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# MGDA II: A direct method for calculating a descent direction common to several criteria

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## MGDA II: A direct method for calculating a descent direction common to several criteria

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Project-Team Opale

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**Abstract:** This report is a sequel of the publications [1] [3] [2]. We consider the multiobjective optimization problem of the simultaneous minimization of  $n$  ( $n \geq 2$ ) criteria,  $\{J_i(Y)\}_{(i=1,\dots,n)}$ , assumed to be smooth real-valued functions of the design vector  $Y \in \Omega \subset \mathbb{R}^N$  ( $n \leq N$ ) where  $\Omega$  is the (open) admissible domain of  $\mathbb{R}^N$  over which these functions admit gradients. Given a design point  $Y^0 \in \Omega$  that is not Pareto-stationary, we introduce the gradients  $\{J'_i\}_{(i=1,\dots,n)}$  at  $Y = Y^0$ , and assume them to be linearly independent. We also consider the possible “scaling factors”,  $\{S_i\}_{(i=1,\dots,n)}$  ( $S_i > 0, \forall i$ ), as specified appropriate normalization constants for the gradients. Then we show that the Gram-Schmidt orthogonalization process, if conducted with a particular calibration of the normalization, yields a new set of orthogonal vectors  $\{u_i\}_{(i=1,\dots,n)}$  spanning the same subspace as the original gradients; additionally, the minimum-norm element of the convex hull corresponding to this new family,  $\omega$ , is calculated explicitly, and the Fréchet derivatives of the criteria in the direction of  $\omega$  are all equal and positive. This direct process simplifies the implementation of the previously-defined *Multiple-Gradient Descent Algorithm (MGDA)*.

**Key-words:** multiobjective optimization, descent direction, convex hull, Gram-Schmidt orthogonalization process

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## MGDA II: Une méthode directe de calcul de direction de descente de plusieurs critères

**Résumé :** Ce rapport est une suite des publications [1] [3] [2]. On considère le problème d'optimisation multiobjectif dans lequel on cherche à minimiser  $n$  ( $n \geq 2$ ) critères,  $\{J_i(Y)\}_{(i=1,\dots,n)}$ , supposés fonctions régulières d'un vecteur de conception  $Y \in \Omega \subset \mathbb{R}^N$  ( $n \leq N$ ) où  $\Omega$  est le domaine (ouvert) admissible, partie de  $\mathbb{R}^N$  dans laquelle les critères admettent des gradients. Étant donné un point de conception  $Y^0 \in \Omega$  qui n'est pas Pareto-stationnaire, on introduit les gradients  $\{J'_i\}_{(i=1,\dots,n)}$  en  $Y = Y^0$ , et on les suppose linéairement indépendants. On considère également un ensemble de "facteurs d'échelles",  $\{S_i\}_{(i=1,\dots,n)}$  ( $S_i > 0, \forall i$ ), spécifiés par l'utilisateur, et considérés comme des constantes appropriées de normalisation des gradients. On montre alors que le processus d'orthogonalisation de Gram-Schmidt, lorsqu'on le conduit avec une calibration bien spécifique de la normalisation, produit un ensemble de vecteurs orthogonaux  $\{u_i\}_{(i=1,\dots,n)}$  qui engendrent le même sous-espace que les gradients d'origine; de plus, l'élément de plus norme de l'enveloppe convexe de cette nouvelle famille,  $\omega$ , se calcule explicitement, et les dérivées de Fréchet des critères dans la direction de  $\omega$  sont égales et positives. Ce processus direct simplifie la mise en œuvre de l'*Algorithme de Descente à Gradients Multiples (MGDA)* défini précédemment.

**Mots-clés :** optimisation multiobjectif, direction de descente, enveloppe convexe, processus d'orthogonalisation de Gram-Schmidt

## 1 Introduction

We consider the context of the simultaneous minimization of  $n$  ( $n \geq 2$ ) criteria,  $\{J_i(Y)\}_{(i=1,\dots,n)}$ , assumed to be smooth real-valued functions of the design vector  $Y \in \Omega \subset \mathbb{R}^N$  ( $n \leq N$ ) where  $\Omega$  is the (open) admissible domain of  $\mathbb{R}^N$  over which these functions admit gradients. Let  $Y^0 \in \Omega$ , and let:

$$J'_i = \nabla J_i(Y^0) \quad (i = 1, \dots, n; J'_i \in \mathbb{R}^N) \quad (1)$$

be the gradients at the design-point  $Y^0$ .

In [1] and [2], we have introduced the local notion of Pareto-stationarity, defined as the existence of a convex combination of the gradients that is equal to zero. We established there that Pareto-stationarity was a necessary condition to Pareto-optimality. If inversely, the Pareto-stationarity condition is not satisfied, vectors having positive scalar products with all the gradients  $\{J'_i\}_{(i=1,\dots,n)}$  exist. We focus on the question of identifying such vectors.

Consider a family  $\{u_i\}_{(i=1,\dots,n)}$  of  $n$  vectors of  $\mathbb{R}^N$ , and recall the definition of their convex hull:

$$\bar{U} = \left\{ u \in \mathbb{R}^N / u = \sum_{i=1}^n \alpha_i u_i; \alpha_i \geq 0 (\forall i); \sum_{i=1}^n \alpha_i = 1 \right\} \quad (2)$$

This set is closed and convex. Hence it admits a unique element  $\omega$  of minimum norm. We established that:

$$\forall u \in \bar{U} : (u, \omega) \geq \|\omega\|^2 \quad (3)$$

By letting

$$u_i = J'_i \quad (i = 1, \dots, n) \quad (4)$$

and identifying the corresponding vector  $\omega$ , we were able to conclude that either  $\omega = 0$  and the design-point  $Y^0$  is Pareto-stationary, or  $-\omega$  is a descent direction common to all criteria.

This observation has led us to propose the *Multiple-Gradient Descent Algorithm (MGDA)* that is an iteration generalizing the classical steepest-descent method to the context of multiobjective optimization. At a given iteration, the design point  $Y^0$  is updated by a step in the direction opposite to  $\omega$ :

$$\delta Y^0 = -\rho \omega \quad (5)$$

Assuming the stepsize  $\rho$  is optimized, *MGDA* converges to a Pareto-stationary design point [1] [2].

Thus, *MGDA* provides a technique to identify Pareto sets when gradients are available, as demonstrated in [3].

We have also shown that the convex hull was isomorphic to the positive part of the sphere of  $\mathbb{R}^{n-1}$  (independently of  $N$ ) and it can be parameterized by  $n-1$  spherical coordinates. Hence, in the first version of our method, when  $n > 2$ , we proposed to identify the vector  $\omega$  by actually finding numerically the minimum of  $\|u\|^2$  in  $\bar{U}$  by optimizing the spherical coordinates, or more precisely, their cosines squared that are  $n-1$  independent parameters varying in the interval  $[0,1]$ . This minimization can be conducted trivially when  $n$  is small, but can become difficult for large  $n$ .

In this new report, we propose a variant of *MGDA* in which the direction  $\omega$  is found by a direct process, assuming the family of gradients  $\{J'_i\}_{(i=1,\dots,n)}$  is linearly independent.

## 2 Direct calculation of the minimum-norm element

In [2], the following remark was made:

**Remark 1**

If the gradients are not normalized, the direction of the minimum-norm element  $\omega$  is expected to be mostly influenced by the gradients of small norms in the family, as the case  $n = 2$  illustrated in Figure 1 suggests. In the course of the iterative optimization, these vectors are often associated with the criteria that have already achieved a fair degree of convergence. If this direction may yield a very direct path to the Pareto front, one may question whether it is adequate for a well-balanced multiobjective iteration. Some on-going research is focused on analyzing various normalization procedures to circumvent this undesirable trend. In these alternatives, the gradient  $J'_i$  is replaced by one of the following formulas:

$$\frac{J'_i}{\|J'_i\|}, \frac{J'_i}{J_i(Y^0)}, \frac{J_i(Y^0)}{\|J'_i\|^2} J'_i, \text{ or } \frac{\max(J_i^{(k-1)}(Y^0) - J_i^{(k)}(Y^0), \delta)}{\|J'_i\|^2} J'_i \quad (6)$$

( $k$ : iteration number;  $\delta > 0$ , small). The first formula is a standard normalization: it has the merit of providing a stable definition; the second realizes equal logarithmic first variations of the criteria whenever  $\omega$  belongs to the interior  $\mathbf{U}$  of the convex hull since then, the Fréchet derivatives  $(u_i, \omega)$  are equal; the last two are inspired from Newton's method (assuming  $\lim J_i = 0$  for the first). This question is still open.

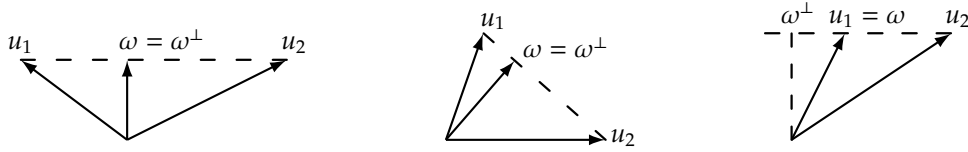


Figure 1: Case  $n = 2$ : possible positions of vector  $\omega$  with respect to the two gradients  $u_1$  and  $u_2$

Thus, accounting for the above remark, we consider a given set of strictly-positive “scaling factors”  $\{S_i\}_{(i=1,\dots,n)}$ , intended to normalize the gradients appropriately. We apply the Gram-Schmidt orthogonalization process to the gradients with the following special calibration of the normalization:

$$u_1 = \frac{J'_1}{A_1} \quad (7)$$

where  $A_1 = S_1$ , and, for  $i = 2, 3, \dots, n$ :

$$u_i = \frac{J'_i - \sum_{k < i} c_{i,k} u_k}{A_i} \quad (8)$$

where:

$$\forall k < i : c_{i,k} = \frac{(J'_i, u_k)}{(u_k, u_k)} \quad (9)$$

and

$$A_i = \begin{cases} S_i - \sum_{k < i} c_{i,k} & \text{if nonzero} \\ \varepsilon_i S_i & \text{otherwise} \end{cases} \quad (10)$$

for some arbitrary, but small  $\varepsilon_i$  ( $0 < |\varepsilon_i| \ll 1$ ).

**Then :**

In general, the vectors  $\{u_i\}_{(i=1,\dots,n)}$  are not of norm unity. But they are orthogonal, and this property

makes the calculation of the minimum-norm element of the convex hull,  $\omega$ , direct. For this, note that:

$$\omega = \sum_{i=1}^n \alpha_i u_i \quad (11)$$

and

$$\|\omega\|^2 = \sum_{i=1}^n \alpha_i^2 \|u_i\|^2. \quad (12)$$

To determine the coefficients  $\{\alpha_i\}_{(i=1,\dots,n)}$ , anticipating that  $\omega$  belongs to the interior of the convex hull, the inequality constraints are ignored, and the following Lagrangian is made stationary:

$$\mathcal{L}(\alpha, \lambda) = \|\omega\|^2 - \lambda \left( \sum_{i=1}^n \alpha_i - 1 \right) = \sum_{i=1}^n \alpha_i^2 \|u_i\|^2 - \lambda \left( \sum_{i=1}^n \alpha_i - 1 \right) \quad (13)$$

This gives:

$$\frac{\partial \mathcal{L}}{\partial \alpha_i} = 2 \|u_i\|^2 \alpha_i - \lambda \implies \alpha_i = \frac{\lambda}{2 \|u_i\|^2} \quad (14)$$

The equality constraint,  $\sum_{i=1}^n \alpha_i = 1$ , then gives:

$$\frac{\lambda}{2} = \frac{1}{\sum_{i=1}^n \frac{1}{\|u_i\|^2}} \quad (15)$$

and finally:

$$\alpha_i = \frac{1}{\|u_i\|^2 \sum_{j=1}^n \frac{1}{\|u_j\|^2}} = \frac{1}{1 + \sum_{j \neq i} \frac{\|u_i\|^2}{\|u_j\|^2}} < 1 \quad (16)$$

which confirms that  $\omega$  does belong to the interior of the convex hull, so that:

$$\forall i : \alpha_i \|u_i\|^2 = \frac{\lambda}{2} \quad (\text{a constant}), \quad (17)$$

and:

$$\forall k : (u_k, \omega) = \alpha_k \|u_k\|^2 = \frac{\lambda}{2}. \quad (18)$$

Now:

$$(J'_i, \omega) = (A_i u_i + \sum_{k < i} c_{i,k} u_k, \omega) = \left( A_i + \sum_{k < i} c_{i,k} \right) \frac{\lambda}{2} = S_i \frac{\lambda}{2}, \quad \text{or } S_i (1 + \varepsilon_i) \frac{\lambda}{2} \quad (\forall i). \quad (19)$$

Lastly, convening that  $\varepsilon_i = 0$  in the regular case ( $S_i \neq \sum_{k < i} c_{i,k}$ ), and otherwise by modifying slightly the definition of the scaling factor according to

$$S'_i = (1 + \varepsilon_i) S_i, \quad (20)$$

the following holds:

$$\boxed{(S_i^{-1} J'_i, \omega) = \frac{\lambda}{2} \quad (\forall i)} \quad (21)$$

that is, the same positive constant.



### 3 Conclusion

We have considered the multiobjective optimization problem of the simultaneous minimization of  $n$  ( $n \geq 2$ ) criteria,  $\{J_i(Y)\}_{(i=1,\dots,n)}$ , assumed to be smooth real-valued functions of the design vector  $Y \in \Omega \subset \mathbb{R}^N$  ( $n \leq N$ ) where  $\Omega$  is the (open) admissible domain of  $\mathbb{R}^N$  over which these functions admit gradients. Given a design point  $Y^0 \in \Omega$  that is not Pareto-stationary, we have introduced the gradients  $\{J'_i\}_{(i=1,\dots,n)}$  at  $Y = Y^0$ , and assumed them to be linearly independent. We have also considered the possible “scaling factors”,  $\{S_i\}_{(i=1,\dots,n)}$  ( $S_i > 0, \forall i$ ), as specified appropriate normalization constants for the gradients. Then we have shown that the Gram-Schmidt orthogonalization process, if conducted with a particular calibration of the normalization yields a new set of orthogonal vectors  $\{u_i\}_{(i=1,\dots,n)}$  spanning the same subspace as the original gradients; additionally, the minimum-norm element of the convex hull corresponding to this new family,  $\omega$ , is calculated explicitly, and the Fréchet derivatives of the criteria in the direction of  $\omega$  are all equal and positive.

This new result has led us to reformulate the definition of the *MGDA*, in which the descent direction common to all criteria is now calculated by a direct process.

Finally, we make the following two remarks:

**Remark 2**

*In the exception case where  $\sum_{k < i} c_{i,k} = S_i$ , for some  $i$ , the corresponding vector  $u_i$  has a large norm. Consequently, this vector has a weak influence on the definition of  $\omega$ .*

**Remark 3**

*As the Pareto set is approached, the family of gradients becomes closer to linear dependence, as the Pareto-stationarity condition requires. At this stage, this variant may experience numerical difficulties, making the more robust former approach preferable.*

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## Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Direct calculation of the minimum-norm element</b>	<b>3</b>
<b>3</b>	<b>Conclusion</b>	<b>6</b>



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