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Complementary notes on Smart Energy-Aware Sensors for Event-Based Control

Nicolas Cardoso de Castro, Daniel E. Quevedo, Federica Garin and Carlos Canudas de Wit

Abstract—This document completes the paper “Smart Energy-Aware Sensors for Event-Based Control” submitted to the 51st IEEE Conference on Decision and Control by the same authors. It is not intended to be self contained; it only gives the proof of Lemma 2.

I. APPENDIX

We recall from [1] the following elements.

Definition 1 The system (5) with the policy (18) is said to be *Input-to-State practically Stable (ISpS)* if there exist a \mathcal{KL} -function γ , and a constant $c \geq 0$, such that, for all $\zeta_0 \in \mathbb{R}^{n_\zeta}$ and for all $m_0 \in \mathbb{M}$:

$$\|\zeta_k(\zeta_0, m_0)\| \leq \gamma(\|\zeta_0\|, k) + c, \quad k \in \mathbb{Z}_{\geq 0}. \quad (20)$$

Definition 2 $V : \mathbb{R}^{n_\zeta} \times \mathbb{M} \rightarrow \mathbb{R}_{\geq 0}$ is called a *ISpS-Lyapunov function* for the system (5) with the policy (18) if:

- there exist a pair of \mathcal{K}_∞ -functions α_1 , α_2 , and a constant $c_1 \geq 0$ such that, for all $\zeta \in \mathbb{R}^{n_\zeta}$ and for all $m \in \mathbb{M}$:

$$\alpha_1(\|\zeta\|) \leq V(\zeta, m) \leq \alpha_2(\|\zeta\|) + c_1, \quad (21)$$

- there exist a suitable \mathcal{K}_∞ -function α_3 and a constant $c_2 \geq 0$ such that, for all $\zeta \in \mathbb{R}^{n_\zeta}$ and for all $m \in \mathbb{M}$:

$$\begin{aligned} \Delta V(\zeta, m) &\triangleq V(g_{v^*}(\zeta, u^*), v^*) - V(\zeta, m) \\ &\leq -\alpha_3(\|\zeta\|) + c_2. \end{aligned} \quad (22)$$

Lemma 2 Under assumption 2, if the closed-loop system formed by (5) and the policy (18) admits an ISpS-Lyapunov function, then it is ISpS.

Proof: This proof is based on the proofs of ISS and ISpS from [2], [3].

We assume that Eq.s (21)-(22) hold, *i.e.* that the system (5) with the policy (18) admits an ISpS-Lyapunov function, denoted $V(\zeta, m)$. Let’s prove that the closed-loop system is ISpS.

Definition 3 A set $\Omega \subset \mathbb{R}^{n_\zeta} \times \mathbb{M}$ is called an *invariant set* for system (5) if there exists a policy (μ, η) such that, for all $(\zeta, m) \in \Omega$, $g_{\eta(\zeta, m)}(\zeta, \mu(\zeta, m)) \in \Omega$.

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1) *Step 1:* First, we prove that the closed-loop system (5),(18) admits an invariant set $\Omega \subset \mathbb{R}^{n_\zeta} \times \mathbb{M}$.

We define $\bar{\alpha}_2(s) \triangleq \alpha_2(s) + s$, then, noting that $c_1 \geq 0$ and $\|\zeta\| \geq 0$, (21) implies:

$$\begin{aligned} V(\zeta, m) &\leq \alpha_2(\|\zeta\| + c_1) + \|\zeta\| + c_1 \\ &= \bar{\alpha}_2(\|\zeta\| + c_1) \\ \Rightarrow \bar{\alpha}_2^{-1}(V(\zeta, m)) &\leq \|\zeta\| + c_1 \end{aligned} \quad (26)$$

Let $\xi(s)$ be a \mathcal{K}_∞ -function.

- If $c_1 \leq \|\zeta\|$

$$\begin{aligned} c_1 \leq \|\zeta\| &\Leftrightarrow \frac{\|\zeta\| + c_1}{2} \leq \|\zeta\| \\ \Rightarrow \alpha_3\left(\frac{\|\zeta\| + c_1}{2}\right) &\leq \alpha_3(\|\zeta\|) \leq \alpha_3(\|\zeta\|) + \xi(c_1) \end{aligned}$$

- If $c_1 > \|\zeta\|$

$$\begin{aligned} \|\zeta\| < c_1 &\Leftrightarrow \frac{\|\zeta\| + c_1}{2} < c_1 \\ \Rightarrow \xi\left(\frac{\|\zeta\| + c_1}{2}\right) &\leq \xi(c_1) \leq \alpha_3(\|\zeta\|) + \xi(c_1) \end{aligned}$$

Let’s define $\underline{\alpha}_3(s) \triangleq \min\{\xi(\frac{s}{2}), \alpha_3(\frac{s}{2})\}$. We notice that $\underline{\alpha}_3 \in \mathcal{K}_\infty$, $\underline{\alpha}_3$ is then strictly increasing, which yields:

$$\begin{aligned} \underline{\alpha}_3(\|\zeta\| + c_1) &\leq \alpha_3(\|\zeta\|) + \xi(c_1) \quad (27) \\ (26), (27) \Rightarrow \underline{\alpha}_3(\bar{\alpha}_2^{-1}(V(\zeta, m))) &\leq \alpha_3(\|\zeta\|) + \xi(c_1) \end{aligned}$$

Let’s define $\alpha_4 \triangleq \underline{\alpha}_3 \circ \bar{\alpha}_2^{-1}$, then:

$$\begin{aligned} \alpha_4(V(\zeta, m)) &\leq \alpha_3(\|\zeta\|) + \xi(c_1) \\ \Rightarrow -\alpha_3(\|\zeta\|) - \xi(c_1) &\leq -\alpha_4(V(\zeta, m)) \\ (22) \Rightarrow \Delta V(\zeta, m) &\leq -\alpha_3(\|\zeta\|) + c_2 - \xi(c_1) + \xi(c_1) \\ \Delta V(\zeta, m) &\leq -\alpha_4(V(\zeta, m)) + c_2 + \xi(c_1) \end{aligned} \quad (28)$$

Let ρ be a \mathcal{K}_∞ -function such that $(id - \rho)$ is also a \mathcal{K}_∞ -function. $\rho(s) = \frac{s}{2}$ is an example. We define $\Omega \subset \mathbb{R}^{n_\zeta} \times \mathbb{M}$:

$$\Omega = \{(\zeta, m) \in \mathbb{R}^{n_\zeta} \times \mathbb{M} : V(\zeta, m) \leq \omega(c_3)\} \quad (29)$$

where $\omega \triangleq \alpha_4^{-1} \circ \rho^{-1}$ and $c_3 \triangleq c_2 + \xi(c_1)$.

We assume that $(id - \alpha_4)$ is a \mathcal{K}_∞ -function. Lemma B.1 in [2] proves that if $(id - \alpha_4)$ is not a \mathcal{K}_∞ -function, there exists a \mathcal{K}_∞ -function $\hat{\alpha}_4$ such that $\hat{\alpha}_4(s) \leq \alpha_4(s)$ and $(id - \hat{\alpha}_4)$ is a \mathcal{K}_∞ -function that can be used hereafter to lead to the same result.

Let's now assume that $(\zeta, m) \in \Omega$:

$$\begin{aligned}
(28) \Rightarrow V(g_{v^*}(\zeta, u^*), v^*) - V(\zeta, m) &\leq -\alpha_4(V(\zeta, m)) + c_3 \\
\Rightarrow V(g_{v^*}(\zeta, u^*), v^*) &\leq (id - \alpha_4)(V(\zeta, m)) + c_3 \\
&\leq (id - \alpha_4)(\omega(c_3)) + c_3 \\
&= \omega(c_3) - \alpha_4(\omega(c_3)) + c_3 \\
&= \omega(c_3) - \alpha_4(\omega(c_3)) + \rho \circ \alpha_4(\omega(c_3)) \\
&= \omega(c_3) - (id - \rho)(\alpha_4(\omega(c_3)))
\end{aligned}$$

Indeed, from the definition of ω , it holds that $\rho \circ \alpha_4(\omega(s)) = s$, and since $(id - \rho)(s) \geq 0$ (being a \mathcal{K}_∞ -function), it yields:

$$V(g_{v^*}(\zeta, u^*), v^*) \leq \omega(c_3) \quad (30)$$

Eq. (30) with the assumption that ζ is in Ω implies that Ω is an invariant set for the policy (18).

2) *Step 2*: Let's now prove that the invariant set Ω is an attractive set, *i.e.* that for any $(\zeta_k, m_k) \notin \Omega$, there exists a finite $\bar{k} \geq k$ such that, $(\zeta_{\bar{k}}, m_{\bar{k}}) \in \Omega$.

We assume that, for any time t from 0 up to time k , $(\zeta_t, m_t) \notin \Omega$. From Eq. (29), and ignoring the time index, the following holds for any time up to k :

$$\begin{aligned}
V(\zeta, m) &> \omega(c_3) = \alpha_4^{-1} \circ \rho^{-1}(c_3) \\
&\Rightarrow \rho \circ \alpha_4(V(\zeta, m)) > c_3 \\
&\Leftrightarrow \rho \circ \alpha_4(V(\zeta, m)) - c_3 > 0
\end{aligned} \quad (31)$$

Moreover:

$$\begin{aligned}
(28) \Rightarrow \Delta V(\zeta, m) &\leq -\alpha_4(V(\zeta, m)) + c_3 \\
&= -(id - \rho) \circ \alpha_4(V(\zeta, m)) - \rho \circ \alpha_4(V(\zeta, m)) + c_3 \\
(31) \Rightarrow \Delta V(\zeta, m) &\leq -(id - \rho) \circ \alpha_4(V(\zeta, m))
\end{aligned} \quad (32)$$

Let's define \bar{k} as the first time index where the system enters Ω , for the initial condition (ζ_0, m_0) :

$$\bar{k} \triangleq \min \{k \in \mathbb{Z}_{\geq 0} : (\zeta_k, m_k) \in \Omega\} \leq \infty \quad (33)$$

Remark 1 We allow \bar{k} to be infinite, which means that the trajectories never enter Ω . To prove that Ω is attractive, we need to prove that \bar{k} is finite.

Eq. (32) holds for all (ζ_0, m_0) and for all (ζ_k, m_k) with $0 \leq k < \bar{k}$. According to [4, Lemma 4.3], there exists a \mathcal{KL} -function $\hat{\gamma}(s, k)$ such that, $\forall (\zeta_0, m_0) \in \mathbb{R}^{n_c} \times \mathbb{M}$:

$$V(\zeta_k, m_k) \leq \hat{\gamma}(V(\zeta_0, m_0), k) \quad \forall k \in \{0, 1, \dots, \bar{k}\}$$

It holds then that, for all $k \in \{0, 1, \dots, \bar{k}\}$, the value function is decreasing and going to 0.

Assume that \bar{k} is not finite, since $\omega(c_3) > 0^1$, there exists a finite time \bar{t} such that $\hat{\gamma}(V(\zeta_0, m_0), \bar{t}) \leq \omega(c_3)$, which contradict the assumption that \bar{k} is not finite. This proves that Ω is an attractive set.

¹Note that $\omega(c_3) = \alpha_4^{-1} \circ \rho^{-1}(\xi(c_1) + c_2)$, and if $c_1 = 0, c_2 = 0, \xi(0) = 0, \rho^{-1}(0) = 0$ and $\alpha_4^{-1}(0) = 0$, then $\omega(c_3) = 0$, but in this situation since $c_1 = 0$ and $c_2 = 0$, we are back in the case studied in [2].

3) *Step 3*: Finally, we want to prove that Eq. (20) holds. We collect the results from the previous steps, $\forall (\zeta_0, m_0) \in \mathbb{R}^{n_c} \times \mathbb{M}, \forall k \in \mathbb{Z}_{\geq 0}$:

- if $(\zeta_k, m_k) \in \Omega, V(\zeta_k, m_k) \leq \omega(c_3)$,
- if $(\zeta_k, m_k) \notin \Omega, V(\zeta_k, m_k) \leq \hat{\gamma}(V(\zeta_0, m_0), k)$.

Eq. (21) implies that $\|\zeta_k\| \leq \alpha^{-1}(V(\zeta_k, m_k))$, we thus obtain:

- if $(\zeta_k, m_k) \in \Omega, \|\zeta_k\| \leq \alpha^{-1}(\omega(c_3))$,
- if $(\zeta_k, m_k) \notin \Omega, \|\zeta_k\| \leq \alpha^{-1}(\hat{\gamma}(V(\zeta_0, m_0), k))$.

In any case, we have:

$$\|\zeta_k\| \leq \alpha^{-1}(\hat{\gamma}(V(\zeta_0, m_0), k)) + \alpha^{-1}(\omega(c_3))$$

Eq. (21) implies that $V(\zeta_0, m_0) \leq \alpha_2(\|\zeta_0\|) + c_1$, which implies:

$$\|\zeta_k\| \leq \alpha^{-1}(\hat{\gamma}(\alpha_2(\|\zeta_0\|) + c_1, k)) + \alpha^{-1}(\omega(c_3)) \quad (34)$$

Then, we notice that, for any function $\alpha(s)$ of class \mathcal{K}_∞ , $\forall (s_1, s_2) \in \mathbb{R}_{\geq 0}$, the following holds:

$$\begin{aligned}
\alpha(s_1 + s_2) &\leq \begin{cases} \alpha(2s_1), & \text{if } s_1 \geq s_2 \\ \alpha(2s_2), & \text{if } s_1 \leq s_2 \end{cases} \\
\Rightarrow \alpha(s_1 + s_2) &\leq \alpha(2s_1) + \alpha(2s_2).
\end{aligned}$$

Since, for a given $k, \alpha^{-1}(\hat{\gamma}(s, k))$ is a function of class \mathcal{K}_∞ , we have:

$$\begin{aligned}
&\alpha^{-1}(\hat{\gamma}(\alpha_2(\|\zeta_0\|) + c_1, k)) \leq \\
&\alpha^{-1}(\hat{\gamma}(2\alpha_2(\|\zeta_0\|), k)) + \alpha^{-1}(\hat{\gamma}(2c_1, k)).
\end{aligned} \quad (35)$$

The function $\alpha^{-1}(\hat{\gamma}(2c_1, k))$ is of class \mathcal{K}_∞ , and thus going to 0 as $k \rightarrow \infty$, which means that it attains its maximum for $k = 0$:

$$\alpha^{-1}(\hat{\gamma}(2c_1, k)) \leq \alpha^{-1}(\hat{\gamma}(2c_1, 0)), \quad \forall k \in \mathbb{Z}_{\geq 0} \quad (36)$$

Eq.s (34)-(36) imply:

$$\begin{aligned}
\|\zeta(\zeta_0, m_0, k)\| &\leq \gamma(\|\zeta_0\|, k) + c \\
&\text{with } \gamma(s, k) = \alpha^{-1}(\hat{\gamma}(2\alpha_2(s), k)) \\
&c = \alpha^{-1}(\omega(c_3)) + \alpha^{-1}(\hat{\gamma}(2c_1, 0)).
\end{aligned}$$

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