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► **To cite this version:**

Juliette Chabassier, Marc Duruflé. Physical parameters for piano modeling. [Technical Report] RT-0425, 2012, pp.24. hal-00688679v1

**HAL Id: hal-00688679**

**<https://inria.hal.science/hal-00688679v1>**

Submitted on 18 Apr 2012 (v1), last revised 8 Aug 2012 (v2)

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# Physical parameters for piano modeling.

Juliette Chabassier, Marc Duruflé

**TECHNICAL  
REPORT**

**N° 425**

April 2012

Project-Teams Poems and Bacchus

ISRN INRIA/RT--425--FR+ENG

ISSN 0249-0803





## Physical parameters for piano modeling.

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Project-Teams Poems and Bacchus

Technical Report n° 425 — April 2012 — 20 pages

**Abstract:** This document lists the physical parameters used by the authors when performing numerical simulations of the piano. We first give the parameters used for the soundboard and the air. Then, the hammer parameters are given. Finally, strings parameters are issued and two cases are considered : with (realistic) or without (virtual) wrapped strings. When the strings are considered wrapped, their length is the effective length measured on the reference piano, but we consider that they are made of a virtual material with a higher density. When the strings are considered unwrapped, the material is steel, and to achieve the very bass notes without increasing inharmonicity too much, we have increased the length up to almost 6 meters.

**Key-words:** piano modeling, physical parameters, damping mechanisms

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## Paramètres physiques pour la modélisation du piano.

**Résumé :** Ce document précise les paramètres physiques utilisés par les auteurs lors des simulations numériques du piano. Nous commençons par donner les paramètres de la table d'harmonie et de l'air. Ensuite, les paramètres des marteaux sont listés. Enfin, nous nous intéressons aux paramètres de cordes, et nous avons considéré deux cas : avec ou sans cordes filées. Dans le cas de cordes filées, les longueurs de cordes sont celles que nous avons mesurées sur le piano de référence, mais nous les considérons composées d'un matériau virtuel très dense. Dans le cas de cordes non filées, nous considérons qu'elles sont faites d'acier, et pour atteindre les notes les plus graves du clavier sans trop accroître l'inharmonicité, les longueurs des cordes sont augmentées jusqu'à presque 6 mètres.

**Mots-clés :** modélisation du piano, paramètres physiques, phénomènes d'amortissement

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## 1 Introduction

This report can be considered as a technical appendix to [1] or [2], which describe in detail how the piano modeling is done. All parameters given here quantitatively for each note of the piano were defined in these references, and will not be defined here. These parameters are based on measurements on the Steinway D piano, seen in a photography from above in figure 1.

The physical parameters given in this report will concern:

- the hammers,
- the strings,
- the soundboard,
- the surrounding air.

## 2 Soundboard and vibroacoustics

### 2.1 Soundboard and referential

The soundboard is considered as a heterogeneous orthotropic bidimensional plate with varying thickness. This is how ribs and bridges are taken into account. We divide the soundboard into four zones, each point belonging to a rib or not, and to a bridge or not, as illustrated in figure 2. The mesh can be downloaded at <http://dl.dropbox.com/u/3984954/PIANO/Maillages/SteinwayChevaletCourtPlaque.mesh>. The following parameters are used for spruce and beech parts of the soundboard.

#### Spruce parameters

$\rho$ kg·m <sup>-3</sup>	$E_x$ GPa	$E_y$ GPa	$E_z$ Gpa	$G_{xy}$ Gpa	$G_{xz}$ GPa	$G_{yz}$ Gpa	$\nu_{xy}$ -	$\nu_{xz}$ -	$\nu_{yz}$ -
390	11.6	0.9	0.5	0.75	0.72	0.039	0.37	0.47	0.43
380	11.0	0.650	-	0.66	1.2	0.042	0.26	-	-

#### Beech parameters

$\rho$ kg·m <sup>-3</sup>	$E_x$ GPa	$E_y$ GPa	$E_z$ Gpa	$G_{xy}$ Gpa	$G_{xz}$ GPa	$G_{yz}$ Gpa	$\nu_{xy}$ -	$\nu_{xz}$ -	$\nu_{yz}$ -
750	13.7	2.24	-	1.61	1.06	0.46	0.3	-	-

Concerning the damping phenomena, the authors have proposed a modal approach which consists in considering that the damping occurs mode per mode. The damping constants are adjusted to the values:

$$\begin{cases} a = 2.00 \times 10^{-7} & (1a) \\ b = 0.07 & (1b) \\ c = 0.00 & (1c) \end{cases}$$

### 2.2 Acoustics

The surrounding air is supposed at rest, with the following homogeneous parameters:

$$\rho_a = 1.29 \text{ kg} \cdot \text{m}^3, \quad c_a = 340 \text{ m} \cdot \text{s}^{-1}, \quad \mu_a = 6.71 \times 10^{-6} \text{ s}^2 \cdot \text{m}^2 \cdot \text{kg}^{-1}$$

The 3D domain is obtained by an extrusion of a planar mesh which can be downloaded at <http://dl.dropbox.com/u/3984954/PIANO/Maillages/SteinwayChevaletCourt.mesh>.

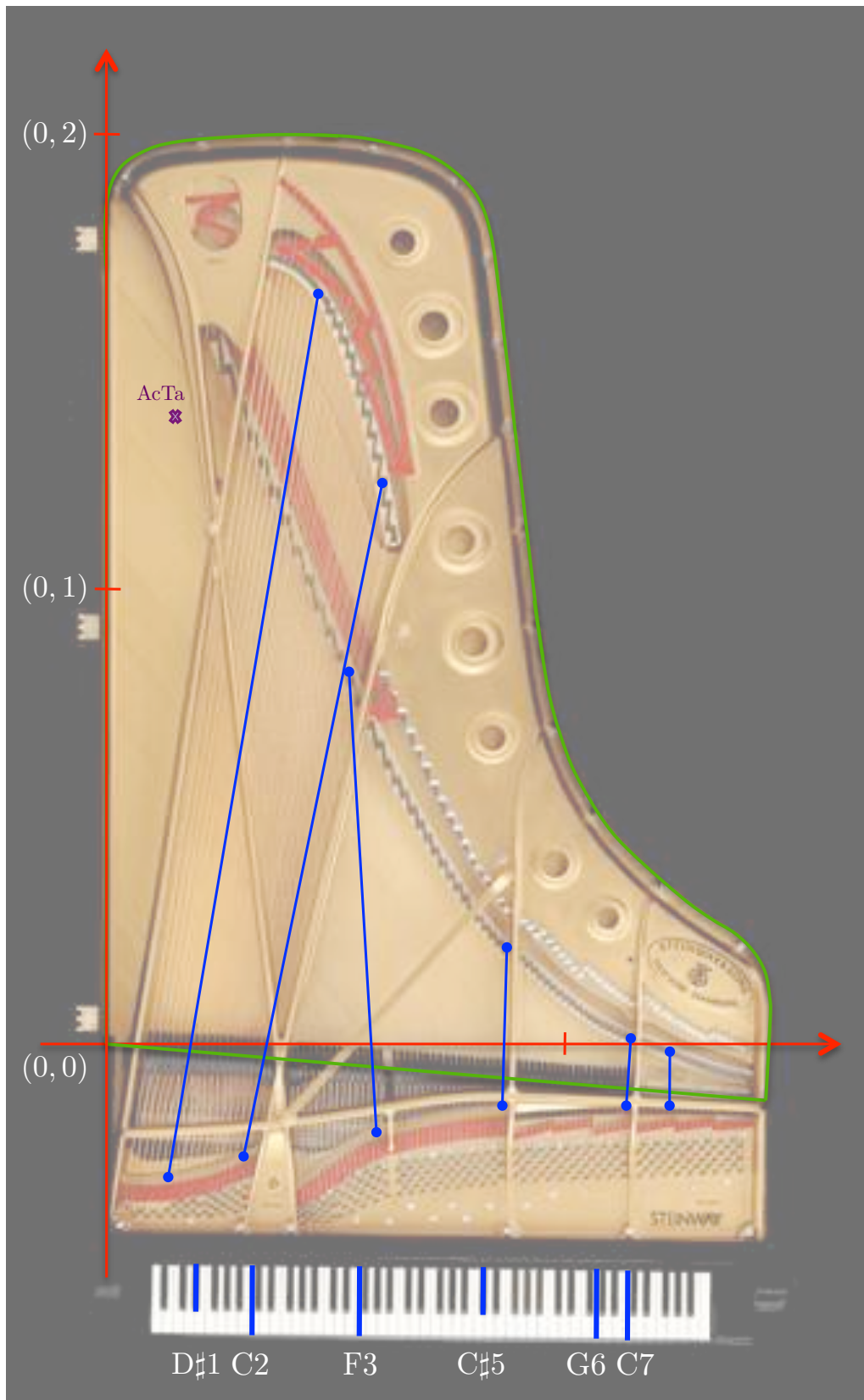
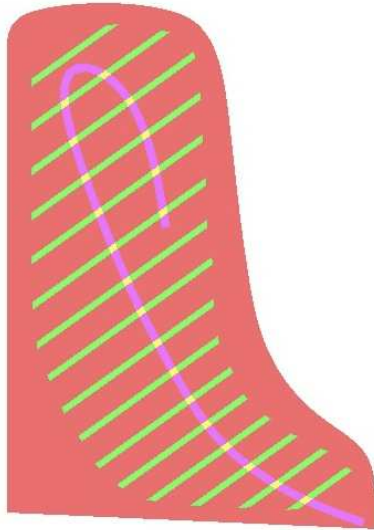
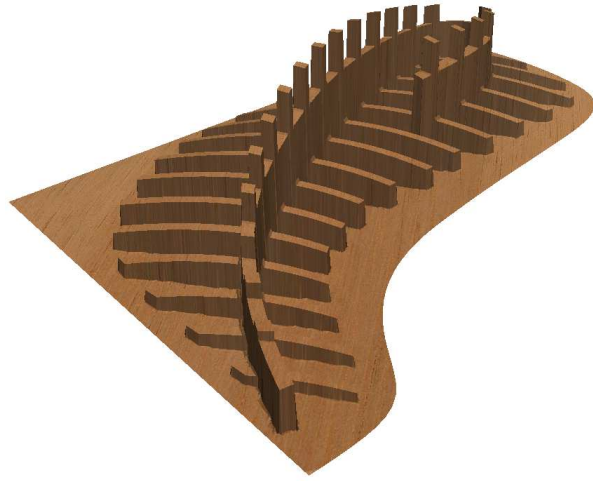


Figure 1: Photography of an open Steinway D (from yost.com). Blue strings were actually measured and used to fit the parameters presented in this report. The soundboard is indicated in green. The red referential will be used to indicate where strings meet the soundboard. All quantities are in meter.

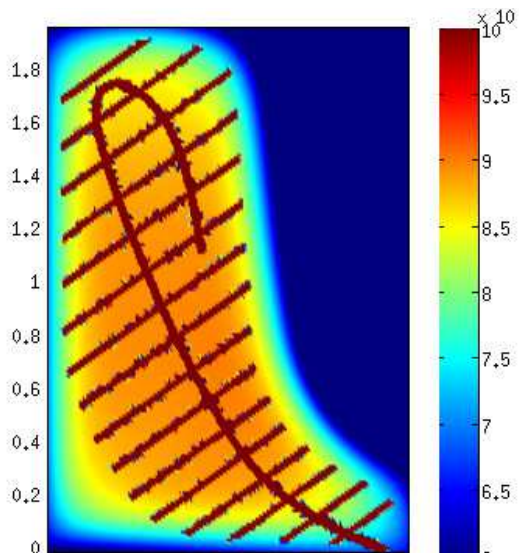




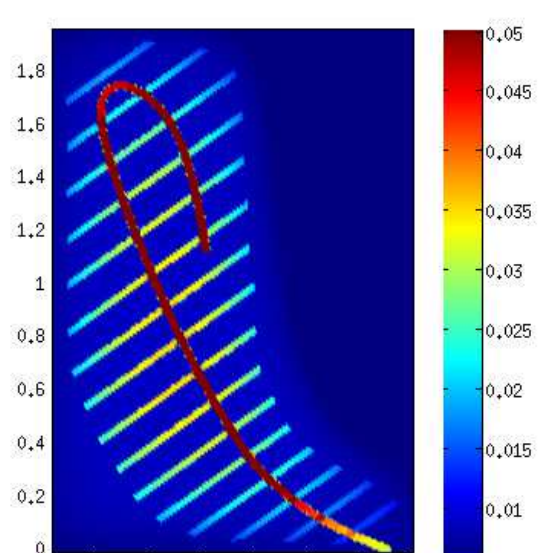
(a) Soundboard zones



(b) Varying thickness



(c) Varying thickness of the plate (AU)



(d) Varying thickness of the ribs (AU)

Figure 2: Different soundboard zones : different materials, orthotropic direction, and thicknesses.

### 2.3 String soundboard coupling : repartition functions

The coupling between strings and soundboard is done via a repartition function on the soundboard, which is centered in a point of the bridge  $(x_0, y_0)$  (where  $(0, 0)$  is the bottom left corner of the soundboard, see figure 1) and rapidly decreasing, as illustrated in figure 3. All points can be visualized in the film [http://dl.dropbox.com/u/3984954/THESE/FILMS/attaches\\_chevalet.avi](http://dl.dropbox.com/u/3984954/THESE/FILMS/attaches_chevalet.avi).

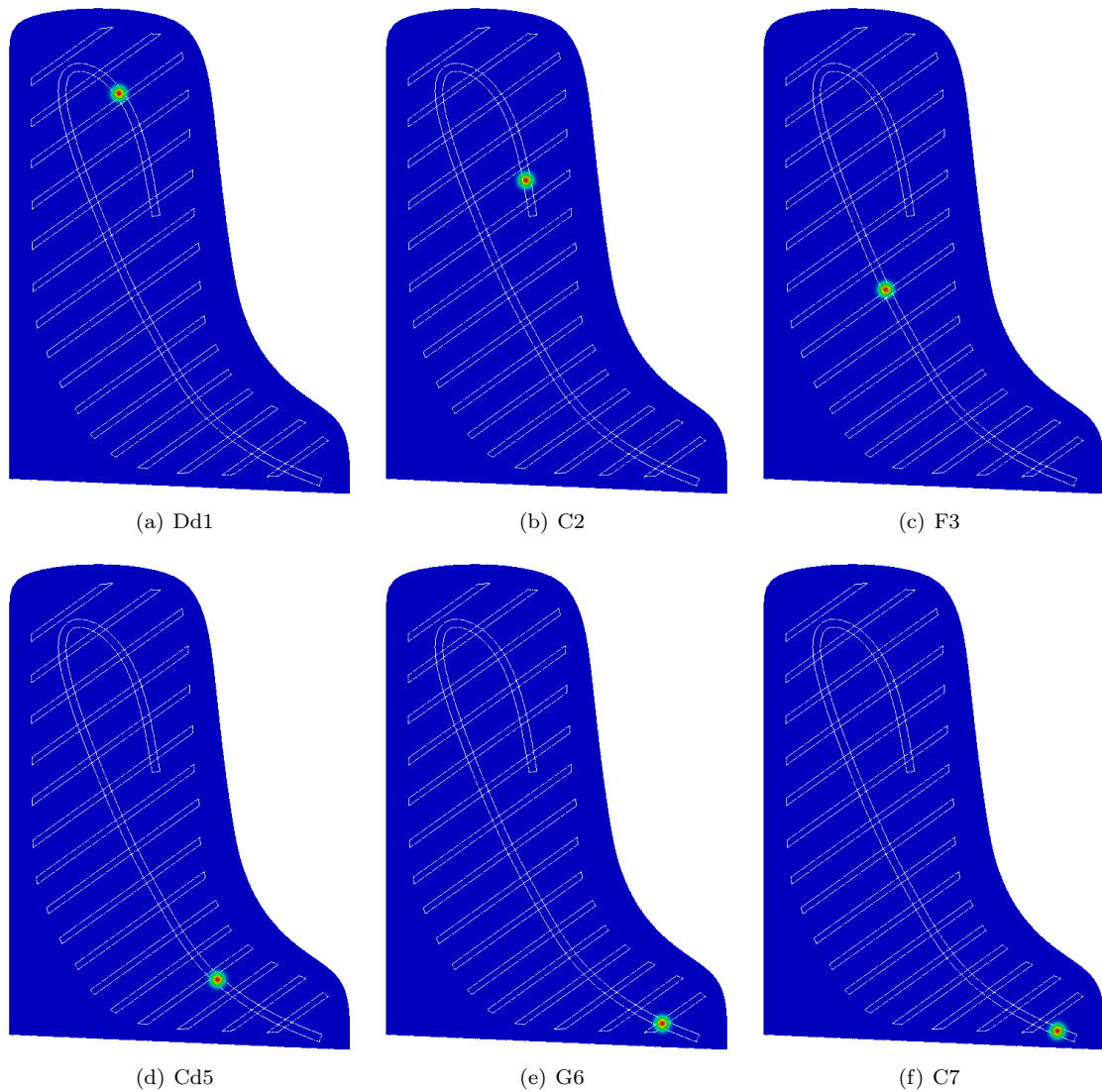


Figure 3: Repartition function for six reference notes.

### 3 Strings and hammers

#### 3.1 Hammers

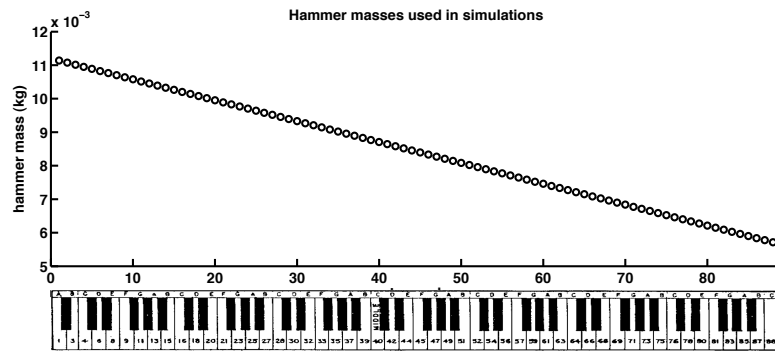
Let  $i \in [1, 88]$  be the number of the string in the keyboard. We propose to use the following values, represented in figure 4 and listed in appendices A and B.

$$M^{\mathcal{H}} = -6.2348 \times 10^{-5} i + 0.0112 \quad (2)$$

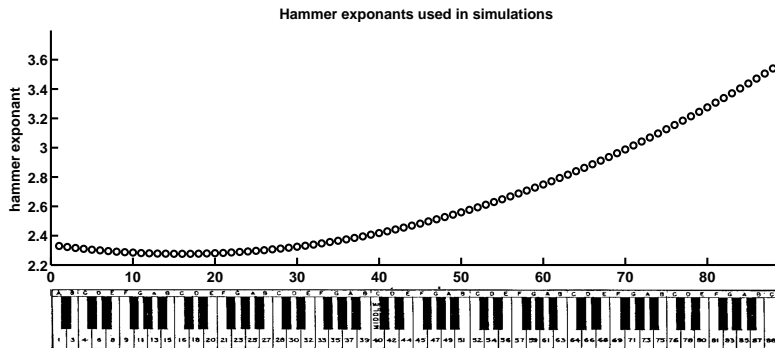
$$p = 2.4295 \times 10^{-4} i^2 - 0.007703 i + 2.337 \quad (3)$$

$$K^{\mathcal{H}} = 10^{(5.3097 \times 10^{-2} i + 7.6425)} \quad (4)$$

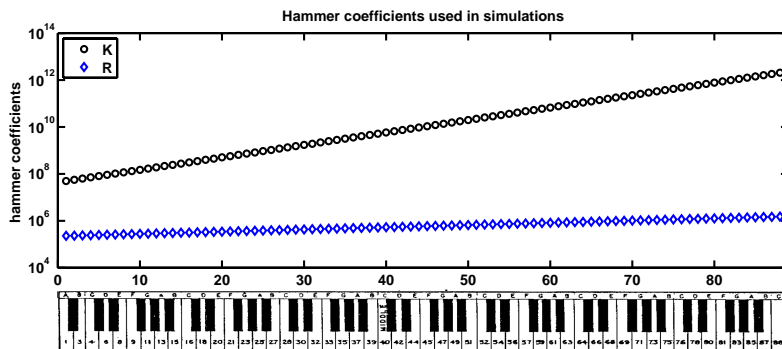
$$R^{\mathcal{H}} = K^{\mathcal{H}} \times 10^{(0.04366 i - 2.294)} \quad (5)$$



(a) Hammer masses



(b) Hammer exponents



(c) Hammer coefficients (logarithmic scale)

Figure 4: Visual representation of the hammer parameters

### 3.2 Strings

In our models, a string is entirely determined with the following parameters :

- length  $L$
- diameter  $d$
- density  $\rho$
- tension  $T_0$
- Young modulus  $E$
- shear modulus  $G$
- damping parameters  $R_u, R_v, R_\varphi, \eta_u, \eta_v, \eta_\varphi$

We have chosen to always use the relations:

$$A = \frac{\pi d^2}{4}, \quad I = \frac{\pi d^4}{64}, \quad E = 2,02 \times 10^{11} \text{ Pa}, \quad G = 8 \times 10^{10} \text{ Pa}$$

which reflects the fact that for wrapped strings, the modeled string is not exactly the real one, but rather an homogenized version with equivalent vibratory properties.

We have considered two different virtual pianos :

- The first one is based on realistic string lengths, which lead us to use very dense virtual materials for the bass strings in order to account for the wrapping,
- The second one is composed of only steel strings, which lead us to use very long lengths in the bass register. This piano could not easily be constructed because the strings would exceed the size of the rim, which would generate a technical impossibility. In our virtual world, this is absolutely not a problem and this scale design will be our favorite.

We will not give here an explicit formula, but all values are listed in appendices A and B for wrapped and unwrapped strings. A visual representation of lengths, diameters, densities, tensions and inharmonicity factors can be seen in figure 5 for both scale designs. They were both made so that the corresponding inharmonicity curves smoothly evolved all along the piano range.

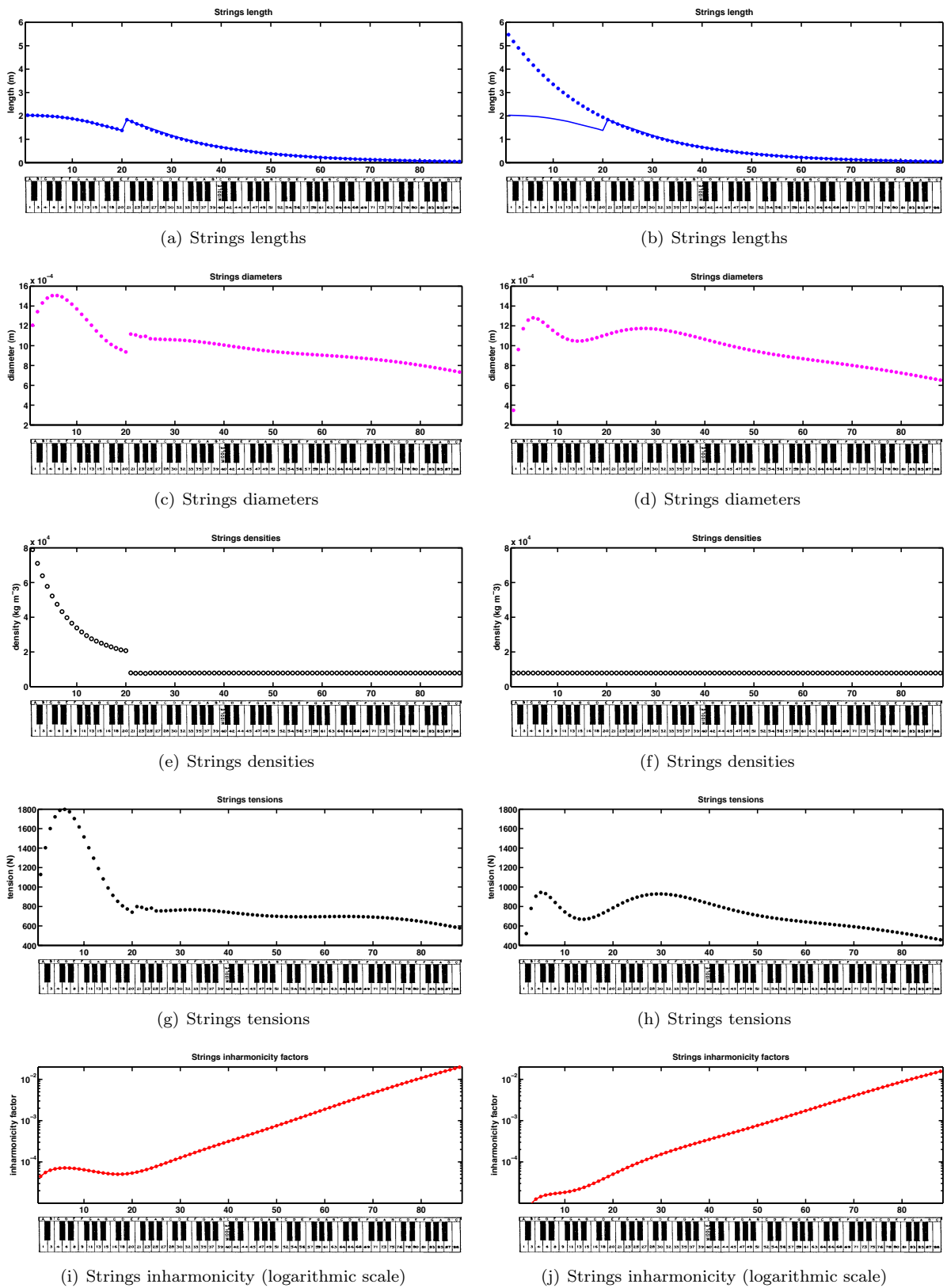


Figure 5: Visual representation of the wrapped (left) and unwrapped (right) strings parameters, with the same y-scale. Values given in appendices A and B.

## References

- [1] J Chabassier. Modélisation et simulation numérique d'un piano par modèles physiques. *PhD dissertation*, Mar 2012.
- [2] J Chabassier and P Joly. Time domain simulation of a piano. *In preparation for Journal of Computational Physics*, 2012.

## A Wrapped strings values

	Note	C1	Cd1	D1	Dd1	E1	F1
$f_0$	(Hz)	32.78	34.73	36.79	38.98	41.30	43.75
L	(m)	2.007	1.997	1.981	1.965	1.938	1.911
d	(mm)	1.480	1.505	1.506	1.492	1.460	1.419
$\rho$	( $\text{kg}\cdot\text{m}^{-3}$ )	57787	52229	47494	43195	39737	36571
$T_0$	(N)	1722	1788	1798	1773	1703	1618
$R_u$	( $\text{s}^{-1}$ )	$5.0 \times 10^{-03}$	$1.0 \times 10^{-02}$	$1.5 \times 10^{-02}$	$2.0 \times 10^{-02}$	$2.5 \times 10^{-02}$	$3.0 \times 10^{-02}$
$R_v$	( $\text{s}^{-1}$ )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	( $\text{s}^{-1}$ )	$5.0 \times 10^{-03}$	$1.0 \times 10^{-02}$	$1.5 \times 10^{-02}$	$2.0 \times 10^{-02}$	$2.5 \times 10^{-02}$	$3.0 \times 10^{-02}$
$\eta_u$	(s)	$1.6 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.8 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$1.6 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.8 \times 10^{-09}$
$p$	(-)	2.310	2.305	2.300	2.295	2.291	2.287
$x^{\mathcal{H}}$	(m)	0.241	0.240	0.238	0.236	0.233	0.229
$M^{\mathcal{H}}$	(g)	10.95	10.89	10.83	10.76	10.70	10.64
$K^{\mathcal{H}}$	( $\text{N}\cdot\text{m}^{-p}$ )	$7.160 \times 10^{07}$	$8.091 \times 10^{07}$	$9.143 \times 10^{07}$	$1.033 \times 10^{08}$	$1.168 \times 10^{08}$	$1.319 \times 10^{08}$
$R^{\mathcal{H}}$	( $\text{N}\cdot\text{m}^{-p}$ )	$2.434 \times 10^{05}$	$2.487 \times 10^{05}$	$2.542 \times 10^{05}$	$2.598 \times 10^{05}$	$2.655 \times 10^{05}$	$2.713 \times 10^{05}$
$x_0$	(m)	0.40	0.42	0.44	0.46	0.48	0.50
$y_0$	(m)	1.69	1.67	1.65	1.63	1.60	1.56

	Note	Fd1	G1	Ab1	A1	Ad1	B1
$f_0$	(Hz)	46.35	49.11	52.03	55.13	58.40	61.88
L	(m)	1.879	1.842	1.805	1.762	1.709	1.655
d	(mm)	1.370	1.316	1.262	1.207	1.148	1.096
$\rho$	( $\text{kg}\cdot\text{m}^{-3}$ )	33851	31521	29376	27588	26243	25043
$T_0$	(N)	1515	1404	1297	1190	1083	991
$R_u$	( $\text{s}^{-1}$ )	$3.5 \times 10^{-02}$	$4.0 \times 10^{-02}$	$4.5 \times 10^{-02}$	$5.0 \times 10^{-02}$	$5.5 \times 10^{-02}$	$6.0 \times 10^{-02}$
$R_v$	( $\text{s}^{-1}$ )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	( $\text{s}^{-1}$ )	$3.5 \times 10^{-02}$	$4.0 \times 10^{-02}$	$4.5 \times 10^{-02}$	$5.0 \times 10^{-02}$	$5.5 \times 10^{-02}$	$6.0 \times 10^{-02}$
$\eta_u$	(s)	$1.8 \times 10^{-09}$	$1.8 \times 10^{-09}$	$1.9 \times 10^{-09}$	$1.9 \times 10^{-09}$	$1.9 \times 10^{-09}$	$1.9 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$1.8 \times 10^{-09}$	$1.8 \times 10^{-09}$	$1.9 \times 10^{-09}$	$1.9 \times 10^{-09}$	$1.9 \times 10^{-09}$	$1.9 \times 10^{-09}$
$p$	(-)	2.284	2.282	2.280	2.278	2.277	2.276
$x^{\mathcal{H}}$	(m)	0.225	0.221	0.217	0.211	0.205	0.199
$M^{\mathcal{H}}$	(g)	10.58	10.51	10.45	10.39	10.33	10.26
$K^{\mathcal{H}}$	( $\text{N}\cdot\text{m}^{-p}$ )	$1.491 \times 10^{08}$	$1.685 \times 10^{08}$	$1.904 \times 10^{08}$	$2.152 \times 10^{08}$	$2.431 \times 10^{08}$	$2.748 \times 10^{08}$
$R^{\mathcal{H}}$	( $\text{N}\cdot\text{m}^{-p}$ )	$2.772 \times 10^{05}$	$2.833 \times 10^{05}$	$2.896 \times 10^{05}$	$2.959 \times 10^{05}$	$3.024 \times 10^{05}$	$3.091 \times 10^{05}$
$x_0$	(m)	0.52	0.53	0.55	0.56	0.57	0.58
$y_0$	(m)	1.52	1.48	1.44	1.39	1.35	1.30

	Note	C2	Cd2	D2	Dd2	E2	F2
$f_0$	(Hz)	65.56	69.45	73.58	77.96	82.59	87.51
L	(m)	1.602	1.548	1.495	1.442	1.378	1.837
d	(mm)	1.051	1.012	0.982	0.960	0.937	1.117
$\rho$	(kg·m <sup>-3</sup> )	23919	22925	21997	21160	20738	7887
$T_0$	(N)	915	853	807	774	741	799
$R_u$	(s <sup>-1</sup> )	$6.5 \times 10^{-02}$	$7.0 \times 10^{-02}$	$7.5 \times 10^{-02}$	$8.0 \times 10^{-02}$	$8.5 \times 10^{-02}$	$9.0 \times 10^{-02}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$6.5 \times 10^{-02}$	$7.0 \times 10^{-02}$	$7.5 \times 10^{-02}$	$8.0 \times 10^{-02}$	$8.5 \times 10^{-02}$	$9.0 \times 10^{-02}$
$\eta_u$	(s)	$2.0 \times 10^{-09}$	$2.0 \times 10^{-09}$	$2.0 \times 10^{-09}$	$2.1 \times 10^{-09}$	$2.1 \times 10^{-09}$	$2.1 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$2.0 \times 10^{-09}$	$2.0 \times 10^{-09}$	$2.0 \times 10^{-09}$	$2.1 \times 10^{-09}$	$2.1 \times 10^{-09}$	$2.1 \times 10^{-09}$
$p$	(-)	2.276	2.276	2.277	2.278	2.280	2.282
$x^{\mathcal{H}}$	(m)	0.192	0.186	0.179	0.173	0.165	0.220
$M^{\mathcal{H}}$	(g)	10.20	10.14	10.08	10.02	9.95	9.89
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$3.105 \times 10^{08}$	$3.509 \times 10^{08}$	$3.965 \times 10^{08}$	$4.481 \times 10^{08}$	$5.063 \times 10^{08}$	$5.722 \times 10^{08}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$3.159 \times 10^{05}$	$3.228 \times 10^{05}$	$3.299 \times 10^{05}$	$3.371 \times 10^{05}$	$3.445 \times 10^{05}$	$3.521 \times 10^{05}$
$x_0$	(m)	0.59	0.60	0.61	0.61	0.62	0.23
$y_0$	(m)	1.26	1.22	1.18	1.14	1.11	1.58

	Note	Fd2	G2	Ab2	A2	Ad2	B2
$f_0$	(Hz)	92.71	98.22	104.06	110.25	116.81	123.75
L	(m)	1.757	1.660	1.591	1.482	1.403	1.329
d	(mm)	1.108	1.091	1.095	1.071	1.067	1.065
$\rho$	(kg·m <sup>-3</sup> )	7734	7772	7589	7850	7850	7850
$T_0$	(N)	791	773	783	754	754	756
$R_u$	(s <sup>-1</sup> )	$9.5 \times 10^{-02}$	$1.0 \times 10^{-01}$	$1.0 \times 10^{-01}$	$1.1 \times 10^{-01}$	$1.2 \times 10^{-01}$	$1.2 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$9.5 \times 10^{-02}$	$1.0 \times 10^{-01}$	$1.0 \times 10^{-01}$	$1.1 \times 10^{-01}$	$1.2 \times 10^{-01}$	$1.2 \times 10^{-01}$
$\eta_u$	(s)	$2.1 \times 10^{-09}$	$2.2 \times 10^{-09}$	$2.2 \times 10^{-09}$	$2.2 \times 10^{-09}$	$2.3 \times 10^{-09}$	$2.3 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$2.1 \times 10^{-09}$	$2.2 \times 10^{-09}$	$2.2 \times 10^{-09}$	$2.2 \times 10^{-09}$	$2.3 \times 10^{-09}$	$2.3 \times 10^{-09}$
$p$	(-)	2.285	2.288	2.292	2.296	2.301	2.306
$x^{\mathcal{H}}$	(m)	0.211	0.199	0.191	0.178	0.168	0.160
$M^{\mathcal{H}}$	(g)	9.83	9.77	9.70	9.64	9.58	9.52
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$6.466 \times 10^{08}$	$7.307 \times 10^{08}$	$8.257 \times 10^{08}$	$9.331 \times 10^{08}$	$1.054 \times 10^{09}$	$1.192 \times 10^{09}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$3.598 \times 10^{05}$	$3.677 \times 10^{05}$	$3.758 \times 10^{05}$	$3.841 \times 10^{05}$	$3.925 \times 10^{05}$	$4.011 \times 10^{05}$
$x_0$	(m)	0.25	0.27	0.30	0.32	0.35	0.37
$y_0$	(m)	1.49	1.41	1.34	1.26	1.19	1.13

	Note	C3	Cd3	D3	Dd3	E3	F3
$f_0$	(Hz)	131.11	138.90	147.17	155.91	165.19	175.01
L	(m)	1.259	1.192	1.129	1.070	1.013	0.960
d	(mm)	1.063	1.061	1.059	1.057	1.053	1.049
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	759	762	764	766	767	766
$R_u$	(s <sup>-1</sup> )	$1.2 \times 10^{-01}$	$1.3 \times 10^{-01}$	$1.4 \times 10^{-01}$	$1.4 \times 10^{-01}$	$1.4 \times 10^{-01}$	$1.5 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$1.2 \times 10^{-01}$	$1.3 \times 10^{-01}$	$1.4 \times 10^{-01}$	$1.4 \times 10^{-01}$	$1.4 \times 10^{-01}$	$1.5 \times 10^{-01}$
$\eta_u$	(s)	$2.3 \times 10^{-09}$	$2.3 \times 10^{-09}$	$2.4 \times 10^{-09}$	$2.4 \times 10^{-09}$	$2.4 \times 10^{-09}$	$2.4 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$2.3 \times 10^{-09}$	$2.3 \times 10^{-09}$	$2.4 \times 10^{-09}$	$2.4 \times 10^{-09}$	$2.4 \times 10^{-09}$	$2.4 \times 10^{-09}$
$p$	(-)	2.312	2.318	2.325	2.332	2.339	2.347
$x^{\mathcal{H}}$	(m)	0.151	0.143	0.136	0.128	0.122	0.115
$M^{\mathcal{H}}$	(g)	9.45	9.39	9.33	9.27	9.20	9.14
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.347 \times 10^{09}$	$1.522 \times 10^{09}$	$1.720 \times 10^{09}$	$1.943 \times 10^{09}$	$2.196 \times 10^{09}$	$2.481 \times 10^{09}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$4.100 \times 10^{05}$	$4.190 \times 10^{05}$	$4.282 \times 10^{05}$	$4.376 \times 10^{05}$	$4.472 \times 10^{05}$	$4.570 \times 10^{05}$
$x_0$	(m)	0.40	0.42	0.45	0.47	0.49	0.51
$y_0$	(m)	1.06	1.01	0.95	0.90	0.85	0.80

	Note	Fd3	G3	Ab3	A3	Ad3	B3
$f_0$	(Hz)	185.41	196.45	208.12	220.49	233.62	247.52
L	(m)	0.909	0.861	0.816	0.773	0.732	0.694
d	(mm)	1.045	1.039	1.033	1.027	1.020	1.013
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	765	763	759	755	751	746
$R_u$	(s <sup>-1</sup> )	$1.5 \times 10^{-01}$	$1.6 \times 10^{-01}$	$1.7 \times 10^{-01}$	$1.7 \times 10^{-01}$	$1.7 \times 10^{-01}$	$1.8 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$1.5 \times 10^{-01}$	$1.6 \times 10^{-01}$	$1.7 \times 10^{-01}$	$1.7 \times 10^{-01}$	$1.7 \times 10^{-01}$	$1.8 \times 10^{-01}$
$\eta_u$	(s)	$2.5 \times 10^{-09}$	$2.5 \times 10^{-09}$	$2.5 \times 10^{-09}$	$2.6 \times 10^{-09}$	$2.6 \times 10^{-09}$	$2.6 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$2.5 \times 10^{-09}$	$2.5 \times 10^{-09}$	$2.5 \times 10^{-09}$	$2.6 \times 10^{-09}$	$2.6 \times 10^{-09}$	$2.6 \times 10^{-09}$
$p$	(-)	2.356	2.365	2.375	2.385	2.395	2.406
$x^{\mathcal{H}}$	(m)	0.109	0.103	0.098	0.093	0.088	0.083
$M^{\mathcal{H}}$	(g)	9.08	9.02	8.96	8.89	8.83	8.77
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$2.804 \times 10^{09}$	$3.169 \times 10^{09}$	$3.581 \times 10^{09}$	$4.047 \times 10^{09}$	$4.573 \times 10^{09}$	$5.168 \times 10^{09}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$4.670 \times 10^{05}$	$4.773 \times 10^{05}$	$4.878 \times 10^{05}$	$4.985 \times 10^{05}$	$5.095 \times 10^{05}$	$5.206 \times 10^{05}$
$x_0$	(m)	0.54	0.56	0.58	0.60	0.62	0.64
$y_0$	(m)	0.75	0.71	0.67	0.63	0.60	0.56

	Note	C4	Cd4	D4	Dd4	E4	F4
$f_0$	(Hz)	262.22	277.81	294.34	311.82	330.36	350.05
L	(m)	0.657	0.622	0.590	0.559	0.529	0.501
d	(mm)	1.006	0.999	0.991	0.984	0.977	0.970
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	741	735	730	725	720	715
$R_u$	(s <sup>-1</sup> )	$1.8 \times 10^{-01}$	$1.9 \times 10^{-01}$	$2.0 \times 10^{-01}$	$2.0 \times 10^{-01}$	$2.0 \times 10^{-01}$	$2.1 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$1.8 \times 10^{-01}$	$1.9 \times 10^{-01}$	$2.0 \times 10^{-01}$	$2.0 \times 10^{-01}$	$2.0 \times 10^{-01}$	$2.1 \times 10^{-01}$
$\eta_u$	(s)	$2.6 \times 10^{-09}$	$2.7 \times 10^{-09}$	$2.7 \times 10^{-09}$	$2.7 \times 10^{-09}$	$2.8 \times 10^{-09}$	$2.8 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$2.6 \times 10^{-09}$	$2.7 \times 10^{-09}$	$2.7 \times 10^{-09}$	$2.7 \times 10^{-09}$	$2.8 \times 10^{-09}$	$2.8 \times 10^{-09}$
$p$	(-)	2.418	2.430	2.442	2.455	2.468	2.482
$x^{\mathcal{H}}$	(m)	0.079	0.075	0.071	0.067	0.064	0.060
$M^{\mathcal{H}}$	(g)	8.71	8.64	8.58	8.52	8.46	8.39
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$5.840 \times 10^{09}$	$6.599 \times 10^{09}$	$7.457 \times 10^{09}$	$8.427 \times 10^{09}$	$9.523 \times 10^{09}$	$1.076 \times 10^{10}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$5.321 \times 10^{05}$	$5.438 \times 10^{05}$	$5.557 \times 10^{05}$	$5.679 \times 10^{05}$	$5.804 \times 10^{05}$	$5.932 \times 10^{05}$
$x_0$	(m)	0.66	0.68	0.70	0.71	0.73	0.75
$y_0$	(m)	0.53	0.50	0.47	0.44	0.41	0.39

	Note	Fd4	G4	Ab4	A4	Ad4	B4
$f_0$	(Hz)	370.80	392.87	416.26	441.00	467.19	494.96
L	(m)	0.475	0.450	0.426	0.404	0.383	0.363
d	(mm)	0.964	0.958	0.952	0.947	0.941	0.937
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	711	707	704	701	699	697
$R_u$	(s <sup>-1</sup> )	$2.1 \times 10^{-01}$	$2.2 \times 10^{-01}$	$2.3 \times 10^{-01}$	$2.3 \times 10^{-01}$	$2.3 \times 10^{-01}$	$2.4 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$2.1 \times 10^{-01}$	$2.2 \times 10^{-01}$	$2.3 \times 10^{-01}$	$2.3 \times 10^{-01}$	$2.3 \times 10^{-01}$	$2.4 \times 10^{-01}$
$\eta_u$	(s)	$2.8 \times 10^{-09}$	$2.8 \times 10^{-09}$	$2.9 \times 10^{-09}$	$2.9 \times 10^{-09}$	$2.9 \times 10^{-09}$	$2.9 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$2.8 \times 10^{-09}$	$2.8 \times 10^{-09}$	$2.9 \times 10^{-09}$	$2.9 \times 10^{-09}$	$2.9 \times 10^{-09}$	$2.9 \times 10^{-09}$
$p$	(-)	2.497	2.512	2.527	2.543	2.559	2.576
$x^{\mathcal{H}}$	(m)	0.057	0.054	0.051	0.048	0.046	0.044
$M^{\mathcal{H}}$	(g)	8.33	8.27	8.21	8.14	8.08	8.02
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.216 \times 10^{10}$	$1.374 \times 10^{10}$	$1.553 \times 10^{10}$	$1.755 \times 10^{10}$	$1.983 \times 10^{10}$	$2.241 \times 10^{10}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$6.062 \times 10^{05}$	$6.195 \times 10^{05}$	$6.331 \times 10^{05}$	$6.470 \times 10^{05}$	$6.612 \times 10^{05}$	$6.758 \times 10^{05}$
$x_0$	(m)	0.76	0.78	0.80	0.81	0.83	0.85
$y_0$	(m)	0.36	0.34	0.32	0.30	0.28	0.27



	Note	C5	Cd5	D5	Dd5	E5	F5
$f_0$	(Hz)	524.48	555.60	588.66	623.69	660.69	699.91
L	(m)	0.344	0.326	0.308	0.292	0.277	0.262
d	(mm)	0.932	0.928	0.924	0.921	0.917	0.914
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	696	695	694	694	694	694
$R_u$	(s <sup>-1</sup> )	$2.4 \times 10^{-01}$	$2.5 \times 10^{-01}$	$2.6 \times 10^{-01}$	$2.6 \times 10^{-01}$	$2.7 \times 10^{-01}$	$2.7 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$2.4 \times 10^{-01}$	$2.5 \times 10^{-01}$	$2.6 \times 10^{-01}$	$2.6 \times 10^{-01}$	$2.7 \times 10^{-01}$	$2.7 \times 10^{-01}$
$\eta_u$	(s)	$3.0 \times 10^{-09}$	$3.0 \times 10^{-09}$	$3.0 \times 10^{-09}$	$3.1 \times 10^{-09}$	$3.1 \times 10^{-09}$	$3.1 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$3.0 \times 10^{-09}$	$3.0 \times 10^{-09}$	$3.0 \times 10^{-09}$	$3.1 \times 10^{-09}$	$3.1 \times 10^{-09}$	$3.1 \times 10^{-09}$
$p$	(-)	2.593	2.611	2.629	2.648	2.668	2.687
$x^{\mathcal{H}}$	(m)	0.041	0.039	0.037	0.035	0.033	0.031
$M^{\mathcal{H}}$	(g)	7.96	7.90	7.83	7.77	7.71	7.65
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$2.532 \times 10^{10}$	$2.862 \times 10^{10}$	$3.234 \times 10^{10}$	$3.655 \times 10^{10}$	$4.130 \times 10^{10}$	$4.667 \times 10^{10}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$6.906 \times 10^{05}$	$7.058 \times 10^{05}$	$7.213 \times 10^{05}$	$7.371 \times 10^{05}$	$7.533 \times 10^{05}$	$7.699 \times 10^{05}$
$x_0$	(m)	0.86	0.88	0.90	0.91	0.93	0.94
$y_0$	(m)	0.25	0.23	0.22	0.21	0.19	0.18

	Note	Fd5	G5	Ab5	A5	Ad5	B5
$f_0$	(Hz)	741.65	785.63	832.54	882.06	934.60	990.20
L	(m)	0.249	0.236	0.223	0.211	0.200	0.190
d	(mm)	0.910	0.907	0.904	0.901	0.898	0.894
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	695	695	696	697	697	697
$R_u$	(s <sup>-1</sup> )	$2.8 \times 10^{-01}$	$2.8 \times 10^{-01}$	$2.8 \times 10^{-01}$	$2.9 \times 10^{-01}$	$2.9 \times 10^{-01}$	$3.0 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$2.8 \times 10^{-01}$	$2.8 \times 10^{-01}$	$2.8 \times 10^{-01}$	$2.9 \times 10^{-01}$	$2.9 \times 10^{-01}$	$3.0 \times 10^{-01}$
$\eta_u$	(s)	$3.1 \times 10^{-09}$	$3.2 \times 10^{-09}$	$3.2 \times 10^{-09}$	$3.2 \times 10^{-09}$	$3.3 \times 10^{-09}$	$3.3 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$3.1 \times 10^{-09}$	$3.2 \times 10^{-09}$	$3.2 \times 10^{-09}$	$3.2 \times 10^{-09}$	$3.3 \times 10^{-09}$	$3.3 \times 10^{-09}$
$p$	(-)	2.708	2.728	2.749	2.771	2.793	2.816
$x^{\mathcal{H}}$	(m)	0.030	0.028	0.027	0.025	0.024	0.023
$M^{\mathcal{H}}$	(g)	7.58	7.52	7.46	7.40	7.33	7.27
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$5.274 \times 10^{10}$	$5.960 \times 10^{10}$	$6.735 \times 10^{10}$	$7.611 \times 10^{10}$	$8.600 \times 10^{10}$	$9.719 \times 10^{10}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$7.868 \times 10^{05}$	$8.040 \times 10^{05}$	$8.217 \times 10^{05}$	$8.398 \times 10^{05}$	$8.582 \times 10^{05}$	$8.771 \times 10^{05}$
$x_0$	(m)	0.96	0.97	0.99	1.01	1.02	1.04
$y_0$	(m)	0.17	0.16	0.15	0.14	0.13	0.12

	Note	C6	Cd6	D6	Dd6	E6	F6
$f_0$	(Hz)	1048.83	1111.06	1177.66	1247.25	1321.31	1399.97
L	(m)	0.180	0.171	0.162	0.153	0.145	0.138
d	(mm)	0.891	0.887	0.883	0.879	0.875	0.870
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	697	697	697	696	695	693
$R_u$	(s <sup>-1</sup> )	$3.0 \times 10^{-01}$	$3.1 \times 10^{-01}$	$3.2 \times 10^{-01}$	$3.2 \times 10^{-01}$	$3.3 \times 10^{-01}$	$3.3 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$3.0 \times 10^{-01}$	$3.1 \times 10^{-01}$	$3.2 \times 10^{-01}$	$3.2 \times 10^{-01}$	$3.3 \times 10^{-01}$	$3.3 \times 10^{-01}$
$\eta_u$	(s)	$3.3 \times 10^{-09}$	$3.3 \times 10^{-09}$	$3.4 \times 10^{-09}$	$3.4 \times 10^{-09}$	$3.4 \times 10^{-09}$	$3.4 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$3.3 \times 10^{-09}$	$3.3 \times 10^{-09}$	$3.4 \times 10^{-09}$	$3.4 \times 10^{-09}$	$3.4 \times 10^{-09}$	$3.4 \times 10^{-09}$
$p$	(-)	2.839	2.863	2.887	2.912	2.937	2.962
$x^{\mathcal{H}}$	(m)	0.022	0.020	0.019	0.018	0.017	0.017
$M^{\mathcal{H}}$	(g)	7.21	7.15	7.09	7.02	6.96	6.90
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.098 \times 10^{11}$	$1.241 \times 10^{11}$	$1.402 \times 10^{11}$	$1.585 \times 10^{11}$	$1.791 \times 10^{11}$	$2.024 \times 10^{11}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$8.963 \times 10^{05}$	$9.160 \times 10^{05}$	$9.361 \times 10^{05}$	$9.567 \times 10^{05}$	$9.777 \times 10^{05}$	$9.992 \times 10^{05}$
$x_0$	(m)	1.05	1.07	1.09	1.10	1.12	1.13
$y_0$	(m)	0.11	0.10	0.09	0.08	0.07	0.06

	Note	Fd6	G6	Ab6	A6	Ad6	B6
$f_0$	(Hz)	1483.31	1571.35	1665.41	1764.21	1869.22	1980.57
L	(m)	0.130	0.124	0.117	0.111	0.105	0.100
d	(mm)	0.866	0.860	0.855	0.850	0.844	0.837
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	691	689	686	683	679	674
$R_u$	(s <sup>-1</sup> )	$3.4 \times 10^{-01}$	$3.4 \times 10^{-01}$	$3.4 \times 10^{-01}$	$3.5 \times 10^{-01}$	$3.5 \times 10^{-01}$	$3.6 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$3.4 \times 10^{-01}$	$3.4 \times 10^{-01}$	$3.4 \times 10^{-01}$	$3.5 \times 10^{-01}$	$3.5 \times 10^{-01}$	$3.6 \times 10^{-01}$
$\eta_u$	(s)	$3.5 \times 10^{-09}$	$3.5 \times 10^{-09}$	$3.5 \times 10^{-09}$	$3.6 \times 10^{-09}$	$3.6 \times 10^{-09}$	$3.6 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$3.5 \times 10^{-09}$	$3.5 \times 10^{-09}$	$3.5 \times 10^{-09}$	$3.6 \times 10^{-09}$	$3.6 \times 10^{-09}$	$3.6 \times 10^{-09}$
$p$	(-)	2.988	3.015	3.042	3.069	3.097	3.126
$x^{\mathcal{H}}$	(m)	0.016	0.015	0.014	0.013	0.013	0.012
$M^{\mathcal{H}}$	(g)	6.84	6.77	6.71	6.65	6.59	6.52
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$2.287 \times 10^{11}$	$2.585 \times 10^{11}$	$2.921 \times 10^{11}$	$3.301 \times 10^{11}$	$3.730 \times 10^{11}$	$4.215 \times 10^{11}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.021 \times 10^{06}$	$1.044 \times 10^{06}$	$1.067 \times 10^{06}$	$1.090 \times 10^{06}$	$1.114 \times 10^{06}$	$1.138 \times 10^{06}$
$x_0$	(m)	1.15	1.16	1.18	1.19	1.21	1.22
$y_0$	(m)	0.06	0.05	0.04	0.04	0.03	0.02

	Note	C7	Cd7	D7	Dd7	E7	F7
$f_0$	(Hz)	2098.32	2222.39	2355.34	2494.63	2643.17	2801.35
L	(m)	0.095	0.090	0.085	0.081	0.076	0.072
d	(mm)	0.831	0.824	0.817	0.810	0.802	0.795
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	670	664	659	652	646	639
$R_u$	(s <sup>-1</sup> )	$3.6 \times 10^{-01}$	$3.7 \times 10^{-01}$	$3.8 \times 10^{-01}$	$3.8 \times 10^{-01}$	$3.9 \times 10^{-01}$	$3.9 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$3.6 \times 10^{-01}$	$3.7 \times 10^{-01}$	$3.8 \times 10^{-01}$	$3.8 \times 10^{-01}$	$3.9 \times 10^{-01}$	$3.9 \times 10^{-01}$
$\eta_u$	(s)	$3.6 \times 10^{-09}$	$3.7 \times 10^{-09}$	$3.7 \times 10^{-09}$	$3.7 \times 10^{-09}$	$3.8 \times 10^{-09}$	$3.8 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$3.6 \times 10^{-09}$	$3.7 \times 10^{-09}$	$3.7 \times 10^{-09}$	$3.7 \times 10^{-09}$	$3.8 \times 10^{-09}$	$3.8 \times 10^{-09}$
$p$	(-)	3.155	3.184	3.214	3.245	3.276	3.307
$x^{\mathcal{H}}$	(m)	0.011	0.011	0.010	0.010	0.009	0.009
$M^{\mathcal{H}}$	(g)	6.46	6.40	6.34	6.27	6.21	6.15
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$4.763 \times 10^{11}$	$5.382 \times 10^{11}$	$6.082 \times 10^{11}$	$6.873 \times 10^{11}$	$7.767 \times 10^{11}$	$8.777 \times 10^{11}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.163 \times 10^{06}$	$1.189 \times 10^{06}$	$1.215 \times 10^{06}$	$1.242 \times 10^{06}$	$1.269 \times 10^{06}$	$1.297 \times 10^{06}$
$x_0$	(m)	1.24	1.25	1.27	1.28	1.29	1.31
$y_0$	(m)	0.02	0.01	0.01	0.00	-0.00	-0.01

	Note	Fd7	G7	Ab7	A7	Ad7	B7
$f_0$	(Hz)	2965.15	3142.89	3330.76	3528.57	3735.94	3959.63
L	(m)	0.069	0.065	0.062	0.058	0.055	0.052
d	(mm)	0.787	0.778	0.770	0.761	0.752	0.743
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	631	623	615	606	598	588
$R_u$	(s <sup>-1</sup> )	$4.0 \times 10^{-01}$	$4.0 \times 10^{-01}$	$4.1 \times 10^{-01}$	$4.1 \times 10^{-01}$	$4.1 \times 10^{-01}$	$4.2 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$4.0 \times 10^{-01}$	$4.0 \times 10^{-01}$	$4.1 \times 10^{-01}$	$4.1 \times 10^{-01}$	$4.1 \times 10^{-01}$	$4.2 \times 10^{-01}$
$\eta_u$	(s)	$3.8 \times 10^{-09}$	$3.8 \times 10^{-09}$	$3.9 \times 10^{-09}$	$3.9 \times 10^{-09}$	$3.9 \times 10^{-09}$	$3.9 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$3.8 \times 10^{-09}$	$3.8 \times 10^{-09}$	$3.9 \times 10^{-09}$	$3.9 \times 10^{-09}$	$3.9 \times 10^{-09}$	$3.9 \times 10^{-09}$
$p$	(-)	3.339	3.371	3.404	3.438	3.471	3.506
$x^{\mathcal{H}}$	(m)	0.008	0.008	0.007	0.007	0.007	0.006
$M^{\mathcal{H}}$	(g)	6.09	6.03	5.96	5.90	5.84	5.78
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$9.919 \times 10^{11}$	$1.121 \times 10^{12}$	$1.267 \times 10^{12}$	$1.431 \times 10^{12}$	$1.617 \times 10^{12}$	$1.828 \times 10^{12}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.325 \times 10^{06}$	$1.354 \times 10^{06}$	$1.384 \times 10^{06}$	$1.415 \times 10^{06}$	$1.446 \times 10^{06}$	$1.477 \times 10^{06}$
$x_0$	(m)	1.32	1.33	1.34	1.36	1.37	1.38
$y_0$	(m)	-0.01	-0.02	-0.02	-0.02	-0.03	-0.03

## B Unwrapped strings values

	Note	C1	Cd1	D1	Dd1	E1	F1
$f_0$	(Hz)	32.78	34.73	36.79	38.98	41.30	43.75
L	(m)	4.647	4.400	4.167	3.945	3.736	3.538
d	(mm)	1.258	1.281	1.268	1.237	1.197	1.156
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	905	945	932	892	841	789
$R_u$	(s <sup>-1</sup> )	$5.0 \times 10^{-03}$	$1.0 \times 10^{-02}$	$1.5 \times 10^{-02}$	$2.0 \times 10^{-02}$	$2.5 \times 10^{-02}$	$3.0 \times 10^{-02}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$5.0 \times 10^{-03}$	$1.0 \times 10^{-02}$	$1.5 \times 10^{-02}$	$2.0 \times 10^{-02}$	$2.5 \times 10^{-02}$	$3.0 \times 10^{-02}$
$\eta_u$	(s)	$1.6 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.8 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$1.6 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.7 \times 10^{-09}$	$1.8 \times 10^{-09}$
$p$	(-)	2.310	2.305	2.300	2.295	2.291	2.287
$x^{\mathcal{H}}$	(m)	0.558	0.528	0.500	0.473	0.448	0.425
$M^{\mathcal{H}}$	(g)	10.95	10.89	10.83	10.76	10.70	10.64
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$7.160 \times 10^{07}$	$8.091 \times 10^{07}$	$9.143 \times 10^{07}$	$1.033 \times 10^{08}$	$1.168 \times 10^{08}$	$1.319 \times 10^{08}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$2.434 \times 10^{05}$	$2.487 \times 10^{05}$	$2.542 \times 10^{05}$	$2.598 \times 10^{05}$	$2.655 \times 10^{05}$	$2.713 \times 10^{05}$
$x_0$	(m)	0.40	0.42	0.44	0.46	0.48	0.50
$y_0$	(m)	1.69	1.67	1.65	1.63	1.60	1.56

	Note	Fd1	G1	Ab1	A1	Ad1	B1
$f_0$	(Hz)	46.35	49.11	52.03	55.12	58.40	61.87
L	(m)	3.350	3.173	3.005	2.845	2.695	2.552
d	(mm)	1.118	1.087	1.065	1.052	1.046	1.048
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	744	708	684	671	669	676
$R_u$	(s <sup>-1</sup> )	$3.5 \times 10^{-02}$	$4.0 \times 10^{-02}$	$4.5 \times 10^{-02}$	$5.0 \times 10^{-02}$	$5.5 \times 10^{-02}$	$6.0 \times 10^{-02}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$3.5 \times 10^{-02}$	$4.0 \times 10^{-02}$	$4.5 \times 10^{-02}$	$5.0 \times 10^{-02}$	$5.5 \times 10^{-02}$	$6.0 \times 10^{-02}$
$\eta_u$	(s)	$1.8 \times 10^{-09}$	$1.8 \times 10^{-09}$	$1.9 \times 10^{-09}$	$1.9 \times 10^{-09}$	$1.9 \times 10^{-09}$	$1.9 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$1.8 \times 10^{-09}$	$1.8 \times 10^{-09}$	$1.9 \times 10^{-09}$	$1.9 \times 10^{-09}$	$1.9 \times 10^{-09}$	$1.9 \times 10^{-09}$
$p$	(-)	2.284	2.282	2.280	2.278	2.277	2.276
$x^{\mathcal{H}}$	(m)	0.402	0.381	0.361	0.341	0.323	0.306
$M^{\mathcal{H}}$	(g)	10.58	10.51	10.45	10.39	10.33	10.26
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.491 \times 10^{08}$	$1.685 \times 10^{08}$	$1.904 \times 10^{08}$	$2.152 \times 10^{08}$	$2.431 \times 10^{08}$	$2.748 \times 10^{08}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$2.772 \times 10^{05}$	$2.833 \times 10^{05}$	$2.896 \times 10^{05}$	$2.959 \times 10^{05}$	$3.024 \times 10^{05}$	$3.091 \times 10^{05}$
$x_0$	(m)	0.52	0.53	0.55	0.56	0.57	0.58
$y_0$	(m)	1.52	1.48	1.44	1.39	1.35	1.30

	Note	C2	Cd2	D2	Dd2	E2	F2
$f_0$	(Hz)	65.55	69.45	73.58	77.96	82.59	87.51
L	(m)	2.417	2.289	2.168	2.053	1.944	1.841
d	(mm)	1.056	1.067	1.081	1.096	1.112	1.126
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	690	710	734	759	786	812
$R_u$	(s <sup>-1</sup> )	$6.5 \times 10^{-02}$	$7.0 \times 10^{-02}$	$7.5 \times 10^{-02}$	$8.0 \times 10^{-02}$	$8.5 \times 10^{-02}$	$9.0 \times 10^{-02}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$6.5 \times 10^{-02}$	$7.0 \times 10^{-02}$	$7.5 \times 10^{-02}$	$8.0 \times 10^{-02}$	$8.5 \times 10^{-02}$	$9.0 \times 10^{-02}$
$\eta_u$	(s)	$2.0 \times 10^{-09}$	$2.0 \times 10^{-09}$	$2.0 \times 10^{-09}$	$2.1 \times 10^{-09}$	$2.1 \times 10^{-09}$	$2.1 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$2.0 \times 10^{-09}$	$2.0 \times 10^{-09}$	$2.0 \times 10^{-09}$	$2.1 \times 10^{-09}$	$2.1 \times 10^{-09}$	$2.1 \times 10^{-09}$
$p$	(-)	2.276	2.276	2.277	2.278	2.280	2.282
$x^{\mathcal{H}}$	(m)	0.290	0.275	0.260	0.246	0.233	0.221
$M^{\mathcal{H}}$	(g)	10.20	10.14	10.08	10.02	9.95	9.89
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$3.105 \times 10^{08}$	$3.509 \times 10^{08}$	$3.965 \times 10^{08}$	$4.481 \times 10^{08}$	$5.063 \times 10^{08}$	$5.722 \times 10^{08}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$3.159 \times 10^{05}$	$3.228 \times 10^{05}$	$3.299 \times 10^{05}$	$3.371 \times 10^{05}$	$3.445 \times 10^{05}$	$3.521 \times 10^{05}$
$x_0$	(m)	0.59	0.60	0.61	0.61	0.62	0.23
$y_0$	(m)	1.26	1.22	1.18	1.14	1.11	1.58

	Note	Fd2	G2	Ab2	A2	Ad2	B2
$f_0$	(Hz)	92.71	98.22	104.06	110.25	116.81	123.75
L	(m)	1.744	1.652	1.564	1.482	1.403	1.329
d	(mm)	1.139	1.151	1.160	1.167	1.171	1.173
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	837	859	879	896	909	919
$R_u$	(s <sup>-1</sup> )	$9.5 \times 10^{-02}$	$1.0 \times 10^{-01}$	$1.0 \times 10^{-01}$	$1.1 \times 10^{-01}$	$1.2 \times 10^{-01}$	$1.2 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$9.5 \times 10^{-02}$	$1.0 \times 10^{-01}$	$1.0 \times 10^{-01}$	$1.1 \times 10^{-01}$	$1.2 \times 10^{-01}$	$1.2 \times 10^{-01}$
$\eta_u$	(s)	$2.1 \times 10^{-09}$	$2.2 \times 10^{-09}$	$2.2 \times 10^{-09}$	$2.2 \times 10^{-09}$	$2.3 \times 10^{-09}$	$2.3 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$2.1 \times 10^{-09}$	$2.2 \times 10^{-09}$	$2.2 \times 10^{-09}$	$2.2 \times 10^{-09}$	$2.3 \times 10^{-09}$	$2.3 \times 10^{-09}$
$p$	(-)	2.285	2.288	2.292	2.296	2.301	2.306
$x^{\mathcal{H}}$	(m)	0.209	0.198	0.188	0.178	0.168	0.160
$M^{\mathcal{H}}$	(g)	9.83	9.77	9.70	9.64	9.58	9.52
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$6.466 \times 10^{08}$	$7.307 \times 10^{08}$	$8.257 \times 10^{08}$	$9.331 \times 10^{08}$	$1.054 \times 10^{09}$	$1.192 \times 10^{09}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$3.598 \times 10^{05}$	$3.677 \times 10^{05}$	$3.758 \times 10^{05}$	$3.841 \times 10^{05}$	$3.925 \times 10^{05}$	$4.011 \times 10^{05}$
$x_0$	(m)	0.25	0.27	0.30	0.32	0.35	0.37
$y_0$	(m)	1.49	1.41	1.34	1.26	1.19	1.13

	Note	C3	Cd3	D3	Dd3	E3	F3
$f_0$	(Hz)	131.11	138.90	147.17	155.91	165.19	175.01
L	(m)	1.259	1.192	1.129	1.070	1.013	0.960
d	(mm)	1.174	1.171	1.168	1.162	1.155	1.146
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	925	929	929	926	921	914
$R_u$	(s <sup>-1</sup> )	$1.2 \times 10^{-01}$	$1.3 \times 10^{-01}$	$1.4 \times 10^{-01}$	$1.4 \times 10^{-01}$	$1.4 \times 10^{-01}$	$1.5 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$1.2 \times 10^{-01}$	$1.3 \times 10^{-01}$	$1.4 \times 10^{-01}$	$1.4 \times 10^{-01}$	$1.4 \times 10^{-01}$	$1.5 \times 10^{-01}$
$\eta_u$	(s)	$2.3 \times 10^{-09}$	$2.3 \times 10^{-09}$	$2.4 \times 10^{-09}$	$2.4 \times 10^{-09}$	$2.4 \times 10^{-09}$	$2.4 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$2.3 \times 10^{-09}$	$2.3 \times 10^{-09}$	$2.4 \times 10^{-09}$	$2.4 \times 10^{-09}$	$2.4 \times 10^{-09}$	$2.4 \times 10^{-09}$
$p$	(-)	2.312	2.318	2.325	2.332	2.339	2.347
$x^{\mathcal{H}}$	(m)	0.151	0.143	0.136	0.128	0.122	0.115
$M^{\mathcal{H}}$	(g)	9.45	9.39	9.33	9.27	9.20	9.14
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.347 \times 10^{09}$	$1.522 \times 10^{09}$	$1.720 \times 10^{09}$	$1.943 \times 10^{09}$	$2.196 \times 10^{09}$	$2.481 \times 10^{09}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$4.100 \times 10^{05}$	$4.190 \times 10^{05}$	$4.282 \times 10^{05}$	$4.376 \times 10^{05}$	$4.472 \times 10^{05}$	$4.570 \times 10^{05}$
$x_0$	(m)	0.40	0.42	0.45	0.47	0.49	0.51
$y_0$	(m)	1.06	1.01	0.95	0.90	0.85	0.80

	Note	Fd3	G3	Ab3	A3	Ad3	B3
$f_0$	(Hz)	185.41	196.45	208.12	220.49	233.62	247.52
L	(m)	0.909	0.861	0.816	0.773	0.732	0.694
d	(mm)	1.136	1.126	1.114	1.102	1.090	1.077
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	905	895	883	870	857	843
$R_u$	(s <sup>-1</sup> )	$1.5 \times 10^{-01}$	$1.6 \times 10^{-01}$	$1.7 \times 10^{-01}$	$1.7 \times 10^{-01}$	$1.7 \times 10^{-01}$	$1.8 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$1.5 \times 10^{-01}$	$1.6 \times 10^{-01}$	$1.7 \times 10^{-01}$	$1.7 \times 10^{-01}$	$1.7 \times 10^{-01}$	$1.8 \times 10^{-01}$
$\eta_u$	(s)	$2.5 \times 10^{-09}$	$2.5 \times 10^{-09}$	$2.5 \times 10^{-09}$	$2.6 \times 10^{-09}$	$2.6 \times 10^{-09}$	$2.6 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$2.5 \times 10^{-09}$	$2.5 \times 10^{-09}$	$2.5 \times 10^{-09}$	$2.6 \times 10^{-09}$	$2.6 \times 10^{-09}$	$2.6 \times 10^{-09}$
$p$	(-)	2.356	2.365	2.375	2.385	2.395	2.406
$x^{\mathcal{H}}$	(m)	0.109	0.103	0.098	0.093	0.088	0.083
$M^{\mathcal{H}}$	(g)	9.08	9.02	8.96	8.89	8.83	8.77
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$2.804 \times 10^{09}$	$3.169 \times 10^{09}$	$3.581 \times 10^{09}$	$4.047 \times 10^{09}$	$4.573 \times 10^{09}$	$5.168 \times 10^{09}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$4.670 \times 10^{05}$	$4.773 \times 10^{05}$	$4.878 \times 10^{05}$	$4.985 \times 10^{05}$	$5.095 \times 10^{05}$	$5.206 \times 10^{05}$
$x_0$	(m)	0.54	0.56	0.58	0.60	0.62	0.64
$y_0$	(m)	0.75	0.71	0.67	0.63	0.60	0.56

	Note	C4	Cd4	D4	Dd4	E4	F4
$f_0$	(Hz)	262.22	277.81	294.34	311.82	330.36	350.05
L	(m)	0.657	0.622	0.590	0.559	0.529	0.501
d	(mm)	1.064	1.051	1.039	1.026	1.014	1.002
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	829	815	801	788	775	762
$R_u$	(s <sup>-1</sup> )	$1.8 \times 10^{-01}$	$1.9 \times 10^{-01}$	$2.0 \times 10^{-01}$	$2.0 \times 10^{-01}$	$2.0 \times 10^{-01}$	$2.1 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$1.8 \times 10^{-01}$	$1.9 \times 10^{-01}$	$2.0 \times 10^{-01}$	$2.0 \times 10^{-01}$	$2.0 \times 10^{-01}$	$2.1 \times 10^{-01}$
$\eta_u$	(s)	$2.6 \times 10^{-09}$	$2.7 \times 10^{-09}$	$2.7 \times 10^{-09}$	$2.7 \times 10^{-09}$	$2.8 \times 10^{-09}$	$2.8 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$2.6 \times 10^{-09}$	$2.7 \times 10^{-09}$	$2.7 \times 10^{-09}$	$2.7 \times 10^{-09}$	$2.8 \times 10^{-09}$	$2.8 \times 10^{-09}$
$p$	(-)	2.418	2.430	2.442	2.455	2.468	2.482
$x^{\mathcal{H}}$	(m)	0.079	0.075	0.071	0.067	0.064	0.060
$M^{\mathcal{H}}$	(g)	8.71	8.64	8.58	8.52	8.46	8.39
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$5.840 \times 10^{09}$	$6.599 \times 10^{09}$	$7.457 \times 10^{09}$	$8.427 \times 10^{09}$	$9.523 \times 10^{09}$	$1.076 \times 10^{10}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$5.321 \times 10^{05}$	$5.438 \times 10^{05}$	$5.557 \times 10^{05}$	$5.679 \times 10^{05}$	$5.804 \times 10^{05}$	$5.932 \times 10^{05}$
$x_0$	(m)	0.66	0.68	0.70	0.71	0.73	0.75
$y_0$	(m)	0.53	0.50	0.47	0.44	0.41	0.39

	Note	Fd4	G4	Ab4	A4	Ad4	B4
$f_0$	(Hz)	370.80	392.87	416.26	441.01	467.19	494.96
L	(m)	0.475	0.450	0.426	0.404	0.383	0.363
d	(mm)	0.990	0.979	0.968	0.958	0.948	0.939
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	750	739	728	718	709	700
$R_u$	(s <sup>-1</sup> )	$2.1 \times 10^{-01}$	$2.2 \times 10^{-01}$	$2.3 \times 10^{-01}$	$2.3 \times 10^{-01}$	$2.3 \times 10^{-01}$	$2.4 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$2.1 \times 10^{-01}$	$2.2 \times 10^{-01}$	$2.3 \times 10^{-01}$	$2.3 \times 10^{-01}$	$2.3 \times 10^{-01}$	$2.4 \times 10^{-01}$
$\eta_u$	(s)	$2.8 \times 10^{-09}$	$2.8 \times 10^{-09}$	$2.9 \times 10^{-09}$	$2.9 \times 10^{-09}$	$2.9 \times 10^{-09}$	$2.9 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$2.8 \times 10^{-09}$	$2.8 \times 10^{-09}$	$2.9 \times 10^{-09}$	$2.9 \times 10^{-09}$	$2.9 \times 10^{-09}$	$2.9 \times 10^{-09}$
$p$	(-)	2.497	2.512	2.527	2.543	2.559	2.576
$x^{\mathcal{H}}$	(m)	0.057	0.054	0.051	0.048	0.046	0.044
$M^{\mathcal{H}}$	(g)	8.33	8.27	8.21	8.14	8.08	8.02
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.216 \times 10^{10}$	$1.374 \times 10^{10}$	$1.553 \times 10^{10}$	$1.755 \times 10^{10}$	$1.983 \times 10^{10}$	$2.241 \times 10^{10}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$6.062 \times 10^{05}$	$6.195 \times 10^{05}$	$6.331 \times 10^{05}$	$6.470 \times 10^{05}$	$6.612 \times 10^{05}$	$6.758 \times 10^{05}$
$x_0$	(m)	0.76	0.78	0.80	0.81	0.83	0.85
$y_0$	(m)	0.36	0.34	0.32	0.30	0.28	0.27

	Note	C5	Cd5	D5	Dd5	E5	F5
$f_0$	(Hz)	524.48	555.60	588.66	623.69	660.69	699.91
L	(m)	0.344	0.326	0.308	0.292	0.277	0.262
d	(mm)	0.929	0.921	0.912	0.904	0.897	0.889
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	692	684	677	670	664	658
$R_u$	(s <sup>-1</sup> )	$2.4 \times 10^{-01}$	$2.5 \times 10^{-01}$	$2.6 \times 10^{-01}$	$2.6 \times 10^{-01}$	$2.7 \times 10^{-01}$	$2.7 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$2.4 \times 10^{-01}$	$2.5 \times 10^{-01}$	$2.6 \times 10^{-01}$	$2.6 \times 10^{-01}$	$2.7 \times 10^{-01}$	$2.7 \times 10^{-01}$
$\eta_u$	(s)	$3.0 \times 10^{-09}$	$3.0 \times 10^{-09}$	$3.0 \times 10^{-09}$	$3.1 \times 10^{-09}$	$3.1 \times 10^{-09}$	$3.1 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$3.0 \times 10^{-09}$	$3.0 \times 10^{-09}$	$3.0 \times 10^{-09}$	$3.1 \times 10^{-09}$	$3.1 \times 10^{-09}$	$3.1 \times 10^{-09}$
$p$	(-)	2.593	2.611	2.629	2.648	2.668	2.687
$x^{\mathcal{H}}$	(m)	0.041	0.039	0.037	0.035	0.033	0.031
$M^{\mathcal{H}}$	(g)	7.96	7.90	7.83	7.77	7.71	7.65
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$2.532 \times 10^{10}$	$2.862 \times 10^{10}$	$3.234 \times 10^{10}$	$3.655 \times 10^{10}$	$4.130 \times 10^{10}$	$4.667 \times 10^{10}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$6.906 \times 10^{05}$	$7.058 \times 10^{05}$	$7.213 \times 10^{05}$	$7.371 \times 10^{05}$	$7.533 \times 10^{05}$	$7.699 \times 10^{05}$
$x_0$	(m)	0.86	0.88	0.90	0.91	0.93	0.94
$y_0$	(m)	0.25	0.23	0.22	0.21	0.19	0.18

	Note	Fd5	G5	Ab5	A5	Ad5	B5
$f_0$	(Hz)	741.65	785.63	832.54	882.06	934.60	990.20
L	(m)	0.249	0.236	0.223	0.211	0.200	0.190
d	(mm)	0.882	0.875	0.868	0.862	0.855	0.848
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	652	647	642	637	632	627
$R_u$	(s <sup>-1</sup> )	$2.8 \times 10^{-01}$	$2.8 \times 10^{-01}$	$2.8 \times 10^{-01}$	$2.9 \times 10^{-01}$	$2.9 \times 10^{-01}$	$3.0 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$2.8 \times 10^{-01}$	$2.8 \times 10^{-01}$	$2.8 \times 10^{-01}$	$2.9 \times 10^{-01}$	$2.9 \times 10^{-01}$	$3.0 \times 10^{-01}$
$\eta_u$	(s)	$3.1 \times 10^{-09}$	$3.2 \times 10^{-09}$	$3.2 \times 10^{-09}$	$3.2 \times 10^{-09}$	$3.3 \times 10^{-09}$	$3.3 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$3.1 \times 10^{-09}$	$3.2 \times 10^{-09}$	$3.2 \times 10^{-09}$	$3.2 \times 10^{-09}$	$3.3 \times 10^{-09}$	$3.3 \times 10^{-09}$
$p$	(-)	2.708	2.728	2.749	2.771	2.793	2.816
$x^{\mathcal{H}}$	(m)	0.030	0.028	0.027	0.025	0.024	0.023
$M^{\mathcal{H}}$	(g)	7.58	7.52	7.46	7.40	7.33	7.27
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$5.274 \times 10^{10}$	$5.960 \times 10^{10}$	$6.735 \times 10^{10}$	$7.611 \times 10^{10}$	$8.600 \times 10^{10}$	$9.719 \times 10^{10}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$7.868 \times 10^{05}$	$8.040 \times 10^{05}$	$8.217 \times 10^{05}$	$8.398 \times 10^{05}$	$8.582 \times 10^{05}$	$8.771 \times 10^{05}$
$x_0$	(m)	0.96	0.97	0.99	1.01	1.02	1.04
$y_0$	(m)	0.17	0.16	0.15	0.14	0.13	0.12

	Note	C6	Cd6	D6	Dd6	E6	F6
$f_0$	(Hz)	1048.83	1111.06	1177.66	1247.25	1321.31	1399.97
L	(m)	0.180	0.171	0.162	0.153	0.145	0.138
d	(mm)	0.842	0.835	0.828	0.822	0.815	0.808
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	623	618	613	608	603	598
$R_u$	(s <sup>-1</sup> )	$3.0 \times 10^{-01}$	$3.1 \times 10^{-01}$	$3.2 \times 10^{-01}$	$3.2 \times 10^{-01}$	$3.3 \times 10^{-01}$	$3.3 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$3.0 \times 10^{-01}$	$3.1 \times 10^{-01}$	$3.2 \times 10^{-01}$	$3.2 \times 10^{-01}$	$3.3 \times 10^{-01}$	$3.3 \times 10^{-01}$
$\eta_u$	(s)	$3.3 \times 10^{-09}$	$3.3 \times 10^{-09}$	$3.4 \times 10^{-09}$	$3.4 \times 10^{-09}$	$3.4 \times 10^{-09}$	$3.4 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$3.3 \times 10^{-09}$	$3.3 \times 10^{-09}$	$3.4 \times 10^{-09}$	$3.4 \times 10^{-09}$	$3.4 \times 10^{-09}$	$3.4 \times 10^{-09}$
$p$	(-)	2.839	2.863	2.887	2.912	2.937	2.962
$x^{\mathcal{H}}$	(m)	0.022	0.020	0.019	0.018	0.017	0.017
$M^{\mathcal{H}}$	(g)	7.21	7.15	7.09	7.02	6.96	6.90
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.098 \times 10^{11}$	$1.241 \times 10^{11}$	$1.402 \times 10^{11}$	$1.585 \times 10^{11}$	$1.791 \times 10^{11}$	$2.024 \times 10^{11}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$8.963 \times 10^{05}$	$9.160 \times 10^{05}$	$9.361 \times 10^{05}$	$9.567 \times 10^{05}$	$9.777 \times 10^{05}$	$9.992 \times 10^{05}$
$x_0$	(m)	1.05	1.07	1.09	1.10	1.12	1.13
$y_0$	(m)	0.11	0.10	0.09	0.08	0.07	0.06

	Note	Fd6	G6	Ab6	A6	Ad6	B6
$f_0$	(Hz)	1483.31	1571.35	1665.41	1764.21	1869.22	1980.57
L	(m)	0.130	0.124	0.117	0.111	0.105	0.100
d	(mm)	0.801	0.794	0.787	0.780	0.772	0.764
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	592	587	581	575	568	562
$R_u$	(s <sup>-1</sup> )	$3.4 \times 10^{-01}$	$3.4 \times 10^{-01}$	$3.4 \times 10^{-01}$	$3.5 \times 10^{-01}$	$3.5 \times 10^{-01}$	$3.6 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$3.4 \times 10^{-01}$	$3.4 \times 10^{-01}$	$3.4 \times 10^{-01}$	$3.5 \times 10^{-01}$	$3.5 \times 10^{-01}$	$3.6 \times 10^{-01}$
$\eta_u$	(s)	$3.5 \times 10^{-09}$	$3.5 \times 10^{-09}$	$3.5 \times 10^{-09}$	$3.6 \times 10^{-09}$	$3.6 \times 10^{-09}$	$3.6 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$3.5 \times 10^{-09}$	$3.5 \times 10^{-09}$	$3.5 \times 10^{-09}$	$3.6 \times 10^{-09}$	$3.6 \times 10^{-09}$	$3.6 \times 10^{-09}$
$p$	(-)	2.988	3.015	3.042	3.069	3.097	3.126
$x^{\mathcal{H}}$	(m)	0.016	0.015	0.014	0.013	0.013	0.012
$M^{\mathcal{H}}$	(g)	6.84	6.77	6.71	6.65	6.59	6.52
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$2.287 \times 10^{11}$	$2.585 \times 10^{11}$	$2.921 \times 10^{11}$	$3.301 \times 10^{11}$	$3.730 \times 10^{11}$	$4.215 \times 10^{11}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.021 \times 10^{06}$	$1.044 \times 10^{06}$	$1.067 \times 10^{06}$	$1.090 \times 10^{06}$	$1.114 \times 10^{06}$	$1.138 \times 10^{06}$
$x_0$	(m)	1.15	1.16	1.18	1.19	1.21	1.22
$y_0$	(m)	0.06	0.05	0.04	0.04	0.03	0.02

	Note	C7	Cd7	D7	Dd7	E7	F7
$f_0$	(Hz)	2098.32	2222.39	2355.34	2494.63	2643.17	2801.35
L	(m)	0.095	0.090	0.085	0.081	0.076	0.072
d	(mm)	0.757	0.749	0.741	0.732	0.724	0.716
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	555	548	541	534	526	518
$R_u$	(s <sup>-1</sup> )	$3.6 \times 10^{-01}$	$3.7 \times 10^{-01}$	$3.8 \times 10^{-01}$	$3.8 \times 10^{-01}$	$3.9 \times 10^{-01}$	$3.9 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$3.6 \times 10^{-01}$	$3.7 \times 10^{-01}$	$3.8 \times 10^{-01}$	$3.8 \times 10^{-01}$	$3.9 \times 10^{-01}$	$3.9 \times 10^{-01}$
$\eta_u$	(s)	$3.6 \times 10^{-09}$	$3.7 \times 10^{-09}$	$3.7 \times 10^{-09}$	$3.7 \times 10^{-09}$	$3.8 \times 10^{-09}$	$3.8 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$3.6 \times 10^{-09}$	$3.7 \times 10^{-09}$	$3.7 \times 10^{-09}$	$3.7 \times 10^{-09}$	$3.8 \times 10^{-09}$	$3.8 \times 10^{-09}$
$p$	(-)	3.155	3.184	3.214	3.245	3.276	3.307
$x^{\mathcal{H}}$	(m)	0.011	0.011	0.010	0.010	0.009	0.009
$M^{\mathcal{H}}$	(g)	6.46	6.40	6.34	6.27	6.21	6.15
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$4.763 \times 10^{11}$	$5.382 \times 10^{11}$	$6.082 \times 10^{11}$	$6.873 \times 10^{11}$	$7.767 \times 10^{11}$	$8.777 \times 10^{11}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.163 \times 10^{06}$	$1.189 \times 10^{06}$	$1.215 \times 10^{06}$	$1.242 \times 10^{06}$	$1.269 \times 10^{06}$	$1.297 \times 10^{06}$
$x_0$	(m)	1.24	1.25	1.27	1.28	1.29	1.31
$y_0$	(m)	0.02	0.01	0.01	0.00	-0.00	-0.01

	Note	Fd7	G7	Ab7	A7	Ad7	B7
$f_0$	(Hz)	2965.15	3142.89	3330.76	3528.57	3735.94	3959.63
L	(m)	0.069	0.065	0.062	0.058	0.055	0.052
d	(mm)	0.707	0.698	0.689	0.680	0.671	0.662
$\rho$	(kg·m <sup>-3</sup> )	7850	7850	7850	7850	7850	7850
$T_0$	(N)	510	502	493	485	476	467
$R_u$	(s <sup>-1</sup> )	$4.0 \times 10^{-01}$	$4.0 \times 10^{-01}$	$4.1 \times 10^{-01}$	$4.1 \times 10^{-01}$	$4.1 \times 10^{-01}$	$4.2 \times 10^{-01}$
$R_v$	(s <sup>-1</sup> )	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$	$5.0 \times 10^{-01}$
$R_\phi$	(s <sup>-1</sup> )	$4.0 \times 10^{-01}$	$4.0 \times 10^{-01}$	$4.1 \times 10^{-01}$	$4.1 \times 10^{-01}$	$4.1 \times 10^{-01}$	$4.2 \times 10^{-01}$
$\eta_u$	(s)	$3.8 \times 10^{-09}$	$3.8 \times 10^{-09}$	$3.9 \times 10^{-09}$	$3.9 \times 10^{-09}$	$3.9 \times 10^{-09}$	$3.9 \times 10^{-09}$
$\eta_v$	(s)	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$	$1.0 \times 10^{-09}$
$\eta_\phi$	(s)	$3.8 \times 10^{-09}$	$3.8 \times 10^{-09}$	$3.9 \times 10^{-09}$	$3.9 \times 10^{-09}$	$3.9 \times 10^{-09}$	$3.9 \times 10^{-09}$
$p$	(-)	3.339	3.371	3.404	3.438	3.471	3.506
$x^{\mathcal{H}}$	(m)	0.008	0.008	0.007	0.007	0.007	0.006
$M^{\mathcal{H}}$	(g)	6.09	6.03	5.96	5.90	5.84	5.78
$K^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$9.919 \times 10^{11}$	$1.121 \times 10^{12}$	$1.267 \times 10^{12}$	$1.431 \times 10^{12}$	$1.617 \times 10^{12}$	$1.828 \times 10^{12}$
$R^{\mathcal{H}}$	(N·m <sup>-p</sup> )	$1.325 \times 10^{06}$	$1.354 \times 10^{06}$	$1.384 \times 10^{06}$	$1.415 \times 10^{06}$	$1.446 \times 10^{06}$	$1.477 \times 10^{06}$
$x_0$	(m)	1.32	1.33	1.34	1.36	1.37	1.38
$y_0$	(m)	-0.01	-0.02	-0.02	-0.02	-0.03	-0.03



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351, Cours de la Libération  
Bâtiment A 29  
33405 Talence Cedex

Publisher  
Inria  
Domaine de Voluceau - Rocquencourt  
BP 105 - 78153 Le Chesnay Cedex  
[inria.fr](http://inria.fr)

ISSN 0249-0803



