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Modular Reasoning about Differential Privacy in a Probabilistic Process Calculus ^{*}

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Abstract. The verification of systems for protecting sensitive and confidential information is becoming an increasingly important issue. Differential privacy is a promising notion of privacy originated from the community of statistical databases, and now widely adopted in various models of computation aiming at the protection of confidential information. We consider a probabilistic process calculus as a specification formalism for concurrent systems, and we establish a framework for reasoning about the degree of differential privacy provided by such systems. We investigate the constructs, specify restricted forms of some particular operators, and prove how they preserve or decrease the degree of privacy under composition. We illustrate our ideas on an anonymity-preservation property for an extension of the Crowds protocol. Finally, we make a first step towards computing the channel matrix and the degree of privacy in a process-algebraic setting.

1 Introduction

The most recent developments and usages of information technologies such as data profiling in databases, or user tracking in pervasive computing, pose serious threats to the confidential information of the users. For instance, the social networks Twitter and Flickr carefully protect their user's data by anonymization, and yet Narayanan and Smatkov [20] were able to conceive a de-anonymization algorithm which could re-identify 30% of the people who have accounts in both of them, with only a 12% error rate. The verification of systems for protecting sensitive and confidential information is becoming an increasingly important issue in the modern world.

Many protocols for protecting confidential information have been proposed in the literature. In order to obfuscate the link between the secret and the public information, randomized mechanisms are often used. Typical examples are the DCNets [8], Crowds [22], Onion Routing [26], Freenet [9] and many others. Another common denominator is that various entities involved in the system to verify occur as concurrent processes and present typically a nondeterministic behavior. It is a natural and standard approach to apply process calculi to specify and reason about security protocols, which

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is also a line of active research. See e.g. the CCS approach [18, 11, 5], as well as the Applied pi-calculus approach for anonymity [1, 19] and for privacy [6]. In this paper, we consider a probabilistic extension of CCS, which we call CCS_p . In addition to the standard parallel composition and nondeterministic choice of CCS, CCS_p provides also a primitive for the probabilistic choice.

In order to properly measure the degree of protection a security system can guarantee, several notions have been proposed. See for instance, strong forms of protection (*strong anonymity* [4, 8], *no interference* [15]), as well as weaker notions of protection (*probable innocence* [22]). Different from the previous notions providing only true-or-false properties, the concepts from Information Theory [10] based on *entropy*, notably *mutual information* and *capacity*, express the degree of protection in a quantitative way.

Differential privacy [12, 14, 13] emerged from the field of statistical databases, is a recent promising definition of confidentiality, that provides strong privacy guarantees, and requires fewer assumptions than the information-theoretical approach. We say that a system is ϵ -differentially private if for every pair of *adjacent* datasets (i.e. datasets which differ in the data of an individual only), the probabilities of obtaining a certain answer differ at most by e^ϵ . Differential privacy captures the intuitive requirement of privacy, that is, a privacy-preserving system does not change much its probability of producing an observable when executing on nearly identical secret inputs. In this paper we consider an extended notion of differential privacy, based on a generic notion of adjacency, to measure the degree of protection of the confidential information.

The main contribution of this work is to investigate differential privacy for concurrent systems in the context of a probabilistic process calculi, and present a modular approach for reasoning about differential privacy, where the modularity is with respect to the constructs of CCS_p . More specifically, the probabilistic choice, the nondeterministic choice, the restriction and a restricted form of parallel composition are safe under composition, i.e. that are guaranteed not to decrease the safety of system. Compositionality results play an important role in the construction and analysis of security systems. Rather than analyzing a complex system as a whole, the safe constructs allow us to split the system in parts, analyze the degree of privacy of each part separately, and combine the results to obtain the global degree of privacy.

We illustrate our compositionality results on an anonymity-preservation property for an extension of the *Crowds* protocol [22]. *Crowds* is an anonymity protocol which allows Internet users to perform web transactions without revealing their private identity. (See detailed descriptions in Section 6) The original protocol assumes that the next forwarder is chosen equally among members from the whole crowd. (See, more precisely, the equations (13) and (14) of symmetry requirements in [7].) Our extension of *Crowds* consists in allowing each member to have a set of trusted users to which they can forward the message. This breaks the symmetry property of the original protocol, and makes the standard analyses from the literature unapplicable. In contrast, our compositional method gives a very simple proof.

Finally, we make a first step towards computing the channel matrix and the degree of privacy in a process-algebraic setting. We consider a simple scenario, that is, a finite process in which the secret choices are only at the top-level, and discuss a compositional method to compute its degree of privacy in a bottom-up way.

Nowadays differential privacy is widely adopted in many models of computation, for instance, it has been investigated in the context of a SQL-like language [17], a linear type system [21], a relational Hoare logic [3], MapReduce for cloud computing [23], and probabilistic labeled transition systems for interactive systems [27]. So far most of the work in applying formal methods for differential privacy concerns the privacy provided by statistical databases or programs. (See further comparisons in Section 8.)

Summary of Contributions To the best of our knowledge, this paper is the first to investigate differential privacy for concurrent systems within the setting of a process algebra. Our contributions are three-fold. We present a modular approach for reasoning about differential privacy on basis of CCS_p constructs. We apply our compositionality result on the proof for the anonymity-preservation of an extended Crowds protocol with member-wise trusted forwarders. We show an algorithm for computing the channel matrix and the degree of privacy of a finite process.

Plan of the paper In Sections 2 - 4 we review and introduce the related notions used throughout this paper. In Section 5 we investigate the compositionality of differential privacy on basis of CCS_p constructs. In Section 6, we apply our framework to the Crowds protocol. In Section 7, we present the algorithm for computing the degree of privacy. Finally, we survey prior art not yet mentioned in the introduction and conclude in Section 8 and 9. Detailed proofs of the theorem in Section 5 are in the Appendix.

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2 Preliminaries

2.1 Probabilistic automata

We recall the formalism of *probabilistic automata* [25], to be used as operational semantics for the probabilistic CCS that will be introduced in the next section.

We denote the set of all discrete probability distributions over a set X by $\text{Disc}(X)$. For $x \in X$, we denote by $\delta(x)$ (the *Dirac measure* on x) the probability distribution that assigns probability 1 to $\{x\}$.

A (*simple*) *probabilistic automaton* is a tuple $\mathcal{M} = (\mathcal{P}, P_{init}, \text{Act}, \mathcal{T})$ where \mathcal{P} is a set of states, $P_{init} \in \mathcal{P}$ is the *initial state*, Act is a set of labels and $\mathcal{T} \subseteq \mathcal{P} \times \text{Act} \times \text{Disc}(\mathcal{P})$ is a *transition relation*. Informally, if $(P, a, \mu) \in \mathcal{T}$ then there is a transition from the state P performing a label a and then leading to a distribution μ over a set of states instead of a single state. Intuitively, the idea is that the transition in \mathcal{T} is chosen nondeterministically, and the target state among the ones allowed by μ is chosen probabilistically.

A *fully probabilistic automaton* is a probabilistic automaton without nondeterminism, at each state only one transition can be chosen.

An *execution* α of a probabilistic automaton is a (possibly infinite) sequence

$$P_{init}a_0P_1a_1P_2a_2P_3 \dots$$

of alternating states and labels, and for each i , a transition $(P_i, a_i, \mu_i) \in \mathcal{T}$ and $\mu_i(P_{i+1}) > 0$ hold. We will use

- $exec^*(\mathcal{M})$ to represent the set of all the finite executions of \mathcal{M} ,
- $exec(\mathcal{M})$ to represent the set of all the executions of \mathcal{M} ,
- and $lst(\alpha)$ to denote the last state of a finite execution $\alpha \in exec^*(\mathcal{M})$.

The *execution tree* of \mathcal{M} , denoted by $etree(\mathcal{M})$, is an automaton $\mathcal{M}' = (\mathcal{P}', P'_{init}, Act, \mathcal{T}')$ such that $\mathcal{P}' \subseteq exec(\mathcal{M})$, $P'_{init} = P_{init}$, and $(\alpha, a, \mu') \in \mathcal{T}'$ if and only if $(lst(\alpha), a, \mu) \in \mathcal{T}$ for some μ and $\mu'(\alpha a P) = \mu(P)$ for all P . Intuitively, $etree(\mathcal{M})$ is produced by unfolding all the executions of \mathcal{M} .

A *scheduler* of a probabilistic automaton \mathcal{M} is a function

$$\zeta : exec^*(\mathcal{M}) \rightarrow \mathcal{T}$$

such that $\zeta(\alpha) = (P, a, \mu) \in \mathcal{T}$ implies that $P = lst(\alpha)$. (We omit \mathcal{M} when it is clear from the context.) The idea is that a scheduler resolves the nondeterminism by selecting a transition among the ones available in \mathcal{T} , basing its decision on the history of the execution.

The *execution tree of \mathcal{M} relative to the scheduler ζ* , denoted by $etree(\mathcal{M}, \zeta)$, is a fully probabilistic automaton $\mathcal{M}'' = (\mathcal{P}'', P''_{init}, Act, \mathcal{T}'')$ which can be obtained from \mathcal{M}' by removing all the transitions in \mathcal{T}' that are not chosen by the scheduler, that is, $(\alpha, a, \mu'') \in \mathcal{T}''$ if and only if $\zeta(\alpha) = (lst(\alpha), a, \mu)$ for some μ and $\mu''(\alpha a P) = \mu(P)$ for all P . Intuitively, $etree(\mathcal{M}, \zeta)$ is produced from $etree(\mathcal{M})$ by resolving all non-deterministic choices using ζ . Note that $etree(\mathcal{M}, \zeta)$ is a simple and fully probabilistic automaton.

3 Probabilistic CCS

In this section, we present a probabilistic version of Milner's CCS [18], that allows for both nondeterministic and probabilistic choice. Following [5] we make a distinction between *observable* and *secret* labels, for the purpose of specifying security systems and protocols.

We consider a countable set Act of labels a , partitioned into a set Sec of *secret labels* s , a set Obs of *observable labels* o , and the silent action τ . For each $o \in Obs$, we assume a complementary label $\bar{o} \in Obs$ with the convention that $\bar{\bar{o}} = o$. The syntax of CCS_p is:

$P ::=$		<i>process term</i>
	$\bigoplus_i p_i P_i$	<i>probabilistic choice</i>
	$\boxplus_i s_i.P_i$	<i>secret choice</i> ($s_i \in Sec$)
	$\boxplus_i r_i.P_i$	<i>nondeterministic choice</i> ($r_i \in Obs \cup \{\tau\}$)
	$P \mid P$	<i>parallel composition</i>
	$(\nu a)P$	<i>restriction</i>
	$!P$	<i>replication</i>

$$\begin{array}{c}
\text{PROB} \frac{}{\bigoplus_i p_i P_i \xrightarrow{\tau} \sum_i p_i \delta(P_i)} \quad \text{ACT} \frac{j \in I}{\lfloor \pm \rfloor_I a_i . P_i \xrightarrow{a_j} \delta(P_j)} \\
\\
\text{PAR1} \frac{P_1 \xrightarrow{a} \mu}{P_1 | P_2 \xrightarrow{a} \mu | P_2} \quad \text{PAR2} \frac{P_2 \xrightarrow{a} \mu}{P_1 | P_2 \xrightarrow{a} P_1 | \mu} \quad \text{REP} \frac{P | ! P \xrightarrow{a} \mu}{! P \xrightarrow{a} \mu | ! P} \\
\\
\text{COM} \frac{P_1 \xrightarrow{a} \delta(P'_1) \quad P_2 \xrightarrow{\bar{a}} \delta(P'_2)}{P_1 | P_2 \xrightarrow{\tau} \delta(P'_1 | P'_2)} \quad \text{RES} \frac{P \xrightarrow{b} \mu \quad b \neq a, \bar{a}}{(\nu a) P \xrightarrow{b} (\nu a) \mu}
\end{array}$$

Table 1: The semantics of CCS_p .

The term $\bigoplus_i p_i P_i$, in which the p_i s are positive rationals that sum up to one, represents a *blind probabilistic choice*, in the sense that the choice of the branch is decided randomly (according to the corresponding probabilities) and there is no visible label associated to the decision. We use the notation $P_1 \oplus_p P_2$ to represent a binary sum with $p_1 = p$ and $p_2 = 1 - p$. Similarly, we use $a_1 . P_1 \lfloor \pm \rfloor a_2 . P_2$ to represent a binary secret or nondeterministic choice. Finally the term $\mathbf{0}$, representing the terminated process, is syntactic sugar for an empty (secret or nondeterministic) choice.

The operational semantics of a CCS_p term P is a probabilistic automaton whose states \mathcal{P} are the processes reachable from P , and whose transition relation is defined according to the rules in the Table 1, where we use $P \xrightarrow{a} \mu$ to represent the transition (P, a, μ) . We denote by $\mu | P$ the measure μ' such that $\mu'(P' | P) = \mu(P')$ for all processes $P' \in \mathcal{P}$ and $\mu'(P'') = 0$ if P'' is not of the form $P' | P$. Similarly $(\nu a)\mu = \mu'$ such that $\mu'((\nu a)P) = \mu(P)$. A transition of the form $P \xrightarrow{a} \delta(P')$, i.e. a transition having for target a Dirac measure, corresponds to a transition of a non-probabilistic automaton. From the rule PROB, we know that all transitions to non-Dirac measures are silent.

Following [5] we assume the secret labels to be the *inputs* of the system. Secrets are given in input to the scheduler and determine completely the secret choices. The scheduler then has to resolve the residual nondeterminism, which is originated by the nondeterministic choice and the parallel operator. From the observer's point of view, only the nondeterministic choices can be observed.

The definition of a scheduler of a CCS_p term is specified more precisely as follows. The notation $X \rightarrow Y$ represents the partial functions from X to Y , and $\alpha|_{\text{Sec}}$ represents the projection of α on Sec .

Definition 1. Let P be a process in CCS_p and \mathcal{M} be the probabilistic automaton generated by P . A scheduler is a function $\zeta : \text{Sec}^* \rightarrow \text{exec}^*(\mathcal{M}) \rightarrow \mathcal{T}$ such that if:

- (i) $s = s_1 s_2 \dots s_n$ and $\alpha|_{\text{Sec}} = s_1 s_2 \dots s_m$ with $m \leq n$, and
- (ii) there exists a transition $(\text{lst}(\alpha), a, \mu)$ such that $a \in \text{Sec} \Rightarrow a = s_{m+1}$

then $\zeta(\mathbf{s})(\alpha)$ is defined, and it is one of such transitions. We will write $\zeta_{\mathbf{s}}(\alpha)$ for $\zeta(\mathbf{s})(\alpha)$.

We now define the *execution tree* of a CCS_p term, in a way similar to what is done in probabilistic automata. The main difference is that in our case the execution tree depends not only on the scheduler, but also on the secret input.

Definition 2. Let $\mathcal{M} = (\mathcal{P}, P_{\text{init}}, \text{Act}, \mathcal{T})$ be the probabilistic automaton generated by a CCS_p process P . Given an input \mathbf{s} and a scheduler ζ , the execution tree of P , denoted by $\text{etree}(P, \mathbf{s}, \zeta)$, is a fully probabilistic automaton $\mathcal{M}' = (\mathcal{P}', P_{\text{init}}, \text{Act}, \mathcal{T}')$ such that:

- (i) $\mathcal{P}' \subseteq \text{exec}(\mathcal{M})$,
- (ii) $(\alpha, a, \mu') \in \mathcal{T}'$ if and only if $\zeta_{\mathbf{s}}(\alpha) = (\text{lst}(\alpha), a, \mu)$ for some μ and $\mu'(\alpha a P) = \mu(P)$

Process terms as channels We now show how CCS_p terms can be used to specify systems manipulating confidential information.

A system can be seen as an information-theoretic channel [10]. Sequences of secret labels constitute the *secret information* (or *secrets*), given in input to the channel, and sequences of observable labels constitute the *public information* (or *observables*), obtained as output from the channel. We denote secrets and observables by \mathcal{S} and \mathcal{O} , and we assume that they have finite cardinality m and n , respectively. We also assume that each sequence in $\mathcal{S} \cup \mathcal{O}$ is finite. Thus, $\mathcal{S} \subseteq_{\text{fin}} \text{Sec}^*$ and $\mathcal{O} \subseteq_{\text{fin}} \text{Obs}^*$. This is enough to model the usual situations in security, where each session is supposed to terminate in finite time.

Given an input $\mathbf{s} \in \mathcal{S}$, a run of the system will produce each $\mathbf{o} \in \mathcal{O}$ with a certain probability $p(\mathbf{o}|\mathbf{s})$ which depends on \mathbf{s} , on the randomized operations performed by the system, and also on the scheduler resolving the nondeterminism. The probabilities $p(\mathbf{o}|\mathbf{s})$, for a given scheduler ζ , constitute a $m \times n$ array M_{ζ} which is called the *matrix* of the channel, where the rows are indexed by the elements of \mathcal{S} and the columns are indexed by the elements of \mathcal{O} . (See some examples of channel matrix in Section 7.2.)

Definition 3 ([5]). Given a term P and a scheduler $\zeta : \mathcal{S} \rightarrow \text{exec}^* \rightarrow \mathcal{T}$, the matrix $M_{\zeta}(P)$ associated to P under ζ is defined as the matrix such that, for each row $\mathbf{s} \in \mathcal{S}$ and column $\mathbf{o} \in \mathcal{O}$, the element at their intersection, $p_{\zeta}(\mathbf{o}|\mathbf{s})$, is the probability of the set of the maximal executions in $\text{etree}(P, \mathbf{s}, \zeta)$ whose projection in Obs is \mathbf{o} .

4 Differential Privacy

Differential Privacy [12] captures the idea that a query on a dataset does not provide too much information about a particular individual, regardless of whether the individual's record is in the dataset or not. In order to achieve this goal, typically some probabilistic noise is added to the answer. The formal definition is the following (where κ denotes the randomized answer, Pr the probability measure, and ϵ a finite non-negative number):

Definition 4 (Differential Privacy, [12]). A mechanism κ provides ϵ -differential privacy iff for all datasets D_1 and D_2 differing in only one record, and for all $S \subseteq \text{Range}(\kappa)$,

$$\Pr[\kappa(D_1) \in S] \leq e^\epsilon \Pr[\kappa(D_2) \in S]$$

Clearly, the smaller the value ϵ is, the higher is the confidentiality protection provided by the mechanism.

We now adapt the notion of differential privacy to our framework. We consider a symmetric adjacency relation \sim between secrets, which extends the dataset-based notion of “differing for only one record” to more general settings. The confidential data can be complex information like sequences of keys or tuples of individual data (see Example 2). Without loss of generality, for complex secrets composed of n elements, for instance (s_1, s_2, \dots, s_n) , we assume that,

$$\text{If } (s_1, s_2, \dots, s_n) \sim (s'_1, s'_2, \dots, s'_n), \text{ then } s_1 \sim s'_1, s_2 \sim s'_2, \dots, s_n \sim s'_n.$$

Example 1 (Anonymity). Anonymity is a particular case of privacy in which the confidential data \mathcal{S} are the agents’ identities. If the identities are just names without any particular structure, it is natural to assume that each name is adjacent to any other. Hence (\mathcal{S}, \sim) is a clique, i.e. for all $s_1, s_2 \in \mathcal{S}$ we have $s_1 \sim s_2$.

Example 2 (Geolocation). In the case of geolocation, the confidential data are the coordinates (*latitude, longitude*) of a point on the earth’s surface. If the purpose is to protect the exact location, a good definition of adjacency is: two points are adjacent if their Manhattan distance is 1, i.e. $(x_1, y_1) \sim (x_2, y_2)$ iff $|x_2 - x_1| = 1$ or $|y_2 - y_1| = 1$.

It can be proved that if the set of answers is discrete (which is our case) Definition 4 can be equivalently stated in terms of singleton S ’s. This leads to the following:

Definition 5. A process P provides ϵ -differential privacy iff for all schedulers ζ , for all secret inputs $s_1, s_2 \in \mathcal{S}$ such that $s_1 \sim s_2$, and for all observable $o \in \mathcal{O}$,

$$p_\zeta(o|s_1) \leq e^\epsilon p_\zeta(o|s_2)$$

We use $dp_\zeta[P]$ to denote the smallest ϵ such that the process term P , under the scheduler ζ , provides ϵ -differential privacy. Furthermore we define

$$dp[P] = \max_{\zeta} dp_\zeta[P]$$

Note that if there are both zero and non-zero probabilities occurring in the same column of the channel matrix of process P , when the respective secrets are connected by the transitive closure of \sim , then the process does not provide differential privacy for any ϵ .

Relation between differential privacy and strong anonymity Strong anonymity for purely probabilistic systems was formalized by Chaum [8] as the property that the observation o does not change the probabilistic knowledge of the culprit’s identity s , i.e.:

$$p(s|o) = p(s) \tag{1}$$

This notion was extended in [4] to probabilistic and nondeterministic systems, essentially by requiring that (1) holds under any scheduler. The following proposition is an immediate consequence of the characterization of strong anonymity given in [4].

Proposition 1. *Under the assumption that (\mathcal{S}, \sim) is a clique, a process P is strongly anonymous iff $dp \llbracket P \rrbracket = 0$.*

5 Modular Reasoning

In this section we investigate the compositional properties of CCS_p constructs with respect to differential privacy and state our first main result. We start by introducing two notions: the *safe component*, and the *sequential replication* which can be used to represent a sequence of sessions re-executing the same protocol. .

Definition 6. *Consider a process P , and observables o_1, o_2, \dots, o_k , we say that $(\nu o_1, o_2, \dots, o_k)P$ is a safe component if*

- (i) P does not contain any secret label, and
- (ii) all the observable labels of P are included in o_1, o_2, \dots, o_k .

Definition 7. *Given a process term P assumed to terminate by performing a specific action \overline{done} , the sequential replication of P n times is defined as*

$$\circlearrowleft^n P = (\nu done)(P \mid !^n done.P)$$

where

$$!^0 P = \mathbf{0} \quad \text{and} \quad !^{n+1} P = P \mid !^n P$$

We now show that the nondeterministic choice, the probabilistic choice, the restriction operator and a restricted form of parallel composition are *safe*, in the sense that combining components in a safe construct does not compromise the privacy of the system, while the sequential replication degrades the privacy in proportion to the number of replication times.

Theorem 1. *Consider a set of processes $\{P_i\}_i$, for $i = 1, 2, \dots$, and assume that for each i , $dp \llbracket P_i \rrbracket = \epsilon_i$. Then:*

- (1) $dp \llbracket \oplus_i o_i.P_i \rrbracket \leq \max_i \{\epsilon_i\}$;
- (2) $dp \llbracket \oplus_i p_i P_i \rrbracket \leq \max_i \{\epsilon_i\}$;
- (3) $dp \llbracket (\nu o)P_i \rrbracket \leq \epsilon_i$;
- (4) Assume that $(\nu o_1, o_2, \dots, o_k)P_i$ is a safe component, that P_i and P_j can communicate with each other only via the labels of the set $\{o_h, \dots, o_k\}$, with $1 \leq h \leq k$, and that $dp \llbracket (\nu o_1, \dots, o_{h-1})P_j \rrbracket = \epsilon_j$. Then $dp \llbracket (\nu o_1, o_2, \dots, o_k)(P_i \mid P_j) \rrbracket \leq \epsilon_j$.
- (5) $dp \llbracket \circlearrowleft^n P_i \rrbracket \leq n \epsilon_i$.

Properties (1) and (2) point out that the degree of privacy of a system, consisting of some subsystems in a nondeterministic or probabilistic choice, is determined by the subsystem with the lowest degree of privacy. Properties (3) and (4) intuitively say that, turning an observable label to be unobservable, and paralleling with a safe component, maintain the level of privacy. Property (5) means that the degree of privacy of a process degrades through multiple runs, since more information may be exposed.

Unfortunately the secret choice and the unrestricted form of parallel composition do not preserve the privacy, essentially due to the presence of nondeterminism. This is illustrated by the following counterexamples taken from [5]. (In Examples 3 - 5, we use the original definition of the adjacency relation, that is, the difference in only one label.)

Example 3 (For the secret choice). Let $Sec = \{s_1, s_2\}$ and assume that \mathcal{S} does not contain the empty sequence. Consider the process $P = o_1.\mathbf{0} \sqcup o_2.\mathbf{0}$. Clearly, P provides 0-differential privacy, because for every sequence $\mathbf{s} \in \mathcal{S}$ we have $p(o_1|\mathbf{s}) = p(o_2|\mathbf{s})$. Consider now a new process $P' = s_1.P \sqcup s_2.P$, and the scheduler ζ for P' which selects o_1 if the secret is s_1 , and o_2 if the secret is s_2 . The resulting matrix under ζ does not preserve differential privacy, since $p(o_1|s_1\mathbf{s}) = p(o_2|s_2\mathbf{s}) = 1$ while $p(o_1|s_2\mathbf{s}) = p(o_2|s_1\mathbf{s}) = 0$.

Example 4 (For the need of condition (i) in Def. 6). Let Sec and \mathcal{S} be as in Example 3. Define $P_1 = s_1.\mathbf{0} \sqcup s_2.\mathbf{0}$ and $P_2 = o_1.\mathbf{0} \sqcup o_2.\mathbf{0}$. Clearly, P_2 provides 0-differential privacy. Consider now the parallel term $P_1 | P_2$ and define a scheduler that first executes a secret label s in P_1 and then, if s is s_1 , it selects o_1 , while if s is s_2 , it selects o_2 . The rest proceeds like in Example 3.

Example 5 (For the need of condition (ii) in Def. 6). Let Sec and \mathcal{S} be as in Example 3. Define $P_1 = o.\mathbf{0}$ and $P_2 = s_1.(o_1.\mathbf{0} \oplus_{.5} o_2.\mathbf{0}) \sqcup s_2.(o_1.\mathbf{0} \oplus_{.5} o_2.\mathbf{0})$. It is easy to see that P_2 provides 0-differential privacy. Consider the term $P_1 | P_2$ and define a scheduler that first executes a secret label s in P_2 and then, if s is s_1 , it selects first P_1 and then the continuation of P_2 , while if s is s_2 , it selects first the continuation of P_2 and then P_1 . Hence, under this scheduler, for every sequence $\mathbf{s} \in \mathcal{S}$, $p(o o_1|s_1\mathbf{s}) = p(o o_2|s_1\mathbf{s}) = 0.5$ and also $p(o_1 o|s_2\mathbf{s}) = p(o_2 o|s_2\mathbf{s}) = 0.5$ while $p(o o_1|s_2\mathbf{s}) = p(o o_2|s_2\mathbf{s}) = 0$ and $p(o_1 o|s_1\mathbf{s}) = p(o_2 o|s_1\mathbf{s}) = 0$. Therefore s_1 and s_2 are disclosed.

Intuitively, the existence of free observables (i.e. o of P_1 in this example) may create different interleavings, which can be used by the scheduler to mark different secrets.

6 A case study: the Crowds protocol

In this section, we apply the compositionality result (4) in Theorem 1 to prove the anonymity-preservation property of an extended Crowds protocol with member-wise trusted forwarders. The novelty of our proof is that it is simple. Instead of requiring the symmetry property that is usually made in the literature in order to simplify the analysis, our proof works with the help of a simple notion, namely the *reputation*.

Crowds is an anonymity protocol which allows Internet users to perform web transactions without revealing their private identity. More specifically, a crowd is a group of n participants constituted by m *honest members* and $c (= n - m)$ *attackers*. The

destination of messages, namely the *server*, may also be malicious and collaborate with other attackers. The way Crowds works is the following:

- When a member, called the *initiator*, wants to send a message to the server, instead of sending it directly to the server, she randomly selects a member (possibly herself) in the crowd and she forwards the message to this member.
- Every member who receives the message, she either delivers the message to the server with a certain probability $1 - p_f$, or randomly selects another member (possibly herself) in the crowd as the new *forwarder* with the probability p_f and relays the message to this new forwarder to repeat the same procedure again.

In this way, even if the message is caught by an attacker, the attacker cannot be sure whether the previous forwarder is the initiator or just a forwarder on behalf of somebody else. Members (including attackers) are assumed to have only access to messages routed through them, so that they only know the identities of their immediate predecessors and successors in the path, and of the destination server.

The above standard Crowds protocol implicitly requires the *symmetry conditions* (see [7]), in the sense that a new forwarder is chosen among all the members in the crowd with uniform probability. In this context, it has been proved in [22] that increasing the number of honest members preserves the privacy of Crowds. In this paper, we consider a more realistic scenario, the so-called *member-wise trusted forwarders* case, in which a current member selects a forwarder only from a subset of Crowds members which she thinks are trustable. We also consider the case of *reputed initiators*. The idea is that a new member has the right to initiate requests only when she has acquired a certain level of *reputation*. This is not a strong restriction, since it is not uncommon that when a new agent wants to join a web conversation, her behavior needs to be examined for a while until she becomes a totally legal member.

It has been shown (see Example 5 in [7]) that, without the symmetry conditions, an addition of a new honest member can diminish the level of anonymity. In contrast to [7], we will show that the addition of a new honest member does not harm anonymity if she does not enjoy a reputation yet, i.e. if she is not allowed to initiate a request. (She can, however, be allowed to be chosen as a forwarder.)

The standard Crowds protocol expressed in CCS_p is stated in the Table 2. For simplicity, we introduce a notation for value-passing in CCS_p , following standard lines.

$$\begin{array}{l} \text{Input } x(i).P = \lfloor \pm \rfloor_j x_j.P[j/i] \\ \text{Output } \bar{x}\langle i \rangle = \bar{x}_i \end{array}$$

We use \mathcal{H} , \mathcal{H}_r and \mathcal{A} to denote the set of honest members, of reputed honest members and of attackers, respectively. $\mathcal{C} = \mathcal{H} \cup \mathcal{A}$ representing the set of all participants in Crowds. We use x_1, x_2, \dots, x_n to range over \mathcal{C} , and \bigoplus^U to represent an uniform distribution. For simplicity we assume that once an attacker receives a message from an honest member, it will terminate after reporting the detection through the observable \overline{detect} . The reason is that by forwarding the message after the first detection, attackers can not gain more useful information. Hence at most one honest member can be detected. Precisely, the set of secret labels is $\{act_o \mid x_o \in \mathcal{H}_r\}$ and the set of observable labels is $\{\overline{detect}\langle i \rangle \mid x_i \in \mathcal{H}\}$.

$$\begin{aligned}
Origin &= \bigoplus_{x_o \in \mathcal{H}_r}^{\mathcal{U}} p_o.act_o.(\bigoplus_{x_j \in \mathcal{C}}^{\mathcal{U}} p_j.\bar{x}_j\langle o \rangle) \\
honest_i &= x_i.((\bigoplus_{x_j \in \mathcal{C}}^{\mathcal{U}} p_j.\bar{x}_j\langle i \rangle.\overline{done}) \oplus_{p_f} \overline{ser}\langle i \rangle.\overline{done}) \\
Honest_i &= \circlearrowleft honest_i \\
Attacker_i &= x_i(j).\overline{detect}\langle j \rangle \\
Server &= ser(j).\overline{detect}\langle j \rangle \\
crowd_n &= Server \mid Origin \mid \prod_{x_i \in \mathcal{H}} Honest_i \mid \prod_{x_j \in \mathcal{A}} Attacker_j \\
Crowd_n &= (\nu ser_{1,\dots,n})(\nu x_1, x_2, \dots, x_n) crowd_n
\end{aligned}$$

Table 2: The CCS_p code for standard Crowds (i.e. with the symmetry conditions).

$$\begin{aligned}
Origin &= \bigoplus_{x_o \in \mathcal{H}_r}^{\mathcal{U}} p_o.act_o.(\bigoplus_{x_j \in \mathcal{T}_o}^{\mathcal{U}} p_j.\bar{x}_j\langle o \rangle) \\
honest_i &= x_i.((\bigoplus_{x_j \in \mathcal{T}_i}^{\mathcal{U}} p_j.\bar{x}_j\langle i \rangle.\overline{done}) \oplus_{p_f} \overline{ser}\langle i \rangle.\overline{done})
\end{aligned}$$

Table 3: The new definitions for Crowds with member-wise trusted forwarders.

We denote by \mathcal{T}_i the subset of Crowds members which the i th honest member trusts. Then the process terms for Crowds with member-wise trusted forwarders are similar to the terms showed in the Table 2, except that the process *Origin* and the process *honest_i* are modified as shown in the Table 3.

Based on the symmetry properties provided by the standard Crowds protocol, it has been proved in [22] that Crowds satisfies probable innocence under certain conditions. In [7] the following equivalent definition of probable innocence is proposed, where $p(o|s)$ denotes the probability of the event, i.e. the member o is the first one detected by attackers when the initiator is s .

Definition 8 ([7]). A Crowds satisfies probable innocence if for all users o, i, j ,

$$(m - 1)p(o|i) \geq p(o|j)$$

where m is the number of honest members.

Since in Crowds the confidential information to conceal is the initiator's identity, the relation over secret inputs is considered as a total adjacency relation. It is straightforward to see the following:

Remark 1 (Relation with probable innocence). A Crowds satisfies probable innocence if and only if it provides $\ln(m - 1)$ -differential privacy.

6.1 The Crowds with member-wise trusted forwarders

Consider the scenario in which there exists a Crowds with n members (shown in Table 2). Assume that from a previous analysis we already know that this Crowds provides

$$crowd_{n+1} = crowd_n \mid Honest_{n+1}$$

$$Crowd_{n+1} = (\nu ser_{1,\dots,n+1})(\nu x_1, x_2, \dots, x_{n+1}) crowd_{n+1}$$

Table 4: CCS_p code expressing the addition of the $n + 1$ th member.

a certain level of differential privacy. Assume that a new honest member is allowed to join the crowd but it does not enjoy the reputation to be an initiator right away. Old members in the crowd can decide by themselves to trust this member or not, which means there is no restriction of how this new member is added, thus resulting in a Crowds with member-wise trusted forwarders. Applying the result in Section 5 we can ensure that the privacy of the Crowds will not be diminished by this addition.

Theorem 2. *The privacy of a Crowds is preserved by the addition of a non-reputed honest member. More precisely, $dp \llbracket Crowd_{n+1} \rrbracket \leq dp \llbracket Crowd_n \rrbracket$.*

Proof. The addition of a new honest member to a Crowds with n participants is presented in the Table 4. Basically, it is a parallel composition of the process term $crowd_n$ of the original crowd, and the process term $Honest_{n+1}$ of the new member. In $crowd_n$ although there is no entity of $Honest_{n+1}$, we assume that the identity x_{n+1} is already available in the set \mathcal{C} as a free label, to be selected as the forwarder.

This new member is known not to be an initiator (because she is not reputed). Therefore her presence will not induce an addition of the secret label act_{n+1} . In the sense that the process $Honest_{n+1}$ does not contain any secret labels. Clearly,

$$(\nu ser_{1,\dots,n+1})(\nu x_1, x_2, \dots, x_{n+1})Honest_{n+1}$$

is a safe component. Note that the observables in $Honest_{n+1}$ through which she communicates with $crowd_n$ are $\{x_{n+1}\} \cup \{\bar{ser}\langle n+1 \rangle\} \cup \{\bar{x}_j\langle n+1 \rangle \mid x_j \in \mathcal{T}_{n+1}\}$. They are free in $Crowd_n$, while restricted in $Crowd_{n+1}$. By Theorem 1(4), $Crowd_{n+1}$ provides at least as much privacy as $Crowd_n$.

7 Computing the degree of privacy

In this section, we study the possibility of computing the degree of privacy in CCS_p in a fine-grained way. We make a first step by considering finite processes in which the secret choices are only at the top-level. We present the algorithm to produce the channel matrix in a bottom-up fashion, from which we obtain its degree of privacy. And we give some toy examples to illustrate the approach.

7.1 The algorithm

Consider a process term $T = \lfloor \pm \rfloor_i s_i.T_i$ starting with a secret choice. For simplicity, in this paper, we assume that there is no secret choice inside any of T_i , and no occurrence of $!$, (while considering $!$ is a real difficult problem, the restriction about the

secret choice can be lifted by using more complicated definitions of how to combine the schedulers of the components. We however leave them for future work). We also assume that every occurrence of the parallel operator is removed by using the *expansion law* [18]. Thus the residual constructs possibly occurring in T_i are the nondeterministic choice, the probabilistic choice, the restriction and the sequential replication. We show the algorithm in an intuitive way and refer the detailed descriptions in the Appendix.

We start by constructing, the set of *all possible schedulers* for a process P , denoted by $\Delta(P)$. We denote its size by $|\Delta(P)|$.

- Case $P = \bigoplus_i p_i P_i$: Intuitively, P 's scheduler selects for each P_i a scheduler from $\Delta(P_i)$. Hence, $|\Delta(P)| = O(\prod_i |\Delta(P_i)|)$.
- Case $P = \bigsqcup_i r_i.P_i$: P 's scheduler first resolves the top nondeterministic choice and then proceeds the continuation of the process corresponding to the selected label. Thus, $|\Delta(P)| = O(\sum_i |\Delta(P_i)|)$.
- Case $P' = (\nu o)P$: Intuitively, a scheduler of P' is obtained from a scheduler in $|\Delta(P)|$ by removing the assignments of the executions containing o . Hence, $|\Delta(P')| = O(|\Delta(P)|)$.
- Case $P' = \circlearrowleft^n P$: Intuitively, a scheduler of P' selects for each run of P a scheduler from $\Delta(P)$, and use them in a sequential way. Hence, $|\Delta(P')| = O(|\Delta(P)|^n)$.

For every term T_i in T , we are able to obtain the scheduler set $\Delta(T_i)$. The corresponding matrix under each scheduler in $\Delta(T_i)$ is computed in a bottom-up way (see appendix). We now turn to the construction of the matrix of T . For every secret label s_i , T 's scheduler ζ chooses a scheduler ζ_i from the set $\Delta(T_i)$. Let $p_\zeta(\mathbf{o}|s_i\mathbf{s})$ (resp. $p_{\zeta_i}^i(\mathbf{o}|\mathbf{s})$) be the probability in the matrix $M(T)_\zeta$ (resp. $M_{\zeta_i}(T_i)$). Hence

$$p_\zeta(\mathbf{o}|s_i\mathbf{s}) = p_{\zeta_i}^i(\mathbf{o}|\mathbf{s}).$$

By the definition of differential privacy we get:

$$dp_\zeta[[T]] = \min\{\epsilon \mid \mathbf{s} \sim \mathbf{s}' \Rightarrow \forall \mathbf{o} \in \mathcal{O}. p_\zeta(\mathbf{o}|\mathbf{s}) \leq e^\epsilon p_\zeta(\mathbf{o}|\mathbf{s}')\}$$

and

$$dp[[T]] = \max_{\zeta \in \Delta(T)} dp_\zeta[[T]].$$

Complexity Analysis The time complexity of the above algorithm is determined by the size of the set of all possible schedulers, that is, the time complexity is $O(|\Delta(T)|)$. However we can make it more efficient in the case in which differential privacy does not hold: Whenever we find a scheduler ζ_i in $\Delta(T_i)$ producing an observable \mathbf{o} which is not included in the set of observables generated by a previous scheduler ζ_j in $\Delta(T_j)$ (with $j < i$ and s_i, s_j connected by the transitive closure of \sim), then we can halt the algorithm and claim that T does not provide differential privacy for any ϵ . In fact, assigning the scheduler ζ_i (resp. ζ_j) to the secret s_i (resp. s_j) differentiates the two secrets by producing a non-null (resp. null) probability in the column of \mathbf{o} .

7.2 Some toy examples

Here we give some simple examples to illustrate the above algorithm.

Example 6. Let $Sec = \{s_1, s_2\}$, $Obs = \{o_1, o_2\}$, and consider the following processes: $P_1 = o_1.\mathbf{0} \oplus_{0.3} o_2.\mathbf{0}$, $P_2 = o_1.\mathbf{0} \oplus_{0.5} o_2.\mathbf{0}$, $P_3 = o_1.\mathbf{0} \oplus_{0.8} o_2.\mathbf{0}$, $P = P_1 \sqcup P_2 \sqcup P_3$ and $P' = s_1.P \sqcup s_2.P$.

For the term P_1 , we have $\Delta(P_1) = \{\emptyset\}$ and $M(P_1) = \begin{array}{|c|c|} \hline o_1 & o_2 \\ \hline 0.3 & 0.7 \\ \hline \end{array}$.

For the term P_2 , we have $\Delta(P_2) = \{\emptyset\}$ and $M(P_2) = \begin{array}{|c|c|} \hline o_1 & o_2 \\ \hline 0.5 & 0.5 \\ \hline \end{array}$.

For the term P_3 , we have $\Delta(P_3) = \{\emptyset\}$ and $M(P_3) = \begin{array}{|c|c|} \hline o_1 & o_2 \\ \hline 0.8 & 0.2 \\ \hline \end{array}$.

For the term P , we have $\Delta(P) = \{\zeta_1, \zeta_2, \zeta_3\}$ with ζ_i representing the choice of P_i . The corresponding matrices are: $M_{\zeta_1}(P) = M(P_1)$, $M_{\zeta_2}(P) = M(P_2)$ and $M_{\zeta_3}(P) = M(P_3)$.

For the term P' we can define the scheduler ζ' which selects ζ_1 and ζ_3 for the secret s_1 and s_2 , respectively. The corresponding matrix is $M_{\zeta'}(P') = \begin{array}{|c|c|c|} \hline & o_1 & o_2 \\ \hline s_1 & 0.3 & 0.7 \\ \hline s_2 & 0.8 & 0.2 \\ \hline \end{array}$, which gives $(\ln 3.5)$ -differential privacy.

Example 7. Let $Sec = \{s_1, s_2\}$, $Obs = \{o, o_1, o_2\}$, and consider the processes $P_1 = o_1.\mathbf{0} \oplus_{0.3} o_2.\mathbf{0}$, $P_2 = o.\mathbf{0}$, $P = P_1 \mid P_2$, and $P' = s_1.P \sqcup s_2.P$.

First we use the expansion law to rewrite P into $(o_1.o.\mathbf{0} \oplus_{0.3} o_2.o.\mathbf{0}) \sqcup (o.o_1.\mathbf{0} \oplus_{0.3} o.o_2.\mathbf{0})$. Through steps similar to the above example, we can find a scheduler producing

a matrix breaking the differential privacy, for example $\begin{array}{|c|c|c|c|c|} \hline & o_1o & o_2o & oo_1 & oo_2 \\ \hline s_1 & 0.3 & 0.7 & 0 & 0 \\ \hline s_2 & 0 & 0 & 0.3 & 0.7 \\ \hline \end{array}$.

Example 8 (A discussion about the recursion operator). Furthermore, for a process including the replication operator while not producing infinite observables, we show a computation of the degree of privacy in this case. To facilitate the analysis, here we use a *fixpoint construct* [18], i.e. $\mu X.P$ where the *agent variable* X may occur in P . Note that $!P$ can be obtained by the definition $\mu X.(P \mid X)$.

Let Sec and Obs be as in Example 6. Consider the processes $P = \mu X.((0.4o_1 \oplus 0.2o_2 \oplus 0.4X) \sqcup (o_1 \oplus_{0.3} o_2) \sqcup (o_1 \oplus_{0.5} o_2))$ and $P' = s_1.P \sqcup s_2.P$. We have $\Delta(P') = \Delta(P) = \{\zeta_1, \zeta_2, \zeta_3\}$, where the respective matrices are $M_{\zeta_1}(P) = \begin{array}{|c|c|c|} \hline o_1 & o_2 & X \\ \hline 0.4 & 0.2 & 0.4 \\ \hline \end{array}$, $M_{\zeta_2}(P) =$

$\begin{array}{|c|c|} \hline o_1 & o_2 \\ \hline 0.3 & 0.7 \\ \hline \end{array}$ and $M_{\zeta_3}(P) = \begin{array}{|c|c|} \hline o_1 & o_2 \\ \hline 0.5 & 0.5 \\ \hline \end{array}$. It can be proved that using the scheduler ζ_1 finitely many times generates the highest (resp. lowest) probability of observing o_1 (resp. o_2), which is 0.67 (resp. 0.33). Using the scheduler ζ_2 generates the lowest (resp. highest) probability of observing o_1 (resp. o_2), which is 0.3 (resp. 0.7). Thus P' enjoys $(\ln 2.22)$ -

differential privacy, determined by the matrix $\begin{array}{|c|c|c|} \hline & o_1 & o_2 \\ \hline s_1 & 0.67 & 0.33 \\ \hline s_2 & 0.3 & 0.7 \\ \hline \end{array}$.

8 Related work

- *Compositionality properties of probabilistic process calculi for security protocols.* In [11] Deng et al. used the notion of relative entropy to measure privacy. In [5] Braun et al. proved that the safety measured by Bayes risk is maintained under the composition of CCS_p constructs with a restricted form of parallel composition and without the secret choice. The compositionality results in our paper are closely related to those of [5], although we use a different measure of protection (differential privacy).
- *Compositionality of Differential Privacy.* As stated in the introduction, there is a vast body of work on formal methods for differential privacy. Compositional methods, as one of the most important features, have been intensively investigated in the field of statistical databases [17] and programs [3, 21]. These works investigate the so-called sequential and parallel compositions of queries (programs), which, in their context, mean a sequence of queries (programs) applied to the same dataset and to disjoint parts of the dataset, respectively. Under this setting, they have proved that the sequential composition decreases the privacy, and the parallel composition maintains the privacy. Our result about the sequential replication and the parallel composition in CCS_p are reminiscent of the above results. But the context is different. In particular, the parallel composition concerned in this paper is different from the above one, in that the parallel operator here represents interactions of concurrent agents. Our restrictions on the parallel composition are incomparable with those of [17, 3, 21] (disjointness of databases).
- *Other extensions on the Crowds protocol.* In [16] Hamadou et al. have taken into account that attackers usually gather additional information correlated to the anonymous agents before attacking the protocol. In [24] Sassone et al. extended Crowds by allowing the possibility for members to turn corrupt, and by considering a trust estimation over participants which is expressed by a known probability distribution over participants. Note that, this kind of trust estimation is shared by all members which implies that the level of a member's reliability is the same from all members' points of views and therefore can still be thought of as the symmetry condition. We consider a more realistic scenario in which users have the right to choose to communicate only with the users they consider reliable, which is the most common case in web transactions in the real world.

9 Conclusion and Future Work

In this paper, we have defined the differential privacy degree of concurrent systems expressed in a probabilistic process calculus, and investigated how the privacy is affected under composition of the CCS_p constructs. We have applied our approach to give a simple proof for the anonymity-preservation of an extended Crowds protocol with member-wise trusted forwarders. Finally, we have presented an algorithm for computing the degree of differential privacy for a finite process.

Fine-grained methods for computing the channel matrix have been studied in [2]. In future work, we plan to optimize our current algorithm, extend it to more general

processes, more precisely, develop an approach that can deduce the global property of differential privacy from local information, w.r.t. the adjacency relation. Another interesting problem is the applicability of our approach to the problem of preserving privacy in geolocation-related applications. More specifically, we intend to use (a possibly extended version of) our probabilistic process calculus to express systems of concurrent agents moving in space and time, and interacting with each other in ways that depend on the adjacency relation. We believe that our compositional method will provide a way to synthesize differentially private mechanisms in a (semi-)automatic way.

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10 Appendix

10.1 Proof of Theorem 1

1. Let $P = \bigsqcup_i o_i.P_i$. Consider an arbitrary scheduler ζ of the process P , according to the transition rule ACT in the Table 1, ζ resolves the nondeterminism in the first step by arbitrarily choosing a label $o_j, j \in \{1, 2, \dots, h\}$. Let ζ_j be the projection of ζ by removing the first state and the first nondeterministic observation o_j from the execution fragments in the domain. Observe that, the scheduler ζ_j is compatible with process term P_j , that is, ζ_j is one of P_j 's schedulers. For every conditional probability $p(o_j \mathbf{o} | \mathbf{s})$ in $M_\zeta(P)$, there exists a corresponding element $p^j(\mathbf{o} | \mathbf{s})$ in $M_{\zeta_j}(P_j)$, such that

$$p(o_j \mathbf{o} | \mathbf{s}) = p^j(\mathbf{o} | \mathbf{s}). \quad (2)$$

From the differential privacy that P_j gives, we derive

$$dp_\zeta \llbracket P \rrbracket = dp_{\zeta_j} \llbracket P_j \rrbracket \leq \epsilon_j \leq \max_i \{\epsilon_i\}$$

which concludes the proof in this case.

2. Let $P = \bigoplus_i p_i P_i$. Consider an arbitrary scheduler ζ for P . After one subprocess is randomly chosen according to its probability, ζ must be compatible with this subprocess, resolving all the nondeterminism in it. It holds that the conditional probability $p(\mathbf{o} | \mathbf{s})$ in the resulting matrix $M_\zeta(P)$ and the corresponding one $p^i(\mathbf{o} | \mathbf{s})$ in each matrix $M_\zeta(P_i)$ satisfy the following relation.

$$p(\mathbf{o} | \mathbf{s}) = \sum_i p_i \cdot p^i(\mathbf{o} | \mathbf{s}) \quad (3)$$

It is easy to see that for all secret inputs \mathbf{s}, \mathbf{s}' such that $\mathbf{s} \sim \mathbf{s}'$,

$$\begin{aligned} \frac{p(\mathbf{o} | \mathbf{s})}{p(\mathbf{o} | \mathbf{s}')} &= \frac{\sum_i p_i \cdot p^i(\mathbf{o} | \mathbf{s})}{\sum_i p_i \cdot p^i(\mathbf{o} | \mathbf{s}')} \\ &\leq \frac{\sum_i p_i \cdot e^{\epsilon_i} p^i(\mathbf{o} | \mathbf{s}')}{\sum_i p_i \cdot p^i(\mathbf{o} | \mathbf{s}')} \quad (\text{since } M_\zeta(P_i) \text{ gives } \epsilon_i\text{-d.p.}) \\ &\leq e^{\max_i \{\epsilon_i\}} \frac{\sum_i p_i \cdot p^i(\mathbf{o} | \mathbf{s}')}{\sum_i p_i \cdot p^i(\mathbf{o} | \mathbf{s}')} \\ &= e^{\max_i \{\epsilon_i\}} \end{aligned}$$

which concludes the proof in this case.

3. Let $P = (\nu o)P_1$. According to the operational semantic rule RES of restriction construct in the Table 1, the process P is not able to perform the label o . Its execution tree $etree(P)$ can be obtained from $etree(P_1)$ by removing all the transitions with the label o and the whole subtrees following those \xrightarrow{o} transitions. Let $p(\mathbf{o} | \mathbf{s})$

denote the conditional probability in the matrix $M_\zeta(P_1)$. Let \mathcal{O}' denote the set of observables of P . Consider an arbitrary scheduler ζ' of P . Observe that there exists a scheduler ζ of the process P_1 , such that, for all the conditional probability $p'(\mathbf{o}'|\mathbf{s})$ in the matrix $M_{\zeta'}(P)$, the following equation holds.

$$p'(\mathbf{o}'|\mathbf{s}) = \sum_{f(\mathbf{o})=\mathbf{o}'} p(\mathbf{o}|\mathbf{s}) \quad (4)$$

where the function $f(\mathbf{o})$ with $\mathbf{o} = o_1, o_2, \dots, o_k$ is

$$f(\mathbf{o}) = \begin{cases} \mathbf{o} & \text{if the label } o \text{ does not appear in the} \\ & \text{sequence } \mathbf{o} \\ o_1, o_2, \dots, o_i & \text{if the label } o \text{ first occurs in the sequence} \\ & \mathbf{o} \text{ at the position } i + 1 \text{ with } i < k. \end{cases} \quad (5)$$

It's easy to get that for all secret inputs $\mathbf{s}_1, \mathbf{s}_2 \in \mathcal{S}$ such that $\mathbf{s}_1 \sim \mathbf{s}_2$, and for all observable $\mathbf{o}' \in \mathcal{O}'$,

$$\frac{p'(\mathbf{o}'|\mathbf{s}_1)}{p'(\mathbf{o}'|\mathbf{s}_2)} = \frac{\sum_{f(\mathbf{o})=\mathbf{o}'} p(\mathbf{o}|\mathbf{s}_1)}{\sum_{f(\mathbf{o})=\mathbf{o}'} p(\mathbf{o}|\mathbf{s}_2)} \leq \frac{\sum_{f(\mathbf{o})=\mathbf{o}'} e^{\epsilon_1} p(\mathbf{o}|\mathbf{s}_2)}{\sum_{f(\mathbf{o})=\mathbf{o}'} p(\mathbf{o}|\mathbf{s}_2)} = e^{\epsilon_1}$$

which shows that $M_{\zeta'}(P)$ enjoys ϵ_1 -differential privacy.

4. Let $P = (\nu o_1, o_2, \dots, o_k) (P_1 | P_2)$. The proof proceeds by reducing the execution tree of P relative to an arbitrary scheduler ζ to a new one $etree'(P, \zeta)$. This new tree enjoys a level of differential privacy which is less safe than the one of the original $etree(P, \zeta)$, while it is isomorphic to the execution tree of $(\nu o_1, \dots, o_{h-1})P_2$ relative to a certain scheduler ζ_2 . We derive,

$$dp[\llbracket etree(P, \zeta) \rrbracket] \leq dp[\llbracket etree'(P, \zeta) \rrbracket] = dp[\llbracket etree((\nu o_1, \dots, o_{h-1})P_2, \zeta_2) \rrbracket] \leq \epsilon_2$$

which proves that the process P enjoys ϵ_2 -differential privacy.

The reduction from $etree(P, \zeta)$ to $etree'(P, \zeta)$ is described in the following. First we give the definitions of P_1 's positions and P_2 's positions. Consider a generic state in $etree(P, \zeta)$, and let $(\nu o_1, o_2, \dots, o_k) (P'_1 | P'_2)$ be the new term in that state. Assume α to be the execution history up to that node. From the assumption that $(\nu o_1, o_2, \dots, o_k)P_1$ is a safe component, there are three possible kinds of transitions performable from each state according to the operational semantics.

- (a-step) $(\nu o_1, o_2, \dots, o_k) (P'_1 | P'_2) \xrightarrow{a} (\nu o_1, o_2, \dots, o_k) (\mu | P'_2)$ due to a transition $P'_1 \xrightarrow{a} \mu$. In this case, a must be τ , because P_1 does not contain secret labels and all its observable labels are included in $\{o_1, o_2, \dots, o_k\}$. Assume that $\mu = \sum_i p_i \delta(P'_{1i})$. Then we have $(\nu o_1, o_2, \dots, o_k) (\mu | P'_2) = \sum_i p_i \delta((\nu o_1, o_2, \dots, o_k) (P'_{1i} | P'_2))$.
- (b-step) $(\nu o_1, o_2, \dots, o_k) (P'_1 | P'_2) \xrightarrow{a} (\nu o_1, o_2, \dots, o_k) (P'_1 | \mu)$ due to a transition $P'_2 \xrightarrow{a} \mu$, with a not included in $\{o_1, o_2, \dots, o_k\}$.

(c-step) $(\nu o_1, o_2, \dots, o_k) (P'_1 | P'_2) \xrightarrow{\tau} (\nu o_1, o_2, \dots, o_k) \delta(P''_1 | P''_2)$ due to the transitions $P'_1 \xrightarrow{a} \delta(P''_1)$ and $P'_2 \xrightarrow{\bar{a}} \delta(P''_2)$. As assumed in the condition a must be an observable in $\{o_h, o_{h+1}, \dots, o_k\}$.

We define P_1 's *positions* (resp. P_2 's *positions*) as the set of states in $etree(P, \zeta)$ where the transition of type (a) (resp. the transition of type (b) or (c)) is chosen by ζ . The tree $etree'(P, \zeta)$ is obtained by replacing each P_1 's position with its subtree $etree(\alpha\tau(\nu o_1, o_2, \dots, o_k) (P'_{1m} | P'_2), \zeta)$ which gives the maximal value of differential privacy among all its subtrees, that is,

$$\begin{aligned} dp \llbracket etree(\alpha\tau(\nu o_1, o_2, \dots, o_k) (P'_{1m} | P'_2), \zeta) \rrbracket \\ = \\ \max_i \{ dp \llbracket etree(\alpha\tau(\nu o_1, o_2, \dots, o_k) (P'_{1i} | P'_2), \zeta) \rrbracket \} \end{aligned}$$

By simple induction on the depth of the tree, we obtain that $dp \llbracket etree(P, \zeta) \rrbracket$ is an increasing function of its subtrees. Then by the previous result about the probabilistic choice, we have

$$dp \llbracket etree(P, \zeta) \rrbracket \leq dp \llbracket etree'(P, \zeta) \rrbracket$$

Note that after the process P_1 's impact on the safety is resolved, all the states left in $etree'(P, \zeta)$ are P_2 's positions. It is easy to find a corresponding scheduler ζ_2 for the execution tree of $(\nu o_1, \dots, o_{h-1})P_2$ such that

- for every b-step in $etree'(P, \zeta)$, ζ_2 chooses the same transition in $etree((\nu o_1, \dots, o_{h-1})P_2, \zeta_2)$, i.e. $P'_2 \xrightarrow{a} \mu$ with $a \notin \{o_1, o_2, \dots, o_k\}$,
- for every c-step in $etree'(P, \zeta)$, ζ_2 chooses the same transition in $etree((\nu o_1, \dots, o_{h-1})P_2, \zeta_2)$, i.e. $P'_2 \xrightarrow{\bar{a}} \delta(P''_2)$ with $a \in \{o_h, o_{h+1}, \dots, o_k\}$.

Observe now that $etree'(P, \zeta)$ is isomorphic to $etree((\nu o_1, \dots, o_{h-1})P_2, \zeta_2)$, which concludes the proof in this case.

5. Rerun the process P_1 n times. Let $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n$ and $\mathbf{o}_1, \mathbf{o}_2, \dots, \mathbf{o}_n$ be the n times replications' secret inputs and observables, respectively. Because of the assumption that the new process starts after the current one terminates, for all inputs sequences $(\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n)$ and $(\mathbf{s}'_1, \mathbf{s}'_2, \dots, \mathbf{s}'_n)$ such that $(\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n) \sim (\mathbf{s}'_1, \mathbf{s}'_2, \dots, \mathbf{s}'_n)$, we have

$$\begin{aligned} & p(\mathbf{o}_1, \mathbf{o}_2, \dots, \mathbf{o}_n | \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n) \\ &= p(\mathbf{o}_1 | \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n) p(\mathbf{o}_2 | \mathbf{o}_1, \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n) \\ & \quad \cdots p(\mathbf{o}_n | \mathbf{o}_1, \mathbf{o}_2, \dots, \mathbf{o}_{n-1}, \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n) \\ &= p(\mathbf{o}_1 | \mathbf{s}_1) p(\mathbf{o}_2 | \mathbf{s}_2) \cdots p(\mathbf{o}_n | \mathbf{s}_n) \quad (\text{sequential replication}) \quad (6) \\ &\leq e^{\epsilon_1} p(\mathbf{o}_1 | \mathbf{s}'_1) e^{\epsilon_1} p(\mathbf{o}_2 | \mathbf{s}'_2) \cdots e^{\epsilon_1} p(\mathbf{o}_n | \mathbf{s}'_n) \quad (\text{since } M_\zeta(P_1) \text{ gives } \epsilon_1\text{-d.p.}) \\ &= e^{n\epsilon_1} p(\mathbf{o}_1 | \mathbf{s}'_1) p(\mathbf{o}_2 | \mathbf{s}'_2) \cdots p(\mathbf{o}_n | \mathbf{s}'_n) \\ &= e^{n\epsilon_1} p(\mathbf{o}_1, \mathbf{o}_2, \dots, \mathbf{o}_n | \mathbf{s}'_1, \mathbf{s}'_2, \dots, \mathbf{s}'_n) \end{aligned}$$

which concludes the proof.

10.2 Computing the channel matrix

We now show the detailed combination of matrices of subprocesses under the non-deterministic choice, the probabilistic choice, the restriction and the sequential replication. Let $p_\zeta(\mathbf{o}|\mathbf{s})$ (resp. $p_\zeta^i(\mathbf{o}|\mathbf{s})$) be the probability in the matrix $M_\zeta(P)$ (resp. $M_\zeta(P_i)$)

- Case $P = \bigoplus_i p_i P_i$: Intuitively, P 's scheduler selects for each P_i a scheduler from $\Delta(P_i)$. Hence $\Delta(P) = \{\bigcup_i \zeta_i \mid \zeta_i \in \Delta(P_i)\}$, and $|\Delta(P)| = \prod_i |\Delta(P_i)|$. For the scheduler $\zeta = \bigcup_i \zeta_i$, where $\zeta_i \in \Delta(P_i)$,

$$p_\zeta(\mathbf{o}|\mathbf{s}) = \sum_i p_i \cdot p_{\zeta_i}^i(\mathbf{o}|\mathbf{s}).$$

- Case $P = \lfloor \lfloor_i r_i . P_i$: Let $\Delta_i(P)$ be the set of all the possible schedulers for P when $r_i . P_i$ is chosen to resolve the nondeterminism. It is easy to see that $\Delta_i(P) = \bigcup\{\{\zeta \mid \zeta(\varepsilon) = r_i \wedge \zeta(r_i \alpha) = \zeta_i(\alpha)\} \mid \zeta_i \in \Delta(P_i)\}$. Thus $\Delta(P) = \bigcup_i \Delta_i(P)$, and $|\Delta(P)| = \sum_i |\Delta(P_i)|$. For the scheduler $\zeta = \{\zeta(\varepsilon) = r_i \wedge \zeta(r_i \alpha) = \zeta_i(\alpha)\}$, where $\zeta_i \in \Delta(P_i)$,

$$p_\zeta(r_i \mathbf{o}|\mathbf{s}) = p_{\zeta_i}^i(\mathbf{o}|\mathbf{s}).$$

- Case $P' = (\nu o)P$: Intuitively, the scheduler $\zeta|_o$ is obtained from ζ by removing the assignments of the executions containing o . Hence $\Delta(P') = \{\zeta|_o \mid \zeta \in \Delta(P)\}$, and $|\Delta(P')| = |\Delta(P)|$. For the scheduler $\zeta' = \zeta|_o$, where $\zeta \in \Delta(P)$,

$$p_{\zeta'}(\mathbf{o}'|\mathbf{s}) = \sum_{f(\mathbf{o})=\mathbf{o}'} p_\zeta(\mathbf{o}|\mathbf{s})$$

where the function $f(\mathbf{o})$ is shown in the formula (5) .

- Case $P' = \bigcirc^n P$: Intuitively, the scheduler $\zeta_1; \zeta_2; \dots; \zeta_n$ selects for each run of P a scheduler from $\Delta(P)$, and use them in a sequential way. Hence $\Delta(P') = \{\zeta_1; \zeta_2; \dots; \zeta_n \mid \zeta_i \in \Delta(P)\}$, and $|\Delta(P')| = |\Delta(P)|^n$. Let $\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n$ and $\mathbf{o}_1, \mathbf{o}_2, \dots, \mathbf{o}_n$ be the n times replications' secret inputs and observables, respectively. For the scheduler $\zeta' = \zeta_1; \zeta_2; \dots; \zeta_n$, where $\zeta_i \in \Delta(P)$ for all i ,

$$p_{\zeta'}(\mathbf{o}_1, \mathbf{o}_2, \dots, \mathbf{o}_n | \mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n) = p_{\zeta_1}(\mathbf{o}_1 | \mathbf{s}_1) p_{\zeta_2}(\mathbf{o}_2 | \mathbf{s}_2) \cdots p_{\zeta_n}(\mathbf{o}_n | \mathbf{s}_n).$$