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# GKW representation theorem and linear BSDEs under restricted information. An application to risk-minimization.

Claudia Ceci\*      Alessandra Cretarola†      Francesco Russo‡

## Abstract

In this paper we provide Galtchouk-Kunita-Watanabe representation results in the case where there are restrictions on the available information. This allows to prove existence and uniqueness for linear backward stochastic differential equations driven by a general càdlàg martingale under partial information. Furthermore, we discuss an application to risk-minimization where we extend the results of Föllmer and Sondermann (1986) to the partial information framework and we show how our result fits in the approach of Schweizer (1994).

*Mathematics Subject Classification (2000):* 60H10, 60H30, 91B28.

*Key words and phrases:* Backward stochastic differential equations, partial information, Galtchouk-Kunita-Watanabe decomposition, predictable dual projection, risk-minimization.

## 1 Introduction

This paper provides two main contributions. First, we prove Galtchouk-Kunita-Watanabe representation results in the case where there are restrictions on the available information and we show an application to risk-minimization. Second, as an important consequence, we prove existence and uniqueness for linear backward stochastic differential equations (in short BSDEs) driven by a general càdlàg martingale under partial information. For BSDEs driven by a general càdlàg martingale beyond the Brownian setting, there exist very few results in literature (see [5] and more recently [1] and [2], as far as we are aware). Here we study for the first time such a general case in the situation where there

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are restrictions on the available information, that represents an interesting issue arising in many financial problems. Mathematically, this means to consider an additional filtration  $\mathbb{H}$  smaller than the full information flow  $\mathbb{F}$ . A typical example arises when  $\mathcal{H}_t = \mathcal{F}_{(t-\tau)^+}$  where  $\tau \in (0, T)$  is a fixed delay and  $(t - \tau)^+ := \max\{0, t - \tau\}$  and  $T$  denotes a time horizon.

We start our investigation by considering BSDEs of the form

$$Y_t = \xi - \int_t^T Z_s dM_s - (O_T - O_t), \quad 0 \leq t \leq T, \quad (1.1)$$

driven by a square-integrable (càdlàg) martingale  $M = (M_t)_{0 \leq t \leq T}$ , where  $T > 0$  is a fixed time horizon,  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ <sup>1</sup> denotes the terminal condition and  $O = (O_t)_{0 \leq t \leq T}$  is a square-integrable  $\mathbb{F}$ -martingale with  $O_0 = 0$ , satisfying a suitable orthogonality condition that we will make more precise in the next section.

We look for a solution  $(Y, Z)$  to equation (1.1) under partial information, where  $Y = (Y_t)_{0 \leq t \leq T}$  is a càdlàg  $\mathbb{F}$ -adapted process and  $Z = (Z_t)_{0 \leq t \leq T}$  is an  $\mathbb{H}$ -predictable process such that  $\mathbb{E} \left[ \int_0^T |Z_t|^2 d\langle M \rangle_t \right] < \infty$ .

To this aim, we prove a Galtchouk-Kunita-Watanabe decomposition in the case where there are restrictions on the available information. More precisely, we obtain that every random variable  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$  can be uniquely written as

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + O_T, \quad \mathbb{P} - \text{a.s.}, \quad (1.2)$$

where  $H^{\mathcal{H}} = (H_t^{\mathcal{H}})_{0 \leq t \leq T}$  is an  $\mathbb{H}$ -predictable process such that  $\mathbb{E} \left[ \int_0^T |H_t^{\mathcal{H}}|^2 d\langle M \rangle_t \right] < \infty$ . To the authors' knowledge such a decomposition has not been proved yet in the existing literature. We will see that decomposition (1.2) allows to construct a solution to the BSDE (1.1) and ensures its uniqueness in this setting.

Moreover, we are able to provide an explicit characterization of the integrand process  $H^{\mathcal{H}}$  given in decomposition (1.2) in terms of the one appearing in the classical Galtchouk-Kunita-Watanabe decomposition, by using  $\mathbb{H}$ -predictable (dual) projections.

Finally, we discuss a financial application. More precisely, we study the problem of hedging a contingent claim in the case where investors acting in the market have partial information. Since the market is incomplete we choose the risk-minimization approach, a quadratic hedging method which keeps the replicability constraint and relaxes the self financing condition, see [6] and [14] for further details. As in [6] and [13], we consider the case where the price process is a martingale under the real world probability measure. In [6], under the case of full information, the authors provide the risk-minimizing hedging strategy in terms of the classical Galtchouk-Kunita-Watanabe decomposition. Here, by using the Galtchouk-Kunita-Watanabe decomposition under partial information, we extend this result to the case where there are restrictions on the available information. Finally, thanks to the explicit representation of the integrand process  $H^{\mathcal{H}}$  appearing in

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<sup>1</sup>The space  $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$  denotes the set of all  $\mathcal{F}_T$ -measurable real-valued random variables  $H$  such that  $\mathbb{E} [|H|^2] = \int_{\Omega} |H|^2 d\mathbb{P} < \infty$ .

decomposition (1.2), we find the same expression for the optimal strategy in terms of the Radon-Nikodym derivative of two  $\mathbb{H}$ -predictable dual projections, that is proved in [13]. The paper is organized as follows. In Section 2 we give the definition of solution to BSDEs under partial information. Section 3 is devoted to prove existence and uniqueness results for the solutions, which are obtained by applying the Galtchouk-Kunita-Watanabe decomposition adapted to the restricted information setting. The explicit representation of the integrand process  $H^{\mathcal{H}}$  appearing in (1.2) can be found in Section 4. Finally, an application to risk-minimization is given in Section 5.

## 2 Setting

Let us fix a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  endowed with a filtration  $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ , where  $\mathcal{F}_t$  represents the full information at time  $t$ . We assume that  $\mathcal{F}_T = \mathcal{F}$ . Then we consider a subfiltration  $\mathbb{H} := (\mathcal{H}_t)_{0 \leq t \leq T}$  of  $\mathbb{F}$ , i.e.  $\mathcal{H}_t \subseteq \mathcal{F}_t$ , for each  $t \in [0, T]$ , corresponding to the available information level. We remark that both filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity, see e.g. [12].

For simplicity we only consider the one-dimensional case. Extensions to several dimensions are straightforward and left to the reader. The data of the problem are:

- an  $\mathbb{R}$ -valued square-integrable (càdlàg)  $\mathbb{F}$ -martingale  $M = (M_t)_{0 \leq t \leq T}$  with  $\mathbb{F}$ -predictable quadratic variation process denoted by  $\langle M \rangle = (\langle M, M \rangle)_{0 \leq t \leq T}$ ;
- a terminal condition  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ .

**Definition 2.1.** *A solution of the BSDE*

$$Y_t = \xi - \int_t^T Z_s dM_s - (O_T - O_t), \quad 0 \leq t \leq T, \quad (2.1)$$

with data  $(\xi, \mathbb{H})$  under partial information, where  $O = (O_t)_{0 \leq t \leq T}$  is a square-integrable  $\mathbb{F}$ -martingale with  $O_0 = 0$ , satisfying the orthogonality condition

$$\mathbb{E} \left[ O_T \int_0^T \varphi_t dM_t \right] = 0, \quad (2.2)$$

for all  $\mathbb{H}$ -predictable processes  $\varphi = (\varphi_t)_{0 \leq t \leq T}$  such that  $\mathbb{E} \left[ \int_0^T |\varphi_t|^2 d\langle M \rangle_t \right] < \infty$ , is a couple  $(Y, Z)$  of processes with values in  $\mathbb{R} \times \mathbb{R}$ , satisfying (2.1) such that

- $Y = (Y_t)_{0 \leq t \leq T}$  is a càdlàg  $\mathbb{F}$ -adapted process;
- $Z = (Z_t)_{0 \leq t \leq T}$  is an  $\mathbb{H}$ -predictable process such that  $\mathbb{E} \left[ \int_0^T |Z_t|^2 d\langle M \rangle_t \right] < \infty$ .

**Remark 2.2.** *The orthogonality condition given in (2.2) is weaker than the classical strong orthogonality condition, see e.g [11] or [12]. Indeed, set  $N_t = \int_0^t \varphi_s dM_s$ , for each  $t \in [0, T]$ , where  $\varphi$  is an  $\mathbb{H}$ -predictable process such that  $\mathbb{E} \left[ \int_0^T |\varphi_t|^2 d\langle M \rangle_t \right] < \infty$ . If*

$$\langle O, M \rangle_t = 0 \quad \mathbb{P} - \text{a.s.}, \quad \forall t \in [0, T],$$

then

$$\langle O, N \rangle_t = \int_0^t \varphi_s d\langle O, M \rangle_s = 0 \quad \mathbb{P} - \text{a.s.}, \quad \forall t \in [0, T].$$

Consequently,  $O \cdot N$  is an  $\mathbb{F}$ -martingale null at zero, that implies

$$\mathbb{E}[O_t N_t] = 0, \quad \forall t \in [0, T],$$

and in particular condition (2.2).

**Remark 2.3.** Since for any  $\mathbb{H}$ -predictable process  $\varphi$ , the process  $\mathbf{1}_{(0,t]}(s)\varphi_s$ , with  $t \leq T$ , is  $\mathbb{H}$ -predictable, condition (2.2) implies that for every  $t \in [0, T]$

$$\mathbb{E}\left[O_T \int_0^t \varphi_s dM_s\right] = 0,$$

and by conditioning with respect to  $\mathcal{F}_t$  (note that  $O$  is an  $\mathbb{F}$ -martingale), we have

$$\mathbb{E}\left[O_t \int_0^t \varphi_s dM_s\right] = \mathbb{E}\left[\int_0^t \varphi_s d\langle M, O \rangle_s\right] = 0 \quad \forall t \in [0, T].$$

From this last equality, we can argue that in the case of full information, i.e.,  $\mathcal{H}_t = \mathcal{F}_t$ , for each  $t \in [0, T]$ , condition (2.2) is equivalent to the strong orthogonality condition between  $O$  and  $M$  (see e.g. Lemma 2 and Theorem 36, Chapter IV, page 180 of [12] for a rigorous proof).

### 3 Existence and uniqueness for linear BSDEs under partial information

Our aim is to investigate existence and uniqueness of a solution to the BSDE (2.1) with data  $(\xi, \mathbb{H})$  driven by the general martingale  $M$  in the sense of Definition 2.1. This requires to prove a Galtchouk-Kunita-Watanabe representation result under restricted information.

We introduce the linear subspace  $\mathcal{L}_T^{\mathcal{H}}$  of  $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$  given by all random variables  $\eta$  of the form

$$\left\{ U_0 + \int_0^T \varphi_t dM_t \mid U_0 \in \mathcal{H}_0, \varphi \text{ is } \mathbb{H} - \text{predictable with } \mathbb{E}\left[\int_0^T |\varphi_t|^2 d\langle M \rangle_t\right] < \infty \right\}. \quad (3.1)$$

**Lemma 3.1.** The set  $\mathcal{L}_T^{\mathcal{H}}$  is a closed subspace of  $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ .

*Proof.* Let  $U_0^n \in \mathcal{H}_0$  and  $(\varphi^n)_{n \in \mathbb{N}}$ , with  $\varphi^n = (\varphi_t^n)_{0 \leq t \leq T}$ , be a sequence of  $\mathbb{H}$ -predictable processes satisfying  $\mathbb{E}\left[\int_0^T |\varphi_t^n|^2 d\langle M \rangle_t\right] < \infty$  such that the sequence

$$\eta^n = U_0^n + \int_0^T \varphi_t^n dM_t, \quad n \in \mathbb{N},$$

converges to some random variable  $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ , as  $n$  goes to infinity. By taking the conditional expectation with respect to  $\mathcal{H}_0$ , we have

$$U_0^n = \mathbb{E} [\eta^n | \mathcal{H}_0] \longrightarrow \mathbb{E} [\eta | \mathcal{H}_0], \quad \text{as } n \rightarrow \infty.$$

We set  $U_0 = \mathbb{E} [\eta | \mathcal{H}_0]$ . Since  $(\eta^n - U_0^n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ , it follows that

$$\mathbb{E} \left[ \int_0^T (\varphi_t^n - \varphi_t^m)^2 d\langle M \rangle_t \right] \longrightarrow 0, \quad \text{as } n, m \rightarrow \infty.$$

Consequently,  $(\varphi^n)_{n \in \mathbb{N}}$  converges in  $L^2(\Omega, d\langle M \rangle \otimes d\mathbb{P})^2$  to some process  $\varphi = (\varphi_t)_{0 \leq t \leq T} \in L^2(\Omega, d\langle M \rangle \otimes d\mathbb{P})$ . Finally, since there is a subsequence converging  $d\langle M \rangle \otimes d\mathbb{P}$ -a.e., the limit  $\varphi$  is necessarily an  $\mathbb{H}$ -predictable process.  $\square$

**Proposition 3.2.** *Let  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ . There exists a unique decomposition of the form*

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + O_T, \quad \mathbb{P} - \text{a.s.}, \quad (3.2)$$

where  $U_0 \in \mathcal{H}_0$ ,  $H^{\mathcal{H}}$  is an  $\mathbb{H}$ -predictable process such that  $\mathbb{E} \left[ \int_0^T |H_t^{\mathcal{H}}|^2 d\langle M \rangle_t \right] < \infty$  and  $O$  is a square-integrable  $\mathbb{F}$ -martingale with  $O_0 = 0$  such that  $\mathbb{E} [O_T \cdot \eta] = 0$ , for every  $\eta \in \mathcal{L}_T^{\mathcal{H}}$ . Moreover  $U_0 = \mathbb{E} [\xi | \mathcal{H}_0]$  and  $\mathbb{E} [O_T | \mathcal{H}_0] = 0$ .

*Proof.* The existence and uniqueness property of decomposition (3.2) is clearly ensured by the orthogonal projection of the random variable  $\xi$  onto the space  $\mathcal{L}_T^{\mathcal{H}}$ , that is closed in virtue of Lemma 3.1. Since  $(U_0 + \int_0^\cdot H_t^{\mathcal{H}} dM_t)$  is an  $\mathbb{F}$ -martingale, by taking the conditional expectation of  $\xi$  with respect to  $\mathcal{H}_0$  in (3.2), we have

$$\begin{aligned} \mathbb{E} [\xi | \mathcal{H}_0] &= \mathbb{E} \left[ U_0 + \int_0^T H_t^{\mathcal{H}} dM_t \middle| \mathcal{H}_0 \right] + \mathbb{E} [O_T | \mathcal{H}_0] \\ &= \mathbb{E} \left[ \mathbb{E} \left[ U_0 + \int_0^T H_t^{\mathcal{H}} dM_t \middle| \mathcal{F}_0 \right] \middle| \mathcal{H}_0 \right] + \mathbb{E} [\mathbb{E} [O_T | \mathcal{F}_0] | \mathcal{H}_0] \\ &= \mathbb{E} [U_0 | \mathcal{H}_0], \end{aligned}$$

where in the last equality we have used the fact that  $\mathbb{E} [O_T | \mathcal{F}_0] = O_0 = 0$ . Consequently  $\mathbb{E} [O_T | \mathcal{H}_0] = 0$  and  $U_0 = \mathbb{E} [U_0 | \mathcal{H}_0] = \mathbb{E} [\xi | \mathcal{H}_0]$ . This concludes the proof.  $\square$

**Theorem 3.3.** *Given data  $(\xi, \mathbb{H})$ , there exists a unique couple  $(Y, Z)$  which solves the BSDE (2.1) according to Definition 2.1.*

<sup>2</sup>The space  $L^2(\Omega, d\langle M \rangle \otimes d\mathbb{P})$  denotes the set of all  $\mathbb{F}$ -adapted processes  $\varphi = (\varphi_t)_{0 \leq t \leq T}$  such that

$$\|\varphi\|_{L^2(\Omega, d\langle M \rangle \otimes d\mathbb{P})} := \left( \mathbb{E} \left[ \int_0^T |\varphi_t|^2 d\langle M \rangle_t \right] \right)^{\frac{1}{2}} < \infty.$$

*Proof. Existence.* Let  $\mathcal{L}_T^{\mathcal{H}}$  be the linear subspace of  $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$  introduced in (3.1). Given  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ , we know by Proposition 3.2 that there exists a unique decomposition of the form

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + A_T, \quad \mathbb{P} - \text{a.s.},$$

where in particular  $A$  is a square-integrable  $\mathbb{F}$ -martingale with  $A_0 = 0$  orthogonal to all the elements in  $\mathcal{L}_T^{\mathcal{H}}$ . We use this result to construct a solution to the BSDE (2.1). We consider the orthogonal projection of  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$  onto this space:

$$P_{\mathcal{L}_T^{\mathcal{H}}}(\xi) := U_0 + \int_0^T H_t^{\mathcal{H}} dM_t.$$

The couple  $(U_0, H)$ , where  $U_0 \in \mathcal{H}_0$  and  $H^{\mathcal{H}}$  is an  $\mathbb{H}$ -predictable process in  $L^2(\Omega, d\langle M \rangle \otimes d\mathbb{P})$ , uniquely identifies the projection, that exists and it is well-defined since  $\mathcal{L}_T^{\mathcal{H}}$  is closed. We set

$$A_T := \xi - P_{\mathcal{L}_T^{\mathcal{H}}}(\xi) \in (\mathcal{L}_T^{\mathcal{H}})^{\perp},$$

where  $(\mathcal{L}_T^{\mathcal{H}})^{\perp}$  denotes the orthogonal subspace of  $\mathcal{L}_T^{\mathcal{H}}$ . Here  $A_T$  corresponds to the final value of a square-integrable  $\mathbb{F}$ -martingale  $A$  with zero initial value, that implies  $\mathbb{E}[\xi - U_0 | \mathcal{F}_0] = 0$ . Clearly, we have

$$\mathcal{L}_T^{\mathcal{H}} \oplus (\mathcal{L}_T^{\mathcal{H}})^{\perp} = L^2(\Omega, \mathcal{F}_T, \mathbb{P}).$$

Now we define the process  $Y$  as follows:

$$\begin{aligned} Y_t &:= \mathbb{E}[\xi | \mathcal{F}_t] \\ &= \mathbb{E}\left[U_0 + \int_0^T H_s^{\mathcal{H}} dM_s + A_T \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}[U_0 | \mathcal{F}_0] + \int_0^t H_s^{\mathcal{H}} dM_s + A_t \\ &= Y_0 + \int_0^t H_s^{\mathcal{H}} dM_s + A_t, \quad 0 \leq t \leq T, \end{aligned}$$

and we set  $Z_t := H_t^{\mathcal{H}}$  and  $O_t := A_t$ , for every  $t \in [0, T]$ . Then we get

$$Y_t = \xi - \int_t^T Z_s dM_s - (O_T - O_t), \quad 0 \leq t \leq T.$$

**Uniqueness.** Let  $(Y, Z)$ ,  $(Y', Z')$  be two solutions to the BSDE (2.1) under partial information associated to the terminal condition  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ . We set  $(\bar{Y}, \bar{Z}) = (Y - Y', Z - Z')$ . Then  $(\bar{Y}, \bar{Z})$  satisfies the BSDE

$$\bar{Y}_t = - \int_t^T \bar{Z}_s dM_s - (\bar{O}_T - \bar{O}_t), \quad 0 \leq t \leq T, \quad (3.3)$$

with final condition  $\bar{Y}_T = 0$ . In addition, we have set  $\bar{O} := O - O'$  in (3.3), where  $O$  and  $O'$  denote the square-integrable  $\mathbb{F}$ -martingales with  $O_0 = O'_0 = 0$  satisfying the orthogonality condition

$$\mathbb{E} \left[ O_T \int_0^T \varphi_t dM_t \right] = \mathbb{E} \left[ O'_T \int_0^T \varphi_t dM_t \right] = 0,$$

for all  $\mathbb{H}$ -predictable processes  $\varphi$  such that  $\mathbb{E} \left[ \int_0^T |\varphi_t|^2 d\langle M \rangle_t \right] < \infty$ . Since  $(\bar{Y}, \bar{Z})$  is a solution of (3.3), then

$$\bar{Y}_t = \bar{Y}_0 + \int_0^t \bar{Z}_s dM_s + \bar{O}_t, \quad 0 \leq t \leq T. \quad (3.4)$$

Since the process  $\bar{Y}$  is an  $\mathbb{F}$ -martingale such that  $\bar{Y}_T = 0$ , we have

$$\bar{Y}_t = \mathbb{E} [\bar{Y}_T | \mathcal{F}_t] = 0, \quad \text{for all } t \in [0, T].$$

Thus  $Y_t = Y'_t$   $\mathbb{P}$ -a.s., for every  $t \in [0, T]$ . Then we can rewrite (3.4) as follows

$$0 = \int_0^t \bar{Z}_s dM_s + \bar{O}_t, \quad 0 \leq t \leq T.$$

By computing the predictable covariation of  $\int_0^t \bar{Z}_s dM_s + \bar{O}$  and  $\bar{O}$  and by taking the expectation of both sides in the equality, for each  $t \in [0, T]$ , we obtain

$$\begin{aligned} 0 &= \int_0^t \bar{Z}_s d\langle M, \bar{O} \rangle_s + \langle \bar{O} \rangle_t \\ &= \mathbb{E} \left[ \int_0^t \bar{Z}_s d\langle M, \bar{O} \rangle_s \right] + \mathbb{E} [\langle \bar{O} \rangle_t]. \end{aligned}$$

Since  $\bar{Z}$  and  $\bar{O}$  are differences of solutions to the BSDE (2.1), then  $\mathbb{E} \left[ \int_0^t \bar{Z}_s d\langle M, \bar{O} \rangle_s \right] = 0$  for  $t \in [0, T]$ , and it follows

$$\mathbb{E} [\langle \bar{O} \rangle_t] = 0, \quad 0 \leq t \leq T. \quad (3.5)$$

By Theorem 4.2 of [11], since  $\bar{O}$  is a square-integrable  $\mathbb{F}$ -martingale null at zero, we have that  $\bar{O}^2 - \langle \bar{O} \rangle$  is an  $\mathbb{F}$ -martingale null at zero. Then by (3.5)

$$\mathbb{E} [\bar{O}_t^2] = \mathbb{E} [\langle \bar{O} \rangle_t] = 0, \quad 0 \leq t \leq T,$$

that implies  $\bar{O}_t^2 = 0$   $\mathbb{P}$ -a.s. for every  $t \in [0, T]$  and then  $O_t = O'_t$   $\mathbb{P}$ -a.s. for every  $t \in [0, T]$ . Now, let  $Y$  be the unique solution of (2.1) for a certain  $\mathbb{H}$ -predictable  $Z$  such that  $\mathbb{E} \left[ \int_0^T |Z_t|^2 d\langle M \rangle_t \right] < \infty$ , i.e.

$$Y_t = Y_0 + \int_0^t Z_s dM_s + O_t, \quad 0 \leq t \leq T. \quad (3.6)$$



It only remains to prove that  $Z$  is unique. For  $t = T$  equation (3.6) becomes

$$Y_T = \xi = Y_0 + \int_0^T Z_s dM_s + O_T.$$

By Proposition 3.2,  $Z_t = H_t^{\mathcal{H}}$   $\mathbb{P}$ -a.s., for each  $t \in [0, T]$  and then  $Z$  is univocally determined. This concludes the proof.  $\square$

## 4 Galtchouk-Kunita-Watanabe representation under partial information

We now wish to provide an explicit characterization of the integrand process  $H^{\mathcal{H}}$  appearing in the representation (3.2) in terms of the one given in the classical Galtchouk-Kunita-Watanabe decomposition, by means in particular of the concept of  $\mathbb{H}$ -predictable dual projection.

Let  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ . We consider the well-known Galtchouk-Kunita-Watanabe decomposition of  $\xi$  with respect to  $M$ :

$$\xi = \tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} dM_t + \tilde{O}_T, \quad \mathbb{P} - \text{a.s.}, \quad (4.1)$$

where  $\tilde{U}_0 \in \mathcal{F}_0$ , the integrand  $H^{\mathcal{F}} = (H_t^{\mathcal{F}})_{0 \leq t \leq T}$  is an  $\mathbb{F}$ -predictable process such that  $\mathbb{E} \left[ \int_0^T |H_t^{\mathcal{F}}|^2 d\langle M \rangle_t \right] < \infty$  and  $\tilde{O} = (\tilde{O}_t)_{0 \leq t \leq T}$  is a square-integrable  $\mathbb{F}$ -martingale with  $\tilde{O}_0 = 0$  such that  $\langle \tilde{O}, M \rangle_t = 0$ , for every  $t \in [0, T]$ . Moreover, let us observe that  $\tilde{U}_0 = \mathbb{E} [\xi | \mathcal{F}_0]$ .

In the sequel we will denote by  ${}^p X$  the  $\mathbb{H}$ -predictable projection of a (generic) integrable process  $X = (X_t)_{0 \leq t \leq T}$ , defined as the unique  $\mathbb{H}$ -predictable process such that

$$\mathbb{E} [X_\tau \mathbf{1}_{\{\tau < \infty\}} | \mathcal{H}_{\tau-}] = {}^p X_\tau \mathbf{1}_{\{\tau < \infty\}} \quad \mathbb{P} - \text{a.s.}$$

for every  $\mathbb{H}$ -predictable stopping time  $\tau$ .

First we give a preliminary result under the additional assumption that the predictable quadratic variation  $\langle M \rangle$  of the  $\mathbb{F}$ -martingale  $M$  is an  $\mathbb{H}$ -predictable process. In Theorem 4.7 we extend such result to the general case.

**Proposition 4.1.** *Let  $(\tilde{U}_0, H^{\mathcal{F}}, \tilde{O}_T)$  be the triplet corresponding to decomposition (4.1) of  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ . Suppose that the predictable quadratic variation  $\langle M \rangle$  of the  $\mathbb{F}$ -martingale  $M$  is an  $\mathbb{H}$ -predictable process. Then*

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + O_T, \quad \mathbb{P} - \text{a.s.},$$

with

$$U_0 = \mathbb{E} \left[ \tilde{U}_0 \middle| \mathcal{H}_0 \right], \quad (4.2)$$

$$H_t^{\mathcal{H}} = {}^p(H_t^{\mathcal{F}}), \quad 0 \leq t \leq T, \quad (4.3)$$

and  $O$  is a square-integrable  $\mathbb{F}$ -martingale with  $O_0 = 0$  such that  $\mathbb{E}[O_T \cdot \eta] = 0$ , for every  $\eta \in \mathcal{L}_T^{\mathcal{H}}$ .

*Proof.* Let

$$\xi = \tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} dM_t + \tilde{O}_T, \quad \mathbb{P} - \text{a.s.}$$

be the classical Galtchouk-Kunita-Watanabe decomposition of  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ . By taking the expectation of  $\xi$  with respect to  $\mathcal{H}_0$ , we have:

$$\mathbb{E}[\xi | \mathcal{H}_0] = \mathbb{E}\left[\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} dM_t \middle| \mathcal{H}_0\right] + \mathbb{E}[\tilde{O}_T | \mathcal{H}_0]. \quad (4.4)$$

Since  $(\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} dM_t)$  is an  $\mathbb{F}$ -martingale, it follows:

$$\begin{aligned} \mathbb{E}\left[\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} dM_t \middle| \mathcal{H}_0\right] &= \mathbb{E}\left[\mathbb{E}\left[\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} dM_t \middle| \mathcal{F}_0\right] \middle| \mathcal{H}_0\right] \\ &= \mathbb{E}[\tilde{U}_0 | \mathcal{H}_0], \end{aligned}$$

so that we can rewrite (4.4) as follows:

$$\mathbb{E}[\xi | \mathcal{H}_0] = \mathbb{E}[\tilde{U}_0 | \mathcal{H}_0] + \mathbb{E}[\tilde{O}_T | \mathcal{H}_0].$$

Moreover, since  $\tilde{O}$  is an  $\mathbb{F}$ -martingale null at zero, we have

$$\mathbb{E}[\tilde{O}_T | \mathcal{H}_0] = \mathbb{E}\left[\mathbb{E}[\tilde{O}_T | \mathcal{F}_0] \middle| \mathcal{H}_0\right] = 0.$$

This implies equality (4.2). To prove equality (4.3), we need to calculate the orthogonal projection of  $\xi$  onto the space  $\mathcal{L}_T^{\mathcal{H}}$ , see (3.1). For the sake of brevity, we suppose that  $\tilde{U}_0 = 0$ . Thanks to Proposition 3.2, this means we need to check the following condition:

$$\mathbb{E}\left[\xi \int_0^T \varphi_t dM_t\right] = \mathbb{E}\left[\int_0^T p(H_t^{\mathcal{F}}) dM_t \int_0^T \varphi_t dM_t\right],$$

for every  $\mathbb{H}$ -predictable process  $\varphi$  such that  $\mathbb{E}\left[\int_0^T |\varphi_t|^2 d\langle M \rangle_t\right] < \infty$ . Taking decomposition (4.1) into account, this corresponds to the following equality:

$$\mathbb{E}\left[\int_0^T H_t^{\mathcal{F}} \varphi_t d\langle M \rangle_t\right] = \mathbb{E}\left[\int_0^T p(H_t^{\mathcal{F}}) \varphi_t d\langle M \rangle_t\right], \quad (4.5)$$

for every  $\mathbb{H}$ -predictable process  $\varphi$  such that  $\mathbb{E}\left[\int_0^T |\varphi_t|^2 d\langle M \rangle_t\right] < \infty$ . If we write the process  $\varphi$  as follows

$$\varphi = \varphi^+ - \varphi^-,$$

where  $\varphi^+$  and  $\varphi^-$  denote the positive and the negative part of  $\varphi$  respectively, and define the  $\mathbb{F}$ -martingales

$$R_t^+ = \int_0^t \sqrt{\varphi_s^+} dM_s, \quad R_t^- = \int_0^t \sqrt{\varphi_s^-} dM_s, \quad 0 \leq t \leq T,$$

equality (4.5) is equivalent to the following relationships:

$$\begin{aligned} \mathbb{E} \left[ \int_0^T H_t^{\mathcal{F}} d\langle R^+ \rangle_t \right] &= \mathbb{E} \left[ \int_0^T p(H_t^{\mathcal{F}}) d\langle R^+ \rangle_t \right] \\ \mathbb{E} \left[ \int_0^T H_t^{\mathcal{F}} d\langle R^- \rangle_t \right] &= \mathbb{E} \left[ \int_0^T p(H_t^{\mathcal{F}}) d\langle R^- \rangle_t \right]. \end{aligned}$$

Hence, we can reduce the problem by assuming directly  $\varphi_t = 1$  in (4.5), for each  $t \in [0, T]$ . Then, it is enough to prove the equality

$$\mathbb{E} \left[ \int_0^T H_t^{\mathcal{F}} d\langle M \rangle_t \right] = \mathbb{E} \left[ \int_0^T p(H_t^{\mathcal{F}}) d\langle M \rangle_t \right]. \quad (4.6)$$

Since  $\langle M \rangle$  is  $\mathbb{H}$ -predictable, Theorem VI.57 in [4] guarantees that equality (4.6) holds, once we have the positivity of the process  $H^{\mathcal{F}}$ . By writing

$$H^{\mathcal{F}} = (H^{\mathcal{F}})^+ - (H^{\mathcal{F}})^-,$$

and applying the above theorem to the positive and negative parts of  $H^{\mathcal{F}}$ ,  $(H^{\mathcal{F}})^+$  and  $(H^{\mathcal{F}})^-$  respectively, and to the associated  $\mathbb{H}$ -predictable projections, we can get the result by setting

$$H^{\mathcal{H}} := p(H^{\mathcal{F}}) = p((H^{\mathcal{F}})^+) - p((H^{\mathcal{F}})^-).$$

□

**Example 4.2.** *Let us consider the particular case where  $M$  is a square-integrable  $\mathbb{F}$ -martingale that is in addition a Lévy process,  $\mathcal{F}_t = \mathcal{F}_t^M$  and  $\mathcal{H}_t = \mathcal{F}_{(t-\tau)^+}^M$ , with  $\tau \in (0, T)$  a fixed delay. We assume  $\xi = h(M_T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ , for some measurable function  $h : \mathbb{R} \rightarrow \mathbb{R}$ .*

*In this framework, by Lemma A.1 (see Appendix), we know that the integrand appearing in the Galtchouk-Kunita-Watanabe decomposition (4.1) can be written as*

$$H_t^{\mathcal{F}} = F(t, M_{t-}), \quad t \in [0, T],$$

*where the function  $F$  is such that the condition  $\mathbb{E} \left[ \int_0^T |F(t, M_{t-})|^2 d\langle M \rangle_t \right] < \infty$  is satisfied. Since in this case  $\langle M \rangle$  is a deterministic process, we can apply Proposition 4.1 and get*

$$H_t^{\mathcal{H}} = pF(t, M_{t-}) = \mathbb{E}[F(t, M_{t-}) | \mathcal{H}_{t-}], \quad t \in [0, T].$$

*Then, it is easy to derive the following:*

$$H_t^{\mathcal{H}} = \begin{cases} c(t, M_{(t-\tau)^-}) & \text{if } t > \tau \\ c(t, M_0) & \text{if } t \leq \tau, \end{cases}$$

where the function  $c$  is given by

$$c(t, y) = \int_{\mathbb{R}} F(t, y + z) d\rho_{t \wedge \tau}(z),$$

with  $\rho_t$  denoting the law of  $M_t$ , for every  $t \in [0, T]$ .

#### 4.1 The $\mathbb{H}$ -predictable dual projection

It is possible to extend the result of Proposition 4.1 by using the concept of  $\mathbb{H}$ -predictable dual projection. For reasons of clarity, we provide a self-contained discussion about this kind of projection in presence of more than one filtration. Let  $G = (G_t)_{0 \leq t \leq T}$  be a càdlàg  $\mathbb{F}$ -adapted process of integrable variation, that is,  $\mathbb{E}[\|G\|_T] < \infty$ . Here the process  $\|G\| = (\|G\|)_{0 \leq t \leq T}$  defined, for each  $t \in [0, T]$ , by

$$\|G\|_t(\omega) = \sup_{\Delta} \sum_{i=0}^{n(\Delta)-1} |G_{t_{i+1}}(\omega) - G_{t_i}(\omega)|,$$

where  $\Delta = \{t_0 = 0 < t_1 < \dots < t_{n(\Delta)} = t\}$  is a partition of  $[0, t]$ , denotes the total variation of the function  $t \rightsquigarrow G_t(\omega)$ .

**Proposition 4.3.** *Let  $G = (G_t)_{0 \leq t \leq T}$  be a càdlàg  $\mathbb{F}$ -adapted process of integrable variation. Then there exists a unique  $\mathbb{H}$ -predictable process  $G^{\mathbb{H}} = (G_t^{\mathbb{H}})_{0 \leq t \leq T}$  of integrable variation, such that*

$$\mathbb{E} \left[ \int_0^T \varphi_t dG_t^{\mathbb{H}} \right] = \mathbb{E} \left[ \int_0^T \varphi_t dG_t \right],$$

for every  $\mathbb{H}$ -predictable (bounded) process  $\varphi$ . The process  $G^{\mathbb{H}}$  is called the  $\mathbb{H}$ -predictable dual projection of  $G$ .

*Proof.* Without loss of generality, we can restrict our attention to the case where  $G$  is an increasing process and prove the statement on the generators  $\varphi$  of the form  $\varphi_u = \mathbf{1}_{(s,t]}(u) \mathbf{1}_B$ , with  $B \in \mathcal{H}_s$  and  $s, t \in [0, T]$  with  $s < t$ . Indeed, decomposing the process  $G$  as  $G = G^+ - G^-$ , where both the positive and negative parts of  $G$  are assumed to be increasing integrable processes, we can suppose  $G$  to be increasing such that

$$\mathbb{E}[G_T] = \mathbb{E}[\|G\|_T] < \infty.$$

If  $G$  is a càdlàg, increasing, integrable  $\mathbb{F}$ -adapted process, we will prove that there exists a unique increasing, integrable  $\mathbb{H}$ -predictable process  $G^{\mathbb{H}}$  such that for every  $s, t \in [0, T]$  with  $s < t$  and  $B \in \mathcal{H}_s$ , the following relationship holds

$$\mathbb{E}[\mathbf{1}_B(G_t - G_s)] = \mathbb{E}[\mathbf{1}_B(G_t^{\mathbb{H}} - G_s^{\mathbb{H}})].$$

Let  $\tilde{G} = (\tilde{G}_t)_{0 \leq t \leq T}$  be the  $\mathbb{H}$ -optional projection of  $G$ , such that for fixed times  $t \in (0, T]$

$$\tilde{G}_t = \mathbb{E}[G_t | \mathcal{H}_t] \quad \mathbb{P} - \text{a.s.}$$

We observe that for every  $s, t \in [0, T]$  with  $s < t$  and  $B \in \mathcal{H}_s$ , we have

$$\mathbb{E} [\mathbf{1}_B(G_t - G_s)] = \mathbb{E} [\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)].$$

Indeed,

$$\begin{aligned} \mathbb{E} [\mathbf{1}_B(G_t - G_s)] &= \mathbb{E} [\mathbb{E} [\mathbf{1}_B(G_t - G_s) | \mathcal{H}_s]] = \mathbb{E} [\mathbf{1}_B (\mathbb{E} [G_t | \mathcal{H}_s] - \tilde{G}_s)] \\ &= \mathbb{E} [\mathbf{1}_B (\mathbb{E} [\tilde{G}_t | \mathcal{H}_s] - \tilde{G}_s)] = \mathbb{E} [\mathbb{E} [\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s) | \mathcal{H}_s]] \\ &= \mathbb{E} [\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)]. \end{aligned}$$

Furthermore, since  $G$  is increasing, then  $\tilde{G}$  is an  $\mathbb{H}$ -submartingale, that is

$$\mathbb{E} [\tilde{G}_t | \mathcal{H}_s] = \mathbb{E} [\mathbb{E} [G_t | \mathcal{H}_t] | \mathcal{H}_s] = \mathbb{E} [G_t | \mathcal{H}_s] \geq \mathbb{E} [G_s | \mathcal{H}_s] = \tilde{G}_s, \quad 0 \leq s \leq t \leq T.$$

Thanks to Doob-Meyer Theorem on decomposition of submartingales, see e.g. Theorem 3.15 of [11], there exists a unique increasing, integrable  $\mathbb{H}$ -predictable process  $G^{\mathbb{H}}$  such that  $\tilde{G} - G^{\mathbb{H}}$  is an  $\mathbb{H}$ -martingale, that is, for every  $s, t \in [0, T]$  with  $s < t$  and  $B \in \mathcal{H}_s$ , we have

$$\mathbb{E} [\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)] = \mathbb{E} [\mathbf{1}_B(G_t^{\mathbb{H}} - G_s^{\mathbb{H}})].$$

□

**Remark 4.4.** *If  $G$  is an  $\mathbb{H}$ -predictable process of integrable variation and  $X$  is an  $\mathbb{F}$ -adapted process satisfying  $\mathbb{E} \left[ \int_0^T X_t dG_t \right] < \infty$ , then*

$$(X_t dG_t)^{\mathbb{H}} = {}^p X_t dG_t, \quad \mathbb{P} - \text{a.s.}, \text{ for every } t \in [0, T].$$

*Indeed, by Theorem VI.57 in [4], for any  $\mathbb{H}$ -predictable (bounded) process  $\varphi$  we can prove that*

$$\mathbb{E} \left[ \int_0^T \varphi_t X_t dG_t \right] = \mathbb{E} \left[ \int_0^T \varphi_t^p X_t dG_t \right].$$

## 4.2 Explicit representation results

We now can apply the results of Subsection 4.1 to extend Proposition 4.1. Let  $\mathcal{P}^{\mathbb{H}}$  and  $\mathcal{P}$  be the  $\mathbb{H}$ -predictable and  $\mathbb{F}$ -predictable  $\sigma$ -field respectively. We consider the measures  $\mu^{\mathbb{H}}$  (respectively  $\mu$ ) defined on  $\mathcal{P}^{\mathbb{H}}$  (respectively  $\mathcal{P}$ ) such that

$$\mu^{\mathbb{H}}((s, t] \times B) = \mathbb{E} [\mathbf{1}_B(A_t^{\mathbb{H}} - A_s^{\mathbb{H}})], \quad B \in \mathcal{H}_s, \quad s, t \in [0, T], \quad s < t, \quad (4.7)$$

where  $A^{\mathbb{H}}$  is the  $\mathbb{H}$ -predictable dual projection of  $A := (\int_0^t H_s^{\mathbb{F}} d\langle M \rangle_s)_{0 \leq t \leq T}$ , that exists thanks to Theorem 4.3, and

$$\mu((u, v] \times F) = \mathbb{E} [\mathbf{1}_F(\langle M \rangle_v^{\mathbb{H}} - \langle M \rangle_u^{\mathbb{H}})], \quad F \in \mathcal{F}_u, \quad u, v \in [0, T], \quad u < v. \quad (4.8)$$

Here  $H^{\mathbb{F}}$  is the integrand appearing in the Galtchouk-Kunita-Watanabe decomposition (4.1).

**Lemma 4.5.** *Let  $\mu^{\mathcal{H}}$  and  $\mu$  measures satisfying conditions (4.7) and (4.8) respectively. Then  $\mu^{\mathcal{H}} \ll \mu$  on  $\mathcal{P}^{\mathcal{H}}$ , that is,  $\mu^{\mathcal{H}}$  is absolutely continuous with respect to the restriction of  $\mu$  on  $\mathcal{P}^{\mathcal{H}}$ .*

*Proof.* By using the definition of absolute continuity, we wish to show that if whenever  $\mu(E) = 0$  for  $E \in \mathcal{P}^{\mathcal{H}}$ , then  $\mu^{\mathcal{H}}(E) = 0$ . Let  $\psi = (\psi_t)_{0 \leq t \leq T}$  be a nonnegative  $\mathbb{H}$ -predictable process such that

$$\mathbb{E} \left[ \int_0^T \psi_t d\langle M \rangle_t^{\mathbb{H}} \right] = 0.$$

Then

$$\mathbb{E} \left[ \int_0^T \psi_t d\langle M \rangle_t \right] = 0,$$

that implies that  $\psi = 0$   $d\langle M \rangle \otimes d\mathbb{P}$  a.e., since  $\psi$  is nonnegative. Finally

$$\mathbb{E} \left[ \int_0^T \psi_t dA_t^{\mathbb{H}} \right] = \mathbb{E} \left[ \int_0^T \psi_t dA_t \right] = \mathbb{E} \left[ \int_0^T \psi_t H_t^{\mathcal{F}} d\langle M \rangle_t \right] = 0.$$

□

Since  $\mu^{\mathcal{H}} \ll \mu$  on  $\mathcal{P}^{\mathcal{H}}$ , thanks to Lemma 4.5, by the Radon-Nikodym theorem there exists a  $\mathcal{P}^{\mathcal{H}}$ -measurable function  $g$  on  $[0, T] \times \Omega$  such that

$$\mu^{\mathcal{H}}(E) = \int_E g(t, \omega) d\mu(t, \omega), \quad \forall E \in \mathcal{P}^{\mathcal{H}}.$$

This allows to identify the process  $H^{\mathcal{H}}$  as the Radon-Nikodym derivative:

$$H_t^{\mathcal{H}}(\omega) := \left. \frac{d\mu^{\mathcal{H}}(t, \omega)}{d\mu(t, \omega)} \right|_{\mathcal{P}^{\mathcal{H}}}, \quad (t, \omega) \in [0, T] \times \Omega. \quad (4.9)$$

Finally, we are ready to state the following theorem.

**Theorem 4.6.** *For any nonnegative  $\mathbb{F}$ -measurable process  $H^{\mathcal{F}}$ , the following equality holds*

$$\mathbb{E} \left[ \int_0^T \varphi_t H_t^{\mathcal{F}} d\langle M \rangle_t \right] = \mathbb{E} \left[ \int_0^T \varphi_t H_t^{\mathcal{H}} d\langle M \rangle_t \right], \quad (4.10)$$

for every  $\mathbb{H}$ -predictable process  $\varphi$  such that  $\mathbb{E} \left[ \int_0^T |\varphi_t|^2 d\langle M \rangle_t \right] < \infty$ . Here  $H^{\mathcal{H}}$  is given by (4.9).

*Proof.* By relationship (4.9) and definition of the measure  $\mu$ , see (4.8), we have for every  $s, t \in [0, T]$  with  $s < t$  and  $B \in \mathcal{H}_s$

$$\mu^{\mathcal{H}}((s, t] \times B) = \int_s^t \int_B H_u^{\mathcal{H}}(\omega) d\mu(u, \omega) = \mathbb{E} \left[ \mathbf{1}_B \int_s^t H_u^{\mathcal{H}} d\langle M \rangle_u^{\mathbb{H}} \right] = \mathbb{E} \left[ \mathbf{1}_B \int_s^t H_u^{\mathcal{H}} d\langle M \rangle_u \right].$$

On the other hand, by (4.7)

$$\mu^{\mathcal{H}}((s, t] \times B) = \mathbb{E} \left[ \mathbf{1}_B(A_t^{\mathbb{H}} - A_s^{\mathbb{H}}) \right] = \mathbb{E} \left[ \mathbf{1}_B \int_s^t H_u^{\mathcal{F}} d\langle M \rangle_u \right].$$

If  $\varphi$  is of the form  $\varphi_u = \mathbf{1}_{(s, t]}(u) \mathbf{1}_B$ , with  $B \in \mathcal{H}_s$  and  $s, t \in [0, T]$  with  $s < t$ , then the statement is proved since relationship (4.10) is verified on the generators of  $\mathcal{P}^{\mathcal{H}}$ .  $\square$

We now give the analogous of Proposition 4.1, without assuming that the process  $\langle M \rangle$  is  $\mathbb{H}$ -predictable.

**Theorem 4.7.** *Let  $(\tilde{U}_0, H^{\mathcal{F}}, \tilde{O}_T)$  be the triplet corresponding to decomposition (4.1) of  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ . Then*

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + O_T, \quad \mathbb{P} - \text{a.s.}, \quad (4.11)$$

with

$$U_0 = \mathbb{E} \left[ \tilde{U}_0 \middle| \mathcal{H}_0 \right],$$

$$H_t^{\mathcal{H}} = \left. \frac{d\mu^{\mathcal{H}}(t, \omega)}{d\mu(t, \omega)} \right|_{\mathcal{P}^{\mathcal{H}}}, \quad (t, \omega) \in [0, T] \times \Omega,$$

where  $\mu^{\mathcal{H}}$  and  $\mu$  are given in (4.7) and (4.8) respectively, and  $O$  is a square-integrable  $\mathbb{F}$ -martingale with  $O_0 = 0$  such that  $\mathbb{E}[O_T \cdot \eta] = 0$ , for every  $\eta \in \mathcal{L}_T^{\mathcal{H}}$ .

*Proof.* We proceed as in the proof of Proposition 4.1 by observing that condition (4.10) plays the same role of condition (4.5).  $\square$

In the next proposition we give a useful result which allows us to compute  $H^{\mathcal{H}}$  as the Radon-Nikodym derivative of the  $\mathbb{H}$ -predictable dual projection  $A^{\mathbb{H}}$  of the process  $A = (\int_0^t H_s^{\mathcal{F}} d\langle M \rangle_s)_{0 \leq t \leq T}$  with respect to the  $\mathbb{H}$ -predictable dual projection  $\langle M \rangle^{\mathbb{H}}$  of the  $\mathbb{F}$ -predictable quadratic variation  $\langle M \rangle$ .

**Proposition 4.8.** *The process  $A^{\mathbb{H}} = (\int_0^t H_s^{\mathcal{F}} d\langle M \rangle_s)^{\mathbb{H}}$  is absolutely continuous with respect to  $\langle M \rangle^{\mathbb{H}}$  and it is given by*

$$A_t^{\mathbb{H}} = \int_0^t H_s^{\mathcal{H}} d\langle M \rangle_s^{\mathbb{H}}, \quad 0 \leq t \leq T.$$

As a consequence

$$H_t^{\mathcal{H}} = \frac{dA_t^{\mathbb{H}}}{d\langle M \rangle_t^{\mathbb{H}}}, \quad 0 \leq t \leq T. \quad (4.12)$$

*Proof.* Set  $\tilde{A}_t := \int_0^t H_s^{\mathcal{H}} d\langle M \rangle_s^{\mathbb{H}}$ , for each  $t \in [0, T]$ . It is sufficient to prove that

$$\mathbb{E} \left[ \int_0^T \varphi_u dA_u \right] = \mathbb{E} \left[ \int_0^T \varphi_u d\tilde{A}_u \right]$$

for every  $\mathbb{H}$ -predictable (bounded) process  $\varphi$ . As before, we can consider  $\varphi$  of the form  $\varphi_u = \mathbf{1}_{(s,t]}(u)\mathbf{1}_B$ , with  $B \in \mathcal{H}_s$  and  $s < t \in [0, T]$ .

Then by the definitions of the measure  $\mu$  and  $\mu^{\mathcal{H}}$ , see (4.8) and (4.7), and recalling (4.9) we get

$$\begin{aligned} \mathbb{E} \left[ \int_0^T \varphi_u d\tilde{A}_u \right] &= \mathbb{E} \left[ \mathbf{1}_B \int_s^t H_u^{\mathcal{H}} d\langle M \rangle_u^{\mathbb{H}} \right] = \int_s^t \int_B H_u^{\mathcal{H}}(\omega) d\mu(u, \omega) = \mu^{\mathcal{H}}((s, t] \times B) = \\ &= \mathbb{E} \left[ \mathbf{1}_B (A_t^{\mathbb{H}} - A_s^{\mathbb{H}}) \right] = \mathbb{E} \left[ \mathbf{1}_B \int_s^t H_u^{\mathcal{F}} d\langle M \rangle_u \right] = \mathbb{E} \left[ \int_0^T \varphi_u dA_u \right] \end{aligned}$$

which concludes the proof.  $\square$

**Example 4.9.** Suppose that the process  $\langle M \rangle$  is of the form

$$\langle M \rangle_t = \int_0^t a_s dG_s, \quad 0 \leq t \leq T$$

for some  $\mathbb{F}$ -predictable process  $a = (a_t)_{0 \leq t \leq T}$  and an increasing deterministic function  $G$ . Then by Remark 4.4

$$\langle M \rangle_t^{\mathbb{H}} = \int_0^t p a_s dG_s, \quad A_t^{\mathbb{H}} = \int_0^t p (H_s^{\mathcal{F}} a_s) dG_s, \quad 0 \leq t \leq T,$$

and as a consequence of Proposition 4.8 we get

$$H_t^{\mathcal{H}} = \frac{p(H_t^{\mathcal{F}} a_t)}{p a_t}, \quad 0 \leq t \leq T.$$

**Remark 4.10.** Let us observe that if the process  $\langle M \rangle$  is  $\mathbb{H}$ -predictable, then again by Remark 4.4

$$\langle M \rangle_t^{\mathbb{H}} = \langle M \rangle_t, \quad A_t^{\mathbb{H}} = \int_0^t p (H_s^{\mathcal{F}}) d\langle M \rangle_s, \quad 0 \leq t \leq T,$$

and by applying Proposition 4.8 we obtain that

$$H_t^{\mathcal{H}} = p(H_t^{\mathcal{F}}), \quad 0 \leq t \leq T.$$

## 5 Risk-minimization under restricted information

In relation to the connection between risk-minimization under full and partial information respectively, we now show how our result obtained in Proposition 4.8 fits in the approach of [13] of risk-minimization under restricted information.

On a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  we consider a financial market with one riskless asset with (discounted) price 1 and one risky asset whose (discounted) price is given by a square-integrable (càdlàg) martingale  $M = (M_t)_{0 \leq t \leq T}$  adapted to a (large) filtration  $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ .

We will study the problem of hedging a contingent claim, whose final payoff is given by



a random variable  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ , in the case where investors acting in the market can access only to the information flow  $\mathbb{H} := (\mathcal{H}_t)_{0 \leq t \leq T}$  with  $\mathcal{H}_t \subseteq \mathcal{F}_t$ , for each  $t \in [0, T]$ . We choose the risk-minimization approach to solve the above hedging problem. In the case of full information, in [6] the authors proved that there exists a unique  $\mathbb{F}$ -risk-minimizing hedging strategy  $\phi^* = (\theta^*, \eta^*)$ , where  $\theta^* = (\theta_t^*)_{0 \leq t \leq T}$  is given by the integrand with respect to  $M$  in the classical Galtchouk-Kunita-Watanabe decomposition of  $\xi$ , i.e.  $\theta^* = H^{\mathcal{F}}$  (see equation (4.1)).

In this section we extend this result to the case where there are restrictions on the available information, by using the Galtchouk-Kunita-Watanabe decomposition under partial information (see equation (3.2)). More precisely, we prove that the  $\mathbb{H}$ -risk-minimizing hedging strategy  $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$  (see Definition 5.2 below) is such that  $\theta^{\mathcal{H}} = H^{\mathcal{H}}$ .

Risk-minimization under restricted information was studied in [13] by using a different approach. We obtain the same explicit representation given in Theorem 3.1 of [13] by applying Proposition 4.8. About risk-minimization under partial information for defaultable markets via nonlinear filtering, we refer to [8]. In particular, they consider the case where the contingent claim  $\xi$  is  $\mathcal{H}_T$ -measurable, in which we can solve the risk-minimization problem by using the classical Galtchouk-Kunita-Watanabe decomposition.

We now assume that the agent has at her/his disposal the information flow  $\mathbb{H}$  about trading in stocks while a complete information about trading in the riskless asset.

**Definition 5.1.** *An  $\mathbb{H}$ -strategy is a pair  $\phi = (\theta, \eta)$  ( $\theta_t$  is the number of shares of the risky asset to be held at time  $t$ , while  $\eta_t$  is the amount invested in the riskless asset at time  $t$ ) where  $\theta$  is  $\mathbb{H}$ -predictable and  $\eta$  is  $\mathbb{F}$ -adapted and such that*

$$\mathbb{E} \left[ \int_0^T \theta_s^2 d\langle M \rangle_s \right] < \infty$$

and the value process  $V(\phi) := \theta M + \eta$  satisfies

$$\mathbb{E} \left[ \left( \sup_{t \in [0, T]} |V_t(\phi)| \right)^2 \right] < \infty.$$

For any  $\mathbb{H}$ -strategy  $\phi$ , the associated cost process  $C(\phi)$  is given by

$$C_t(\phi) := V_t(\phi) - \int_0^t \theta_r dM_r, \quad 0 \leq t \leq T.$$

Finally the  $\mathbb{H}$ -risk process of  $\phi$  is defined by

$$R_t(\phi) := \mathbb{E} \left[ (C_T(\phi) - C_t(\phi))^2 \mid \mathcal{H}_t \right], \quad 0 \leq t \leq T.$$

**Definition 5.2.** *An  $\mathbb{H}$ -strategy  $\phi = (\theta, \eta)$  is called  $\mathbb{H}$ -risk-minimizing if  $V_T(\phi) = \xi$   $\mathbb{P}$ -a.s. and if for any  $\mathbb{H}$ -strategy  $\psi$  such that  $V_T(\psi) = \xi$   $\mathbb{P}$ -a.s., we have  $R_t(\phi) \leq R_t(\psi)$   $\mathbb{P}$ -a.s. for every  $t \in [0, T]$ .*

**Remark 5.3.** By Corollary 3.1 in [13] we have that if  $\phi = (\theta, \eta)$  is an  $\mathbb{H}$ -risk-minimizing strategy then  $\phi$  is mean-self-financing, i.e. the cost process  $C(\phi)$  is an  $\mathbb{F}$ -martingale. Moreover, if  $\phi = (\theta, \eta)$  is a mean-self-financing  $\mathbb{H}$ -strategy, then  $V(\phi)$  is an  $\mathbb{F}$ -martingale, hence for a given  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ , we have that  $V_t(\phi) = \mathbb{E}[\xi | \mathcal{F}_t]$ , for every  $t \in [0, T]$ .

To prove the main result of this section, we need the following Lemma.

**Lemma 5.4.** Let  $O = (O_t)_{0 \leq t \leq T}$  be a square-integrable  $\mathbb{F}$ -martingale with  $O_0 = 0$ , satisfying the orthogonality condition

$$\mathbb{E} \left[ O_T \int_0^T \varphi_t dM_t \right] = 0$$

for all  $\mathbb{H}$ -predictable processes  $\varphi = (\varphi_t)_{0 \leq t \leq T}$  such that  $\mathbb{E} \left[ \int_0^T |\varphi_t|^2 d\langle M \rangle_t \right] < \infty$ . Then for any  $t \in [0, T]$

$$\mathbb{E} \left[ (O_T - O_t) \int_t^T \varphi_s dM_s \middle| \mathcal{H}_t \right] = 0 \quad \mathbb{P} - \text{a.s.}$$

*Proof.* Since for any  $\mathbb{H}$ -predictable process  $\varphi$

$$\mathbf{1}_{(t, T]}(s) \mathbf{1}_B \varphi_s, \quad B \in \mathcal{H}_t, \quad t \in [0, T],$$

is  $\mathbb{H}$ -predictable, we get

$$\mathbb{E} \left[ O_T \mathbf{1}_B \int_t^T \varphi_s dM_s \right] = \mathbb{E} \left[ \mathbf{1}_B \mathbb{E} \left[ O_T \int_t^T \varphi_s dM_s \middle| \mathcal{H}_t \right] \right] = 0, \quad \forall B \in \mathcal{H}_t,$$

and then

$$\mathbb{E} \left[ O_T \int_t^T \varphi_s dM_s \middle| \mathcal{H}_t \right] = 0 \quad \mathbb{P} - \text{a.s.}$$

Finally, let us observe that

$$\mathbb{E} \left[ O_t \int_t^T \varphi_s dM_s \middle| \mathcal{H}_t \right] = \mathbb{E} \left[ O_t \int_0^T \varphi_s dM_s \middle| \mathcal{H}_t \right] - \mathbb{E} \left[ O_t \int_0^t \varphi_s dM_s \middle| \mathcal{H}_t \right] = 0$$

since

$$\mathbb{E} \left[ O_t \int_0^t \varphi_s dM_s \middle| \mathcal{H}_t \right] = \mathbb{E} \left[ \mathbb{E} \left[ O_t \int_0^t \varphi_s dM_s \middle| \mathcal{F}_t \right] \middle| \mathcal{H}_t \right] = \mathbb{E} \left[ O_t \int_0^t \varphi_s dM_s \middle| \mathcal{H}_t \right],$$

and this concludes the proof.  $\square$

We are now in the position to provide an alternative proof to that given in [13], concerning the explicit representation for an  $\mathbb{H}$ -risk-minimizing strategy, by applying the Galtchouk-Kunita-Watanabe decomposition under partial information and the representation result given in Proposition 4.8.

**Theorem 5.5.** For every  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ , there exists a unique  $\mathbb{H}$ -risk-minimizing strategy  $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$  such that  $\theta^{\mathcal{H}} = H^{\mathcal{H}}$ , where  $H^{\mathcal{H}}$  is given by (4.12) and  $\eta_t^{\mathcal{H}} = \mathbb{E}[\xi | \mathcal{F}_t] - \theta_t^{\mathcal{H}} M_t$ , for every  $t \in [0, T]$ .

*Proof.* The proof is similar to that of Theorem 2.4 of [14] performed in the full information case. Let  $\phi = (\theta, \eta)$  be a mean-self-financing  $\mathbb{H}$ -strategy such that  $V_T(\phi) = \xi$   $\mathbb{P}$ -a.s.. Hence, by computing the Galtchouk-Kunita-Watanabe decomposition under partial information, see (4.11), we have

$$\begin{aligned} C_T(\phi) - C_t(\phi) &= V_T(\phi) - V_t(\phi) - \int_t^T \theta_s dM_s = \xi - V_t(\phi) - \int_t^T \theta_s dM_s \\ &= U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + O_T - \int_t^T \theta_s dM_s - V_t(\phi), \end{aligned}$$

where  $H^{\mathcal{H}}$  is given by (4.12). Since  $V_t(\phi) = E[\xi | \mathcal{F}_t]$ , for every  $t \in [0, T]$ , see Remark 5.3, we get that

$$V_t(\phi) = U_0 + \int_0^t H_s^{\mathcal{H}} dM_s + O_t$$

and

$$C_T(\phi) - C_t(\phi) = \int_t^T \{H_s^{\mathcal{H}} - \theta_s\} dM_s + O_T - O_t.$$

By similar computation we get that

$$C_T(\phi^{\mathcal{H}}) - C_t(\phi^{\mathcal{H}}) = O_T - O_t.$$

Finally

$$\begin{aligned} (C_T(\phi) - C_t(\phi))^2 &= \left(C_T(\phi^{\mathcal{H}}) - C_t(\phi^{\mathcal{H}})\right)^2 + \left(\int_t^T \{H_s^{\mathcal{H}} - \theta_s\} dM_s\right)^2 \\ &\quad + 2(O_T - O_t) \int_t^T \{H_s^{\mathcal{H}} - \theta_s\} dM_s \end{aligned}$$

and by Lemma 5.4 we obtain that

$$R_t(\phi) = R_t(\phi^{\mathcal{H}}) + \mathbb{E} \left[ \left( \int_t^T \{H_s^{\mathcal{H}} - \theta_s\} dM_s \right)^2 \middle| \mathcal{H}_t \right] \geq R_t(\phi^{\mathcal{H}}).$$

Hence  $\phi^{\mathcal{H}}$  is  $\mathbb{H}$ -risk-minimizing. If some other  $\phi$  is also  $\mathbb{H}$ -risk-minimizing then

$$\mathbb{E} \left[ \left( \int_0^T \{H_s^{\mathcal{H}} - \theta_s\} dM_s \right)^2 \middle| \mathcal{H}_0 \right] = \mathbb{E} \left[ \int_0^T \{H_s^{\mathcal{H}} - \theta_s\}^2 d\langle M \rangle_s \middle| \mathcal{H}_0 \right] = 0,$$

which implies  $H^{\mathcal{H}} = \theta$ . Since  $V_t(\phi) = V_t(\phi^{\mathcal{H}}) = \mathbb{E}[\xi | \mathcal{F}_t]$  for each  $t \in [0, T]$ , we also obtain  $\phi = \phi^{\mathcal{H}}$ .  $\square$

In the rest of the section we investigate the case where there is a relationship between the information flow  $\mathbb{H}$  and the filtration generated by the stock price  $M$ , that we denote by  $\mathbb{F}^M$ . A possible choice is the assumption that investors acting in the market have access only to the information contained in past asset prices, that is  $\mathbb{H} = \mathbb{F}^M$ . Such a situation has been studied for instance in [7] and [3] for stock price dynamics with jumps. In the sequel we will assume  $\mathbb{H} \subseteq \mathbb{F}^M$ , which takes also into account, for instance, the case where the asset price is only observed at discrete times or with a fixed delay  $\tau \in (0, T)$ , i.e.  $\mathcal{H}_t = \mathcal{F}_{(t-\tau)^+}^M$ , for every  $t \in (0, T)$ .

In a such particular case, when in addition  $\xi \in L^2(\Omega, \mathcal{F}_T^M, \mathbb{P}; \mathbb{R})$ , we can find an  $\mathbb{H}$ -risk minimizing strategy,  $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$ , where  $\theta^{\mathcal{H}}$  is  $\mathbb{H}$ -predictable and  $\eta^{\mathcal{H}}$  is  $\mathbb{F}^M$ -adapted, while in the general case  $\eta^{\mathcal{H}}$  has been taken  $\mathbb{F}$ -adapted. This means that we study the situation where the agent has at her/his disposal the information flow  $\mathbb{H} \subseteq \mathbb{F}^M$  about trading in stocks and the filtration  $\mathbb{F}^M$  about trading in the riskless asset, and when  $\mathbb{H} = \mathbb{F}^M$  the same information flow.

More precisely, from now on we restrict ourself to consider  $\mathbb{H}$ -strategies  $\phi = (\theta, \eta)$  as in Definition 5.1 where  $\eta$  is chosen  $\mathbb{F}^M$ -adapted.

**Remark 5.6.** *Let us observe that given an  $\mathbb{H}$ -strategy  $\phi = (\theta, \eta)$ , the associated value process  $V(\phi) := \theta M + \eta$  turns out to be  $\mathbb{F}^M$ -adapted. By Corollary 3.1 in [13], we have that if  $\phi = (\theta, \eta)$  is an  $\mathbb{H}$ -risk-minimizing strategy according to this new definition, then  $\phi$  is mean-self-financing, i.e. the cost process  $C(\phi)$  is an  $\mathbb{F}^M$ -martingale. Moreover, if  $\phi = (\theta, \eta)$  is a mean-self-financing  $\mathbb{H}$ -strategy, then  $V(\phi)$  is an  $\mathbb{F}^M$ -martingale, hence  $V_t(\phi) = E[\xi | \mathcal{F}_t^M]$ , for every  $t \in [0, T]$ .*

We are now ready to give the following result.

**Theorem 5.7.** *For every  $\xi \in L^2(\Omega, \mathcal{F}_T^M, \mathbb{P}; \mathbb{R})$ , there exists a unique  $\mathbb{H}$ -risk-minimizing strategy  $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$  such that  $\theta^{\mathcal{H}} = H^{\mathcal{H}}$ , where  $H^{\mathcal{H}}$  is given by (4.12) and  $\eta^{\mathcal{H}} = E[\xi | \mathcal{F}_t^M] - \theta_t^{\mathcal{H}} M_t$ , for every  $t \in [0, T]$ .*

*Proof.* Since  $\xi$  is  $\mathcal{F}_T^M$ -measurable, by decomposition (4.11) we obtain that

$$\xi = \mathbb{E}[\xi | \mathcal{F}_T^M] = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + \mathbb{E}[O_T | \mathcal{F}_T^M],$$

where  $H^{\mathcal{H}}$  is given by (4.12). Set  $\hat{O}_t := \mathbb{E}[O_T | \mathcal{F}_t^M]$ , for each  $t \in [0, T]$ . It is known that  $\hat{O}$  is an  $\mathbb{F}^M$ -martingale and

$$\mathbb{E} \left[ \hat{O}_T \int_0^T H_t^{\mathcal{H}} dM_t \right] = \mathbb{E} \left[ \mathbb{E} \left[ \hat{O}_T | \mathcal{F}_T^M \right] \int_0^T H_t^{\mathcal{H}} dM_t \right] = \mathbb{E} \left[ O_T \int_0^T H_t^{\mathcal{H}} dM_t \right] = 0.$$

Therefore we obtain the Galtchouk-Kunita-Watanabe decomposition of  $\xi$  under restricted information with respect to the filtration  $\mathbb{F}^M$ , that is

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} dM_t + \hat{O}_T, \quad \mathbb{P} - \text{a.s.}$$

The rest of the proof follows from Theorem 5.5 by replacing the filtration  $\mathbb{F}$  by  $\mathbb{F}^M$  and the  $\mathbb{F}$ -martingale  $O$  by the  $\mathbb{F}^M$ -martingale  $\hat{O}$ .  $\square$

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## APPENDIX

### A Technical results

Recall that  $M = (M_t)_{0 \leq t \leq T}$  is a square-integrable  $\mathbb{F}$ -martingale and assume that  $\mathbb{F} = \mathbb{F}^M := (\mathcal{F}_t^M)_{0 \leq t \leq T}$ , i.e. the information flow  $\mathbb{F}$  coincides with the canonical filtration  $\mathbb{F}^M$  of  $M$ .

**Lemma A.1.** *Let  $M$  be a Lévy process and  $\xi = h(M_T) \in L^2(\Omega, \mathcal{F}_T^M, \mathbb{P}; \mathbb{R})$  for some measurable function  $h : \mathbb{R} \rightarrow \mathbb{R}$ . Then, there exists a measurable function  $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  such that*

$$\xi = \mathbb{E}[\xi] + \int_0^T F(s, M_{s-}) dM_s + \tilde{O}_T, \quad \mathbb{P} - a.s.,$$

where  $\tilde{O} = (\tilde{O}_t)_{0 \leq t \leq T}$  is a square-integrable  $\mathbb{F}^M$ -martingale null at zero such that  $\langle \tilde{O}, M \rangle_t = 0$ , for every  $t \in [0, T]$ . Moreover, the following integrability condition is satisfied

$$\mathbb{E} \left[ \int_0^T |F(s, M_{s-})|^2 d\langle M \rangle_s \right] < \infty.$$

*Proof.* If  $\xi$  is given as a Fourier transform of  $M_T$ , that is, the function  $h$  is of the form

$$h(x) = \int_{\mathbb{R}} e^{iax} d\nu(a), \quad \text{for all } x \in \mathbb{R}, \quad (\text{A.1})$$

where  $\nu$  is a finite measure, the result is contained in Proposition 4.3 of [9], which was an adaptation of [10].

As a consequence, the thesis follows once we show the existence of a sequence  $(h_n)_{n \in \mathbb{N}}$  of functions of the kind (A.1) such that

$$\mathbb{E} \left[ |h_n(M_T) - h(M_T)|^2 \right] \xrightarrow[n \rightarrow \infty]{} 0. \quad (\text{A.2})$$

To see that, denoting by  $F_n(t, M_{t-})$  the integrand in the Galtchouk-Kunita-Watanabe decomposition of  $h_n(M_T)$ ,  $n \in \mathbb{N}$ , we can proceed as in the proof of Lemma 3.1 and we get that the sequence  $(F_n(t, M_{t-}))_{n \in \mathbb{N}}$  converges in  $L^2(\Omega, d\langle M \rangle \otimes d\mathbb{P})$  to the integrand  $H^{\mathcal{F}}$  in the Galtchouk-Kunita-Watanabe decomposition of  $h(M_T)$ . Now, there is a subsequence converging  $d\langle M \rangle \otimes d\mathbb{P}$ -a.e. to the  $\mathbb{F}$ -predictable process  $(H_t^{\mathcal{F}})_{0 \leq t \leq T}$  and for almost all  $t \in [0, T]$ ,  $H_t^{\mathcal{F}}$  is  $\sigma(M_{t-})$ -measurable. Finally this implies the existence of a measurable function  $F : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $H_t^{\mathcal{F}} = F(t, M_{t-})$ .

It remains to show the existence of a sequence  $(h_n)_{n \in \mathbb{N}}$  of functions of type (A.1) verifying (A.2). If  $\rho_T$  is the law of  $M_T$ , (A.2), translates into

$$\int_{\mathbb{R}} (h_n(y) - h(y))^2 d\rho_T(y) \xrightarrow{n \rightarrow \infty} 0. \quad (\text{A.3})$$

Since  $\rho_T$  is a finite non-negative measure, it is well-known that the space of smooth functions with compact support is dense in  $L^2(\rho_T)$ . This implies that the Schwartz space  $\mathcal{S}(\mathbb{R})$  of the fast decreasing functions is dense in  $L^2(\rho_T)$ . Let  $(h_n)_{n \in \mathbb{N}}$  belong to  $\mathcal{S}(\mathbb{R})$  such that (A.3), and consequently, (A.2) holds. Since the inverse Fourier transform  $\mathcal{F}^{-1}$  maps  $\mathcal{S}(\mathbb{R})$  into itself, then we observe that  $h_n$  are of the type (A.1) with  $\nu(da) = \mathcal{F}^{-1}h_n(a)da$ .  $\square$

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