problem has a unique solution $\bar{u} = (0, 0)^T$, $\bar{y} = (0, 0)^T$. The associated costate $p = (p_1, p_2)$ is constant, given by $p_1 = 0$ and $p_2 = 1$. Thus,

$$H[p](u, v, \bar{y}_t) = 2(v - \theta)^2 - u^2.$$

This Hamiltonian has been “designed” in a way to have a unique minimizer when $\theta = 0$, but two minimizers $(\pm 2\theta, 2\theta)$ when $\theta > 0$. Let us focus on optimal solutions to the problem when $\theta > 0$. Let $u, v \in L^\infty([0, T], \mathbb{R})$, we have

$$\begin{aligned} y_{2,T}[u, v] &= \int_0^T y_{1,t}[u, v]^2 + 2(v_t - \theta)^2 - u_t^2 dt \\ &= \int_0^T y_{1,t}[u, v]^2 + 2v_t^2 - 4\theta v_t + 2\theta^2 - u_t^2 dt \\ &= \int_0^T y_{1,t}[u, v]^2 + (v_t^2 - u_t^2) + (v_t - 2\theta)^2 - 2\theta^2 dt \\ &\geq -2\theta^2 T. \end{aligned}$$

This last inequality is satisfied if for almost all t in $[0, T]$, $y_{1,t}[u, v] = 0$, $v_t = 2\theta$, $|u_t| = v_t$. As a consequence, the problem does not have classical solutions, but has a unique relaxed one, $\mu^\theta = ((\delta_{-2\theta} + \delta_{2\theta})/2, 2\theta)$. Moreover,

$$\frac{\mu^\theta \ominus \bar{u}}{\theta} = ((\delta_{-2} + \delta_2)/2, \delta_2).$$

A Appendix on Young measures

A.1 First definitions

Weak-* topology on bounded measures Let X be a closed measurable subset of \mathbb{R}^m . We say that a real function ψ on $[0, T] \times X$ vanishes at infinity if for all $\varepsilon > 0$, there exists a compact subset K of $[0, T] \times X$ such that for all (t, u) in $([0, T] \times X) \setminus K$, $|\psi(t, u)| \leq \varepsilon$. We denote by $C^0([0, T] \times X)$ the set of continuous real functions vanishing at infinity. The set $\mathcal{M}_b([0, T] \times X)$ of bounded measures on $[0, T] \times X$ is the topological dual of $C^0([0, T] \times X)$. We endow this dual pair with the associated *weak-* topology*. Note that this topology is metrizable since $[0, T] \times X$ is separable.

Young measures Let us denote by P the projection from $[0, T] \times X$ to $[0, T]$. Let μ be in $\mathcal{M}_b^+([0, T] \times X)$, μ is said to be a *Young measure*, if $P_{\#}\mu$ is the Lebesgue measure on $[0, T]$. We denote by $\mathcal{M}^Y(X)$ the set of Young measures, it is weakly-* compact [21, theorem 1].

Disintegrability Let us denote by $\mathcal{P}(X)$ the set of probability measures on X . To all measurable mapping $\nu : [0, T] \rightarrow \mathcal{P}(X)$ (see the definition in [21, page 157]), we associate a unique Young measure μ defined by: for all ψ in $C^0([0, T] \times X)$,

$$\int_{[0, T] \times X} \psi(t, u) d\mu(t, u) = \int_0^T \int_X \psi(t, u) d\nu_t(u) dt.$$

This mapping defines a bijection from $L([0, T]; \mathcal{P}(X))$ to $\mathcal{M}^Y(X)$. This property is called *disintegrability*. Note that $L([0, T]; \mathcal{P}(X)) \subset L^\infty([0, T]; \mathcal{M}_b(X))$, which is the dual of $L^1([0, T]; C^0(X))$. On $\mathcal{M}^Y(X)$, the weak-* topology of this dual pair is equivalent to the weak-* topology previously defined [21, theorem 2]. In the article, we always write Young measures in a disintegrated form.

Denseness To all u in $L([0, T]; X)$, we associate the unique Young measure μ defined by for almost all t in $[0, T]$, $\mu_t = \delta_{u_t}$. The space $L([0, T]; X)$ is dense in $\mathcal{M}^Y(X)$ for the weak-* topology [22, proposition 8].

Lower semi-continuity of integrands We say that $\psi : [0, T] \times X \rightarrow \overline{\mathbb{R}}$ is a *positive normal integrand* if ψ is measurable, $\psi \geq 0$ and if for almost all t in $[0, T]$, $\psi(t, \cdot)$ is lower semi-continuous. If ψ is a positive normal integrand, then the mapping

$$\mu \in \mathcal{M}^Y(X) \mapsto \int_0^T \int_X \psi(t, u) d\mu_t(u) dt$$

is lower semi-continuous for the weak-* topology [21, theorem 4].

Narrow topology We say that $\psi : [0, T] \times X \rightarrow \mathbb{R}$ is a *bounded Caratheodory integrand* if for almost all t in $[0, T]$, $\psi(t, \cdot)$ is continuous and bounded and if $\|\psi(t, \cdot)\|_\infty$ is integrable. The *narrow topology* on $\mathcal{M}^Y(X)$ is the weakest topology such that for all bounded Caratheodory integrand ψ ,

$$\mu \in \mathcal{M}^Y(X) \mapsto \int_0^T \int_X \psi(t, u) d\mu_t(u) du$$

is continuous. This topology is finer than the weak-* topology.

Wasserstein distance We denote by P^1 and P^2 the two projections from $[0, T] \times X \times X$ to $[0, T] \times X$ defined by $P^1(t, u, v) = (t, u)$ and $P^2(t, u, v) = (t, v)$. Let μ^1 and μ^2 be in $\mathcal{M}^Y(X)$, then π in $\mathcal{M}_b^+([0, T] \times X \times X)$ is said to be a *transportation plan* between μ^1 and μ^2 if $P_{\#}^1 \pi = \mu^1$ and $P_{\#}^2 \pi = \mu^2$. Note that a transportation plan is disintegrable in time, like Young measures. We denote by $\Pi(\mu^1, \mu^2)$ the set of transportation plans between μ^1 and μ^2 . It is never empty, since it contains the measure π defined by $\pi_t = \mu_t^1 \otimes \mu_t^2$ for almost all t . We define the L^p -distance between μ^1 and μ^2 by

$$d_p(\mu^1, \mu^2) = \left[\inf_{\pi \in \Pi(\mu^1, \mu^2)} \int_0^T \int_{X \times X} |v - u|^p d\pi_t(u, v) dt \right]^{1/p}. \quad (\text{A.1})$$

This distance is called the *Wasserstein distance* [8, section 3.4]. The set $\Pi(\mu^1, \mu^2)$ is narrowly closed and if $d_p(\mu^1, \mu^2)$ is finite, any minimizing sequence of the problem associated with (A.1) possesses a limit point by Prokhorov's theorem [21, theorem 11], thus by lower-semi-continuity of the duality product with a positive normal integrand, we obtain the existence of an optimal transportation plan.

If μ^1 is the Young measure associated to $u^1 \in L([0, T]; X)$, then for all μ^2 in $\mathcal{M}^Y(X)$, there is only one transportation plan π in $\Pi(\mu^1, \mu^2)$, which is, for almost all t in $[0, T]$, for all u and v in X , $\pi_t(u, v) = \delta_{u_t^1}(u) \mu_t^2(v)$, therefore,

$$d_s(\mu^1, \mu^2) = \left[\int_0^T \int_{U_R} |v - u_t^1|^s d\mu_t^2(v) dt \right]^{1/s}. \quad (\text{A.2})$$

If μ^1 and μ^2 are both associated with u_1 and u_2 in $L([0, T]; X)$, then $d_p(\mu^1, \mu^2) = \|u_2 - u_1\|_p$.

A.2 Young measures on U_R

We suppose here that X is equal to U_R , the ball of \mathbb{R}^m with radius R and center 0. We denote $\mathcal{M}_R^Y = \mathcal{M}^Y(U_R)$. The set U_R being compact, \mathcal{M}_R^Y is weakly-* compact [21, theorem 1]. Moreover, the weak-* topology and the narrow topology are equivalent [21, theorem 4].

Differential systems controlled by Young measures Let $x^0 \in \mathbb{R}^n$, let $g : [0, T] \times X \rightarrow \mathbb{R}^n$ be Lipschitz continuous (with modulus A), then for all μ in \mathcal{M}_R^Y , the differential system

$$\begin{cases} \dot{x}_t = \int_{U_R} f(x_t, u) d\mu_t(u) \\ x_0 = x^0 \end{cases}$$

has a unique solution. This solution is denoted by $x[\mu]$.

Lemma 56. *Let us equip $L^\infty([0, T]; \mathbb{R}^n)$ with the uniform norm. The mapping $\mu \in \mathcal{M}_R^Y \mapsto x[\mu] \in L^\infty([0, T]; \mathbb{R}^n)$ is weakly-* continuous and Lipschitz continuous for the L^1 -distance of Young measures.*

Proof. \triangleright *Weak-* continuity.*

Let $\mu \in \mathcal{M}_R^Y$, let $(\mu_k)_k$ be a sequence in \mathcal{M}_R^Y converging to μ for the weak-* topology. The sequence $(g^k)_k$ defined by

$$g_t^k = \int_0^t \int_{U_R} f(x_s[\mu], u) (d\mu_s^k(u) - d\mu_s(u)) ds$$

converges pointwise to 0. We can show with the Arzelà-Ascoli theorem that this convergence is uniform. For all t in $[0, T]$,

$$\begin{aligned} |x_t[\mu^k] - x_t[\mu]| &\leq \int_0^t \int_{U_R} |f(x_s[\mu^k], u) - f(x_s[\mu], u)| d\mu_s^k(u) ds \\ &\quad + \left| \int_0^t \int_{U_R} f(x_s[\mu], u) (d\mu_s^k(u) - d\mu_s(u)) ds \right| \\ &= \int_0^t O(|x_s[\mu^k] - x_s[\mu]|) ds + o(1), \end{aligned}$$

where the estimate $o(1)$ is uniform in time. The uniform convergence of $x[\mu^k]$ follows with Gronwall's lemma.

▷ *L^1 -Lipschitz continuity.*

Let μ^1 and μ^2 in \mathcal{M}_Y^R , let π be an optimal transportation plan between μ^1 and μ^2 for the L^1 -distance. For all t in $[0, T]$,

$$\begin{aligned} |x_t[\mu^2] - x_t[\mu^1]| &\leq \left| \int_0^t \int_{U_R \times U_R} f(x_s[\mu^2], v) - f(x_s[\mu^1], u) d\pi_s(u, v) ds \right| \\ &\leq \int_0^t \int_{U_R \times U_R} A(|x_s[\mu^2] - x_s[\mu^1]| + |v - u|) d\pi_s(u, v) ds \\ &\leq \int_0^t A|x_s[\mu^2] - x_s[\mu^1]| ds + Ad_1(\mu^1, \mu^2). \end{aligned}$$

The Lipschitz continuity follows with Gronwall's lemma. \square

A.3 Young measures on \mathbb{R}^m

We suppose here that $X = \mathbb{R}^m$. We equip $\mathcal{M}^Y := \mathcal{M}^Y(\mathbb{R}^m)$ with the narrow topology. In the article, elements of \mathcal{M}^Y are denoted by ν . For p in $[1, \infty)$, we denote by \mathcal{M}_p^Y the set of Young measures ν in \mathcal{M}^Y with a finite L^p -norm, defined by $\|\nu\|_p = d_p(0, \nu)$, where d_p is the Wasserstein distance. We denote by \mathcal{M}_∞^Y the set of Young measures with a bounded support and we define the L^∞ -norm as follows:

$$\|\nu\|_\infty = \inf \{a \in \mathbb{R}, \nu([0, T] \times B(0, a)) = \nu([0, T] \times \mathbb{R}^m)\}.$$

Note the inclusion $\mathcal{M}_\infty^Y \subset \mathcal{M}_2^Y \subset \mathcal{M}_1^Y$.

Lemma 57. *The unit ball B_2^Y of \mathcal{M}_2^Y is narrowly compact.*

Proof. Let us prove that B_2^Y is tight ie, for all $\varepsilon > 0$, there exists a compact subset K of \mathbb{R}^m such that for all ν in B_2^Y , $\nu([0, T] \times (\mathbb{R}^m \setminus K)) \leq \varepsilon$. Let $\varepsilon > 0$, let K be the ball of \mathbb{R}^m with centre 0 and radius $1/\sqrt{\varepsilon}$. For all ν in B_2^Y ,

$$\nu([0, T] \times (\mathbb{R}^m \setminus K)) = \varepsilon \int_0^T \int_{\mathbb{R}^m \setminus K} \frac{1}{\varepsilon} d\nu_t(u) dt \leq \varepsilon \int_0^T \int_{\mathbb{R}^m \setminus K} |u|^2 d\nu_t(u) dt \leq \varepsilon.$$

Then B_2^Y is tight and by Prokhorov's theorem [21, theorem 11], B_2^Y is narrowly precompact in \mathcal{M}^Y . The mapping $(t, u) \mapsto |u|^2$ being a positive normal integrand, the L^2 -norm is lower semi-continuous and therefore, B_2^Y is closed for the narrow topology. Finally, B_2^Y is narrowly compact. \square

Lemma 58. *The mapping $\mu \in \mathcal{M}_2^Y \mapsto x[\mu] \in L^\infty([0, T], \mathbb{R}^n)$ is well-defined and Lipschitz continuous.*

Proof. The proof is similar to the proof of lemma 56. \square

Lemma 59. *Let $\Psi : [0, T] \times X \rightarrow \mathbb{R}^m$ a measurable mapping be such that for almost all t in $[0, T]$, $\Psi(t, \cdot)$ is continuous and such that*

$$\operatorname{ess\,sup}_{t \in [0, T]} |\Psi(t, u)| = o_{|u| \rightarrow \infty}(|u|^2).$$

Then the mapping

$$\nu \in \mathcal{M}_2^Y \mapsto \int_0^T \int_{\mathbb{R}^m} \psi(t, u) \, d\nu_t(u) \, dt \tag{A.3}$$

is sequentially continuous for the narrow topology.

Proof. The proof is inspired from [1, remark 5.3]. Let $(\nu^k)_k$ be a sequence in \mathcal{M}_2^Y converging to ν in \mathcal{M}_2^Y for the narrow topology. Let A be a bound on $\|\nu^k\|_2$. Let $\varepsilon > 0$. Let $B \geq 0$ be such that for almost all t in $[0, T]$, for all u in \mathbb{R}^m ,

$$\psi(t, u) \leq \varepsilon|u|^2 + B.$$

Then, $-\psi(t, u) + \varepsilon|u|^2 + B$ is a positive normal integrand. Thus,

$$\int_0^T \int_{\mathbb{R}^m} -\psi(t, u) + \varepsilon|u|^2 + B \, d\nu_t(u) \, dt \leq \liminf_{k \rightarrow \infty} \int_0^T \int_{\mathbb{R}^m} -\psi(t, u) + \varepsilon|u|^2 + B \, d\nu_t^k(u) \, dt$$

and therefore,

$$\int_0^T \int_{\mathbb{R}^m} -\psi(t, u) \, d\nu_t(u) \, dt \leq \liminf_{k \rightarrow \infty} \int_0^T \int_{\mathbb{R}^m} -\psi(t, u) \, d\nu_t^k(u) \, dt + 2\varepsilon A^2.$$

To the limit when $\varepsilon \downarrow 0$, we obtain that

$$\int_0^T \int_{\mathbb{R}^m} \psi(t, u) \, d\nu_t(u) \, dt \geq \limsup_{k \rightarrow \infty} \int_0^T \int_{\mathbb{R}^m} \psi(t, u) \, d\nu_t^k(u) \, dt,$$

which proves the upper semi-continuity of the mapping (A.3). We prove similarly the lower semi-continuity. \square

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