



A reconstruction of Felkel's tables of primes and factors (1776)

Denis Roegel

► To cite this version:

Denis Roegel. A reconstruction of Felkel's tables of primes and factors (1776). [Research Report] 2011. hal-00654425

HAL Id: hal-00654425

<https://inria.hal.science/hal-00654425>

Submitted on 21 Dec 2011

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A reconstruction of
Felkel's tables of primes and factors
(1776)

Introduction

Denis Roegel

28 October 2011

(...) *certainly the most remarkable-looking table we have seen.*
J. W. L. Glaisher [37, p. 36]

[Felkel's] table, both on account of its rarity and its confused arrangement,
is practically useless.
James Glaisher [36, p. 25]

1 Lambert's tables project

In 1770, Johann Heinrich Lambert (1728–1777) published a collection of tables containing in particular a table giving the smallest factor of all integers up to 102000 and not divisible by 2, 3, or 5 [53].

In that table, Lambert wrote that “it would in fact be desirable if we could obtain the factors of a number from 1 to 1000 000 and even beyond, merely by opening a table.”¹ This note had some tangible effects, first in that a number of persons sent to Lambert various tables they had made before, and second in that others started to extend Lambert's table, often seeking some fame. Felkel was one of those.

2 Anton Felkel (1740–ca. 1800?)

Anton Felkel² was born in 1740 in Kamenz (Kamieniec Żąbkowski in Poland). At the age of seven, he went to the piarist school in Weißwasser (Bílá Voda in Czech Republic) and this is where he learned German and Latin. He decided to become a priest and in 1765 he therefore went to the monastery of Sagan (Żagań in Poland). But soon afterwards, he gave up this idea and decided to travel. He first went to Rome, and then to Vienna. He studied law, then became teacher in 1767 at the *St. Stephan Bürgerschule* in Vienna. He then returned to Sagan in order to learn about Felbiger's³ school methods. When he came back to Vienna, he applied these methods in the orphan school headed by Ignaz Parhamer (1715–1786) [24]. In 1771, he became teacher in the newly opened *Normalschule* in Vienna, where he was teaching at least until 1776 [69, pp. 21–34].

According to Luca's contemporary biographical notice, Felkel's first works were tables, printed after 1771. It was at the beginning of the 1770s that Felkel studied mathematics with Lukas Ebe, a teacher in Linz, and Wilhelm Bauer, who was professor in Vienna.⁴ This led him, in 1775–1777 to work on the tables described here.

Felkel also wrote a book on the theory of parallels in 1781 [31], perhaps inspired by Lambert.

¹*Es wäre in der That erwünscht, wenn wir von 1 bis auf 1000000 und noch weiter die Theiler der Zahlen durch blosses Aufschlagen einer Tafel haben könnten* [53, p. 9].

²The main sources on Felkel are the biographical notice by Luca [24], Felkel's letters in Lambert's correspondence [14, 15], the introduction to the translation of Lambert's tables [55], and the school records.

³Johann Ignaz von Felbiger (1724–1788) was abbot of the Augustine monastery in Sagan.

⁴On Ebe (1753–181?), see [71], and on Bauer (1742–1825) see [69, pp. 55–60]. Both Ebe and Bauer published textbooks of mathematics.

T a f e l
aller
S i n f a c h e n F a c t o r e n
der durch 2, 3, 5 nicht theilbaren Zahlen
von 1 bis 10 000 000.

Entworfen
von
A n t o n F e l k e l,
Lehrer an der k. k. Normalschule.

I. T h e i l.
Enthaltend die Factoren von 1 bis 144000.



W i e n,
mit von Ghelenschen Schriften gedruckt 1776.

Figure 1: The engraving on the first page of Felkel's table to 144000. This presumably depicts Felkel [15, pp. 222–223], [9, p. 337]. (source: Göttinger Digitalisierungszentrum)

In 1784, he resumed his work on factor tables after having laid it aside for eight years. In 1787, he was director of the *gräfl. Thun'schen Armenschulen* in Bohemia, a charity school [15, pp. v–vi].

In 1791, he went to Lisbon [55, preface, p. VII], [40, pp. 306–307]. There he became the head of the Casa Pia charity school.⁵ During that period, he continued to work on tables and was in touch with the Portuguese mathematician Garção Stockler (1759–1829). Stockler wrote him a letter about the computation of factors and it sheds some light on Felkel's research at that time [35, 6].⁶

According to Cantor, Felkel invented several machines, in particular a reading and speaking machine [19]. It is presumed that Felkel died in Lisbon around 1800.

3 Felkel's first tables

Felkel's projects evolved quickly between 1774 and 1777. Felkel writes that he started to read Euclid in 1774 [15, p. 489]. In October 1775, he started to read Euler's algebra [15, p. 491]. Felkel started his work on table of factors in November 1775 [15, p. 156] and eventually, especially following the race with Hindenburg, his aim was to give a table of factors to 10 millions. Felkel recounted this period as that of the “factor disease”⁷ [15, p. 489].

Felkel set out to give all simple factors of all integers not divisible by 2, 3, or 5. In order to achieve this goal, he had to save as much space as possible, and he naturally decided to use symbols for various primes. Felkel also decided to publish his table by subscription.

Felkel wrote that the first table was computed to 2 millions from scratch in 18 months,⁸ but war troubles, deaths and other problems prevented the printing of going beyond 408000, and that at the same time it had to become destroyed [15, pp. 123–124], [38, p. 118].⁹

Felkel's table was published in three parts.¹⁰ The first part to 144000 was first pub-

⁵The Casa Pia school may have archives on Felkel, and this should be investigated.

⁶The letter is undated, but was probably written around 1795. In 1799, Felkel also wrote to Joseph Banks, the president of the Royal Society in London, about his studies of factors [23, p. 325], and the content of the letter is possibly related.

⁷*Faktorenkrankheit*

⁸In 1798, Felkel wrote of 16 months [55, preface, p. VII].

⁹Palamà writes that the manuscript to 2 millions passed into the possession of the Imperial Treasury in Vienna [59]. But Zach, who is cited by Palamà, writes that Felkel obtained the manuscript back from the war office [73]. Gauss also mentions Felkel's work in the preface to Dase's seventh million and writes that Felkel had his table ready in manuscript up to 2 millions [22].

¹⁰Archibald knew of complete copies of Felkel's table in the Graves library in London and in the library of the University of Tübingen. Moreover, the first two parts are also at Göttingen and the first part is at the Royal Society in London [9]. Archibald also mentioned a photographic copy of the Tübingen copy at Brown University. Digitized versions of the tables to 336000 are available on the web. The only actual copy that we saw is located in the library of the observatory of the University of Vienna, and it does extend to 408000. The volumes have the approximate dimensions 28.7 cm × 42 cm. In this copy the second and third parts have been bound together, with no intermediate title for the third part. The second volume bears the same title as the second copy in Goettingen, namely stating only that the range

lished in German in 1776, then in Latin in 1777. The two other parts (from 144001 to 336000 and from 336001 to 408000) were published in Latin, probably in 1777. Felkel stopped working on the tables around April 1777, and he testified himself that he worked for 16 or 18 months on that topic. Since very few copies of Felkel’s table were sold, the whole printing, with only a few exceptions, had to be destroyed, although the exact circumstances are somewhat obscure. In 1784, Felkel explained that because of war troubles, deaths, and other fates, the copies had to suffer destruction [15, pp. 121–124]. It was Zach who wrote in 1800 that the printing was converted to “Patronen-Papier” for the last war with the Turks [73]. But there should be doubts about this assertion, as the Austro-Turkish war took place from 1787 to 1792. It is therefore not totally clear which war troubles Felkel had in mind in 1784, but they were not the war with the Turks.

Felkel’s layout appears close to the one proposed by Euler in 1775 [25], but he was not influenced by it, as Euler’s article was probably not yet published when Felkel started his work [15, pp. 488–489]. Moreover, Felkel himself wrote that by 1785 he had still not yet seen Euler’s article [15, p. 501].

Felkel then left the tables of factor aside for eight years. He only resumed working on factor tables in 1784, chiefly as a consequence of the edition of Lambert’s correspondence (see § 6.1).

3.1 The coding of the primes

Felkel only published his table to 408000. In our reconstruction, we have extrapolated Felkel’s scheme and named all primes up to 10 millions (and a little more) and used these names in the decompositions. Felkel’s table is difficult to use, as it requires to look up primes either in the main table itself, or in an auxiliary table which is quite long. Felkel’s auxiliary table of primes was only one page long for the interval 1–144000, but Felkel made use of primes which are not in this auxiliary table. The complete auxiliary table for the primes to 10 millions is 289 pages long.

In Felkel’s published tables to 144000, the only primes which appear with a multiplicity greater than 1 are those of the first column of the auxiliary table (figure 2), namely primes up to 523. But in the table to 408000, there are cases where a greater prime is squared. These primes have names made of more than one symbol. For instance, $292681 = 541^2$ and 541 is encoded with αa . The question arose whether Felkel wrote $(\alpha a)^2$ or αa^2 . The inspection of his table reveals that he merely adds an exponent, but does not use parentheses. This may seem strange, but it is in fact not ambiguous, because Felkel on the one hand separates consecutive factors with a small space, and on the other hand that $\alpha \cdot a^2$ is not a valid decomposition, because α can not stand for an isolated prime.

The primes until 523 are indicated by a unique symbol. Starting with 541, prefixes are used. The first series of prefixes runs the lowercase Greek alphabet from α to ω , and so we have $\alpha a = 541$, $\alpha b = 547$, etc. The next series runs the uppercase Greek alphabet ($Aa = 22307$, $\dots$, $Ba = 23321$, etc.). Then follows the lowercase Greek alphabet prefixed by a parenthesis (“ $(\alpha a) = 47527$, etc.), the uppercase Greek alphabet prefixed

is 144001–336000. The copy at the Graves library in London is probably similar, except that the first part is in Latin.

by a parenthesis, the Greek alphabets postfixed by a parenthesis, the Greek alphabets prefixed by a bracket, the Greek alphabets postfixed by a bracket, the Greek alphabets between a pair of parentheses, the Greek alphabets between a pair of brackets, and the Greek alphabets prefixed by a half-bracket $\lceil$. We assume that Felkel would have continued with the Greek alphabets postfixed by a half-bracket $\rceil$, but what would have come next is not known. If a group of prefixes $\dots\alpha\dots\Omega$ is called an alphabet, he would have needed $\lceil \frac{23 \times 289 - 49}{48} \rceil = 138$ alphabets, which are difficult to envision with only one common symbol.¹¹ Felkel may have planned to use the brackets and parentheses $(,), [,], \{, \}, \lceil, \rceil, \lfloor, \rfloor, \llbracket, \rrbracket$ and perhaps other special characters. With a pair of 12 such symbols, 144 combinations are possible, which would be sufficient to name the 138 alphabets, but these symbols would have had a drawback, in that primes often need to be located, and locating them requires locating the alphabet in a sequence. This is much easier if the prefixes follow a regular pattern. Therefore, although we could have used such combinations of patterns in our reconstruction, we have actually started to use a simple 3-symbol prefix from 523049 on. This prefix is made of a digit followed by a pair of Greek letters. The first two symbols define the alphabet and we can have $10 \times 48 = 480$ alphabets, of which we only use 130, not counting the initial alphabets used by Felkel. Our extended prefixes are $0\alpha\alpha, 0\alpha\beta, \dots, 0\alpha\omega, 0\alpha A, \dots, 0\alpha\Omega, 0\beta\alpha, 0\beta\beta, \dots, 0\beta\omega, 0\beta A, \dots, 0\beta\Omega, \dots, 0\Omega\Omega, 1\alpha\alpha, \dots$. Our last but one prefix is $2K\chi$ starting with 10 000 439, and our last prefix is $2K\psi$.

The first prime with a 3-symbols name is $(\alpha\alpha = 47527$ and it never occurs with a multiplicity greater than 1. Only primes smaller or equal to $3137 = \delta\omega$ can occur with a multiplicity greater than 1 between 1 and 10 002 000.

We should observe that Felkel's interest was also in counting the primes, and Felkel's naming scheme makes it relatively easy. In order to find the position of a prime, one merely needs to know the position of the alphabet in the sequence of alphabets and the position of the prime within the alphabet.

In the fifth volume of Lambert's correspondence, Bernoulli appears critical of the use of letters by Felkel, and concludes that the demand for a different arrangement is warranted [15, pp. 230–231].

Several other tables made use of symbols for numbers. This was in particular the case of Jakob Philipp Kulik, who may have been inspired by Felkel, at least indirectly [50]. However, for each composite number, Kulik only gives the smallest factor, whereas Felkel gives the whole decomposition. Moreover, Felkel gives names to all the primes, and Kulik only named those that he needed, and the greatest prime that could occur in his tables was 10009.

3.2 The table of primes

In his table giving the factors for numbers up to 144000, Felkel gave on one page the prime numbers from 1 to 20353 (figure 2). This table names all the primes which are

¹¹Nevertheless, an alternative would be the use of the number notation developed in the late 13th century by Cistercian monks and described by David King [46]. With this notation, we could express a number up to 9999 by a unique symbol.

factors in the table, but it does not show Felkel’s scheme beyond 20353. Of course, this table is not even enough for naming all factors up to 10 millions and beyond.

In our reconstruction, we have chosen to extend Felkel’s table of primes so as to cover all prime numbers up to 10 001 921. Using 100 lines and 23 columns, this list covers exactly 289 pages. There is only one prime number which appears in the complete table to 10 002 000 and not in the first table, namely 10 001 963, which is named $2K\psi a$ in our naming scheme (figure 3). It didn’t seem necessary to add a whole page of primes for that sole case.

3.3 The basic layout of a factor page

When Felkel published the first part of his table, he also gave a one-page table showing the layout of his main table and useful to locate a given number (figure 4).

An examination of Felkel’s table gives a feeling of irregularity in that the content of a cell can have three different forms:

- if a number is composed, all the symbols for all factors are given in sequence, from the smallest to the largest, and separated by a small space; these entries are justified to the right;
- the second form is made of a number followed by a dash and a letter, roman or gothic, lowercase or uppercase; this refers to a prime and the complete symbol for the prime is made of the current prefix (in the margin) and the suffix; these entries are justified to the left;
- the third form is a special case of the second form, for those primes using a prefix for the first time; the number for the prime is then followed by a dot, followed by the prefix, followed by the letter; these entries are also justified to the left.

In 1873, Glaisher gave a slightly incorrect description of Felkel’s table. He wrote that Felkel only gave the last three digits of a composed number when there was enough space for it [37, p. 36]. In fact, Felkel only gives the last three digits on the first page (for all numbers) and for the primes on the subsequent pages. This is of course redundant, as the “coordinates” would be sufficient to identify the numbers.

3.4 The 1776 factor table and its completion

We have reconstructed Felkel’s projected table of factors up to 10 002 000, but it has only been made with certainty from 1 to 408000. Afterwards, it is not totally clear what Felkel’s conventions would have been for the prime alphabets. Towards the end of his table, Felkel used “[” as a prefix in an alphabet, and he would then presumably have used “]” as a postfix, but this is only a guess.

There are also anomalies in the page numbering. In the first part of the table (1–144000), Felkel started with 1 and numbered each page separately. In the second part, he

A. Tafel der Primzahlen von 1 bis 20353.

	α	β	γ	δ	ϵ	ζ	η	θ	ι	κ	λ	μ	ν	ξ	$\omicron$	π	ρ	σ	τ	υ	ϕ	χ		
a	1	541	1223	1987	2741	3571	4409	5279	6133	6997	7919	8831	9733	10657	11657	12553	13499	14519	15401	16381	17389	18313	19423	
b	2	47	29	93	49	81	21	81	43	7001	27	37	39	63	77	69	13513	33	13	16411	93	29	27	
c	3	57	31	97	53	83	23	97	51	13	33	39	43	67	81	77	23	37	47	17401	41	49	29	
d	5	63	37	99	67	93	41	5303	63	19	37	49	49	87	89	83	37	43	39	21	17	53	33	
e	7	69	49	2003	77	3607	47	9	73	27	49	61	67	91	99	89	53	49	43	27	19	67	41	
f	11	71	59	14	89	13	51	23	17	39	51	63	69	10709	11701	12601	1367	51	33	31	71	47		
g	13	77	77	17	91	17	57	33	99	43	63	67	81	11	17	11	77	57	61	47	43	79	57	
h	17	87	79	27	97	23	63	47	6203	57	93	87	87	23	19	13	91	61	67	51	49	97	63	
i	19	93	83	29	2801	31	81	51	11	69	8009	93	91	29	31	19	97	63	73	53	67	18401	69	
k	23	99	89	39	3	37	83	81	17	79	11	8923	9803	33	43	37	13013	91	93	77	71	13	71	
l	29	601	91	53	19	43	93	87	21	7103	17	29	11	39	77	41	19	93	97	81	77	27	77	
m	31	7	97	63	33	59	4507	93	29	9	39	33	17	53	79	47	27	14621	15511	87	83	33	53	
n	37	13	1301	69	37	71	13	99	47	21	43	41	29	71	83	53	33	27	27	93	89	39	89	
p	41	17	3	81	43	73	17	5407	57	27	59	51	33	81	89	59	49	29	41	16519	91	43	19501	
q	43	19	7	83	51	77	19	13	63	29	69	63	39	89	11801	71	69	33	51	29	97	51	7	
r	47	31	19	87	57	91	23	17	69	51	81	69	51	99	7	89	79	39	59	47	17509	57	31	
s	53	41	21	89	61	97	47	19	71	59	87	71	57	10831	13	97	81	53	69	53	19	61	41	
t	59	43	27	99	79	3701	49	31	77	77	89	99	59	37	21	12703	87	57	81	61	39	81	43	
u	61	47	61	2111	87	9	61	37	87	87	93	9001	71	47	27	13	91	69	83	67	51	93	53	
v	67	53	67	13	97	19	47	44	99	93	8101	7	83	53	31	21	93	83	15601	73	69	18503	59	
w	71	59	73	29	2903	27	83	43	6301	7207	11	11	87	59	33	39	57	99	97	16603	31	17	71	
x	73	61	9	31	9	33	91	49	11	11	13	9901	61	39	43	13709	14713	19	7	79	21	77		
y	79	73	99	37	17	39	97	71	17	13	23	29	7	67	63	57	11	17	29	19	81	23	83	
z	83	77	1409	41	27	61	4603	77	23	19	47	41	23	83	67	63	21	23	41	31	97	39	97	
a	89	83	23	43	39	67	21	70	29	29	61	43	29	89	87	81	23	31	43	33	99	41	19603	
b	97	a	27	a	53	a	69	a	83	a	37	a	67	a	91	a	97	a	91	a	29	a	37	a
c	101	701	29	61	57	79	39	5501	43	43	71	59	41	10903	11903	99	51	41	49	51	23	83	61	
d	3	9	29	63	79	93	43	3	43	47	79	67	49	9	12809	57	47	61	57	27	87	81	81	
e	7	19	39	2203	69	97	49	7	59	53	91	91	67	37	23	21	59	53	67	61	57	73	87	
f	9	27	47	7	71	3803	51	19	61	83	8209	9103	73	29	27	23	63	59	71	73	59	18617	97	
g	13	31	53	21	3001	23	63	27	73	7307	21	27	9	57	39	41	89	71	83	93	81	61	19709	
h	17	43	59	37	11	33	73	31	79	9	31	33	37	73	41	53	99	79	15727	99	83	71	17	
i	19	51	71	39	19	47	79	57	89	21	33	37	39	79	53	89	13807	83	31	16703	17707	79	27	
k	23	57	81	43	23	51	91	63	97	31	37	51	61	87	59	93	29	97	33	29	13	91	39	
l	29	61	83	51	37	53	4703	69	6421	33	43	57	67	109	1	69	1	99	31	14813	41	29	18701	
m	31	69	87	67	41	63	21	73	27	49	63	61	67	11003	71	12907	41	21	37	47	37	13	53	
n	37	73	89	69	49	77	23	81	49	51	69	71	79	27	81	11	59	47	81	47	19	59	59	
o	43	87	93	73	61	81	29	91	51	69	73	81	91	47	87	17	73	31	61	63	49	31	63	
p	47	97	99	81	67	89	33	5623	69	93	87	87	93	57	12007	19	77	43	67	87	61	43	77	
q	73	909	1511	87	79	3907	51	4	39	73	7411	91	99	99	99	59	11	23	79	51	73	16811	83	49
r	79	11	23	93	83	11	59	41	81	17	93	9203	10103	69	37	41	83	67	87	27	57	19801	53	
s	81	21	31	97	89	17	83	47	91	33	97	9	11	71	41	53	13901	69	91	29	91	73	13	
t	91	23	43	2309	3109	19	87	51	6521	51	8311	21	33	83	43	59	3	79	97	31	17807	87	19	
u	93	27	49	11	19	23	89	53	29	57	17	27	39	87	49	67	87	15803	43	27	93	41		
v	97	b	53	b	33	b	21	b	93	b	57	b	47	b	93	b	73	b	13	b	91	b	43	
w	99	39	59	37	31	39	31	99	51	77	53	41	51	11113	73	79	21	97	17	79	39	18803	53	
x	101	53	67	41	63	43	4801	69	53	81	63	57	59	17	97	83	31	14923	23	83	51	39	61	
y	103	57	71	47	67	47	13	83	63	87	69	77	63	19	12101	13001	33	29	59	89	63	59	67	
z	107	59	79	51	69	67	17	89	69	89	77	81	69	31	7	3	63	39	77	16901	81	69	89	
a	109	a	53	a	57	a	81	a	89	a	31	a	93	a	71	a	99	a	7	a	67	a	91	
b	113	77	97	71	87	4001	61	5701	77	7507	89	93	81	59	13	9	97	51	87	21	17903	18911	19913	
c	117	81	1601	77	91	3	71	11	81	17	8419	9311	93	61	19	33	99	57	89	27	9	13	19	
d	121	83	7	81	3203	7	77	17	99	23	23	19	10211	71	43	37	14009	69	15901	31	11	17	27	
e	125	87	9	83	9	13	89	37	6607	29	23	23	73	49	43	11	83	7	37	21	19	37		
f	129	F	907	F	13	F	89	F	19	4903	F	41	F	37	F	43	F	77	F	49	F	29	F	
g	133	G	11	19	93	21	21	9	43	37	41	43	41	47	97	61	63	33	17	19	63	29	61	
h	137	H	19	21	99	29	27	19	49	53	47	47	43	53	11213	63	93	51	31	23	79	73	63	
i	141	I	71	29	27	2411	51	49	31	79	59	49	61	49	59	39	97	99	57	53	37	81	57	
k	145	K	77	37	17	53	41	33	83	61	59	67	71	67	43	12203	13103	71	61	59	87	59	19001	
l	149	L	81	41	L	57	L	37	L	91	L	73	L	71	L	11	L	9	L	81	L	73	L	
m	153	M	83	47	63	37	59	73	43	5801	79	73	13	77	73	57	27	21	83	77	73	17011	77	
n	157	N	87	53	67	41	71	79	51	7	89	77	81	97	89	61	41	39	27	87	91	21	81	
o	161	O	93	67	69	47	99	81	13	91	83	27	9403	10301	73	41	47	14107	91	16001	27	87	37	
p	165	P	11	71	69	59	3301	93	67	21	6701	89	37	13	3	79	51	51	43	15101	7	29	89	
q	169	Q	13	77	Q	67	Q	7	Q	99	Q	69	Q	27	Q	3	Q	19	Q	13	Q	18013	Q	
r	173	R	17	83	97	73	13	1111	73	39	9	7603	43	21	21	99	63	63	53	21	57	41	73	
s	177	S	31	91	1709	77	19	27	87	43	19	7	63	31	31	11311	69	71	59	31	61	47	79	
t	181	T	37	97	21	2503	23	29	93	49	33	21	73	33	33	17	77	77	73	37	63	53	47	
u	185	U	47	109	23	21	29	33	99	51	37	39	81	37	37	21	81	83	77	67	77	49	87	
v	189	V	49	113	33	31	31	39	5003	57	61													

89	$g\ 0\delta\Pi\Pi\mathfrak{R}$	$q\ [\text{Xe}]$	$f\ 0\zeta\Psi P$	$\delta\mathfrak{i}\ \delta\tilde{F}$	$i\ w\ \iota w$	$659\text{-}\mathfrak{E}$	$e\ o\ Nw$	$F\ \Pi\mathfrak{Z}$
90	$p\ \xi \mathfrak{u}$	$e\ 0\xi\Lambda q$	$\alpha O\ \mu V$	$r\ [\text{BH}]$	$687\text{-}\mathfrak{F}$	$H\ \text{Op}$	$g\ u\ \nu\mathfrak{R}$	$v\ \text{K})\mathfrak{D}$
91	$701\text{-}\mathfrak{G}$	$707\text{-}\mathfrak{D}$	$k\ [\text{EB}]$	$713\text{-}\mathfrak{Z}$	$f\ 0\zeta\Psi Q$	$e\ g\ \mathfrak{h}\ \alpha\mathfrak{w}$	$l\ \eta V$	$h\ n\ \sigma D$
92	$z\ \mathfrak{e}\ \alpha Y$	$\mathfrak{L}\ \text{Am}$	$q\ D\ \alpha D$	$t\ \mathfrak{f}\ \beta\mathfrak{f}$	$e\ m\ \Psi T$	$y\ \rho)\mathfrak{q}$	$O\ \text{K}\mathfrak{X}$	$759\text{-}\mathfrak{R}$
93	$e\ f\ \text{A})B$	$\alpha^2\ \alpha\mathfrak{E}$	$g\ i\ \Sigma\mathfrak{v}$	$\mathfrak{p}\ \text{XM}$	$777\text{-}\mathfrak{L}$	$779\text{-}\mathfrak{M}$	$f\ 0\zeta\Psi R$	$e\ r\ \text{EI}$
94	$791\text{-}\mathfrak{R}$	$g\ h\ \mathfrak{p}\ I$	$801\text{-}\mathfrak{D}$	$e\ k^2\ n\ w$	$807\text{-}\mathfrak{P}$	$i\ m\ \tau I$	$813\text{-}\mathfrak{D}$	$819\text{-}\mathfrak{R}$
95	$821\text{-}\mathfrak{E}$	$f\ o\ u\ S$	$e^2\ h\ \xi\mathfrak{p}$	$833\text{-}\mathfrak{Z}$	$837\text{-}\mathfrak{U}$	$l\ u\ \beta\mathfrak{C}$	$p\ o\ \beta\mathfrak{a}$	$f\ g\ k\ \delta\mathfrak{m}$
96	$Y\ \text{E}\mathfrak{Z}$	$s\ [\kappa F]$	$861\text{-}\mathfrak{R}$	$\alpha\mathfrak{c}\ \pi O$	$\mathfrak{R}\ \text{A}\mathfrak{F}$	$\alpha G\ \nu\mathfrak{i}$	$e\ 0\xi\Lambda r$	$\mathfrak{f}\ (\vartheta q$
97	$881\text{-}\mathfrak{R}$	$e\ \mathfrak{v}\ \iota\mathfrak{b}$	$891\text{-}\mathfrak{X}$	$f\ \alpha h\ \beta\mathfrak{u}$	$l\ \eta W$	$h\ 0\beta\beta\mathfrak{m}$	$903\text{-}\mathfrak{D}$	$o\ z\ \delta a$
98	$C\ \text{T}\mathfrak{Z}$	$\beta\mathfrak{P}\ \eta f$	$921\text{-}\mathfrak{Z}$	$i\ \mathfrak{l}\ \delta\mathfrak{u}$	$g^2\ (\lambda\mathfrak{R})$	$e^2\ p\ q\ \mathfrak{v}$	$h\ m\ \varphi I$	$\mathfrak{w}\ (\gamma I$
99	$k\ [\text{EC}]$	$\delta\mathfrak{i}\ \delta R$	$n\ Y W$	$g\ x\ \mu\mathfrak{b}$	$e\ 0\xi\Lambda s$	$f\ \mathfrak{i}\ \vartheta G$	$963.2K\psi a$	$\mathfrak{f}\ (\text{N}\mathfrak{D})$
100	$e\ 0\xi\Lambda t$	$\beta q\ \iota O$	$f^2\ \mathfrak{h}\ \alpha q$	$v\ \mathfrak{r}\ \alpha\mathfrak{o}$	$k\ t\ \iota p$	$\beta t\ \iota\mathfrak{m}$	$\alpha C\ \nu X$	$e\ i\ n\ \mathfrak{E}$

2K ψ

Figure 3: The end of the reconstructed table. This excerpt illustrates the three forms of an entry. Most of the entries are of the first form. For instance, the entry at the upper left gives a decomposition in two factors g and $0\delta\Pi\Pi\mathfrak{R}$. Two lines below, $701\text{-}\mathfrak{G}$ represents a prime. The only occurrence of the third form is on the last but one line, with $963.2K\psi a$. Note that we have compressed the entry to make it fit the cell.

started again at 1, but numbered each column. We have reproduced this style faithfully, and extended it until the end of the table.¹²

Felkel has also changed the header of the pages. In the first part, the headers were in German (Factoren von ... bis ...). After 144000, the headers are in Latin (Factores ab ... usque ...). This too has been reproduced.

Felkel's table is certainly one of the best illustrations of the usefulness of a reconstruction. Checking Felkel's table without reconstructing it is extremely difficult, and spotting errors with our reconstruction has been much easier, although we have not gone into all details.

¹²Bernoulli remarked that if Felkel's first table is the first part of a table to 10 millions, then there would have been 70 such parts ($10\,000\,000/144000 = 69.44$) [15, pp. 222–223].

B. Tafel der auf jeder Seite wiederkehrenden Endesfiguren.

+	a	b	c	d	e	f	g	h		a	b	c	d	e	f	g	h	+
000	101	007	011	013	017	019	023	029	1	101	007	011	013	017	019	023	029	3000
031	037	041	043	047	049	053	059		2	031	037	041	043	047	049	053	059	
061	067	071	073	077	079	083	089		3	061	067	071	073	077	079	083	089	
091	097	101	103	107	109	113	119		4	091	097	101	103	107	109	113	119	
121	127	131	133	137	139	143	149		5	121	127	131	133	137	139	143	149	
151	157	161	163	167	169	173	179		6	151	157	161	163	167	169	173	179	
181	187	191	193	197	199	203	209		7	181	187	191	193	197	199	203	209	
211	217	221	223	227	229	233	239		8	211	217	221	223	227	229	233	239	
241	247	251	253	257	259	263	269		9	241	247	251	253	257	259	263	269	
271	277	281	283	287	289	293	299		10	271	277	281	283	287	289	293	299	
301	307	311	313	317	319	323	329		11	301	307	311	313	317	319	323	329	
331	337	341	343	347	349	353	359		12	331	337	341	343	347	349	353	359	
361	367	371	373	377	379	383	389		13	361	367	371	373	377	379	383	389	
391	397	401	403	407	409	413	419		14	391	397	401	403	407	409	413	419	
421	427	431	433	437	439	443	449		15	421	427	431	433	437	439	443	449	
451	457	461	463	467	469	473	479		16	451	457	461	463	467	469	473	479	
481	487	491	493	497	499	503	509		17	481	487	491	493	497	499	503	509	
511	517	521	523	527	529	533	539		18	511	517	521	523	527	529	533	539	
541	547	551	553	557	559	563	569		19	541	547	551	553	557	559	563	569	
571	577	581	583	587	589	593	599		20	571	577	581	583	587	589	593	599	
601	607	611	613	617	619	623	629		21	601	607	611	613	617	619	623	629	
631	637	641	643	647	649	653	659		22	631	637	641	643	647	649	653	659	
661	667	671	673	677	679	683	689		23	661	667	671	673	677	679	683	689	
691	697	701	703	707	709	713	719		24	691	697	701	703	707	709	713	719	
721	727	731	733	737	739	743	749		25	721	727	731	733	737	739	743	749	
751	757	761	763	767	769	773	779		26	751	757	761	763	767	769	773	779	
781	787	791	793	797	799	803	809		27	781	787	791	793	797	799	803	809	
811	817	821	823	827	829	833	839		28	811	817	821	823	827	829	833	839	
841	847	851	853	857	859	863	869		29	841	847	851	853	857	859	863	869	
871	877	881	883	887	889	893	899		30	871	877	881	883	887	889	893	899	
901	907	911	913	917	919	923	929		31	901	907	911	913	917	919	923	929	
931	937	941	943	947	949	953	959		32	931	937	941	943	947	949	953	959	
961	967	971	973	977	979	983	989		33	961	967	971	973	977	979	983	989	
991	997	1001	1003	1007	1009	1013	1019		34	991	997	1001	1003	1007	1009	1013	1019	
1000	027	031	033	037	039	043	049		35	027	031	033	037	039	043	049		4000
051	057	061	063	067	069	073	079		36	051	057	061	063	067	069	073	079	
081	087	091	093	097	099	103	109		37	081	087	091	093	097	099	103	109	
111	117	121	123	127	129	133	139		38	111	117	121	123	127	129	133	139	
141	147	151	153	157	159	163	169		39	141	147	151	153	157	159	163	169	
171	177	181	183	187	189	193	199		40	171	177	181	183	187	189	193	199	
201	207	211	213	217	219	223	229		41	201	207	211	213	217	219	223	229	
231	237	241	243	247	249	253	259		42	231	237	241	243	247	249	253	259	
261	267	271	273	277	279	283	289		43	261	267	271	273	277	279	283	289	
291	297	301	303	307	309	313	319		44	291	297	301	303	307	309	313	319	
321	327	331	333	337	339	343	349		45	321	327	331	333	337	339	343	349	
351	357	361	363	367	369	373	379		46	351	357	361	363	367	369	373	379	
381	387	391	393	397	399	403	409		47	381	387	391	393	397	399	403	409	
411	417	421	423	427	429	433	439		48	411	417	421	423	427	429	433	439	
441	447	451	453	457	459	463	469		49	441	447	451	453	457	459	463	469	
471	477	481	483	487	489	493	499		50	471	477	481	483	487	489	493	499	
501	507	511	513	517	519	523	529		51	501	507	511	513	517	519	523	529	
531	537	541	543	547	549	553	559		52	531	537	541	543	547	549	553	559	
561	567	571	573	577	579	583	589		53	561	567	571	573	577	579	583	589	
591	597	601	603	607	609	613	619		54	591	597	601	603	607	609	613	619	
621	627	631	633	637	639	643	649		55	621	627	631	633	637	639	643	649	
651	657	661	663	667	669	673	679		56	651	657	661	663	667	669	673	679	
681	687	691	693	697	699	703	709		57	681	687	691	693	697	699	703	709	
711	717	721	723	727	729	733	739		58	711	717	721	723	727	729	733	739	
741	747	751	753	757	759	763	769		59	741	747	751	753	757	759	763	769	
771	777	781	783	787	789	793	799		60	771	777	781	783	787	789	793	799	
801	807	811	813	817	819	823	829		61	801	807	811	813	817	819	823	829	
831	837	841	843	847	849	853	859		62	831	837	841	843	847	849	853	859	
861	867	871	873	877	879	883	889		63	861	867	871	873	877	879	883	889	
891	897	901	903	907	909	913	919		64	891	897	901	903	907	909	913	919	
921	927	931	933	937	939	943	949		65	921	927	931	933	937	939	943	949	
951	957	961	963	967	969	973	979		66	951	957	961	963	967	969	973	979	
981	987	991	993	997	999	1003	1009		67	981	987	991	993	997	999	1003	1009	
1000	017	021	023	027	029	033	039		68	017	021	023	027	029	033	039		5000
041	047	051	053	057	059	063	069		69	041	047	051	053	057	059	063	069	
071	077	081	083	087	089	093	099		70	071	077	081	083	087	089	093	099	
101	107	111	113	117	119	123	129		71	101	107	111	113	117	119	123	129	
131	137	141	143	147	149	153	159		72	131	137	141	143	147	149	153	159	
161	167	171	173	177	179	183	189		73	161	167	171	173	177	179	183	189	
191	197	201	203	207	209	213	219		74	191	197	201	203	207	209	213	219	
221	227	231	233	237	239	243	249		75	221	227	231	233	237	239	243	249	
251	257	261	263	267	269	273	279		76	251	257	261	263	267	269	273	279	
281	287	291	293	297	299	303	309		77	281	287	291	293	297	299	303	309	
311	317	321	323	327	329	333	339		78	311	317	321	323	327	329	333	339	
341	347	351	353	357	359	363	369		79	341	347	351	353	357	359	363	369	
371	377	381	383	387	389	393	399		80	371	377	381	383	387	389	393	399	
401	407	411	413	417	419	423	429		81	401	407	411	413	417	419	423	429	
431	437	441	443	447	449	453	459		82	431	437	441	443	447	449	453		

Figure 5: A page of Felkel’s table. (source: Göttinger Digitalisierungszentrum)

4 Felkel's machine

In order to compute the factors of his table, Felkel made use of a “machine” with which he could work faster. Felkel himself never described this machine in sufficient detail, although it is depicted on the cover of the first part of his table (figure 6).

An advertisement for this machine was published by Wilhelm Bauer in the *Leipziger Zeitung* on 29 July 1776, and in final notes of the fifth volume of Lambert's correspondence, Bernoulli writes that Felkel once attributed the invention of his machine to Wilhelm Bauer, but that it is not clear how much of it is true [15, p. 236]. The announcement by Bauer was mentioned by Hindenburg in the letter he sent on 3 August 1776 to Lambert.

The machine was described in more detail in the review published in the *Allgemeine deutsche Bibliothek* in 1778 [3] and which was reproduced by Bernoulli in the second appendix of his correspondence of Lambert [15, p. 232–234]. Recently, Weiss made a reconstruction of this “machine” which should help clarify its principle [74]. It appears that Felkel used rods. A set of 30 rods was needed to find the multiples of all numbers, and 8 were needed to find the multiples of the numbers not divisible by 2, 3, or 5. These numbers correspond to the possible remainders q when a number N is written in the form $30p + q$. The 8 values correspond to the 8 columns in Felkel's table.

Felkel printed 30 columns of numbers on paper, starting with 1 to 30. Of these columns, 8 were in red.

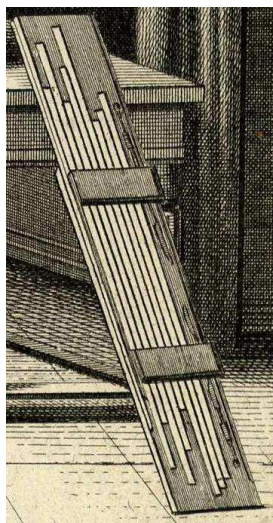


Figure 6: Felkel's machine as it appears on the frontispice of Felkel's table (excerpt of figure 1).

The numbers in each column increased by 30. The 7th rod, for instance, had the numbers 7, 37, 67, 97, until 2977. The thousands were not indicated. Each column had 100 numbers. The paper was glued on cardboard, and cut, and this gave the rods. If we want to find the multiples of 47 not divisible by 2, 3, and 5, we take the rod marked with 47 in the upper third. Then, we multiply 47 by 7, 11, 13, 17, 19, 23, and 29, giving

329, 517, ..., 1081, and 1363. The first product is 329, and we take the rod marked with 329 in the upper third. This is the rod marked with 29, 59, 89, 119, 149, 170, 209, 239, 269, 299, 329, etc. The corresponding rods are taken for the other multiples of 47. For the last two products, 1081 and 1363, we take the rods marked with 81 and 363 in the middle part.

Felkels Factoren-Tafel

Tab. II

The table contains a grid of numbers. The first row starts with 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100. The subsequent rows follow a similar pattern, with numbers increasing in a structured manner. The table is used for finding factors and products in the context of Felkel's machine.

Figure 8: The 30 rods of Felkel's machine, as illustrated by Bischoff [16] (courtesy: Stephan Weiss)

All these rods are put next to each other, and after the first rod, and we now have on a same line the numbers 47, 329, 517, etc., that is 47 , 47×7 , 47×11 , 47×13 , ..., 47×29 . The rods are put on a board and are kept in that arrangement. This is shown on the cover of Felkel's table, for instance. A ruler can be put over this line of numbers. The next multiple of 47 is $47 \cdot 31 = 1457$. The value 457 is located in the middle part of the first rod, the one with 47. Corresponding to *739, *927, **021, **209, **303, **491, and **773. These are the products of 47 by $30 + 7$, $30 + 11$, $30 + 13$, $30 + 17$, $30 + 19$, $30 + 23$, and $30 + 29$. A ruler can be put at this position.

This procedure is repeated by putting a third ruler at the same distance below the second ruler as the second is below the first (47 lines) and this will give the next multiples. If there is no longer enough space below, the ruler can be moved up by 53 lines, but in general it seems that the first product was computed by hand and the others were found with the ruler. This procedure may provide up to 8 multiples for every ruler position. The symbol for 47 then had to be copied in the corresponding cells of the manuscript.

The author of the review added that he gave a detailed description of the machine, because Felkel's own description was not very clear.¹³

5 Errors in Felkel's table

There are very few erratas for Felkel's table, no doubt because locating errors was particularly difficult. Besides the errors given by Felkel himself in a letter to Lambert dated 22 November 1776 [15, p. 110] and in the first part of his table, we know of Florian Ulbrich (1738–1800) who is said to have found several errors in Felkel's table and as a consequence constructed his own table of factors.¹⁴ We also know that Vega used Felkel's table for his table of factors and primes. In fact, Vega did not make it totally clear if he used Felkel's table for his table of factors published in 1797 [72], but he used it for his table of primes. As explained in the next section, however, the errors found in Vega's table show that he did not copy the table of factors from Felkel.

We give here all the known errors, both of content and of sequencing, the latter having been found while analyzing Felkel's table.

5.1 Errors of content

In 1820, a comparison between Vega's table of factors and Ulbrich's was published, and this revealed a number of errors in Felkel's table [7].¹⁵

First, a number of errors were found in which Vega gave incorrect decompositions. These errors concern the numbers 27293, 43921, 57103, 82943, 90983, and 93137. The article published in 1820 merely stated that all errors (these and the ones below) appeared also in Felkel's table, but we have examined Felkel's table and found the correct decompositions for these six numbers. This suggests that Vega did not copy the first part of his table of factors to 102000 from Felkel, but merely took it from Lambert's table, as we initially understood it.

The other errors concern the list of primes given by Vega, and this list ranges from 102001 to 400031. This list was taken from Felkel's table and the errata published coincides with errors in Felkel's table. These errors are the following.

¹³For more background on the mechanization of factor search, see chapter 7 of Williams [75].

¹⁴Florian Ulbrich was Stiftsherr in Klosterneuburg from 1773 to 1791 [47, p. 164], [68], [34], [4]. He computed a table of factors until 1 071 800. This table was never published. Darnaut *et al.* also mention a table of logarithms made by Ulbrich in 1781 [21, p. 105]. According to [5, 58, 8], Ulbrich had actually computed a table to 1.5 millions and had computed auxiliary tables up to 20 millions. His table of factors is now held at the Austrian National Library (Cod. 10684 and 10685), but we have not yet examined it.

¹⁵According to Lindner, some of these errors had long been known [56].

Felkel (and therefore Vega) gave decompositions for the following numbers which are prime: 148303, 148669, 168559, 168677, 173309, 177347, 189913, 194167, 216569, 232103, 235003, 242639, 247609, 326119, 330509, 331921, 336437, 339671, 357817, 357883, 371299, and 397427.

On the other hand, Felkel and Vega gave the following composite numbers as primes: 148363, 148699, 168647, 168859, 173279, 177377, 194107, 210473, 216599, 232163, 235303, 242939, 247669, 326419, 331927, 336467, 339971, 357853, 358837, 359741, 371269, and 397457.

It will be noticed that these two lists often go in pairs, as a misplacement of one factor naturally led to a complementary error nearby. This is similar to some of the errors observed in Kulik's table [50].

The above lists are probably not complete, as the numbers from 400031 to 408000 in Felkel's table have not been subject to a comparison with Vega's table, and might contain other errors.

5.2 Sequencing errors

In addition to the errors mentioned in the previous section, there are others errors involving the names of the primes. We list here the errors we have found, but we have by no means analyzed the whole table.

5.2.1 First part (1–144000)

- Felkel puts “827- $\mathfrak{E}$ ” in the cell of prime number 74827, but he should have put “827- $\mathfrak{D}$ ”. As a consequence, all later primes up to the next sequencing error are misnamed.
- 88741 is named 741- $\mathfrak{B}$ but should have been called 741- $\mathfrak{B}$ if there were only the 74827 error, and 741- $\mathfrak{U}$ if there were no errors at all. So, from 88741 on, we have a shift of two letters.
- It is interesting to note that Bernoulli's example of location of a number is in that range and he was unaware of Felkel's sequencing errors. His example is that of 89371, but if there were no errors, the number 89387 would have been meant [15, p. 230].
- For 101789, Felkel writes “789- $\mathfrak{z}$ ” and for 101797, he writes “797- $\ast\mathfrak{z}$ ”. The asterisk seems to mean that Felkel is bringing the shift back to one unit by introducing a variant letter. Such deviations appear several times, and the most likely explanation seems to be that Felkel compared his list of primes with another one, probably that of Marci published in 1772 and which went up to 400000 [57].
- Felkel gave $112699 = 251 \times 449$ as prime, but he gives the correct decomposition in his errata. Then, for 112799 which is prime, Felkel gives a decomposition, and this error happens to reduce the shift to 0, so that from then on until 144000, the names of the primes are correct (except possibly if two errors cancel out within

some pages that we haven't checked). This error in 112799 was also corrected the errata in which Felkel replaced the content of the cell by "799-*" (with no adjoined symbol).

5.2.2 Second part (144001–336000)

After 144000, in the second part of the table, there are also several errors, which are easy to identify by comparing our reconstruction with Felkel's table. We have located the following ones:

- 144773 and 160709 are indicated as prime, but Felkel skips one letter in each case, and he has therefore by then a discrepancy of two letters.
- 168673 and 168697 are given as prime successors (and are not), but Felkel skipped two letters in the naming.
- 174067 was named irregularly as "067*u" to distinguish it from the previous value "061-u".
- At the end of the interval 174001–180000, Felkel's count of primes is in excess by 4 units.
- Felkel corrects his count for 182123 which is named "123*k" and at 186000 the excess is only three units.
- The greatest rectification occurs in the interval 186001–192000 (figure 9). Here, Felkel corrects the count by nine units between 186647 and 186709. This is a sequence of nine consecutive primes. As a result, there is now a shortage of 6 units in the count of primes.
- Felkel gave 189913 as composed, but it is a prime. This error was given in [7]. The shortage is now seven units.
- 193031 was named "031-m", like the previous prime "013-m". The shortage is now eight units.
- Felkel skipped one letter for the prime 201473, bringing the shortage back to seven units.
- Other errors occur and eventually the shortage ends with eleven units by 336000.

5.2.3 Third part (336001–408000)

In the third part, the shortage starts with 11 units, and jumps from 11 for 342000 to 16 units for 345000. Eventually, at the end of the last printed page of Felkel's table, for the interval 402001–408000, there is a shift of 13 units between Felkel's numbering of primes, and the exact one. This means that if the problems were only omissions of primes, Felkel would have forgotten 13 primes. In reality, as we have seen, things are a little more

complex, since some numbers have been given as primes when they were not, there were some sequencing errors, and there were some intentional shifts, presumably to catch up errors found by comparison with another table.

311-V	f τ E	317-V	f α D	e p α p	k α u	f g β o	11
343-X	b α f	e ² ϵ e	O α m	S α d	n α B	12	36
377-Y	i α f	f τ F	379-Z	l β m	e E u	13	39
437-D	r δ M	e m α j	g ² α L	i μ l	h β e	419-E	14
	o ζ r	e E v	g π h	f b α T	n ζ M	q ϵ j	15
	i β e	f ² k u	w α K	G α c	e b β x	479-G	16
	e E w	p ϵ S	E α h	M α v	l s e	h β M	17
	n v ²	B ζ	i μ m	f m α b	e E x	c β S	18
	b β j	551-J	k α v	e l α H	v α M	g f g	569-R
	p ϵ S	581-L	583-M	587-N	g m Ω	f τ G	e i k t
	t α Q	f α Y	e r β B	s δ r	619-P	j β R	629-Q
	f i ² q	e ² g N	b v i	647* β h	649* β j	653* β R	f v C
	g y q	671* β L	l o n	b x t	679* β M	e E y	689* β R
	e l r	701* β D	f ² β t	707* β P	709* β Q	i m R	g r I
	727-R	l q i	733-S	k ² W	e ² n c	743-S	p ² b
	757-U	761-B	763- β B	e E z	f τ H	773-X	b β f
	m β d	f τ I	793-Y	g π R	799-Z	Y β	e E b
	g β i	i α t	e g γ l	i μ o	k α x	y γ l	F α e
	b l M	e E c	s δ y	f τ K	859-b	u δ f	869-c
	877-e	b v l	883-f	n ζ B	889-g	e E d	m η M
	e E e	P α l	D β j	917-b	o q a	f τ L	b β X
	h β a	i μ p	v γ h	947-i	e b β y	g m v	959-k
	f k α g	s δ j	f α Z	e E f	g i α f	b ² α t	f z t
							31
							32
							33
							34

Figure 9: A string of anomalies probably corresponding to errors that Felkel wanted to rectify on the page for the interval 186001 to 192000. The anomalies are the use of ‘*’ in the names of the primes. 649* β , for instance, repeats the naming of 551- β and therefore shifts the letters. Here, no less than nine symbols are inserted between 629- Ω and 727- $\mathfrak{M}$.

6 Felkel's later work

Apparently Felkel stopped working on his table of factors in 1777, and took up again the research in 1784 when the correspondence with Lambert began to be printed [15, p. 122], [38, p. 118].

6.1 New computation (1784)

In 1784, Felkel published a tract about a new projected table, with a first part going to 1 008 000, a second part going to 2 millions and with plans to reach 10 millions [15, pp. 121–134]. Moreover, in the preface of the 1798 translation of Lambert's *Zusätze*, Felkel wrote that he computed the table anew from 1 to $2\,856\,000 = 7 \times 408\,000$ [55, preface, p. VII]. He actually probably computed it up to 3 millions [38, pp. 120–121].¹⁶ It is possible that he had to recompute everything if he no longer had the initial calculations made in 1776.

The new table was going to be very similar to the one published in 1776, the main difference being that Felkel no longer gave all the factors. Felkel now recognized that the use of letters is inconvenient and he proposed to give the smallest factor explicitly, as suggested by Lambert. More precisely, if a number is composed, all the factors except the largest one are given. The first factor is given explicitly, but the other factors are given with letters.

With these new restrictions, for the numbers up to 2 millions, and given that the smallest explicit factor is 7, Felkel only needed symbols for primes up to $\sqrt{\frac{2\,000\,000}{7}} = 534.5$.

The largest prime below this limit is 523. This is also true if the table goes up to 2 016 000 and Felkel was therefore able to use the first column of primes from his first table, where the primes 1 to 523 were encoded by a single letter.

The primes are encoded as in the first table from 1776 (figure 2), namely with a prefix and a letter from the roman or gothic alphabet. Figure 10 shows several primes, for instance on the first line, “–M”, whose complete name is “(μ–M”, and corresponds to 1 006 507. The prefixes are “(μ,” “(ν,” and the first appearance of the prefix “(ν” is in column *b*. A comparison with our reconstruction and extension of the 1776 table will show that the beginnings of the prefixes differ from ours. It is moreover not clear which sequence of prefixes was used by Felkel in this new table. The prefix “(μ” was used in the first table for the primes 59359 to 60493. One may be tempted to include the dot in the prefix (which would then be “(μ.”), but the dot does not seem to appear in the first use of the prefix “(ν”, so that we will leave this problem open for the moment.

Another difference with the first table is that each page now only covers an interval of 3000 integers, instead of 6000 initially. In the table published in 1776, each page was made of two columns, each covering 3000 integers (figure 4). Each page contained exactly 100 lines. In the new project, only one column was taken, and split in two parts, so that each page now has only 50 lines, and the first line contains the suffixes 001, 007, 011, 013,

¹⁶Felkel actually wrote that he computed the table from 1 to 5 millions [55, introduction, p. XIV], but Glaisher views this as a misprint, and believes that Felkel did not go beyond three millions [38, pp. 108, 120].

017, 019, 023, 029, and then 501, 507, 511, 513, 517, 519, 523, and 529. This appears clearly on figure 11. The number 1007539, which was given in 1785 as an example by Felkel in his article on periods of fractions [33], appears towards the bottom of column *f* in figure 10 and its factors except the largest 97 are $13.hq = 13 \times 17 \times 47$.

The entire table to 2016 000 would have occupied 672 pages. Pursuing this scheme to 10 millions would have required 3334 pages, but then Felkel would not have been able to use only one-letter symbols.

Felkel again started a subscription. He planned to first issue the first million, then installments of 200000 four times a year. But Felkel's project apparently failed again, perhaps for lack of time on his side, or because he moved to Lisbon. His project was only fulfilled on a small scale in Felkel's translation of Lambert's table in 1798.

isque 1008000

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	
1005	13. z	— M	11. n	— N	179	17	7. r	823	1006
(λ	— O	7	67. y	— P	— Q	23. b	17	— R	(μ
9. t	71	331	787	11. g	7	— S	— T		
41	73	19. m	13	7 ²	— U	— V	29. c		
fh	439	283	— W	— X	19	11	7		
— Y	241	— Z	7. m	101	257	41. p	89		
3. f	11. k ²	7	47	79	677	661	11. v		
— 91	59. f	— 28	17	373	127	7. g ² k	— 6		
199	7	— 2	11. i	17	13. p	97 ²	— 6		
457	139	— 8	— 6	31. q	7	29. l	— 9		
541	37	13	137	7	11	281	19 ²		
7	13. o	11 ² . r	167	— 3	31	— 8	7. h		
— 8	19. b	13	7. y	— 28	— 28	— 28	13. w		
— 9	— 2	7 ²	181	11. C	29	229	41 ²		
313	17. t	43	— 28	— 6	443	7. iu	— 2		
11	7. a	17	23	13. l	— 11	11. m	— 28		
19	— 28	— 28	13. v	131	7 ² *	379	23		
641	11. p	— 9	— 3	7. G	17. n *	173	11. y	1007	
7. N	(v a	— b	53	19	— c	17	7. q	(z	
3 ² . s	67	— d	7. f ² l	137	— e	521	— f		
23	449	7	619	— g	— h	13	— i		
17	— k	29	191	23	11. g	7	31. r		
— l	7	11. it	— m	37. o	— n	257	43. s		
(μ	107	283	13	— o	563	7. i	89	773	
31	13	— p	17. r	7. t ² c	23	— q	— r		
7. n	29. q	197	251	17. u	— s	101	7. g		
11	977	223	7 ² . t	— t	— u	11	— v		
509	— w	7. m	19	73	41. x	139	— x		
347	11	47	— y	13	— z	7	11. h		
3. y	7. k	— a	13	— b	149	853	19. n		
— c	31	41	11	— b	7. v	23	— e		
* 29	17. i	— f	103	7. p	277	421	— 8		
1006	7. g	— h	17	37. w	53	11. u	— i	7 ²	
127	— f	11	7. o	593	503	— i	— l		
131	— m	7. m	43	61. y	13. hq	59	— n		
19 ²	— o	23. v	349	11 ²	89	7. h	— p		
642	7 ²	12. r	21	— o	— r	— r	— c		

Figure 10: A fragment of the last page of the first part of the new projected table (1784). The page was scanned by Google without unfolding the table, so that the left part (with another set of columns *a* to *h*) is not visible, except for the marginal inscriptions 1005 and 1006. The lower part of the table is also not visible. Part of it is visible in figure 11.

Zu Seite 126.

	a	b	c	d	e	f	g	h
001	007	011	13	517	519	523	529	
031	037	041	43	547	549	553	550	
061	067	071	73	577	579	583	589	
091	097	101	03	607	609	613	619	
121	127	131	33	637	639	643	649	
151	157	161	63	667	669	673	679	
181	187	191	93	697	699	703	709	
211	217	221	23	727	729	733	739	
241	247	251	53	757	759	763	769	
271	277	281	83	787	789	793	799	
301	307	311	13	817	819	823	829	
331	337	341	43	847	849	853	859	
361	367	371	73	877	879	883	889	
391	397	401	03	907	909	913	919	
421	427	431	33	937	939	943	949	
451	457	461	63	967	969	973	979	
481	487	491	93	997	999	003	009	
511	517	521	23	027	029	033	039	
541	547	551	53	057	059	063	069	
571	577	581	83	087	089	093	099	
601	607	611	13	117	119	123	129	
631	637	641	43	147	149	153	159	
661	667	671	73	177	179	183	189	
691	697	701	03	207	209	213	219	
721	727	731	33	237	239	243	249	
751	757	761	63	267	269	273	279	
781	787	791	93	297	299	303	309	
811	817	821	23	327	329	333	339	
841	847	851	53	357	359	363	369	
871	877	881	83	387	389	393	399	
901	907	911	13	417	419	423	429	
931	937	941	43	447	449	453	459	
961	967	971	73	477	479	483	489	
991	997	001	03					

Zu Seite 126.

	a	b	c	d	e	f	g	h
001	007	011	13	517	519	523	529	
031	037	041	43	547	549	553	550	
061	067	071	73	577	579	583	589	
091	097	101	03	607	609	613	619	
121	127	131	33	637	639	643	649	
151	157	161	63	667	669	673	679	
181	187	191	93	697	699	703	709	
211	217	221	23	727	729	733	739	
241	247	251	53	757	759	763	769	
271	277	281	83	787	789	793	799	
301	307	311	13	817	819	823	829	
331	337	341	43	847	849	853	859	
361	367	371	73	877	879	883	889	
391	397	401	03	907	909	913	919	
421	427	431	33	937	939	943	949	
451	457	461	63	967	969	973	979	
481	487	491	93	997	999	003	009	

	a	b	c	d	e	f	g	h
001	007	011	13	517	519	523	529	
031	037	041	43	547	549	553	550	
061	067	071	73	577	579	583	589	
091	097	101	03	607	609	613	619	
121	127	131	33	637	639	643	649	
151	157	161	63	667	669	673	679	
181	187	191	93	697	699	703	709	
211	217	221	23	727	729	733	739	
241	247	251	53	757	759	763	769	
271	277	281	83	787	789	793	799	
301	307	311	13	817	819	823	829	
331	337	341	43	847	849	853	859	
361	367	371	73	877	879	883	889	
391	397	401	03	907	909	913	919	
421	427	431	33	937	939	943	949	
451	457	461	63	967	969	973	979	
481	487	491	93	997	999	003	009	

Figure 11: The layout of a page in the 1784 project. These are two different scans, but in both cases the page was scanned by Google without unfolding the table. On the right scan, we can see the bottom of the last page of the projected table, but folded.

6.2 Felkel's new method (1793–1794)

In the preface of the 1798 translation of Lambert's table, Felkel writes that he was working on a more portable scheme and he constructed 15 "bases" by the aid of which numbers up to 24.6 millions could be factored, when the least factor does not exceed 400 [55, preface, p. vii], [38, p. 121]. Felkel also found a method to factor numbers up to 100 millions on 65 "bases." Zach also mentioned a table up to 24 600 000 [73].

There is an account of Felkel's new method in a letter sent by Stockler to Felkel [35], probably around 1795. In order to decompose the numbers from 40 000 to 1 200 000, Felkel proceeds as follows. He first assumes that he has the decompositions of all integers up

to 40000 and he considers a table of the numbers $30n + 1$, $30n + 7$, $30n + 11$, $30n + 13$, $30n + 17$, $30n + 19$, $30n + 23$, $30n + 29$, for $n < 1200$. Now, if $N = 30n + a$ is a number to decompose, Felkel considers the double sequence

$$\begin{array}{ccccccc} a & 30 + a & 60 + a & 90 + a & \dots & X & \dots \\ n & n - 1 & n - 2 & n - 3 & \dots & Y & \dots \end{array}$$

If X and Y have a common factor, then it is easy to show that this is also a common factor of N . For instance, for $N = 40003 = 1333 \times 30 + 13$, we have the double sequence

$$\begin{array}{ccccccc} 13 & 43 & 73 & \dots & 763 & \dots \\ 1333 & 1332 & 1331 & \dots & 1308 & \dots \end{array}$$

and 763 and 1308 have the common factor 109, which divides $40003 = 109 \times 367$. The common factors are found using the known decompositions until 40000, and this sets the limit of n to 40000, and therefore of N to 1 200 000. This technique can be extended for larger intervals.

The bases alluded to by Felkel are apparently tables of numbers not divisible by 2, 3, or 5 in certain ranges. For instance, Felkel calls “base A” the table of the numbers from 1 to 36000 not divisible by 2, 3, or 5. “Base B” contains the numbers not divisible by 2, 3, or 5 from 2 364 000 to 2 400 000 [35, p. 392].

6.3 Felkel’s edition of Lambert’s table (1798)

Although Lambert had a difficult time with Felkel, as testified by Lambert’s correspondence, Felkel somehow had his revenge in publishing a Latin translation of Lambert’s *Zusätze* with an improved table of factors in 1798 [55]. This table was clearly influenced by Felkel’s 1784 table, in that Felkel also tried to give all the factors, except the largest one.

7 Chronology 1770–1784

The period between the publication of Lambert’s tables in 1770 and the publication of Felkel’s table and Hindenburg’s method was a period of intense development and uncompleted computations which must have been a source of frustration for all the involved parties.

Thanks to the publication of Lambert’s correspondence by Johann Bernoulli,¹⁷ we have a fairly detailed picture of these developments. The correspondence contains in particular the letters exchanged with Felkel and other table computers. Bernoulli organized the letters by correspondents, and then by date. In this section, we have summarized the parts of each letter relevant to the computation of factor tables, and we have ordered them strictly chronologically. This seemed adequate, since there are often allusions to one correspondence in another, and the development and interactions between Lambert, Felkel and others can be followed more easily that way.¹⁸ We have also added a few other events in this chronology.

The chronology starts in 1770, after the publication of Lambert’s table [53].

- On 27 September 1770, Lambert publishes a notice in the *Allgemeine deutsche Bibliothek* on the tables that he received from Oberreit, without naming him [54]. Oberreit has extended the table to 150000, and plans to extend it at least to 204000. Lambert published this notice in order to avoid duplicate work. Lambert also mentions a second computer, but it is not clear if he refers to Wolfram or someone else.
- In a letter sent to Kant in 1770 [12, p. 366],¹⁹ Lambert writes that he found someone to extend his table of factors to 204000. He writes that the table of factors will perhaps be extended to one million. (In 1782, Bernoulli adds that this table to 204000 was computed by Oberreit,²⁰ that it was extended to 500000, and that it was now in the hands of professor Schulze in Berlin. Bernoulli also mentions a work giving the natural logarithms of all prime numbers up to 100000 [12, p. 368].)
- 5 March 1772, Wolfram²¹ to Lambert [14, p. 436]

Wolfram writes that he received Lambert’s 1770 book on February 28, and he informs Lambert that he had computed for his personal use a table giving the smallest factor of all numbers up to 300000 not divisible by 2, 3, and 5. This table occupies 25 pages. (see also [15, p. 140])

¹⁷Johann III Bernoulli (1744–1807).

¹⁸Lambert’s correspondence about tables of factors was also described by Glaisher [38] and Bullynck [17, 18].

¹⁹This letter is also reproduced in Kant’s published correspondence [44].

²⁰Ludwig Ober(r)eit was an Ober-Finanz-Buchhalter in Dresden. He was born in Lindau in 1734 and died in Dresden in 1803. He was the brother of Jacob Hermann Ober(r)eit (1725–1798) who rediscovered one of the main manuscripts of the Nibelungenlied. See also *Neue deutsche Biographie*, volume 19, 1999, pp. 382–384.

²¹Isaac Wolfram. A summary of what is known on Wolfram was given by Archibald [10].

- 21 March 1772, Lambert to Wolfram [14, p. 441]

Lambert mentions Oberreit who planned to extend the table of factors to one million, giving all factors. Lambert writes that he published a notice about Oberreit's computations on 27 September 1770, in order to prevent duplicate work.

- 3 August 1772, Wolfram to Lambert [14, p. 445]

Wolfram mentions the work of Adolph Marci who had factored the numbers up to 300000 by 1756. Wolfram had known about this table since 1758. Wolfram also had plans to extend the table to a million in 1756.

- 5 April 1773, Wolfram to Lambert [14, p. 493]

Wolfram mentions the table “Tafelen over de Primgetallen van 1 tot 400000” published by Marci in 1772 or 1773 [57].

- 26 July 1773, Wolfram to Lambert [14, p. 497]

Among other things, Wolfram writes that among the 266666 numbers smaller than one million and having their smallest factor greater than 5, there are 78458 primes, 121852 numbers with two prime factors, 55244 numbers with three prime factors, 10160 numbers with four prime factors, 907 numbers with five prime factors, 44 numbers with six prime factors, and only one number with seven prime factors.²²

- 18 May 1774 : Lambert to von Stamford [15, p. 5]

Lambert mentions to von Stamford²³ that the table of factors had been computed from 1 to 72000 and from 100000 to 504000. This is the work done by Oberreit.

- 24 May 1774: von Stamford to Lambert [15, p. 8]

von Stamford doesn't understand why there is a gap between 72000 and 100000, given that there is none in Lambert's 1770 table. He asks for clarifications to Lambert. In fact, in 1770, Lambert gave only the smallest factors, whereas Oberreit gave all factors.

- 7 June 1774: Lambert to von Stamford [15, p. 10]

Lambert explains how to fill the gap between 72000 and 100000.

- Hindenburg²⁴ started to work on the tables in 1774 [15, pp. 156, 240].

- 11 August 1774, Lambert to Wolfram [14, p. 509]

²²This number is $7^7 = 823543$.

²³Heinrich Wilhelm von Stamford (1740–1807), officer in Potsdam. From 1772 to 1775, he was teaching French at the monastic school of Ilfeld in Germany [60].

²⁴Carl Friedrich Hindenburg (1741–1808) developed the combinatorial school [20].

Lambert mentions the work of von Stamford. Lambert received from Oberreit the factors of the numbers from 1 to 72000 and from 100000 to 500000,²⁵ with the notice that Oberreit could no longer complete it. von Stamford is now working on filling this gap.

- 2 April 1775 : von Stamford to Lambert [15, p. 15]

von Stamford announces that he filled the gap from 72000 to 100000. He announces that he agreed with Rosenthal²⁶ to split the computation from 504000 to one million. von Stamford will take the range 504000 to 750000 and Rosenthal will go from 750000 to one million.

- 12 August 1775 : Lambert to von Stamford [15, p. 20]

Lambert writes that the limit should be pushed to 1002000 in order to fill the last page.

- 2 December 1775, von Schönberg to Lambert

Curt Friedrich von Schönberg, a student of Hindenburg, writes that Hindenburg has been working on factor tables and that his method would allow him to recompute Lambert's table to 102000 in 8 days [13, p. 310].

- 26 December 1775, Lambert to von Schönberg

Lambert gives a list of tables he already has, and he mentions the table giving all factors to 500000, and that he had a pledge that this table would be extended to one million [13, p. 312].

- 15 January 1776: Felkel to Lambert [15, p. 33]

Felkel announces the completion of his table to 144000, on 12 sheets (24 pages, with a range of 6000 per page). He says that he tried to build such a table, without knowing if such a table did already exist. He writes that he spent six days to find all factors of odd numbers from 1 to 10000, and it occurred to him that the limits of 100000, or even 1 million, were not unreachable.

On one side of a half-sheet, he put an interval of 3000 numbers, and he had such a page printed in 400 copies. We can therefore assume that he had pages for $400 \times 6000 = 2\,400\,000$ numbers. He then describes his first sieve, a “scale that could be turned into a machine.”

The interval of 144000 took Felkel two months and he hoped to reach 10 millions in several years. He wrote of 840 sheets, that is 1680 pages, which is the number

²⁵In fact, probably 504000. According to [15, p. 140], Oberreit had completed the table up to 260000 by the Summer 1771, and the next year he had reached 339000, both having been sent to Lambert. In 1775, and perhaps earlier, the table was ready to be printed until 500000 (actually 504000) and the plan was to go up to a million. There was however apparently a gap between 72000 and 100000 (or 102000?).

²⁶Gottfried Erich Rosenthal (1745–1813) was a pioneer of meteorology and measuring instruments. He was *Bergkommissarius*.

of pages if the initial scheme is continued. Felkel was looking for the support of Lambert and the Academy.

The numbers with factors 2, 3, and 5 were not included in Felkel's table.

Felkel briefly mentions that he provides two sieves, one for numbers not divisible by 2, 3, and 5, and which provides the factors of 8 numbers for every move; the second sieve applies to all numbers and identifies the factors of 30 numbers at a time.

- 3 February 1776, Felkel to Lambert [15, p. 44]

In this letter, Felkel advertises the making of the table until one million by subscription, on 84 sheets. He writes that factors can be found 40 times faster than by hand with his machine. Felkel also sells his machines and claims their usefulness for people who would like to find factors of numbers not within his table. Finally, Felkel estimates that he can complete one million every six months and that completing a 10 million table should not be given up.

- 12 February 1776, Lambert to Felkel [15, p. 51]

Lambert informed Lagrange of Felkel's project, and writes that Lagrange only has an incomplete understanding of Felkel's machine. Lagrange is not convinced that the machine can be used with large numbers.

- 21 February 1776, Felkel to Lambert [15, p. 53]

Felkel writes that he takes up the task of computing the factors until 2 millions (or beyond). One year should suffice. At that point he had already computed more than half a million. Some errors are mentioned but they probably concern a preliminary version of his table to 144000, for the mentioned errors do not occur in the published version. Felkel writes that he is preparing a description of his machine. Felkel also writes that his work has been examined by 15 mathematicians.

- March 1776, Lambert to Felkel [15, p. 57]

Lambert encourages the computation from 1 million to 2 millions. Lambert mentions his other two correspondents who were working on the interval from 500000 to 1 million, namely Stamford and Rosenthal. Another correspondent, Ludwig Oberreit, had extended the table to 500000. Lambert writes that he does not object the publication of Felkel's table to 144000 which will find some use, but he questions Felkel's choice of a subscription for the table. But at the same time, Lambert thinks that a subscription is the best way to sell the machine.

- February or March 1776: Lambert to Rosenthal [15, p. 24]

Lambert writes to Rosenthal because he has had notice of Felkel's work. Lambert says that he advised Felkel to continue with the second million.

Lambert hopes that the works of each part will fit together.

- 6 March 1776: Rosenthal to Lambert [15, p. 26]

Now, Rosenthal writes that he was informed by Stamford that the latter's illness does no longer allow him to continue the work, and Rosenthal agreed to do all the work himself.

Rosenthal asks Lambert if he can obtain for him a machine by Felkel.

- 30 March 1776, Felkel to Lambert [15, p. 62]

Felkel writes that he finds the factors of 240 numbers in one hour. He writes that a 10 year old child could operate the machine and dictate its results. He estimates that the numbers from 1 million to 2 millions could be factored in at most 2927 hours, and in fact faster, in particular by the use of several computers.

Felkel proposes to complete the table until 1 008 000 for Easter 1777, to have it printed on 42 sheets, and to have the factors up to 2 000 000 ready for print. (but shouldn't the first million be on 84 sheets?)

Felkel writes that it would be good to print the table until 300000, and that it would occupy 12.5 sheets. So, on one sheet, there would be four pages, since $300000 = 12.5 \times 4 \times 6000$.

Felkel plans to work on parts of 300000 numbers and seven such parts amount to a little more than 2 millions. This would make it easier for Felkel to distribute the work among aides.

- 2 April 1776, Wolfram to Lambert [14, p. 520]

Wolfram writes that he had factored the numbers from 1 to 126000 in 1743 in Danzig, and continued up to 300000 in Holland. He also gives a list of 30 errors in Marci's table up to 300000. He writes that Marci had planned to go to 1 200 000.

According to Wolfram, the prime numbers until 100000 in Marci's book are printed like those of Jäger published by Krüger in 1746.

- 24 April 1776, Felkel to Lambert [15, p. 70]

Felkel explains that he is now compelled to publish a table in order to satisfy his patrons. The printing will start in May 1776. This may explain why the table to 144000 was printed.

Felkel observes that he computes the tables as fast as the printer can print them, and he works without any aide. Felkel is confident that by the following Easter, the first two millions will be computed, and a large part of the second million will be printed.

- 24 Juni 1776, Felkel to Lambert [15, p. 73]

Felkel writes that his computations go faster than the setting by two printers. Felkel now plans to write an introduction to the first table.

- 30 July 1776: Lambert to Rosenthal [15, p. 27]

Lambert writes that he was compelled not to answer Felkel's letters, as Felkel is not willing to cooperate with other computers. Lambert writes that he has already

three unanswered letters by Felkel. Lambert's letter is quite critical of Felkel's projects and arrogance.

Lambert now suggests to Rosenthal to wait for Felkel's completion of the two millions and for the description of his machine, until further decisions be taken.

- 3 August 1776: Hindenburg to Lambert [15, p. 137]

Hindenburg informs Lambert of a published announcement to go beyond Lambert's table. Hindenburg mentions a machine advertised by Wilhelm Bauer, professor in Vienna.

Hindenburg knows that Lambert has already obtained all factors of the numbers up to 500000.²⁷

Hindenburg writes that he plans to provide a million, but only with the smallest factor. He also wants to provide a million, to make sure to extend beyond other competitors.

This letter contains a printed notice from 24 May 1776 by the publisher Siegfried Leberecht Crusius about Hindenburg's project of giving the smallest factor of the numbers between 1 and 5 millions [15, pp. 142–150].

Crusius' notice is also an answer to Lambert's call. The notice insists on the purely mechanical and reliable process of finding the primes and factors.

According to this notice, the computation of the interval from 1 to 102000 took a little less than 14 days. This table contains 27200 number positions.²⁸ There are 17433 factors and 9767 prime numbers. Hindenburg only had to search the factors of 11427 numbers.

The notice observes that the way the factors are computed is very regular, and it makes it possible to predict the time of completion of the table. It is also mentioned that Hindenburg would be happy to have others do the computation, if there were an unexpensive way to check the other computers' results. And this way must be easier than computing the factors by himself.

At the beginning, Hindenburg had planned to issue intervals of 200000 at a time, but he now plans to issue one million at a time.

For reasons unrelated with the computation, the first million cannot be printed before Easter 1777.

The layout will follow that of Lambert, with some changes, and there will be two fifths more numbers on a page than on two consecutive pages by Lambert. (There are 800 numbers on two pages in Lambert's arrangement and so there might be about 1100 numbers on one page of Hindenburg's project) It seems that there will be intervals of 5000 numbers per page.

One octavo sheet will cover an interval of 40000 and 25 sheets will cover a million.

²⁷On details of the communication of this table to Lambert, see [15, p. 140].

²⁸Each double page contains 800 values and there are 34 double pages, hence $34 \times 800 = 27200$.

- 13 August 1776: Lambert to Hindenburg [15, p. 151]

Lambert informs Hindenburg of Felkel's projects. He writes that he had already told Felkel that it was not necessary to compute the table from 1 to 1 million, because this had already been computed, and that he could instead compute the table from 1 million to 2 millions. Felkel consequently altered his plans to extend the first table to 300000, and to announce the continuation to 2 millions. Lambert complains that Felkel asked him to support his endeavour, although Felkel did not describe his machine. Consequently, Lambert did not answer the last three letters by Felkel. Lambert now expects Felkel to extend his plans further, since Felkel mentioned the limit of 10 millions in a printed notice.

Lambert writes that Felkel does not realize that his method and machine could make him more famous than an unnecessary use of them.

Lambert even writes that before what happens with Felkel becomes clear, Lambert will not publish anything on factors, as he considers that Felkel might claim that he already knew everything better.

Lambert mentions that such duplicate work occurred before, for instance for the translation of foreign works.

Felkel is not intimidated by 5 millions, and has already found people to assist him.

Lambert says that because Felkel does not listen to him, he is happy to see Hindenburg's work progress.

- 13 August 1776: Lambert to Rosenthal [15, p. 30]

Lambert informs Rosenthal that Hindenburg, after learning from Felkel's proposal, decided to extend the table up to 5 millions, but only indicating the smallest divisors. Lambert says that Hindenburg and Felkel want to compete, instead of cooperating. Lambert concludes that this affair will in the future occupy a certain space in the history of mathematics.

- 13 August 1776, Lambert to Felkel [15, p. 81]

Lambert mentions the work of Hindenburg, without naming him, after the letter Lambert received of him above (dated 3 August).

- 17 August 1776, Hindenburg to Lambert [15, p. 155]

Hindenburg estimates that he can compute faster than Felkel with his machine.

He writes that it seems that Felkel does not follow Lambert's layout. (In a footnote, Bernoulli adds that Felkel followed Euler's arrangement)

Hindenburg observes that no cooperation is possible with Felkel.

- 24 August 1776: Lambert to Hindenburg [15, p. 161]

Lambert tells Hindenburg that he only sent to Felkel Hindenburg's printed notice.

Lambert recalls, without naming them, the works of Oberreit, von Stamford and Rosenthal.

Lambert mentions Euler’s 1775 article [25] and his proposed layout.

- 10 September 1776, Felkel to Lambert [15, p. 82]

In this letter, a conflict with Hindenburg is mentioned for the first time. Felkel writes that he could also promise 20 millions.

This letter also contains a long “Nachricht”, dated 1st June 1776, but actually probably 1 September 1776. (see Hindenburg’s letter from 27 September 1776 and also Bernoulli’s comment [15, p. 235]; the same text does exist with two different dates) This text announces two tables, one giving all factors of all numbers up to 1 million, and another giving all factors of all numbers not divisible by 2, 3, and 5 from 1 to 10 millions.

This text also gives the names of 16 “mathematicians” who testified about Felkel’s work.²⁹ Felkel then writes that the first million has been completed for some time. He writes again that one year is enough to compute two millions, assuming that no more than four hours are spent every day on it.

So, his plan is to compute two millions every year until 10 millions. But he also writes that he now has two computers by his side.

Felkel recollects the influence of Lambert and that he had started his computations, not knowing that others had also partly extended Lambert’s table. Felkel writes that Lambert encouraged Felkel to go beyond the first million.

- 27 September 1776, Hindenburg to Lambert [15, p. 166]

Hindenburg writes that for priority reasons he had to publish a notice of his method.

He also mentions a notice by Felkel, dated 1 September 1776. This notice must be the one Felkel sent to Lambert on 10 September. Hindenburg accuses Felkel of having borrowed words from Hindenburg’s notice.

Hindenburg mentions the publication of his “Beschreibung einer ganz neuen Art etc.” [41].³⁰

Hindenburg writes again that he puts 5000 numbers per page. He now gives for every divisible number n the smallest factor p , as well as the remaining factor, that is p and n/p . He claims again to find the factors faster than Felkel.

Hindenburg now also presses Lambert for his opinion on his work.

- 2 October 1776: Hindenburg to Lambert [15, p. 173]

Hindenburg gives additional details. (Hindenburg’s table also does not contain multiples of 2, 3, and 5) He writes that for every million 60000 factors can be printed in advance. (I fail to see why, as one million is not a multiple of any factor that is found in the table) Hindenburg writes that Felkel’s machine does not have this advantage, however it is made. Hindenburg also stresses the usefulness of

²⁹See Felkel’s letter of 21 February, above.

³⁰Reviews of Hindenburg’s publication appeared in 1777 [2] and 1778 [3]. For a critique of Hindenburg’s text, see Knuth [48].

adding the multiplicity of the smallest factor and the co-factor (that is, every non prime number is written as $a^p \cdot b$). The co-factor could be written in italics when it is not a prime number. He writes that this almost as good, and in a certain sense even better than giving all the factors.

- 5 October 1776: Lambert to Hindenburg [15, p. 179]

Lambert informs Hindenburg that Felkel wrote him that Hindenburg’s notice was useful to him.

Lambert makes some observations on the layout of the table, for instance considering that $3556553 = 7 \cdot 508079 = 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. The first decomposition takes a lot of space, but less than the complete one. Lambert also suggests to put an asterisk after a prime number in a decomposition.

- 18 October 1776, Wolfram to Lambert [14, p. 523]

- 8 November 1776, Felkel to Lambert [15, p. 106]

In this letter, Felkel even writes that his plan would need no modifications until 1000 millions. He writes that he can find all factors of 500 numbers in one hour.³¹

- 22 November 1776, Felkel to Lambert [15, p. 110]

Felkel mentions someone who, being stimulated by Felkel’s work, decided to compute the logarithms of all prime numbers up to a million with 15 decimals.

Felkel also gives an errata to his table. This errata is for the table to 144000, but does not account for all errors.

- On 29 November 1776, Hindenburg published a notice of his projected table in the *Wittenbergsches Wochenblatt* [1].

- 30 November 1776, Lambert to Wolfram [14, p. 524]

Lambert mentions Felkel and Hindenburg, but without naming them. He writes that the one in Leipzig (Hindenburg) wants to go to 5 millions, and the one in Vienna (Felkel) to 2, or even to 10 millions. Lambert observes that the tables can be organised in columns of intervals of 300, that is, of 80 values.

- 7 December 1776: Hindenburg to Lambert [15, p. 184]

Hindenburg now sends his complete 1776 text “Beschreibung ...” [41]

- 10 December 1776: Lambert to Hindenburg [15, p. 188]

This is an answer to Hindenburg’s letter from 2 October, and Lambert had apparently not yet received the letter from 7 December.

Lambert writes that Felkel sent him his table up to 144000 and he is glad that he will be able to tell him his opinion (and this he did on 12 December).

³¹In 1785, he wrote that he could factor up to 800 numbers hourly [15, p. 492].

Lambert gives a description of Felkel's table. The use of letters is criticized, but I think that Felkel's table is still the only one that could go so far in a compact way. Kulik's table goes beyond, but by giving only the smallest factor.

Lambert says that he will suggest to Felkel giving explicitly the first factor.

Lambert also describes Felkel's machine. It is made of 8 small rods and 30 big ones. Lambert observes that a circle or a cylinder could be used, and even clock or gearwork, which anticipates the work of Lehmer and others.

Lambert now understands that Felkel could not have started at 1 million.

Lambert also writes that Wolfram is still unaware of the latest developments.

- 12 December 1776, Lambert to Felkel [15, p. 114]

This is Lambert's first critical letter to Felkel. Lambert observes first that laying out the numbers in eight columns $30n + 1$, $30n + 7$, $30n + 11$, $30n + 13$, $30n + 17$, $30n + 19$, $30n + 23$, $30n + 29$ is a good idea. Lambert understands that the letters are used to save space, but he believes that the factors could have been written in full, without losing too much space. (I don't think this is true, though.) Lambert also considers that the use of letters creates a separation between Felkel's work and the work of others, so that neither can be helpful to the other.

Lambert is also somewhat critical of Felkel's machine, writing that it is not so new. Lambert observes that everything reduces to 1) laying out the tables in the best way, and 2) finding in the shortest possible way the positions where any factor is to be inserted. Lambert alludes to some useful formulæ by Wolfram for that purpose.

Lambert refers to Hindenburg's work in Leipzig, and the publication of his "Beschreibung einer ganz neuen Art etc." [41]. Lambert writes that Hindenburg's work differs a lot from Felkel's and contains a lot of theory.

He mentions again the person who computed the table to 500000 and another person who also did some work, but they are not named by Lambert.

Lambert considers that most readers will understand that these tables are useful, but will not consider them necessary to them.

Regarding the computation of logarithms, Lambert considers that the most important is to have tables without errors, and that it would be good to have a table of logarithms from 990000 to 1 000 000, with their differences.

- 14 December 1776: Lambert to Hindenburg [15, p. 194]

Lambert now answers Hindenburg's letter from 7 December.

Having two copies of Hindenburg's treatise, Lambert sent one of them to Lagrange. He also sent him the copy of Felkel's work.

Among Lambert's observations to Hindenburg, he makes some comments on the decimal periods given in Hindenburg's work (pp. 113–115).

- 22 December 1776: Hindenburg to Lambert [15, p. 199]

This is the last letter from Hindenburg to Lambert.

Hindenburg now comments on Felkel's table, which he has also obtained. He observes that Euler had already used the same arrangement as Felkel.

Hindenburg observes that Wolfram probably lives in a place where few news arrive.

The rest of the letter is concerned with decimal periods.

- 8 March 1777, Lambert to Wolfram [14, p. 531]

Bernoulli mentions here an additional note by Lambert with the number of primes in a number of intervals.

- 9 August 1777, Lambert's last letter to Wolfram [14, p. 536]; Lambert died on 25 September 1777.
- Simultaneous accounts of the publication of Hindenburg's method and Felkel's table eventually appeared in 1778 in the *Allgemeine deutsche Bibliothek* [3].
- In 1782, Hindenburg published again a notice about the upcoming publication of his table [42], but the table was never published.
- In November 1783, Felkel published a short notice about his machine [32], in which he also wished that Hindenburg's tables be at least deposited in a library in Leipzig.
- Bernoulli writes that he saw Hindenburg's first million in September 1784 in Leipzig, where Hindenburg was professor [15, p. 242].

8 Conclusion

Felkel's table of factors was one of several interesting projects spawned by Lambert's call for tables. Felkel chose an uneasy road and created a quite unique and beautiful table of factors.

But Felkel's table was little used. As mentioned above, it was chiefly used by Vega as a basis for his table of primes from 102001 to 400031 published in 1797 [72]. But this table was included in the *Sammlung mathematischer Tafeln* edited by Hülke, in 1840 and 1849 [43]. Some of Vega's errors were corrected, but not all of them. $173279 = 241 \times 719$, for instance, is given as prime in both editions, and this error goes back directly to Felkel.

Vega's tables, and later Burckhardt's, made it unnecessary for anyone to delve into the intricacies of Felkel's rare opus, and it is likely that Vega was Felkel's only heir. We can only hope that the wider availability of Felkel's original works and our reconstruction will contribute to a better recognition of his work.

References

The following list covers the most important references³² related to Felkel’s table. Not all items of this list are mentioned in the text, and the sources which have not been seen are marked so. We have added notes about the contents of the articles in certain cases.

- [1] Anonymous. (Review of Hindenburg’s book). *Wittenbergsches Wochenblatt*, 9(48):385–387, 1776. [issue dated 29 November 1776]
- [2] Anonymous. (Review of Hindenburg’s book). *Zugabe zu den Göttingischen Anzeigen von gelehrte Sachen*, 1:29–31, 1777.
- [3] Anonymous. (Reviews of Hindenburg and Felkel’s books). *Allgemeine deutsche Bibliothek*, 33(2):494–497, 1778.
- [4] Anonymous. Des Hrn. Florian Ulbrich’s regulirten Chorherrn und Stadtpfarrer zu Korneuburg in Niederösterreich. *Archiv für Geographie und Statistik, ihre Hilfswissenschaften u. Litteratur*, 1(3):194–196, 1800. [obituary, reproduced from the Wiener-Zeitung of 17 May 1800]
- [5] Anonymous. (Obituary of Florian Ulbrich). *Annalen der österreichischen Literatur*, 28:217, 1802.
- [6] Anonymous. Obras, & i.e. *the Works of Francis de Borja Stockler* (review). *The Monthly review; or, Literary journal*, 51:490–494, 1806.
- [7] Anonymous. (On errors in Felkel’s and Vega’s tables). *Oesterreichischer Beobachter*, 52:251–252, 1820. [See [56] for a critical answer.]
- [8] Anonymous. Ulbrich, Florian. In Johann Jakob Heinrich Czikann and Franz Gräffer, editors, *Oesterreichische National-Encyklopädie, oder alphabetische Darlegung der wissenschaftigsten Eigenthümlichkeiten des österreichischen Kaiserthumes, etc.*, volume 5, page 464. Wien: Friedr. Beck’schen Universitäts-Buchhandlung, 1836.
- [9] Raymond Clare Archibald. Felkel’s table. *Scripta Mathematica*, 4:336–338, 1936.
- [10] Raymond Clare Archibald. New information concerning Isaac Wolfram’s life and calculations. *Mathematical Tables and other Aids to Computation*, 4(32):185–200, 1950.

³²**Note on the titles of the works:** Original titles come with many idiosyncrasies and features (line splitting, size, fonts, etc.) which can often not be reproduced in a list of references. It has therefore seemed pointless to capitalize works according to conventions which not only have no relation with the original work, but also do not restore the title entirely. In the following list of references, most title words (except in German) will therefore be left uncanceled. The names of the authors have also been homogenized and initials expanded, as much as possible.

The reader should keep in mind that this list is not meant as a facsimile of the original works. The original style information could no doubt have been added as a note, but we have not done it here.

- [11] Extrait de la correspondance de M. Bernoulli. *Nouveaux mémoires de l'Académie royale des sciences et belles-lettres, année 1781*, pages 31–35, 1783.
- [12] Johann Bernoulli, editor. *Johann Heinrich Lamberts deutscher gelehrter Briefwechsel*, volume 1. Berlin, 1782.
- [13] Johann Bernoulli, editor. *Johann Heinrich Lamberts deutscher gelehrter Briefwechsel*, volume 2. Berlin, 1782.
- [14] Johann Bernoulli, editor. *Johann Heinrich Lamberts deutscher gelehrter Briefwechsel*, volume 4. Berlin, 1784.
- [15] Johann Bernoulli, editor. *Johann Heinrich Lamberts deutscher gelehrter Briefwechsel*, volume 5. Berlin: Franz de la Garde, 1785–1787.
- [16] Johann Paul Bischoff. *Versuch einer Geschichte der Rechenmaschine, Ansbach 1804*. München: Systhema Verlag, 1990. [edited by Stephan Weiss]
- [17] Maarten Bullynck. Factor tables 1657–1817, with notes on the birth of number theory. *Revue d'histoire des mathématiques*, 16(2):133–216, 2010.
- [18] Maarten Bullynck. Stages towards a German mathematical journal (1750–1800). In Jeanne Peiffer and Jean-Pierre Vittu, editors, *Les journaux savants dans l'Europe des XVIIe et XVIIIe siècles. Formes de la communication et agents de la construction des savoirs*, page ?? Amsterdam: Brepols, 2011? [to appear]
- [19] Moritz Cantor. Felkel, Anton. In Historische Kommission bei der Bayerischen Akademie der Wissenschaften, editor, *Allgemeine Deutsche Biographie*, volume 6, page 612. Leipzig: Duncker & Humblot, 1877.
- [20] Moritz Cantor. Hindenburg, Karl Friedrich. In Historische Kommission bei der Bayerischen Akademie der Wissenschaften, editor, *Allgemeine Deutsche Biographie*, volume 12, pages 456–457. Leipzig: Duncker & Humblot, 1880.
- [21] Vincenz Darnaut, Aloys Edem von Bergenstamm, and Aloys Schützenberger. *Kirchliche Topographie von Österreich: Ein Beytrag zur Kirchen-, Staats- und Culturgeschichte des Landes*. Wien: Anton Strauß, 1819.
- [22] Johann Martin Zacharias Dase. *Factoren-tafeln für alle Zahlen der siebenten Million etc.* Hamburg: Perthes-Besser & Mauke, 1862. [reconstructed in [61]]
- [23] Warren Royal Dawson. *The Banks letters: a calendar of the manuscript correspondence of Sir Joseph Banks, preserved in the British Museum, the British Museum (Natural History) and other collections in Great Britain*. London: Printed by order of the trustees of the British Museum, 1958. [not seen, but Felkel's letter is summarized on page 325 and should still exist among Banks' letters]
- [24] Ignaz de Luca. *Das gelehrte Oesterreich. Ein Versuch*, volume 1. Wien: von Ehelenschen, 1776. [biographical notice on Felkel on pages 122–124]

- [25] Leonhard Euler. De tabula numerorum primorum usque ad millionem et ultra continuanda; in qua simul omnium numerorum non primorum minimi divisores exprimantur. *Novi Commentarii academiae scientiarum Petropolitanae*, 19:132–183, 1775.
- [26] Leonhard Euler. *Commentationes Arithmeticae*, volume 2. Leipzig: B. G. Teubner, 1917.
- [27] Anton Felkel. *Tafel aller einfachen Factoren der durch 2, 3, 5 nicht theilbaren Zahlen von 1 bis 10 000 000. I. Theil. Enthaltend die Factoren von 1 bis 144000*. Wien: von Ehelenschen, 1776. [There is also a Latin edition [28] of this first part.]
- [28] Anton Felkel. *Tabula omnium factorum simplicium numerorum per 2, 3, 5 non divisibilium, ab 1 usque 10 000 000. Pars I. Exhibens factores ab 1 usque 144000*. Wien: A. Gheleniana, 1777. [Latin version of [27].] [not seen]
- [29] Anton Felkel. *Tabula factorum. Pars II. Exhibens factores numerorum ab 144001 usque 336000*. Wien: A. Gheleniana, 1777?
- [30] Anton Felkel. *Tabula factorum. Pars III. Exhibens factores numerorum ab 336001 usque 408000*. Wien: A. Gheleniana, 1777?
- [31] Anton Felkel. *Neueröffnetes Geheimniß der Parallellinien, enthaltend verschiedene wichtige Zusätze zur Proportion- und Körperlehre*. Wien: von Ehelenschen, 1781.
- [32] Anton Felkel. Nachricht von der Felckelschen Factorenmaschine. *Leipziger Intelligenz-Blatt*, (48):406, 1783. [issue dated 1 November 1783]
- [33] Anton Felkel. Verwandlung der Bruchperioden nach den Gesetzen verschiedener Zahlensysteme. *Abhandlungen der Böhmischen Gesellschaft der Wissenschaften auf das Jahr 1785*, pages 135–174, 1785. [pp. x–xiii introduces Felkel’s article and also describes Felkel’s new table of factors]
- [34] Franz Anton de Paula Gaheis. *Spazierfahrten in die Gegenden um Wien*. 1794. [not seen]
- [35] Francisco de Borja Garção Stockler. *Obras de Francisco de Borja Garção Stockler*, volume 1. Lisbon, 1805. [volume 1, pp. 389–409 contains a “Lettre à M. Felkel ou exposition abrégée des principes de la methode de ce géomètre pour la détermination des facteurs des nombres naturels”]
- [36] James Glaisher. *Factor table for the fourth million etc*. London: Taylor and Francis, 1879. [reconstructed in [62]]
- [37] James Whitbread Lee Glaisher. *Report of the committee on mathematical tables*. London: Taylor and Francis, 1873. [Also published as part of the “Report of the forty-third meeting of the British Association for the advancement of science,” London: John Murray, 1874. A review by R. Radau was published in the *Bulletin des sciences mathématiques et astronomiques*, volume 11, 1876, pp. 7–27]

- [38] James Whitbread Lee Glaisher. On factor tables, with an account of the mode of formation of the factor table for the fourth million. *Proceedings of the Cambridge Philosophical Society*, 3(4):99–138, 1878.
- [39] James Whitbread Lee Glaisher. Table, mathematical. In Hugh Chisholm, editor, *The Encyclopædia Britannica, 11th edition*, volume 26, pages 325–336. Cambridge, England: at the University Press, 1911.
- [40] Georg Christoph Hamberger and Johann Georg Meusel. *Das gelehrte Teutschland oder Lexikon der jetzt lebenden teutschen Schriftsteller*, volume 2. Lemgo: Meyerschen Buchhandlung, 1796. [biographical notice on Felkel on pages 306–307]
- [41] Carl Friedrich Hindenburg. *Beschreibung einer ganz neuen Art, nach einem bekannten Gesetze fortgehende Zahlen, durch Abzählen oder Abmessen bequem und sicher zu finden ; nebst Anwendung der Methode auf verschiedene Zahlen, besonders auf eine darnach zu fertigende Factorentafel, mit eingestreueten, die Zahlenberechnung überhaupt betreffenden Anmerkungen*. Leipzig: Siegfried Lebrecht Crusius, 1776.
- [42] Carl Friedrich Hindenburg. (Notice on the publication of the table of factors). *Leipziger Magazin zur Naturkunde, Mathematik und Oekonomie*, page 151, 1782.
- [43] Julius Ambrosius Hülße, editor. *Sammlung mathematischer Tafeln*. Leipzig: Weidmann, 1849.
- [44] Immanuel Kant. *I. Kants sämtliche kleine Schriften*, volume 3. Königsberg, 1797. [see pp. 127–130 for Lambert’s letter]
- [45] Abraham Gotthelf Kästner. *Fortsetzung der Rechenkunst in Anwendungen auf mancherley Geschäfte*. Göttingen: Wittwe Vandenhöck, 1786. [pages 565–567 describe Felkel’s machine]
- [46] David A. King. *The ciphers of the monks: A forgotten number notation of the Middle Ages*. Stuttgart: Franz Steiner, 2001.
- [47] *Historische und topographische Darstellung von Klosterneuburg und seiner Umgegend diesseits der Donau, mit besonderer Rücksicht auf Pfarren, Stifte, Klöster, milde Stiftungen und Denkmäler*. Wien: Anton Doll, 1824.
- [48] Donald Ervin Knuth. *The art of computer programming, volume 4A: Combinatorial algorithms, part 1*. Upper Saddle River, N.J.: Addison-Wesley, 2011.
- [49] Karl Christian Friedrich Krause. *Factoren und Primzahlentafel von 1 bis 100000 neuberechnet und zweckmäßig eingerichtet nebst einer Gebrauchsanleitung und Abhandlung der Lehre von Factoren und Primzahlen*. Jena: Christian Ernst Gabler, 1804. [reconstructed in [63]]

- [50] Jakob Philipp Kulik. Magnus Canon Divisorum pro omnibus numeris per 2, 3 et 5 non divisibilibus, et numerorum primorum interjacentium ad Millies centena millia accuratius ad 100330201 usque, ca. 1825–1863. [7 manuscript volumes deposited in the Library of the Academy of Sciences, Vienna] [reconstructed in [64]]
- [51] Ernst Laas. Lambert, Johann Heinrich. In Historische Kommission bei der Bayerischen Akademie der Wissenschaften, editor, *Allgemeine Deutsche Biographie*, volume 17, pages 552–556. Leipzig: Duncker & Humblot, 1883.
- [52] Johann Heinrich Lambert. *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, volume 2. Berlin, 1770. [pp. 42–53 are about making a table of divisors and contain a table of factors from 1 to 10200]
- [53] Johann Heinrich Lambert. *Zusätze zu den Logarithmischen und Trigonometrischen Tabellen zur Erleichterung und Abkürzung der bey Anwendung der Mathematik vorkommenden Berechnungen*. Berlin: Haude und Spener, 1770. [the table of factors was reconstructed in [66]; [55] is a Latin translation of this book]
- [54] Johann Heinrich Lambert. (Announcement of the computation of factor tables). *Allgemeine deutsche Bibliothek*, 14(1):304–305, 1771.
- [55] Johann Heinrich Lambert and Anton Felkel. *Supplementa tabularum logarithmicarum et trigonometricarum*. Lisbon, 1798. [Latin translation of [53]; the table of factors was reconstructed in [65]]
- [56] Ignaz Lindner. (On errors in Vega’s table). *Oesterreichischer Beobachter*, 86:422, 1820. [This is a critical answer to [7].]
- [57] Adolf Frederik Marci. *Uitvoerige tafelen van de ondeelbaare of prim-getallen van 1 tot 400000*. Amsterdam, 1772. [not seen]
- [58] Johann Georg Megerle von Mühlfeld. *Memorabilien des Österreichischen Kaiserstaates, etc*. Wien: J. P. Sollinger, 1825. [pp. 319–321 on Florian Ulbrich]
- [59] Giuseppe Palamà. Une grande impresa: continuazione della tavola dei numeri primi di Lehmer a mezzo delle tavole del Kulik, del Poletti e del Porter. *Bollettino dell’Unione Matematica Italiana, serie 3*, 5(3-4):343–360, 1950.
- [60] H. Pröhle. Stamford, Heinrich Wilhelm v. In Historische Kommission bei der Bayerischen Akademie der Wissenschaften, editor, *Allgemeine Deutsche Biographie*, volume 35, pages 424–426. Leipzig: Duncker & Humblot, 1893.
- [61] Denis Roegel. A reconstruction of Dase’s table of factors (seventh million, 1862). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table in [22].]
- [62] Denis Roegel. A reconstruction of Glaisher’s table of factors (fourth million, 1879). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table in [36].]

- [63] Denis Roegel. A reconstruction of Krause’s table of factors (1804). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of [49].]
- [64] Denis Roegel. A reconstruction of Kulik’s “Magnus Canon Divisorum” (ca. 1825–1863): Introduction. Technical report, LORIA, Nancy, 2011. [This is a reconstruction of [50].]
- [65] Denis Roegel. A reconstruction of Lambert and Felkel’s table of factors (1798). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table of factors in [55].]
- [66] Denis Roegel. A reconstruction of Lambert’s table of factors (1770). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the table of factors in [53].]
- [67] Denis Roegel. A reconstruction of Vega’s table of primes and factors (1797). Technical report, LORIA, Nancy, 2011. [This is a partial reconstruction of [72].]
- [68] Floridus Röhrig and Herbert Fasching. *Stift Klosterneuburg und seine Kunstschatze*. Verlag Niederösterreichisches Pressehaus, 1984. [not seen]
- [69] Renate Seebauer. *Lehrerbildung in Porträts : Von der Normalschule bis zur Gegenwart*. Münster: LIT Verlag, 2011.
- [70] Paul Peter Heinrich Seelhoff. Geschichte der Factorentafeln. *Archiv der Mathematik und Physik*, 70:413–426, 1884.
- [71] Stadt Linz. *Historisches Jahrbuch der Stadt Linz*. Archiv der Stadt Linz, 1978.
- [72] Georg Vega. *Logarithmisch-trigonometrische Tafeln nebst andern zum Gebrauch der Mathematik eingerichteten Tafeln und Formeln*. Leipzig: Weidmannischen Buchhandlung, 1797. [2 volumes] [partial reconstruction in [67]]
- [73] Franz Xaver von Zach. J. H. Lambert Supplementa Tabularum logar. et trigon. cet. (review). *Monatliche Correspondenz zur Beförderung der Erd- und Himmels-Kunde*, 2:222–223, 1800.
- [74] Stephan Weiss. Nachbau und Gebrauch der Faktorenstäbe von Felkel 1776, 2011. [available on <http://www.mechrech.info/publikat/FelkelRekon.pdf>]
- [75] Hugh C. Williams. *Édouard Lucas and primality testing*. New York: John Wiley & Sons, 1998. [chapter on early devices for mechanization, where Felkel’s and Hindenburg’s machines are mentioned]