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Stochastic integration with respect to multifractional Brownian motion *via* tangent fractional Brownian motions

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Abstract

Stochastic integration w.r.t. fractional Brownian motion (fBm) has raised strong interest in recent years, motivated in particular by applications in finance and Internet traffic modelling. Since fBm is not a semi-martingale, stochastic integration requires specific developments. Multifractional Brownian motion (mBm) generalizes fBm by letting the local Hölder exponent vary in time. This is useful in various areas, including financial modelling and biomedicine. The aim of this work is twofold: first, we prove that an mBm may be approximated in law by a sequence of "tangent" fBms. Second, using this approximation, we show how to construct stochastic integrals w.r.t. mBm by "transporting" corresponding integrals w.r.t. fBm. We illustrate our method on examples such as the Hitsuda-Skohorod and Wick-Itô stochastic integrals.

Keywords: Fractional Brownian motion, multifractional Brownian motion, Gaussian processes, white noise theory, Wick-Itô integral, Hitsuda-Skohorod integral.

1 Introduction and Background

Fractional Brownian motion (fBm) is a centered Gaussian process with features that make it a useful model in various applications such as financial and Internet traffic modeling, image analysis and synthesis, physics, geophysics and more. These features include self-similarity, long range dependence and the ability to match any prescribed constant local regularity. Its covariance function R_H reads:

$$R_H(t, s) := \frac{\gamma_H}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$

where γ_H is a positive constant and H , which is usually called the Hurst exponent, belongs to $(0, 1)$. An fBm with $\gamma_H = 1$ is termed normalized. Obviously, when $H = \frac{1}{2}$, fBm reduces to standard Brownian motion. Various integral representations of fBm are known, including the harmonizable and moving average ones [20], as well as representations by integrals over a finite domain [2, 9].

The fact that most of the properties of fBm are governed by the single real H restricts its application in some situations. In particular, its Hölder exponent remains the same all along its trajectory. This does not seem to be adapted to describe adequately natural terrains, for instance. In addition, long range dependence requires $H > 1/2$, and thus imposes paths smoother than the ones of Brownian motion. Multifractional Brownian motion was introduced to overcome these limitations. The basic idea is to replace the real H by a function $t \mapsto h(t)$ ranging in $(0, 1)$.

Several definitions of multifractional Brownian motion exist. The first ones were proposed in [17] and in [4]. A more general one was introduced in [21]. In this work, we shall use a new definition that includes all previously known ones and which is, in our opinion, both more flexible and retains the essence of this class of processes. We first need to define a fractional Brownian field:

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Definition 1.1 (Fractional Brownian field). *Let (Ω, \mathcal{F}, P) be a fixed probability space. A fractional Brownian field on $\mathbb{R} \times (0, 1)$ is a Gaussian field, noted $(\mathbf{B}(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$, such that, for every H in $(0, 1)$, the process $(B_t^H)_{t \in \mathbb{R}}$ defined by $B_t^H := \mathbf{B}(t, H)$ is a fractional Brownian motion¹ with Hurst parameter H .*

Note that this definition puts no constraint on the "inter-line" behaviour of the field, *i.e.* the relation between $(B_t^H)_{t \in \mathbb{R}}$ and $(B_t^{H'})_{t \in \mathbb{R}}$ for $H \neq H'$. In order to obtain a useful model, we need to control to some extent these relations. It turns out that the following condition is sufficient to prove all the results we will need in this section. ($\mathbb{E}[Y]$ denotes the expectation of a random variable Y in $L^1(\Omega, \mathcal{F}, P)$):

$$(\mathcal{H}_1) : \forall [a, b] \subset \mathbb{R}, \forall [c, d] \subset (0, 1), \exists (\Lambda, \delta) \in (\mathbb{R}_+^*)^2, \text{ such that } \mathbb{E}[(\mathbf{B}(t, H) - \mathbf{B}(t, H'))^2] \leq \Lambda |H - H'|^\delta, \\ \text{for every } (t, H, H') \text{ in } [a, b] \times [c, d]^2.$$

Note that assumption (\mathcal{H}_1) is equivalent to the following one:

$$(\mathcal{H}) : \forall [a, b] \times [c, d] \subset \mathbb{R} \times (0, 1), \exists (\Lambda, \delta) \in (\mathbb{R}_+^*)^2, \text{ s.t. } \mathbb{E}[(\mathbf{B}(t, H) - \mathbf{B}(s, H'))^2] \leq \Lambda (|t - s|^{2c} + |H - H'|^\delta), \\ \text{for every } (t, s, H, H') \in [a, b]^2 \times [c, d]^2.$$

Indeed, since \mathbf{B} is a fractional field, for every (t, s, H) in $[a, b]^2 \times [c, d]$, $\mathbb{E}[(\mathbf{B}(t, H) - \mathbf{B}(s, H))^2] = |t - s|^{2H}$. Then if (\mathcal{H}_1) is true we have, for all $(t, s, H, H') \in [a, b]^2 \times [c, d]^2$,

$$\mathbb{E}[(\mathbf{B}(t, H) - \mathbf{B}(s, H'))^2] \leq 2\Lambda (1 + |b - a|^{2(d-c)}) (|t - s|^{2c} + |H - H'|^\delta).$$

Thus, we will refer either to assumption (\mathcal{H}_1) or (\mathcal{H}) in the sequel.

Remark 1.2. (i) *Assumption (\mathcal{H}) entails that the map $C : (t, s, H, H') \mapsto \mathbb{E}[\mathbf{B}(t, H) \mathbf{B}(s, H')]$ is continuous on $\mathbb{R}^2 \times (0, 1)^2$.*

(ii) *Let p be an integer such that $p(2c \wedge \delta) > 2$. Let κ_p denote the constant such that $\mathbb{E}[Y^{2p}] = \kappa_p (\mathbb{E}[Y^2])^p$ where Y is a centered Gaussian random variable. Hypothesis (\mathcal{H}) entails that for all $(t, s, H, H') \in [a, b]^2 \times [c, d]^2$,*

$$\mathbb{E}[(\mathbf{B}(t, H) - \mathbf{B}(s, H'))^{2p}] = \kappa_p (\mathbb{E}[(\mathbf{B}(t, H) - \mathbf{B}(s, H'))^2])^p \leq \kappa_p \Lambda^p (|t - s|^{2c} + |H - H'|^\delta)^p \\ \leq \kappa_p (2\tilde{\Lambda})^p \max\{|t - s|; |H - H'|^\delta\}^{p(\delta \wedge 2c)},$$

where $\tilde{\Lambda} := \Lambda (1 + b - a)^{\delta \vee 2c - \delta \wedge 2c}$, $x \vee y := \max\{x; y\}$ and $x \wedge y := \min\{x; y\}$. Kolmogorov's criterion implies that the field \mathbf{B} has a d -Hölder continuous version for any d in $(0, \frac{2p}{p(\delta \wedge 2c) - 2})$. In the sequel we will always work with this version.

An mBm is defined as follows:

Definition 1.3 (Multifractional Brownian motion). *Let $h : \mathbb{R} \rightarrow (0, 1)$ be a deterministic continuous function. A multifractional Brownian motion with functional parameter h is the Gaussian process $B^h := (B_t^h)_{t \in \mathbb{R}}$ defined by $B_t^h := \mathbf{B}(t, h(t))$ for every t in \mathbb{R} .*

A word on notation: B^H or $B^{h(t)}$ will always denote an fBm with Hurst index H or $h(t)$, while B^h will stand for an mBm. Note that $B_t^h := \mathbf{B}(t, h(t)) = B_t^{h(t)}$, for every real t .

Furthermore, we say that h is the *regularity function* of the mBm. We will say that the fractional field $(\mathbf{B}(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$ is normalized when, for all H in $(0, 1)$, the fractional Brownian motion $(B_t^H)_{t \in \mathbb{R}}$, resulting from the fractional field $(\mathbf{B}(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$, is a normalized fBm. In this case we will also say that the mBm $B^h := (B_t^h)_{t \in \mathbb{R}}$ is normalized.

¹One might also want to start from a family of fBms $(B^H)_{H \in (0, 1)}$ (*i.e.* $B^H := (B_t^H)_{t \in \mathbb{R}}$ is an fBm for every H in $(0, 1)$) and define from it the field $(\mathbf{B}(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$ by $\mathbf{B}(t, H) := B_t^H$. However it is not true, in general, that the field $(\mathbf{B}(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$ obtained in this way is Gaussian.

It is straightforward to check that an mBm in the sense of [21, def.1.1] is also an mBm with our definition. [21] provides the correlation structure of various mBms. In general, it is clear that the correlation structures of two mBms derived from different fractional Brownian fields will differ whenever the "inter-line" correlations of the fields differ.

Fractional fields $(\mathbf{B}(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$ leading to previously considered mBms include:

$$\begin{aligned} \mathbf{B}_1(t, H) &:= \frac{1}{c_H} \int_{\mathbb{R}} \frac{e^{itu} - 1}{|u|^{H+1/2}} \widetilde{\mathbb{W}}_1(du), & \mathbf{B}_2(t, H) &:= \int_{\mathbb{R}} (|t-u|^{H-1/2} - |u|^{H-1/2}) \mathbb{W}_2(du), \\ \mathbf{B}_3(t, H) &:= \int_{\mathbb{R}} ((t-u)_+^{H-1/2} - (-u)_+^{H-1/2}) \mathbb{W}_3(du), & \mathbf{B}_4(t, H) &:= \int_0^T \mathbb{1}_{\{0 \leq u < t \leq T\}}(t, u) K_H(t, u) \mathbb{W}_4(du), \end{aligned}$$

where $c_H := \left(\frac{2 \cos(\pi H) \Gamma(2-2H)}{H(1-2H)} \right)^{\frac{1}{2}}$, $d_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(\frac{1}{2}+H)\Gamma(2-2H)} \right)^{\frac{1}{2}}$ and

$$K_H(t, s) := d_H (t-s)^{H-\frac{1}{2}} + c_H \left(\frac{1}{2} - H \right) \int_s^t (u-s)^{H-\frac{3}{2}} \left(1 - \left(\frac{s}{u} \right)^{\frac{1}{2}-H} \right) du,$$

and where \mathbb{W}_i denotes an independently scattered standard Gaussian measure on \mathbb{R} , $i \in \{2; 3; 4\}$, and $\widetilde{\mathbb{W}}_1$ denotes the complex-valued Gaussian measure which can be associated in a unique way to any independently scattered standard Gaussian measure \mathbb{W}_1 on \mathbb{R} (see [21] p.203-204 and [20] p.325-326 for more details on the meaning of $\int_{\mathbb{R}} f(u) \mathbb{W}(du)$ and $\int_{\mathbb{R}} f(u) \widetilde{\mathbb{W}}(du)$ for a real or complex-valued function f in $L^2(\mathbb{R}, du)$). Replacing H with $h(t)$ in $\mathbf{B}_1(t, H)$ and $\mathbf{B}_3(t, H)$ leads to the so-called harmonisable mBm, first considered in [4]. The same operation on $\mathbf{B}_2(t, H)$ yields the moving average mBm defined in [17]. Both are particular cases of mBms in the sense of [21]. Finally, $\mathbf{B}_4(t, h(t))$ corresponds to the Volterra multifractional Gaussian process studied in [9]. This last process is not an mBm in the sense of [21], but is one in our sense.

The remaining of this paper is organized as follows. Section 2 proves that an mBm may be approximated in law, as well as in the L^2 and almost sure senses, by a sequence of tangent fBms. Section 3 uses the result of section 2 to define a stochastic integral w.r.t. mBm as a limit of integrals w.r.t. approximating fBms. We provide a condition on the stochastic integral w.r.t. fBm that guarantees convergence of the sequence of approximations. In other words, as soon as a method of integration w.r.t. fBm verifies this condition, then our method allows to "transport" it into an integral w.r.t. mBm. We apply this construction to the case of the Skohorod integral in section 4, and of the Wick-Itô integral in section 5.

2 Approximation of multifractional Brownian motion

Since an mBm is just a continuous path traced on a fractional Brownian field, a natural question is to enquire whether it may be approximated by patching adequately chosen fBms, and in which sense. Heuristically, for $a < b$, we divide $[a, b]$ into "small" intervals $[t_i, t_{i+1})$, and replace on each of these B^h by the fBm B^{H_i} where $H_i = h(t_i)$. It seems reasonable to expect that the resulting process $\sum_i B_t^{H_i} \mathbb{1}_{[t_i, t_{i+1})}(t)$ will converge, in a sense to be made precise, to B^h when the size of the intervals $[t_i, t_{i+1})$ goes to 0. This idea is further supported by the fact that the moving average, harmonisable and Volterra mBms are all "tangents" to fBms in the following sense: for every real u ,

$$\left\{ \frac{B_{u+rt}^h - B_u^h}{r^{h(u)}}; t \in [a, b] \right\} \xrightarrow[r \rightarrow 0^+]{\text{law}} \{B_t^{h(u)}; t \in [a, b]\}$$

Our aim in this section is to make this line of thought rigorous.

2.1 Approximation of mBm by piecewise fBms

In the sequel, we fix a fractional Brownian field \mathbf{B} and a continuous function h , thus an mBm. We aim to prove that this mBm can be approximated on every compact interval $[a, b]$ by patching together fractional Brownian motions defined on a sequence of partitions of $[a, b]$. In that view, define an increasing sequence

$(q_n)_{n \in \mathbb{N}}$ of integers such that $q_0 := 1$, $2^n \leq q_n \leq 2^{2^n}$ for every n in \mathbb{N}^* . For a compact interval $[a, b]$ of \mathbb{R} and n in \mathbb{N} , let $x^{(n)} := \{x_k^{(n)}; k \in \llbracket 0, q_n \rrbracket\}$ where $x_k^{(n)} := a + k \frac{(b-a)}{q_n}$ for every k in $\llbracket 0, q_n \rrbracket$ (for integers p and q with $p < q$, $\llbracket p, q \rrbracket$ denotes the set $\{p; p+1; \dots; q\}$). Now, if we define, for every n in \mathbb{N} , the partition $\mathcal{A}_n := \{[x_k^{(n)}, x_{k+1}^{(n)}]; k \in \llbracket 0, q_n - 1 \rrbracket\} \cup \{x_{q_n}^{(n)}\}$, it is clear that $\mathcal{A} := (\mathcal{A}_n)_{n \in \mathbb{N}}$ is a decreasing nested sequence of subdivisions of $[a, b]$ (i.e. $\mathcal{A}_{n+1} \subset \mathcal{A}_n$, for every n in \mathbb{N}).

For every t in $[a, b]$ and n in \mathbb{N} there exists a unique integer p in $\llbracket 0, q_n - 1 \rrbracket$ such that $x_p^{(n)} \leq t < x_{p+1}^{(n)}$. We will note $x_t^{(n)}$ the real $x_p^{(n)}$ in the sequel. The sequence $(x_t^{(n)})_{n \in \mathbb{N}}$ is increasing and converges to t as n tends to $+\infty$. Besides, define for every n in \mathbb{N} , the function $h_n : [a, b] \rightarrow (0, 1)$ by setting $h_n(b) = h(b)$ and, for any t in $[a, b)$, $h_n(t) := h(x_t^{(n)})$. The sequence of functions $(h_n)_{n \in \mathbb{N}}$ converges pointwise to h on $[a, b]$. Define, for t in $[a, b]$ and n in \mathbb{N} , the process

$$B_t^{h_n} := \mathbf{B}(t, h_n(t)) = \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \mathbf{B}(t, h(x_k^{(n)})) + \mathbb{1}_{\{b\}}(t) \mathbf{B}(b, h(b)). \quad (2.1)$$

Note that, despite the notation, the process B^{h_n} is not an mBm, as h_n is not continuous in general. We believe however there is no risk of confusion in using this notation. B^{h_n} is almost surely càdlàg and discontinuous at times $x_k^{(n)}$, k in $\llbracket 0, q_n \rrbracket$.

Theorem 2.1 (Approximation theorem). *Let \mathbf{B} be a fractional Brownian field, $h : \mathbb{R} \rightarrow (0, 1)$ be a continuous deterministic function and B^h be the associated mBm. Let $[a, b]$ be a compact interval of \mathbb{R} , \mathcal{A} be a sequence of partitions as defined above, and consider the sequence of processes defined in (2.1). Then:*

1. *If \mathbf{B} is such that the map C (defined in (i) of remark 1.2) is continuous on $[a, b]^2 \times h([a, b])^2$ then the sequence of processes $(B^{h_n})_{n \in \mathbb{N}}$ converges in $L^2(\Omega)$ to B^h , i.e*

$$\forall t \in [a, b], \lim_{n \rightarrow +\infty} E \left[(B_t^{h_n} - B_t^h)^2 \right] = 0.$$

2. *If \mathbf{B} satisfies assumption (\mathcal{H}) and if h is β -Hölder continuous for some positive real β , then the sequence of processes $(B^{h_n})_{n \in \mathbb{N}}$ converges*

$$(i) \text{ in law to } B^h, \text{ i.e.} \quad \{B_t^{h_n}; t \in [a, b]\} \xrightarrow[n \rightarrow +\infty]{law} \{B_t^h; t \in [a, b]\}.$$

$$(ii) \text{ almost surely to } B^h, \text{ i.e.} \quad P \left(\forall t \in [a, b], \lim_{n \rightarrow +\infty} B_t^{h_n} = B_t^h \right) = 1.$$

Proof. 1. Let $t \in [a, b]$. For any n in \mathbb{N} , and using the notations of (i) in remark 1.2, one has

$$E \left[(B_t^{h_n} - B_t^h)^2 \right] = C(t, t, h(x_t^{(n)}), h(x_t^{(n)})) - 2 C(t, t, h(x_t^{(n)}), h(t)) + C(t, t, h(t), h(t)).$$

The continuity of the maps $h, (t, H, H') \mapsto C(t, t, H, H')$ and the fact that $\lim_{n \rightarrow +\infty} x_t^{(n)} = t$ entail that $\lim_{n \rightarrow \infty} E \left[(B_t^{h_n} - B_t^h)^2 \right] = 0$.

2. By assumption, there exists η in \mathbb{R}_+^* such that for all (s, t) in $[a, b]$,

$$|h(s) - h(t)| \leq \eta |s - t|^\beta. \quad (2.2)$$

(i) We proceed as usual in two steps (see for examples [8, 19]), **a)**: finite-dimensional convergence and **b)**: tightness of the sequence of probability measures $(P \circ B^{h_n})_{n \in \mathbb{N}}$.

a) Finite dimensional convergence

Since the process B^h and, for every n in \mathbb{N} , the process B^{h_n} defined by (2.1) are centered and Gaussian it is sufficient to prove that $\lim_{n \rightarrow \infty} E \left[B_t^{h_n} B_s^{h_n} \right] = E \left[B_t^h B_s^h \right]$ for every (s, t) in $[a, b]^2$.

The cases where $t = b$ or $s = t$ are consequences of point 1. above. We now assume that $a \leq s < t < b$.

For every integer n in \mathbb{N} , we get

$$E \left[B_t^{h_n} B_s^{h_n} \right] = \sum_{(k,j) \in \llbracket 0, q_n - 1 \rrbracket^2} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) \mathbb{1}_{[x_j^{(n)}, x_{j+1}^{(n)}]}(s) E \left[\mathbf{B}(t, h_n(t)) \mathbf{B}(s, h_n(s)) \right].$$

We hence get $E \left[B_t^{h_n} B_s^{h_n} \right] = E \left[\mathbf{B}(t, h(x_t^{(n)})) \mathbf{B}(s, h(x_s^{(n)})) \right]$ for all large enough integers n , (i.e such that $x_s^{(n)} \leq s < x_t^{(n)} \leq t$).

The continuity of h , (i) of remark 1.2, and the fact that $\lim_{n \rightarrow \infty} (x_t^{(n)}, x_s^{(n)}) = (t, s)$, entail that

$$\lim_{n \rightarrow \infty} E \left[B_t^{h_n} B_s^{h_n} \right] = E \left[B_t^h B_s^h \right].$$

b) Tightness of the sequence of probability measures $(P \circ B^{h_n})_{n \in \mathbb{N}}$.

We are in the particular case where a sequence of *càdlàg* processes converges to a continuous one. Theorem page 92 of [18] applies to this situation: it is sufficient to show that, for every positive reals ε and τ , there exist an integer m and a grid $\{t_i\}_{i \in \llbracket 0, m \rrbracket}$ such that $a = t_0 < t_1 < \dots < t_m = b$ that verify

$$\limsup_{n \rightarrow +\infty} P \left(\left\{ \max_{0 \leq i \leq m} \sup_{t \in [t_i, t_{i+1}]} |B_t^{h_n} - B_{t_i}^{h_n}| > \tau \right\} \right) < \varepsilon. \quad (2.3)$$

Let us then fix (ε, τ) in $(\mathbb{R}_+^*)^2$. Define $[H_1, H_2] := [\inf_{u \in [a, b]} h(u), \sup_{u \in [a, b]} h(u)]$ and set, until the end of this proof,

$q_n := 2^{2^n}$, $n \in \mathbb{N}$. Define $F := [a, b] \times [H_1, H_2]$. The process $(\mathbf{B}(t, H))_{(t, H) \in F}$ is Gaussian and the space of continuous real-valued functions defined on F endowed with the sup-norm is a separable Banach space. Fernique theorem (see [10, theorem 2.6 p.37]) applies to the effect that there exists a positive real α such that $A_\alpha := E \left[\exp \left\{ \alpha \sup_{(t, H) \in F} \mathbf{B}(t, H)^2 \right\} \right] < +\infty$. Set $G := L \sum_{p=0}^{+\infty} \frac{2^{p/2}}{q_p^{\delta/2}}$, where L is the universal constant in [22, Theorem 2.1.1 p. 33]. Define also $D := \Lambda (\eta (b - a)^\beta)^\delta$ where Λ and δ are the constants appearing in (\mathcal{H}) and η in (2.2). Let N be the smallest integer n such that

$$\max \left\{ A_\alpha (1 + q_n) \exp \left\{ \frac{-\alpha \tau^2}{4|b-a|2H_2} q_n^{2H_2} \right\}; 4 (1 + q_n) \exp \left\{ \frac{-\tau^2}{2\tau D} q_n^{\delta\beta} \right\}; \frac{b-a}{q_n}; \frac{G D^{1/2}}{q_n^{\frac{\delta\beta}{2}}} \right\} < 1 \wedge \frac{\tau}{8} \wedge \frac{\varepsilon}{2}. \quad (2.4)$$

Set $m := m(\tau, \varepsilon) = q_N$ and $t_i := x_i^{(m)}$ for i in $\llbracket 0, m \rrbracket$. Note that this entails that, for all n larger than N , $h_n(t_i) = h(t_i)$. Besides, also as soon as $n \geq N$, $h_n(t)$ belongs to the set $h([t_i, t_{i+1}])$ when $t \in [t_i, t_{i+1}]$ (draw a picture). Let $J_n^{\tau, m} := P \left(\left\{ \max_{0 \leq i \leq m} \sup_{t \in [t_i, t_{i+1}]} |B_t^{h_n} - B_{t_i}^{h_n}| > \tau \right\} \right)$. Then

$$J_n^{\tau, m} \leq (1 + m) \max_{0 \leq i \leq m} P \left(\left\{ \sup_{t \in [t_i, t_{i+1}]} |\mathbf{B}(t, h_n(t)) - \mathbf{B}(t_i, h_n(t_i))| > \tau \right\} \right) \leq (1 + m) \left(\max_{0 \leq i \leq m} L_n^{\tau, i} + \max_{0 \leq i \leq m} Q_n^{\tau, i} \right), \quad (2.5)$$

where

$$L_n^{\tau, i} := P \left(\left\{ \sup_{t \in [t_i, t_{i+1}]} |\mathbf{B}(t, h_n(t)) - \mathbf{B}(t_i, h_n(t_i))| > \tau/2 \right\} \right)$$

and

$$Q_n^{\tau, i} := P \left(\left\{ \sup_{t \in [t_i, t_{i+1}]} |\mathbf{B}(t_i, h_n(t)) - \mathbf{B}(t_i, h_n(t_i))| > \tau/2 \right\} \right).$$

- Upper bound for $(1 + m) \max_{0 \leq i \leq m} L_n^{\tau, i}$:

The couple (i, n) being fixed in $\llbracket 0, m \rrbracket \times \mathbb{N}$, the process $(\mathbf{B}(s, h_n(t)))_{s \in [t_i, t_{i+1}]}$ is a fractional Brownian motion of Hurst index $h_n(t)$. Using increment-stationarity and self-similarity of fractional Brownian motion yields:

$$\begin{aligned}
L_n^{\tau,i} &= P \left(\sup_{t \in [t_i, t_{i+1}]} |\mathbf{B}(t - t_i, h_n(t))| > \tau/2 \right) \\
&= P \left(\sup_{u \in [0, t_{i+1} - t_i]} |t_{i+1} - t_i|^{h_n(u+t_i)} \left| \mathbf{B} \left(\frac{u}{t_{i+1} - t_i}, h_n(u+t_i) \right) \right| > \tau/2 \right) \\
&\leq P \left(\sup_{v \in [0,1]} |\mathbf{B}(v, h_n(t_i + v(t_{i+1} - t_i)))| > \frac{\tau}{2|t_{i+1} - t_i|^{H_2}} \right). \tag{2.6}
\end{aligned}$$

Using Markov identity and Fernique's theorem,

$$\begin{aligned}
L_n^{\tau,i} &\leq P \left(\sup_{(v,H) \in F} |\mathbf{B}(v, H)| > \frac{\tau}{2|t_{i+1} - t_i|^{H_2}} \right) = P \left(\exp \left\{ \alpha \sup_{(v,H) \in F} |\mathbf{B}(v, H)|^2 \right\} > \exp \left\{ \frac{\alpha \tau^2}{4|t_{i+1} - t_i|^{2H_2}} \right\} \right) \\
&\leq E \left[\exp \left\{ \alpha \sup_{(t,H) \in F} \mathbf{B}(t, H)^2 \right\} \right] \exp \left\{ \frac{-\alpha \tau^2}{4|t_{i+1} - t_i|^{2H_2}} \right\} = A_\alpha \exp \left\{ \frac{-\alpha \tau^2 q_N^{2H_2}}{4|b-a|^{2H_2}} \right\} \\
&< \frac{\varepsilon}{2(1+q_N)} = \frac{\varepsilon}{2(1+m)}.
\end{aligned}$$

We have shown that

$$\forall i \in \llbracket 0, q_N \rrbracket, \forall n \geq N, \quad (1+m) \max_{0 \leq i \leq m} L_n^{\tau,i} < \frac{\varepsilon}{2}. \tag{2.7}$$

- Upper bound for $(1+m) \max_{0 \leq i \leq m} Q_n^{\tau,i}$:

Fix a couple (i, n) in $\llbracket 0, m \rrbracket \times \mathbb{N}$. Recall that $h_n(t)$ belongs to $h(\llbracket t_i, t_{i+1} \rrbracket) =: \mathcal{K}^{(i)}$ for every t in $[t_i, t_{i+1}]$. We hence have,

$$Q_n^{\tau,i} \leq P \left(\left\{ \sup_{H \in \mathcal{K}^{(i)}} |\mathbf{B}(t_i, H) - \mathbf{B}(t_i, h(t_i))| > \tau/2 \right\} \right) \leq 2 P \left(\left\{ \sup_{H \in \mathcal{K}^{(i)}} \overbrace{\mathbf{B}(t_i, H) - \mathbf{B}(t_i, h(t_i))}^{=: X_i(H)} > \tau/4 \right\} \right). \tag{2.8}$$

Observe that the right hand side of the previous inequality does not depend on n any more.

Our aim is to apply [1, (2.6) p.43]. In that view, we first prove the following estimate:

Lemma 2.2. *For all i in $\llbracket 0, q_N \rrbracket$,* $\mu_i := E \left[\sup_{H \in \mathcal{K}^{(i)}} X_i(H) \right] < \frac{G D^{1/2}}{q_N^{\delta\beta/2}} < \frac{\tau}{8}$.

Proof of lemma 2.2: Fix i in $\llbracket 0, q_N \rrbracket$. We recall some notions from [22]. A sequence $C^{(i)} := (C_n^{(i)})_{n \in \mathbb{N}}$ of partitions of $\mathcal{K}^{(i)}$ is called admissible if it is increasing and such that $\text{card}(C_n^{(i)}) \leq 2^{2^n}$, for every n in \mathbb{N} . Let d_i denote the pseudo-distance associated to the Gaussian process $(X_i(H))_{H \in \mathcal{K}^{(i)}}$, (i.e $d_i(H, H') := (\mathbb{E}[(X_i(H) - X_i(H'))^2])^{1/2}$, for every (H, H') in $\mathcal{K}^{(i)} \times \mathcal{K}^{(i)}$).

For (H, p) in $\mathcal{K}^{(i)} \times \mathbb{N}$, let $C_p^{(i)}(H)$ be the unique element of the partition $C_p^{(i)}$ which contains H . The diameter of $C_p^{(i)}(H)$ is by definition $\Delta_i(C_p^{(i)}(H)) := \sup_{(H, H') \in C_p^{(i)}(H)^2} d_i(H, H')$. [22, Theorem 2.1.1] entails that:

$$\mu_i \leq L \gamma_2(\mathcal{K}^{(i)}, d_i), \tag{2.9}$$

where $\gamma_2(\mathcal{K}^{(i)}, d_i) := \inf \sup_{H \in \mathcal{K}^{(i)}} \sum_{p \geq 0} 2^{p/2} \Delta_i(C_p^{(i)}(H))$ and where the infimum is taken over all admissible

sequences of partitions of $\mathcal{K}^{(i)}$. Let $H_1^{(i)}$ and $H_2^{(i)}$ be such that $\mathcal{K}^{(i)} = [H_1^{(i)}, H_2^{(i)}]$. Consider the sequence of partitions $C^{(i)} := (C_n^{(i)})_{n \in \mathbb{N}}$ of $\mathcal{K}^{(i)}$ defined, for every integer n , by setting $C_n^{(i)} := \{[y_k^{(n)}, y_{k+1}^{(n)}]; k \in \llbracket 0, q_n - 1 \rrbracket\} \cup \{y_{q_n}^{(n)}\}$, where $y_0^{(n)} = H_1^{(i)}$ and $y_{k+1}^{(n)} - y_k^{(n)} = \frac{H_2^{(i)} - H_1^{(i)}}{q_n}$, for (n, k) in $\mathbb{N} \times \llbracket 0, q_n - 1 \rrbracket$.

It is clear that $C^{(i)}$ is a decreasing nested sequence of partitions of $\mathcal{K}^{(i)}$ and is hence admissible. For (H_0, p) in $\mathcal{K}^{(i)} \times \mathbb{N}$, denote $[y_{k_0}^{(p)}, y_{k_0+1}^{(p)})$ the unique element of $C_p^{(i)}$ which contains H_0 . Then, using (\mathcal{H}) ,

$$\begin{aligned} \mu_i &\leq L \sum_{p \geq 0} 2^{p/2} \sup_{H_0 \in \mathcal{K}^{(i)}(H, H') \in [y_{k_0}^{(p)}, y_{k_0+1}^{(p)})^2} \sup_{(H, H') \in [y_{k_0}^{(p)}, y_{k_0+1}^{(p)})^2} (\mathbb{E}[(X_i(H) - X_i(H'))^2])^{1/2} \\ &\leq L \sum_{p \geq 0} 2^{p/2} \sup_{k \in \llbracket 0, q_p - 1 \rrbracket} \sup_{(H, H') \in [y_k^{(p)}, y_{k+1}^{(p)})^2} (\mathbb{E}[(X_i(H) - X_i(H'))^2])^{1/2} \\ &\leq L \Lambda^{1/2} \sum_{p \geq 0} 2^{p/2} \sup_{k \in \llbracket 0, q_p - 1 \rrbracket} |y_{k+1}^{(p)} - y_k^{(p)}|^{\delta/2} = G \Lambda^{1/2} (H_2^{(i)} - H_1^{(i)})^{\delta/2}. \end{aligned}$$

By Hölder continuity of h , $H_2^{(i)} - H_1^{(i)} \leq \eta |t_{i+1} - t_i|^\beta = \eta \frac{|b-a|^\beta}{q_N^\beta}$. Using (2.4), we finally get $\mu_i < \frac{GD^{1/2}}{q_N^{\delta\beta/2}} < \tau/8$. \square

Note that lemma 2.2 implies that $(\tau/8)^2 < (\tau/4 - \mu_i)^2$. Let's go back to the proof of tightness. [1, (2.6) p.43] yields, for all $n \geq N$,

$$Q_n^{\tau, i} \leq 4 \exp \left\{ \frac{-(\tau/4 - \mu_i)^2}{2\sigma_i^2} \right\},$$

where $\sigma_i^2 := \sup_{H \in \mathcal{K}^{(i)}} \mathbb{E}[X_i(H)^2]$. By definition of $(X_i(H))_{H \in \mathcal{K}^{(i)}}$ and assumption (\mathcal{H}) :

$$\sigma_i^2 \leq \Lambda \sup_{H \in \mathcal{K}^{(i)}} |H - h(t_i)|^\delta \leq \Lambda \eta^\delta |t_{i+1} - t_i|^{\beta\delta} = \Lambda \eta^\delta \frac{|b-a|^{\beta\delta}}{q_N^{\beta\delta}} = \frac{D}{q_N^{\beta\delta}}.$$

This yields that $Q_n^{\tau, i} \leq 4 \exp \left\{ \frac{-\tau^2 q_N^{\delta\beta}}{2^r D} \right\}$ and finally:

$$\forall i \in \llbracket 0, q_N \rrbracket, \forall n \geq N, \quad (1+m) \max_{0 \leq i \leq m} Q_n^{\tau, i} < \frac{\varepsilon}{2}. \quad (2.10)$$

Using (2.7) and (2.10), inequality (2.5) then becomes $J_n^{\tau, m} < \varepsilon$ for every $n \geq N$, which ends the proof. \square

(ii) Almost sure convergence

Denote $\tilde{\Omega}$ the measurable subset of Ω , verifying $P(\tilde{\Omega}) = 1$, such that for every ω in $\tilde{\Omega}$, $(t, H) \mapsto B(t, H)(\omega)$ is continuous on $[a, b] \times [H_1, H_2]$. Then, for every ω in $\tilde{\Omega}$, we get:

$$|B_t^{h_n}(\omega') - B_t^h(\omega')| = |B(t, h_n(t))(\omega') - B(t, h(t))(\omega')| = |B(t, h(x_i^{(n)}))(\omega') - B(t, h(t))(\omega')| \xrightarrow[n \rightarrow +\infty]{} 0.$$

This ends the proof. \square

The four fractional fields $(\mathbf{B}_i(\cdot, \cdot))_{i \in \llbracket 1, 4 \rrbracket}$ defined at the end of section 1 verify assumption (\mathcal{H}) , as shown by the next proposition:

Proposition 2.3. *The fractional Brownian fields $\mathbf{B}_i := (\mathbf{B}_i(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$, $i \in \llbracket 1, 4 \rrbracket$, fulfill Assumption (\mathcal{H}_1) .*

Proof: The case of \mathbf{B}_1 and \mathbf{B}_2 :

(3.10) of [21] yields that $\mathbf{B}_2(t, H) \stackrel{(law)}{=} \frac{2 \cos((H+1/2)\frac{\pi}{2})}{\sqrt{2\pi}} \Gamma(H+1/2) c_H \mathbf{B}_1(t, H)$. It is thus sufficient to establish (\mathcal{H}_1) for \mathbf{B}_1 . Let $[a, b] \times [c, d]$ be fixed in $\mathbb{R} \times (0, 1)$. For all (t, H, H') in $[a, b] \times [c, d]^2$,

$$\begin{aligned} I_t^{H, H'} &:= \mathbb{E} \left[(B(t, H) - B(t, H'))^2 \right] = \int_{\mathbb{R}} \left| \frac{e^{it\xi} - 1}{c_H |\xi|^{H+1/2}} - \frac{e^{it\xi} - 1}{c_{H'} |\xi|^{H'+1/2}} \right|^2 d\xi \\ &= \int_{\mathbb{R}} \left| \frac{e^{it\xi} - 1}{\xi} \right|^2 \left| \frac{1}{c_H} |\xi|^{1/2-H} - \frac{1}{c_{H'}} |\xi|^{1/2-H'} \right|^2 d\xi. \end{aligned} \quad (2.11)$$

For ξ in \mathbf{R}^* , the map $f_\xi : [c, d] \rightarrow \mathbf{R}_+$, defined by $f_\xi(H) := \frac{1}{c_H} |\xi|^{1/2-H}$ is C^1 since $H \mapsto c_H$ is C^∞ on $(0, 1)$. Thus there exists a positive real D such that

$$\forall (\xi, H) \in \mathbf{R}^* \times [c, d], |f'_\xi(H)| \leq D |\xi|^{1/2-H} (1 + |\ln(|\xi|)|) \leq D (|\xi|^{1/2-c} + |\xi|^{1/2-d}) (1 + |\ln(|\xi|)|).$$

Thanks to the mean-value theorem, (2.11) then reads

$$\begin{aligned} I_t^{H, H'} &\leq D^2 |H - H'|^2 \int_{\mathbf{R}} \frac{|e^{it\xi} - 1|^2}{|\xi|^2} (|\xi|^{1/2-c} + |\xi|^{1/2-d})^2 (1 + |\ln(|\xi|)|)^2 d\xi \\ &\leq D^2 |H - H'|^2 \left(2^3 \int_{|\xi| > 1} \frac{(1 + \ln|\xi|)^2}{|\xi|^{1+2c}} d\xi + (2t)^2 \int_{|\xi| \leq 1} |\xi|^{1-2d} (1 + |\ln(|\xi|)|)^2 d\xi \right) \\ &\leq (2^3 + T^2) D^2 \left(\int_{|\xi| > 1} \frac{(1 + \ln|\xi|)^2}{|\xi|^{1+2c}} d\xi + \int_{|\xi| \leq 1} |\xi|^{1-2d} (1 + |\ln(|\xi|)|)^2 d\xi \right) |H - H'|^2. \end{aligned}$$

Since the two integrals in the last line are finite, assumption (\mathcal{H}_1) is verified with $\delta = 2$.

The case of \mathbf{B}_3 is settled in [17, proof of theorem 4, (2.10)] and the one of \mathbf{B}_4 in [9, Proposition 3, (5)]. \square

2.2 Particular cases of kernel integral representations of \mathbf{B}

In many cases of interest, \mathbf{B} will be specified through an integral representation. It is thus relevant to recast assumption (\mathcal{H}) in terms of the kernel used in these representations. We need to distinguish between two situations: the case where the integral is over a compact interval, and where it is over \mathbb{R} .

2.2.1 On a compact set $[0, T]$

In this case (see, e.g [2]), the fractional field $(\mathbf{B}(t, H))_{(t, H) \in [0, T] \times (0, 1)}$ is defined by

$$\mathbf{B}(t, H) := \int_0^T K(t, u, H) \mathbb{W}(du),$$

where \mathbb{W} is a Gaussian measure, K is defined on $[0, T]^2 \times (0, 1)$ and is such that $u \mapsto K(t, u, H)$ belongs to $L^2([0, T], du)$, for all (t, H) in $[0, T] \times (0, 1)$. This case is for instance the one of \mathbf{B}_4 . As one can easily see, the following condition (C_K) entails (\mathcal{H}) .

$$(C_K) : \forall (c, d) \text{ with } 0 < c < d < 1, H \mapsto K(t, u, H) \text{ is Hölder continuous on } [c, d], \text{ uniformly in } (t, u) \text{ in } [0, T]^2, \\ \text{i.e } \exists (M, \delta) \in (\mathbb{R}_+^*)^2, \forall (t, u) \text{ in } [0, T]^2, |K(t, u, H) - K(t, u, H')| \leq M |H - H'|^\delta.$$

Condition (C_K) is fulfilled by the kernel K defining \mathbf{B}_4 (see [9, Proposition 3, (5)]).

2.2.2 On \mathbb{R}

A representation with an integral over \mathbb{R} is used for instance in [7, 11]. The fractional field $(\mathbf{B}(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$ is then defined by $\mathbf{B}(t, H) := \int_{\mathbb{R}} M(t, u, H) \mathbb{W}(du)$ where M is defined on $\mathbb{R}^2 \times (0, 1)$, and is such that $u \mapsto M(t, u, H)$ belongs to $L^2(\mathbb{R}, du)$, for every (t, H) in $\mathbb{R} \times (0, 1)$. This is the case for the fields \mathbf{B}_1 , \mathbf{B}_2 and \mathbf{B}_3 . Condition (C_M) entails (\mathcal{H}) :

$$(C_M) : \forall [a, b] \subset \mathbb{R}, \forall [c, d] \subset (0, 1), \exists \delta \in \mathbb{R}_+^*, \forall t \in [a, b], \exists \Phi_t \in L^2(\mathbb{R}, du), \text{ verifying } \sup_{t \in [a, b]} \int_{\mathbb{R}} |\Phi_t(u)|^2 du < +\infty, \\ \text{s.t. } \forall (u, H, H') \in \mathbb{R} \times [c, d]^2, |M(t, u, H) - M(t, u, H')| \leq \Phi_t(u) |H - H'|^\delta.$$

Condition (C_M) is fulfilled by the kernel M defining \mathbf{B}_1 and \mathbf{B}_2 (see [9, Proposition 3, (5)]).

3 Stochastic integrals w.r.t. mBm as limits of integrals w.r.t. fBm

The results of the previous section suggest that one may define stochastic integrals with respect to mBm as limits of integrals with respect to approximating fBms. We formalize this intuition in the present section.

We consider as above a fractional field $(\mathbf{B}(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$, but assume in addition that the field is C^1 in H on $(0, 1)$ in the $L^2(\Omega)$ sense, *i.e.* we assume that the map $H \mapsto \mathbf{B}(t, H)$, from $(0, 1)$ to $L^2(\Omega)$, is C^1 for every real t . We will denote $\frac{\partial \mathbf{B}}{\partial H}(t, H')$ the $L^2(\Omega)$ -derivative at point H' of the map $H \mapsto \mathbf{B}(t, H)$. The field $(\frac{\partial \mathbf{B}(t, H)}{\partial H})_{(t, H) \in \mathbb{R} \times (0, 1)}$ is of course Gaussian. We will need that the derivative field satisfies the same assumption (\mathcal{H}_1) as $\mathbf{B}(t, H)$. More precisely, from now on, we assume that $\mathbf{B}(t, H)$ satisfies (\mathcal{H}_2) :

$(\mathcal{H}_2) : \forall [a, b] \times [c, d] \subset \mathbb{R} \times (0, 1), H \mapsto \mathbf{B}(t, H)$ is C^1 in the $L^2(\Omega)$ sense from $(0, 1)$ to $L^2(\Omega)$ for every t in $[a, b]$ and $\exists (\Delta, \alpha, \lambda) \in (\mathbb{R}_+^*)^3$ s.t. $\mathbb{E} \left[\left(\frac{\partial \mathbf{B}}{\partial H}(t, H) - \frac{\partial \mathbf{B}}{\partial H}(s, H') \right)^2 \right] \leq \Delta (|t - s|^\alpha + |H - H'|^\lambda)$, for every (t, s, H, H') in $[a, b]^2 \times [c, d]^2$.

Proposition 3.1. *The fractional Brownian fields $\mathbf{B}_i := (\mathbf{B}_i(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$, $i \in \llbracket 1, 4 \rrbracket$, verify Assumption (\mathcal{H}_2) .*

Proof: The proof of this proposition in the case of \mathbf{B}_1 and \mathbf{B}_2 may be found in Appendix A. The ones for \mathbf{B}_3 and \mathbf{B}_4 are easily obtained using results from [17] and [9] and are left to the reader. \square

In the remaining of this paper, we consider a C^1 deterministic function $h : \mathbb{R} \rightarrow (0, 1)$, a fractional field \mathbf{B} which fulfills assumptions (\mathcal{H}_1) and (\mathcal{H}_2) , and the associated mBm $B_t^h := \mathbf{B}(t, h(t))$.

We now explain in a heuristic way how to define an integral with respect to mBm using approximating fBms. Write the “differential” of $\mathbf{B}(t, H)$:

$$d\mathbf{B}(t, H) = \frac{\partial \mathbf{B}}{\partial t}(t, H) dt + \frac{\partial \mathbf{B}}{\partial H}(t, H) dH.$$

Of course, this is only formal as $t \mapsto \mathbf{B}(t, H)$ is not differentiable in the L^2 -sense nor almost surely with respect to t . It is, however, in the white noise sense, but we are not interested in these aspects at this stage. With a differentiable function h in place of H , this (again formally) yields

$$d\mathbf{B}(t, h(t)) = \frac{\partial \mathbf{B}}{\partial t}(t, h(t)) dt + h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt. \quad (3.1)$$

The second term on the right-hand side of (3.1) is defined for almost every ω and every real t by assumption. Moreover, it is almost surely continuous as fonction of t and thus Riemann integrable on compact intervals. On the other hand, the first term of (3.1) has no meaning *a priori* since mBm is not differentiable with respect to t . However, since stochastic integrals with respect to fBm do exist, we are able to give a sense to $t \mapsto \frac{\partial \mathbf{B}}{\partial t}(t, H)$ for every fixed H in $(0, 1)$. Continuing with our heuristic reasoning, we then approximate $\frac{\partial \mathbf{B}}{\partial t}(t, h(t))$ by $\lim_{n \rightarrow +\infty} \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) \frac{\partial \mathbf{B}}{\partial t}(t, h_n(t))$. This formally yields:

$$d\mathbf{B}(t, h(t)) \approx \lim_{n \rightarrow +\infty} \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) \frac{\partial \mathbf{B}}{\partial t}(t, h_n(t)) dt + h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt. \quad (3.2)$$

Assuming we may exchange integrals and limits, we would thus like to define, for suitable processes Y ,

$$\int_0^1 Y_t d\mathbf{B}(t, h(t)) = \lim_{n \rightarrow +\infty} \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} Y_t dB_t^{h(x_k^{(n)})} + \int_0^1 Y_t h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt, \quad (3.3)$$

where the first term of (3.3) is a limit, in a sense to be made precise depending on the method of integration, of a sum of integrals with respect to fBms and the second term is, for almost every ω , a Riemann integral. For ease of reading, we define the following notation:

Notation (integral with respect to lumped fBms) Let $Y := (Y_t)_{t \in [0,1]}$ be a real-valued process on $[0, 1]$ which is integrable with respect to all fBms of index H in $h([0, 1])$ in the sense of method (\mathcal{M}) . We denote the integral with respect to lumped fBms in the sense of method (\mathcal{M}) by:

$$\int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n} := \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) Y_t d^{(\mathcal{M})} B_t^{h(x_k^{(n)})}, \quad n \in \mathbb{N}. \quad (3.4)$$

With this convention, our tentative definition of an integral w.r.t. to mBm (3.3) reads:

$$\int_0^1 Y_t d\mathbf{B}(t, h(t)) = \lim_{n \rightarrow +\infty} \sum_{k=0}^{q_n-1} \int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n} + \int_0^1 Y_t h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt, \quad (3.5)$$

The interest of (3.3) is that it allows to use any of the numerous definitions of stochastic integrals with respect to fBm, and automatically obtain a corresponding integral with respect to mBm. It is worthwhile to note that, with this approach, an integral with respect to mBm is a sum of two terms: the first one seems to depend only on the chosen method for integrating with respect to fBm, (for instance, a white noise or pathwise Riemann integral), while the second is an integral which appears to depend only on the field used to define the chosen mBm, *i.e.* essentially on its correlation structure. This second term will imply that the integral with respect to the moving average mBm, for instance, is different from the one with respect to the harmonisable mBm. However, as the example of simple processes in the next subsection will show, the second term does also depend on the integration method with respect to fBm.

Let us now make the above ideas a bit more precise. We first fix some notations. (\mathcal{M}) denotes a given method of integration with respect to fBm (*e.g.* Skohorod, white noise, pathwise, \dots). For the sake of notational simplicity, we will consider integrals over the interval $[0, 1]$. For H in $(0, 1)$, denote $\int_0^1 Y_t d^{(\mathcal{M})} B_t^H$ the integral of $Y := (Y_t)_{t \in [0,1]}$ on $[0, 1]$ with respect to the fBm B^H , in the sense of method (\mathcal{M}) , assuming it does exist. With the same notations as in subsection 2.1, we consider a sequence $(q_n)_{n \in \mathbb{N}}$, a family $x^{(n)} := \{x_k^{(n)}; k \in \llbracket 0, q_n \rrbracket\}$ (which is defined on $[0, 1]$ by $x_k^{(n)} := \frac{k}{q_n}$ for k in $\llbracket 0, q_n \rrbracket$) and the family of partition $\mathcal{A}_n := \{[x_k^{(n)}, x_{k+1}^{(n)}]; k \in \llbracket 0, q_n - 1 \rrbracket\} \cup \{x_{q_n}^{(n)}\}$ of $[0, 1]$. The following notation will be useful:

Note that the nature of $\int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n}$ depends on the method (\mathcal{M}) of integration. For example, $\int_0^1 Y_t d^{(\mathcal{M})} B_t^H$ and hence $\int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n}$ will belong to $L^2(\Omega)$ if (\mathcal{M}) denotes the Skohorod integral, whereas $\int_0^1 Y_t d^{(\mathcal{M})} B_t^H$ and hence $\int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n}$ belong to the space $(S)^*$ of stochastic distributions when (\mathcal{M}) denotes the integral in the sense of white noise theory.

We will write $\int_0^1 Y_t d^{(\mathcal{M})} B_t^h$ for the integral of Y on $[0, 1]$ with respect to mBm in the sense of method (\mathcal{M}) (which is yet to be defined). When we do not want to specify a particular method (\mathcal{M}) but instead wish to refer to all methods at the same time, we will write $\int_0^1 Y_t dB_t^{h_n}$ and $\int_0^1 Y_t dB_t^h$ instead of $\int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n}$ and $\int_0^1 Y_t d^{(\mathcal{M})} B_t^h$.

In order to gain a better understanding of our approach, we explore in the following subsection the particular cases of simple deterministic and then random integrands.

3.1 Example: simple integrands

3.1.1 Deterministic simple integrands

Any reasonable definition of an integral with respect to mBm must be linear. Thus it suffices to consider the case of $Y = 1$. Obviously, we should find that $\int_0^1 1 dB_t^h = B_1^h$.

In order to verify this fact, let us compute the limit of the sequence $(\int_0^1 1 dB_t^{h_n})_{n \in \mathbb{N}}$. In that view, it is convenient to use the S -transform. For a function η in the Schwartz space $\mathcal{S}(\mathbb{R})$, we will denote $S(\Phi)(\eta)$ the S -transform of the L^2 -random variable Φ at point η (the reader who is not familiar with the S -transform may refer to section 5 of this paper where this notion is recalled in the more general setting of Hida spaces). For a field \mathbf{B} and (t, H, η) in $\mathbb{R} \times (0, 1) \times \mathcal{S}(\mathbb{R})$, we shall write $g_\eta(t, H)$ for the S -transform at point η of the random variable $\mathbf{B}(t, H)$. In other words, $g_\eta(t, H) := S(\mathbf{B}(t, H))(\eta)$.

Proposition 3.2. Assume that the map h is C^2 , that the map $H \mapsto g_\eta(t, H)$ is C^2 for every (t, η) in $\mathbb{R} \times \mathcal{S}(\mathbb{R})$ and that $(t, H) \mapsto \frac{\partial^i g_\eta}{\partial H^i}(t, H)$ is continuous on $\mathbb{R} \times (0, 1)$ for i in $\{1; 2\}$. The sequence $(\int_0^1 1 dB_t^{h_n})_{n \in \mathbb{N}}$ then converges in $L^2(\Omega)$ to $\mathbf{B}(1, h(1)) - \int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt$, where the second term is a pathwise integral.

Proposition 3.2 implies that, for regular enough fields \mathbf{B} and h functions, formula (3.3) does indeed yield $\int_0^1 1 dB_t^h = B_1^h$.

Proof:

Let us first show that $\int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt$ belongs to $L^2(\Omega)$. Cauchy-Schwarz inequality and assumption (\mathcal{H}_2) entail:

$$\begin{aligned} \mathbb{E} \left[\left(\int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt \right)^2 \right] &\leq \underbrace{\left(\int_0^1 |h'(t)|^2 dt \right)}_{:=\mu} \int_0^1 \mathbb{E} \left[\left(\frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \right)^2 \right] dt \\ &\leq \mu \Delta \left(\int_0^1 (|t|^\alpha + |h(t)|^\lambda) dt \right) < +\infty. \end{aligned}$$

Let $I_n := \int_0^1 1 dB_t^{h_n}$. Thanks to (3.4) we have:

$$\begin{aligned} I_n &= \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) 1 dB_t^{h(x_k^{(n)})} = \sum_{k=0}^{q_n-1} \left(B_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} - B_{x_k^{(n)}}^{h(x_k^{(n)})} \right) \\ &= B_1^{h(x_{q_n-1}^{(n)})} - \underbrace{\sum_{k=1}^{q_n-1} \left(B_{x_k^{(n)}}^{h(x_k^{(n)})} - B_{x_k^{(n)}}^{h(x_{k-1}^{(n)})} \right)}_{=:J_n} \end{aligned} \quad (3.6)$$

It is thus sufficient to show that the sequence $(J_n)_{n \in \mathbb{N}}$ converges in $L^2(\Omega)$ to $\int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt$. Thanks to [6, theorem 2.3], this is equivalent to showing that the sequence $((S(J_n)(\eta))_{n \in \mathbb{N}}$ (respectively $(\mathbb{E}[J_n^2])_{n \in \mathbb{N}}$) converges to $S(\int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt)(\eta)$ (respectively to $\mathbb{E} \left[\left(\int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt \right)^2 \right]$) for all η in $\mathcal{S}(\mathbb{R})$. Now, for every integer n and η in $\mathcal{S}(\mathbb{R})$, taking the S -transform of J_n at point η and using two times the finite increment theorem, there exist $(r_k^{(n)}, w_k^{(n)})$ in $[x_{k-1}^{(n)}, x_k^{(n)}]^2$ such that

$$\begin{aligned} S(J_n)(\eta) &= \sum_{k=1}^{q_n-1} S \left(B_{x_k^{(n)}}^{h(x_k^{(n)})} \right) (\eta) - S \left(B_{x_k^{(n)}}^{h(x_{k-1}^{(n)})} \right) (\eta) = \sum_{k=1}^{q_n-1} g_\eta(x_k^{(n)}, h(x_k^{(n)})) - g_\eta(x_k^{(n)}, h(x_{k-1}^{(n)})) \\ &= \sum_{k=1}^{q_n-1} \frac{\partial g_\eta}{\partial H}(x_k^{(n)}, h(r_k^{(n)})) (h(x_k^{(n)}) - h(x_{k-1}^{(n)})) = \sum_{k=1}^{q_n-1} \frac{\partial g_\eta}{\partial H}(x_k^{(n)}, h(r_k^{(n)})) h'(w_k^{(n)})(x_k^{(n)} - x_{k-1}^{(n)}) \\ &= \underbrace{\frac{1}{q_n} \sum_{k=1}^{q_n-1} \left(\frac{\partial g_\eta}{\partial H}(x_k^{(n)}, h(r_k^{(n)})) h'(w_k^{(n)}) - \frac{\partial g_\eta}{\partial H}(x_k^{(n)}, h(x_k^{(n)})) h'(x_k^{(n)}) \right)}_{=:K_n} + \underbrace{\frac{1}{q_n} \sum_{k=1}^{q_n-1} \frac{\partial g_\eta}{\partial H}(x_k^{(n)}, h(x_k^{(n)})) h'(x_k^{(n)})}_{=:L_n}. \end{aligned} \quad (3.7)$$

By definition of the Riemann integral,

$$\lim_{n \rightarrow +\infty} L_n = \int_0^1 h'(t) \frac{\partial g_\eta}{\partial H}(t, h(t)) dt = \int_0^1 h'(t) S \left(\frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \right) (\eta) dt = S \left(\int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt \right) (\eta). \quad (3.8)$$

Besides, the assumptions made on the map g_η and the mean value theorem entail that there exists a couple $(t_k^{(n)}, v_k^{(n)})$ in $[x_k^{(n)}, x_{k+1}^{(n)}]^2$ such that

$$|K_n| \leq \frac{1}{q_n} \sum_{k=1}^{q_n-1} \left(|h(r_k^{(n)}) - h(x_k^{(n)})| |h'(w_k^{(n)})| \left| \frac{\partial^2 g_\eta}{\partial H^2}(x_k^{(n)}, h(t_k^{(n)})) \right| \right. \\ \left. + |h''(v_k^{(n)})| |w_k^{(n)} - x_k^{(n)}| \left| \frac{\partial g_\eta}{\partial H}(x_k^{(n)}, h(x_k^{(n)})) \right| \right).$$

We hence have,

$$|K_n| \leq A_1 \frac{1}{q_n} \sum_{k=1}^{q_n-1} \left(\eta |r_k^{(n)} - x_k^{(n)}|^\beta + |w_k^{(n)} - x_k^{(n)}| \right) \leq A_1 (1 + \eta) \left(\frac{1}{q_n^\beta} + \frac{1}{q_n} \right) \xrightarrow{n \rightarrow +\infty} 0, \quad (3.9)$$

where $A_1 := \sup_{u \in [0,1]} |h'(u)| \sup_{(u,v) \in [0,1] \times h([0,1])} \left| \frac{\partial^2 g_\eta}{\partial H^2}(u, v) \right| + \sup_{u \in [0,1]} |h''(u)| \sup_{(u,v) \in [0,1] \times h([0,1])} \left| \frac{\partial g_\eta}{\partial H}(u, v) \right|$.

Thanks to (3.8) and (3.9) we get $\lim_{n \rightarrow +\infty} S(J_n)(\eta) = S\left(\int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt\right)(\eta)$ for every η in $\mathcal{S}(\mathbb{R})$. It

then remains to show that $\lim_{n \rightarrow +\infty} \mathbb{E}[J_n^2] = \mathbb{E}\left[\left(\int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt\right)^2\right]$. Using Fubini's theorem,

$$Q := \mathbb{E} \left[\left(\int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt \right)^2 \right] = \int_0^1 \int_0^1 h'(t) h'(s) \mathbb{E} \left[\frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \frac{\partial \mathbf{B}}{\partial H}(s, h(s)) \right] ds dt. \\ =: \Theta((t, s, h(t)), h(s))$$

For the same reason as in (i) of remark 1.2, assumption (\mathcal{H}_2) entails that the function Θ is continuous on $\mathbb{R}^2 \times (0, 1)^2$. Thus,

$$Q := \lim_{n \rightarrow +\infty} \frac{1}{q_n^2} \sum_{1 \leq k, j \leq q_n-1} h'(x_j^{(n)}) h'(x_k^{(n)}) \mathbb{E} \left[\frac{\partial \mathbf{B}}{\partial H}(x_j^{(n)}, h(x_j^{(n)})) \frac{\partial \mathbf{B}}{\partial H}(x_k^{(n)}, h(x_k^{(n)})) \right]. \quad (3.10)$$

Besides for any n , thanks to (3.7),

$$J_n = \underbrace{\frac{1}{q_n} \sum_{k=1}^{q_n-1} \left(\frac{\partial \mathbf{B}}{\partial H}(x_k^{(n)}, h(r_k^{(n)})) h'(w_k^{(n)}) - \frac{\partial \mathbf{B}}{\partial H}(x_k^{(n)}, h(x_k^{(n)})) h'(x_k^{(n)}) \right)}_{=: J_n^{(1)}} \\ + \underbrace{\frac{1}{q_n} \sum_{k=1}^{q_n-1} \frac{\partial \mathbf{B}}{\partial H}(x_k^{(n)}, h(x_k^{(n)})) h'(x_k^{(n)})}_{=: J_n^{(2)}}. \quad (3.11)$$

In view of (3.10), it is clear that the sequence $(\mathbb{E}[(J_n^{(2)})^2])_{n \in \mathbb{N}}$ tends to $\mathbb{E}[Q^2]$. Let us now show that $(J_n^{(1)})_{n \in \mathbb{N}}$ tends to 0 in $L^2(\Omega)$. In a way similar to the computations leading to (3.9), we get

$$J_n^{(1)} = \frac{1}{q_n} \sum_{k=1}^{q_n-1} \left(h''(v_k^{(n)}) (w_k^{(n)} - x_k^{(n)}) \frac{\partial \mathbf{B}}{\partial H}(x_k^{(n)}, h(x_k^{(n)})) \right) \\ + \frac{1}{q_n} \sum_{k=1}^{q_n-1} h'(w_k^{(n)}) \left(\frac{\partial \mathbf{B}}{\partial H}(x_k^{(n)}, h(r_k^{(n)})) - \frac{\partial \mathbf{B}}{\partial H}(x_k^{(n)}, h(x_k^{(n)})) \right) =: M_n + N_n. \quad (3.12)$$

Using (3.9), we get on the one hand

$$\begin{aligned}
\mathbb{E} [M_n^2] &= \frac{1}{q_n^2} \sum_{1 \leq k, j \leq q_n - 1} h''(v_k^{(n)}) h''(v_j^{(n)}) (w_k^{(n)} - x_k^{(n)}) (w_j^{(n)} - x_j^{(n)}) \Theta(x_k^{(n)}, x_j^{(n)}, h(x_k^{(n)}), h(x_j^{(n)})) \\
&\leq \frac{A_2}{q_n^2} \sum_{1 \leq k, j \leq q_n - 1} (w_k^{(n)} - x_k^{(n)}) (w_j^{(n)} - x_j^{(n)}) \leq \frac{A_2}{q_n^2} \xrightarrow{n \rightarrow +\infty} 0,
\end{aligned} \tag{3.13}$$

where we set $A_2 := \sup_{u \in [0,1]} |h''(u)|^2 \sup_{(u,v) \in [0,1]^2} |\Theta(u, v, h(u), h(v))|$. On the other hand, setting $A_3 := \sup_{u \in [0,1]} |h'(u)|$ and using Cauchy-Schwarz's inequality and assumption (\mathcal{H}_2) , we get

$$\begin{aligned}
\mathbb{E} [N_n^2] &\leq \frac{A_3}{q_n^2} \sum_{1 \leq k, j \leq q_n - 1} \left\| \frac{\partial \mathbf{B}}{\partial H}(x_k^{(n)}, h(r_k^{(n)})) - \frac{\partial \mathbf{B}}{\partial H}(x_k^{(n)}, h(x_k^{(n)})) \right\|_{L^2(\Omega)} \left\| \frac{\partial \mathbf{B}}{\partial H}(x_j^{(n)}, h(r_j^{(n)})) - \frac{\partial \mathbf{B}}{\partial H}(x_j^{(n)}, h(x_j^{(n)})) \right\|_{L^2(\Omega)} \\
&\leq \frac{A_3 \Delta}{q_n^2} \sum_{1 \leq k, j \leq q_n - 1} (|h(r_j^{(n)}) - h(x_j^{(n)})| |h(r_k^{(n)}) - h(x_k^{(n)})|)^{\lambda/2} \\
&\leq \frac{A_3 \Delta (1 + \eta^\lambda)}{q_n^2} \left(\sum_{1 \leq k \leq q_n - 1} |r_k^{(n)} - x_k^{(n)}|^{\beta\lambda/2} \right)^2 \leq \frac{A_3 \Delta (1 + \eta^\lambda)}{q_n^2} \left(\sum_{1 \leq k \leq q_n - 1} \frac{1}{q_n^{\beta\lambda/2}} \right)^2 \\
&\leq \frac{A_3 \Delta (1 + \eta^\lambda)}{q_n^{\beta\lambda}} \xrightarrow{n \rightarrow +\infty} 0.
\end{aligned} \tag{3.14}$$

Inequalities (3.13) and (3.14) show that $(J_n^{(1)})_{n \in \mathbb{N}}$ tends to 0 in $L^2(\Omega)$ which ends the proof. \square

Remark 3.3. (i) One can also show that the sequence $(\int_0^1 1 dB_t^{h_n} dt)_{n \in \mathbb{N}}$ converges almost surely.

(ii) Proposition (3.2) applies to the four fields considered in the introduction, since they all satisfy the required assumptions: in the case of \mathbf{B}_1 , it has been proved in [14, lemma 5.5], that $(t, H) \mapsto g_\eta(t, H)$ belongs to $C^\infty(\mathbb{R} \times (0, 1))$ for every η in $\mathcal{S}(\mathbb{R})$. Moreover, the result is similar when $\mathbf{B} = \mathbf{B}_3$, since [21] example 2 shows that there exist two C^∞ maps $H \mapsto \alpha_H$ and $H \mapsto v_H$ from $(0, 1)$ to \mathbb{R} such that

$$\mathbf{B}_3(t, H) = \alpha_H \mathbf{B}_1(t, H) + v_H \underbrace{\int_{\mathbb{R}} \frac{e^{it\xi} - 1}{i\xi|\xi|^{H-1/2}} \widetilde{\mathbb{W}}_1(d\xi)}_{=: Z_H(t)}.$$

η in $\mathcal{S}(\mathbb{R})$ the conclusion follows. The case of \mathbf{B}_4 is similar and left to the reader.

3.1.2 Simple processes

We consider a particular case of a simple process that will show that formula (3.3) does not always yield the expected result, and must be modified in certain situations. We take $Y_t = Y_0$ with Y_0 a centered Gaussian random variable which is $\mathcal{F}(B^h)$ -measurable where $\mathcal{F}(B^h)$ denotes the σ -field generated by the random variables in the first Wiener chaos of the mBm B^h .

Case of the integral in the sense of white noise theory

The reader who is not familiar with the integral with respect to fBm in the sense of white noise theory (also called fractional Wick-Itô integral) may refer to [11] and [6]. We denote this integral $\int Y_i d^\circ B_t^H$, with a similar notation for the integral w.r.t. mBm. Denote $(W_t^H)_{t \in [0, T]}$ the fractional White Noise process (see section 5 and references therein for more details). Using (3.6) we get, for all $n \in \mathbb{N}$,

$$\begin{aligned}
S_n &:= \int_0^1 Y_t \, d^\diamond B_t^{h_n} = \sum_{k=1}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} Y_0 \, d^\diamond B_t^{h(x_k^{(n)})} \\
&= \sum_{k=1}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} Y_0 \diamond W_t^{h(x_k^{(n)})} \, dt = \sum_{k=1}^{q_n-1} Y_0 \diamond \left(\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} W_t^{h(x_k^{(n)})} \, dt \right) \\
&= Y_0 \diamond \left(\sum_{k=1}^{q_n-1} \left(B_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} - B_{x_k^{(n)}}^{h(x_k^{(n)})} \right) \right) = Y_t \diamond \left(\sum_{k=1}^{q_n-1} \left(B_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} - B_{x_k^{(n)}}^{h(x_k^{(n)})} \right) \right). \tag{3.15}
\end{aligned}$$

Using exactly the same method used to show that $(J_n)_{n \in \mathbb{N}}$ converges in $L^2(\Omega)$ to $\int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt$ in the proof of Proposition 3.2, one shows that the sequence $\left(\sum_{k=1}^{q_n-1} \left(B_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} - B_{x_k^{(n)}}^{h(x_k^{(n)})} \right) \right)_{n \in \mathbb{N}}$ converges in $L^2(\Omega)$ to $B_1^h - \int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt$ when n tends to infinity. By continuity of the Wick product,

$$\begin{aligned}
\lim_{n \rightarrow \infty} S_n &= Y_0 \diamond B_1^h - Y_0 \diamond \int_0^1 h'(t) \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt = Y_0 \diamond B_1^h - \int_0^1 h'(t) Y_0 \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt, \\
&= Y_t \diamond B_1^h - \int_0^1 h'(t) Y_0 \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt. \tag{3.16}
\end{aligned}$$

where the limit holds in $L^2(\Omega)$. Now, for the Wick-Itô integral w.r.t. mBm defined in [14], one has:

$$\int_0^1 Y_0 \, d^\diamond B_t^h = Y_0 \diamond B_1^h.$$

We see that, for this integration method, formula (3.3) should be modified into

$$\int_0^1 Y_t \, d^\diamond B_t^h := \lim_{n \rightarrow \infty} \int_0^1 Y_t \, d^\diamond B_t^{h_n} + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt. \tag{3.17}$$

The natural spaces of white noise theory are the spaces (\mathcal{S}_{-p}) . Equality (3.17) will be used in section 5 to define the integral of an (\mathcal{S}_{-p}) -valued process $Y := (Y_t)_{t \in [0,1]}$ with respect to mBm, where the limit, the last integral (which is in the sense of Bochner), assuming they both exist, and the equality will hold in (\mathcal{S}_{-q}) for some integer q in \mathbb{N} .

Case of the integral in the sense of Skohorod

We denote this integral with the symbol δ . Thus, for instance, $\int_0^1 X \, \delta B_t^H$ denotes the Skohorod integral of the process X w.r.t. the fractional Brownian motion B^H .

Continuing with our example $Y_t = Y_0$, where Y_0 is a centered Gaussian random variable, we use theorem 7.40 in [12, section 7], that yields the general form of a Skorohod integral with respect to a Gaussian process. In our very simple case, this reads:

$$\int_0^1 Y_0 \, \delta B_t^h = Y_0 \diamond \int_0^1 1 \, \delta B_t^h = Y_0 \diamond B_1^h.$$

Thus, we find that we should define our Skohorod integral w.r.t. to mBm using a Wick product as in the case of the integral in the sense of white noise theory, *i.e.*, use a formula analogous to (3.17):

$$\int_0^1 Y_t \, \delta B_t^h := \lim_{n \rightarrow \infty} \int_0^1 Y_t \, \delta B_t^{h_n} + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt, \tag{3.18}$$

where the equality and limit hold in $L^2(\Omega)$ and the last integral is pathwise for almost every ω .

A notable advantage of these definitions is that they will ensure, by construction, the equality $\int_0^1 Y_t \, d^\diamond B_t^h = \int_0^1 Y_t \, \delta B_t^h$ as soon as Y is integrable w.r.t. mBm in the sense of Skohorod. This result generalizes the case

of integral w.r.t. fBm (see [15, proposition 8] and [6, corollary 3.5]).

Case of the Pathwise integrals

In the case of the pathwise fractional integral in the sense of Zähle, denoted $\int X_t dB_t^h$, the use of formula (22) in [23] with g an mBm and $f = Y_0$ yields

$$\int_0^1 Y_0 dB_t^h = Y_0 B_1^h.$$

Thus the correct way to define our integral w.r.t. mBm in this case is to use a standard product, *i.e.* to set:

$$\int_0^1 Y_t dB_t^h := \lim_{n \rightarrow \infty} \int_0^1 Y_t dB_t^{h_n} + \int_0^1 h'(t) Y_t \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt. \quad (3.19)$$

3.2 Integral with respect to mBm through approximating fBms

We now define in a precise way our integral with respect to mBm. Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two normed linear spaces, endowed with their Borel σ -field $\mathcal{B}(E)$ and $\mathcal{B}(F)$. Let $Y := (Y_t)_{t \in [0,1]}$ be an E -valued process (i.e Y_t belongs to E for every real t in $[0, 1]$ and $t \mapsto Y_t$ is measurable from $(0, 1)$ to $(E, \mathcal{B}(E))$). Fix an integration method (\mathcal{M}) . As explained in the previous subsection, we wish to define the integral w.r.t. an mBm B^h in the sense of (\mathcal{M}) by a formula of the kind:

$$\int_0^1 Y_t d^{(\mathcal{M})} B_t^h := \lim_{n \rightarrow \infty} \int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n} + \int_0^1 h'(t) Y_t * \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt. \quad (3.20)$$

where $*$ denotes the ordinary (in the case of pathwise integrals) or Wick (in other cases) product depending on (\mathcal{M}) . For this formula to make sense, it is certainly necessary that Y be (\mathcal{M}) -integrable w.r.t. fBm of all exponents in $h([0, 1])$.

We thus define, for $\alpha \in (0, 1)$,

$$\mathcal{H}_E^\alpha := \left\{ Y \in E^{[0,1]} : \int_{[0,1]} Y_t d^{(\mathcal{M})} B_t^\alpha \text{ exists and belongs to } F \right\},$$

and

$$\mathcal{H}_E = \bigcap_{\alpha \in h([0,1])} \mathcal{H}_E^\alpha.$$

We will always assume that there exists a subset Λ_E of \mathcal{H}_E (maybe equal to \mathcal{H}_E) which may be endowed with a norm $\|\cdot\|_{\Lambda_E}$ which satisfies the following property: there exists $M > 0$ such that for all disjoint Borel subsets A_1, \dots, A_n of $[0, 1]$,

$$\|Y \cdot \mathbb{1}_{A_1}\|_{\Lambda_E} + \dots + \|Y \cdot \mathbb{1}_{A_n}\|_{\Lambda_E} \leq M \|Y\|_{\Lambda_E}. \quad (3.21)$$

When Y belongs to \mathcal{H}_E , definition (3.20) will be a valid one as soon as the limit and the last term on the right hand side exist. While it does not seem practical to get a general criterion for the existence of $\int_0^1 h'(t) Y_t * \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt$, it turns out that a simple sufficient condition allows to guarantee the existence of the limit of the integral w.r.t. lumped fBms. Define, for $n \in \mathbb{N}$, the map

$$L_n : \Lambda_E \rightarrow F$$

$$Y \mapsto \int_{[0,1]} Y_t d^{(\mathcal{M})} B_t^{h_n}. \quad (3.22)$$

The following theorem provides a necessary condition under which $(L_n(Y))_{n \in \mathbb{N}}$ converges in F .

Theorem 3.4. Let $(a_n)_{n \in \mathbb{N}}$ be an increasing sequence of positive integers such that $2^n \leq \prod_{0 \leq k \leq n-1} a_k \leq 2^{2^n}$ for every n in \mathbb{N} and such that $\lim_{n \rightarrow +\infty} (n(a_n - 1)) \left(\prod_{0 \leq k \leq n-1} a_k \right)^{-1} = 0$. Choose the sequence $(q_n)_{n \in \mathbb{N}}$ used in (3.4) such that $q_0 = 1$ and $q_{n+1} = a_n q_n$ for all n in \mathbb{N} . Assume that the function $\mathcal{I} : \Lambda_E \times (0, 1) \rightarrow F$ defined by

$$\forall Y \in \Lambda_E, \forall \alpha \in (0, 1), \quad \mathcal{I}(Y, \alpha) := \int_{[0,1]} Y_t d^{(\mathcal{M})} B_t^\alpha,$$

is θ -Hölder continuous with respect to α uniformly in Y for some positive real θ , i.e. there exists $K > 0$ such that

$$\forall (\alpha, \alpha') \in (0, 1)^2, \quad \sup_{\|Y\|_{\Lambda_E} \leq 1} \|\mathcal{I}(Y, \alpha) - \mathcal{I}(Y, \alpha')\|_F \leq K |\alpha - \alpha'|^\theta. \quad (3.23)$$

Then the sequence of functions $(L_n)_{n \in \mathbb{N}}$ defined in (3.22) converges pointwise to a function $L : \Lambda_E \rightarrow F$.

Proof: For the sake of simplicity, we will establish the result in the case where $a_n \equiv 2$ which obviously fulfils the required conditions. The general case is similar. For $n \in \mathbb{N}$ and $Y \in \Lambda_E$, $L_n(Y)$ may be decomposed as

$$L_n(Y) = \sum_{k=0}^{2^n-1} \left(\int_{[2k \cdot 2^{-(n+1)}, (2k+1) \cdot 2^{-(n+1)}]} Y_t d^{(\mathcal{M})} B_t^{h(k \cdot 2^{-n})} + \int_{[(2k+1) \cdot 2^{-(n+1)}, (2k+2) \cdot 2^{-(n+1)}]} Y_t d^{(\mathcal{M})} B_t^{h(k \cdot 2^{-n})} \right).$$

Now,

$$\begin{aligned} L_{n+1}(Y) &= \sum_{k=0}^{2^n-1} \left(\int_{[2k \cdot 2^{-(n+1)}, (2k+1) \cdot 2^{-(n+1)}]} Y_t d^{(\mathcal{M})} B_t^{h(2k \cdot 2^{-(n+1)})} \right. \\ &\quad \left. + \int_{[(2k+1) \cdot 2^{-(n+1)}, (2k+2) \cdot 2^{-(n+1)}]} Y_t d^{(\mathcal{M})} B_t^{h((2k+1) \cdot 2^{-(n+1)})} \right). \end{aligned}$$

Using assumptions (3.21) and (3.23) we obtain

$$\begin{aligned} \|L_n(Y) - L_{n+1}(Y)\|_F &= \left\| \sum_{k=0}^{2^n-1} \left(\mathcal{I}(Y \cdot \mathbb{1}_{\left[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}\right]}, h\left(\frac{k}{2^n}\right)) - \mathcal{I}(Y \cdot \mathbb{1}_{\left[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}\right]}, h\left(\frac{2k+1}{2^{n+1}}\right)) \right) \right\|_F \\ &\leq \sum_{k=0}^{2^n-1} \left\| \left(\mathcal{I}(Y \cdot \mathbb{1}_{\left[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}\right]}, h\left(\frac{k}{2^n}\right)) - \mathcal{I}(Y \cdot \mathbb{1}_{\left[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}\right]}, h\left(\frac{2k+1}{2^{n+1}}\right)) \right) \right\|_F \\ &\leq K \sum_{k=0}^{2^n-1} \|Y \cdot \mathbb{1}_{[(2k+1) \cdot 2^{-(n+1)}, (k+1) \cdot 2^{-n}]}\|_{\Lambda_E} \left| h((2k+1) \cdot 2^{-(n+1)}) - h(k \cdot 2^{-n}) \right|^\theta \\ &\leq K 2^{-\theta(n+1)} \sup_{t \in [0,1]} |h'(t)|^\theta \sum_{k=0}^{2^n-1} \|Y \cdot \mathbb{1}_{[(2k+1) \cdot 2^{-(n+1)}, (k+1) \cdot 2^{-n}]}\|_{\Lambda_E} \\ &\leq K M \|Y\|_{\Lambda_E} 2^{-\theta(n+1)} \sup_{t \in [0,1]} |h'(t)|^\theta. \end{aligned} \quad (3.24)$$

It follows that the series $\sum_{n \in \mathbb{N}} (L_{n+1}(Y) - L_n(Y))$ converges absolutely for any fixed $Y \in \Lambda_E$, and consequently $(L_n(Y))_{n \in \mathbb{N}}$ converges to a limit $L(Y)$ as n goes to infinity. \square

For a process Y in \mathcal{H}_E , we will say that $t \mapsto h'(t) Y_t * \frac{\partial \mathbf{B}}{\partial H}(t, h(t))$ is integrable on $[0, 1]$ if

- $h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t))$ is almost surely Riemann integrable and $\int_0^1 h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt$ belongs to $L^2(\Omega)$ in the case of the Skohorod integral,
- $\int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt$ exists in the sense of Bochner and belongs to F in the case of the Wick-Itô integral (in this situation, $L^2(\Omega) \subset F$). The reader who is not familiar with the Bochner integral may refer to section 5 below and references therein,
- $\int_{[0,1]} h'(t) Y_t \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt$ exists for almost every ω in the case of a pathwise integral.

Definition 3.5 (Integral with respect to mBm in the sense of (\mathcal{M})). Assume (\mathcal{M}) fulfils condition (3.23) and let $Y := (Y_t)_{t \in [0,1]}$ be an element of \mathcal{H}_E such that the map $t \mapsto h'(t) Y_t * \frac{\partial \mathbf{B}}{\partial H}(t, h(t))$ is integrable. The integral of Y with respect to B^h in the sense of (\mathcal{M}) is defined as:

$$\int_0^1 Y_t d^{(\mathcal{M})} B_t^h := \lim_{n \rightarrow \infty} \int_0^1 Y_t d^{(\mathcal{M})} B_t^{h_n} + \int_0^1 h'(t) Y_t * \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt, \quad (3.25)$$

where the limit and equality hold in F .

Remark 3.6. (i) Contrary to what one might expect in view of proposition 3.2, the second term on the right-hand side of (3.25) does not only depend on the choice of the fractional field \mathbf{B} but also on the method (\mathcal{M}) . Of course, the same is true of the first term on the right-hand side of (3.25).

(ii) The main advantage of the above definition is that any known stochastic integral with respect to fBm (e.g. pathwise integrals, Skohorod or Wick-Itô integral) gives rise to a corresponding stochastic integral with respect to mBm.

(iii) Once again, note that E is not necessary a space of random variables (e.g. $E := (\mathcal{S}_{-p})$ for some positive integer p ; see section 5 below) and that E may be different from F (this will be the case in section 5).

4 Skohorod integral with respect to mBm through approximating fBms

In this section, we apply Theorem 3.4 to define a Skohorod-type integral with respect to mBm. Our reference method of integration with respect to fBm here is the one based on Malliavin calculus, as exposed in [2]. We assume throughout this section that $H > 1/4$ and that h ranges in $(1/4, 1)$.

Our notations are as follows (for a presentation of Malliavin calculus, see e.g. [3, 16]). Let:

$$\mathcal{S} = \left\{ R := f(W(h_1), W(h_2), \dots, W(h_n)), f \in C_b^\infty(\mathbf{R}^n), h_i \in L^2([0, T]), i = 1, \dots, n \right\}$$

where $W(h_i) := \int_{[0, T]} h_i(s) dW_s$ with $W := (W_s)_{s \in [0, T]}$ a Brownian motion, $C_b^\infty(\mathbf{R}^n)$ is the set of functions which are bounded along with all their derivatives. For an element of \mathcal{S} , one defines the derivative operator D as:

$$DR = \sum_{i=1}^n \partial_i f(W(h_1), W(h_2), \dots, W(h_n)) h_i.$$

D extends to the domain \mathbb{D} which is the completion of \mathcal{S} with respect to the norm:

$$\|R\|_{1,2} = \left(\mathbf{E}(R^2) + \mathbf{E}(\|DR\|_{L^2([0, T])}^2) \right)^{\frac{1}{2}}.$$

We denote by δ the adjoint of D , and by $\text{Dom}(\delta)$ its domain. More precisely, $\text{Dom}(\delta)$ is the set of $u \in L^2(\Omega, [0, T])$ such that:

$$|\mathbf{E}(\langle DR, u \rangle)| \leq c_u \mathbf{E}(R^2)$$

for all $R \in \mathbb{D}$ (we use $\langle \cdot, \cdot \rangle$ to denote the scalar product on $L^2([0, T])$), and δ is defined on $\text{Dom}(\delta)$ by the relation:

$$\mathbf{E}(R\delta(u)) = \mathbf{E}(\langle DR, u \rangle).$$

The operator δ coincides with the Skohorod integral.

Let us now recall briefly the approach of [2] for the construction of a stochastic integral w.r.t. a class of Gaussian processes. Assume the continuous Gaussian process X may be written:

$$X_t = \int_0^t K(t, s) dW_s, \quad (4.1)$$

where the kernel $K(t, s)$ is defined for $0 < s < t < T$ and verifies

$$\sup_{t \in [0, T]} \int_0^t K(t, s)^2 ds < \infty. \quad (4.2)$$

Define the operator K^* on the set of step functions on $[0, T]$:

$$(K^* \varphi)(s) := \varphi(s)K(T, s) + \int_s^T (\varphi(t) - \varphi(s)) K(dt, s). \quad (4.3)$$

Then the stochastic integral w.r.t. X is defined for processes in $\text{Dom}(\delta_X)$ ([2], formula (12)):

$$\text{Dom}(\delta_X) := (K^*)^{-1}(\text{Dom}(\delta)).$$

For a process v in $\text{Dom}(\delta_X)$, one sets:

$$\delta_X(v) := \int_0^T v(s) \delta X(s) := \int_0^T (K^* v)(s) \delta W(s).$$

In the case of fBm, one has:

$$B_t^H = \int_0^t K_H(t, s) dW_s,$$

where

$$K_H(t, s) = d_H (t - s)^{H - \frac{1}{2}} + c_H \left(\frac{1}{2} - H \right) \int_s^t (u - s)^{H - \frac{3}{2}} \left(1 - \left(\frac{s}{u} \right)^{\frac{1}{2} - H} \right) du \quad (4.4)$$

and

$$d_H = \left(\frac{2H\Gamma(\frac{3}{2} - H)}{\Gamma(\frac{1}{2} + H)\Gamma(2 - 2H)} \right)^{\frac{1}{2}}.$$

We will index our operators and sets with H , *i.e.* we will write:

$$\text{Dom}(\delta_H) = (K_H^*)^{-1}(\text{Dom}(\delta))$$

for the domain of the Skorhod integral with respect to B^H and

$$\delta_H(v) := \int_0^T v(s) \delta B^H(s) := \int_0^T (K_H^* v)(s) \delta W(s)$$

for the integral. In other words, $\delta_H(v) = \delta(K_H^* v)$. We will also denote:

$$\mathbb{D}_H := (K_H^*)^{-1}(\mathbb{D}).$$

Since $\mathbb{D} \subset \text{Dom}(\delta)$ ([2] p. 769), we have that $\mathbb{D}_H \subset \text{Dom}(\delta_H)$.

As the methodology of [2] works in a general framework, one may wonder whether it is possible to apply it directly to mBm. The prerequisite is to exhibit a kernel K such that (4.1) holds when X is an mBm. This is indeed the case for \mathbf{B}_4 , and the work [9] develops this approach. In contrast, it does not seem to be an easy task to find such a kernel for the moving average and harmonizable mBms, as they have very different correlation structures (see [21]). As a matter of fact, we conjecture that such a kernel does not exist for the harmonizable mBm, based on the following fact: its covariance verifies, for all t_1, t_2 in \mathbb{R} and H_1, H_2 in $(0, 1)$,

$$\mathbb{E} [\mathbf{B}(t_1, H_1)\mathbf{B}(t_2, H_2)] = \mathbb{E} [\mathbf{B}(t_1, H_2)\mathbf{B}(t_2, H_1)].$$

This will be investigated in a forthcoming work.

We now seek to apply Theorem 3.4 in order to define a Skorohod integral w.r.t. mBm through approximating Skorohod integrals w.r.t. fBms. In that view, we set $F = L^2(\Omega)$ and $\mathcal{I}(Y, \alpha) = \delta_\alpha(Y)$.

It is straightforward to check that, for every (t, s) in $[0, T]^2$, the function $H \mapsto K_H(t, s)$ is C^1 . We denote its derivative by G_H , *i.e.* G_{H_1} is the derivative of the function $H \mapsto K_H$ evaluated at H_1 . We associate to G_H an operator G_H^* in a way similar to (4.3). Note that G_H^* is the derivative of the function $H \mapsto K_H^*$. One easily verifies that G_H fulfils (4.2), so that one may define as above $\delta_G(\cdot) := \delta(G_H^*\cdot)$ for a suitable class of processes. Let:

$$\mathcal{D} := \bigcap_{H \in h([0,1])} \text{Dom}(\delta_H),$$

and

$$\mathcal{F} := \bigcap_{H \in h([0,1])} \mathcal{F}_H,$$

where

$$\mathcal{F}_H := (G_H^*)^{-1}(\text{Dom}(\delta)).$$

In principle, our approach should allow to define the integral w.r.t. mBm of any process in $\mathcal{D} \cap \mathcal{F}$. However, since we need a norm in order to apply Theorem 3.4, we shall restrict to a subset Λ of $\mathcal{D} \cap \mathcal{F}$ defined as follows:

$$\Lambda := \left(\bigcap_{H \in h([0,1])} \mathbb{D}_H \right) \cap \left(\bigcap_{H \in h([0,1])} (G_H^*)^{-1}(\mathbb{D}) \right).$$

We equip Λ with the norm:

$$\|v\|_\Lambda = \max \left(\sup_{H \in h([0,1])} \|(K_H^*v)\|_{1,2}, \sup_{H \in h([0,1])} \|(G_H^*v)\|_{1,2} \right).$$

Clearly, this norm satisfies condition (3.21). By definition, $\delta_H(v)$ exists for all $v \in \Lambda$. Moreover, $\int_0^T (G_H^*v)(s)\delta W(s)$ also exists for all H . Fix (v, s, H, H') in $\Lambda \times [0, T] \times h([0, 1])^2$ with that $H < H'$. Consider the function $\varphi : \Omega \times [H, H'] \rightarrow \mathbb{R}$ defined by:

$$\varphi(\omega, H_1) := (K_{H_1}^*v(\omega))(s) - (K_H^*v(\omega))(s) - (H_1 - H) \frac{(K_{H'}^*v(\omega))(s) - (K_H^*v(\omega))(s)}{H' - H}.$$

For every ω in Ω , $\varphi(\omega, \cdot)$ is C^1 and $\varphi(\omega, H) = \varphi(\omega, H') = 0$. As a consequence, there exists H''_ω in $[H, H']$ such that $\varphi'(H''_\omega) = 0$. Thus, the set $A_\omega := \{H_1 \in [H, H'] : \varphi'(\omega, H_1) = 0\}$ is a non-empty closed subset of $[H, H']$. It has a minimum, that we denote $H_0(\omega)$. The map $\omega \mapsto H_0(\omega)$ is measurable (*i.e.* H_0 is a random variable), as is the map $(v, s, H, H', \omega) \mapsto H_0(v, s, H, H', \omega)$ is measurable.

We wish to estimate $\|\mathcal{I}(v, H) - \mathcal{I}(v, H')\|_F$ for v in Λ . As we have just seen, there exists a measurable function $H_0 = H_0(H, H', v, s, \omega)$ such that:

$$\begin{aligned} u(s) &:= (K_{H'}^*v)(s) - (K_H^*v)(s) \\ &= (H' - H)(G_{H_0}^*v)(s). \end{aligned}$$

Thus:

$$\begin{aligned} \mathcal{I}(v, H') - \mathcal{I}(v, H) &= \delta(u) \\ &= (H' - H) \int_0^T (G_{H_0}^*v)(s)\delta W(s). \end{aligned}$$

Now, $\int_0^T (G_{H_0}^*v)(s)\delta W(s) = \delta(G_{H_0}^*v)$ and there exists a constant C such that:

$$\|\delta(G_{H_0}^*v)\|_{L^2(\Omega)} \leq C \|(G_{H_0}^*v)\|_{1,2}$$

[2], formula (5)), which entails that

$$\|\mathcal{I}(v, H') - \mathcal{I}(v, H)\|_{L^2(\Omega)} \leq |H - H'| \|v\|_\Lambda$$

i.e. (3.23) holds with $\theta = 1$, $E = F := L^2(\Omega)$ and $\Lambda_E := \Lambda$.

We may thus set the following definition and theorem:

Theorem-Definition 4.1. *Let $Y \in \Lambda$ be such that $h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t))$ is almost surely Riemann integrable and such that $\int_0^1 h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt$ belongs to $L^2(\Omega)$. Then the Skorohod integral of Y with respect to mBm is well-defined and given by:*

$$\int_0^1 Y_t \delta B_t^h := \lim_{n \rightarrow \infty} \int_0^1 Y_t \delta B_t^{h_n} + \int_0^1 h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) dt. \quad (4.5)$$

where the equality holds in $L^2(\Omega)$.

Remark 4.1. *It is easily verified that, for $\mathbf{B} = \mathbf{B}_4$, our integral coincides with the one studied in [9] when they are both defined. Comparing their domains would be an interesting task.*

5 Wick-Itô integral with respect to mBm through approximating $fBms$

Our aim in this section is to construct a Wick-Itô integral w.r.t. mBm using approximating $fBms$, and to compare this integral with the one defined in [14]. For definiteness, we will use the field \mathbf{B}_1 (which was the one used in [14]), but any other field would lead to similar developments.

We first briefly recall some basic facts about white noise theory and the Bochner integral, as well as on the construction of the integral w.r.t. fBm in the spirit of [5, 6, 7, 11].

5.1 Recalls on White Noise theory and the Bochner integral

5.1.1 White Noise theory

Define the measurable space (Ω, \mathcal{F}) by setting $\Omega := \mathcal{S}'(\mathbb{R})$ and $\mathcal{F} := \mathcal{B}(\mathcal{S}'(\mathbb{R}))$, where \mathcal{B} denotes the σ -algebra of Borel sets. There exists a unique probability measure μ on (Ω, \mathcal{F}) such that, for every f in $L^2(\mathbb{R})$, the map $\langle \cdot, f \rangle: \Omega \rightarrow \mathbb{R}$ defined by $\langle \cdot, f \rangle(\omega) = \langle \omega, f \rangle$ (where $\langle \omega, f \rangle$ is by definition $\omega(f)$, i.e the action of the distribution ω on the function f) is a centered Gaussian random variable with variance equal to $\|f\|_{L^2(\mathbb{R})}^2$ under μ . For every n in \mathbb{N} , define $e_n(x) := (-1)^n \pi^{-1/4} (2^n n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2})$ the n -th Hermite function. Let $(\|\cdot\|_p)_{p \in \mathbb{Z}}$ be the family norms defined by $\|f\|_p^2 := \sum_{k=0}^{+\infty} (2k+2)^{2p} \langle f, e_k \rangle_{L^2(\mathbb{R})}^2$, for all (p, f) in $\mathbb{Z} \times L^2(\mathbb{R})$. The operator A defined on $\mathcal{S}'(\mathbb{R})$ by $A := -\frac{d^2}{dx^2} + x^2 + 1$ admits the sequence $(e_n)_{n \in \mathbb{N}}$ as eigenfunctions and the sequence $(2n+2)_{n \in \mathbb{N}}$ as eigenvalues.

We denote (L^2) the space $L^2(\Omega, \mathcal{G}, \mu)$ where \mathcal{G} is the σ -field generated by $(\langle \cdot, f \rangle)_{f \in L^2(\mathbb{R})}$. For every random variable Φ of (L^2) there exists, according to the Wiener-Itô theorem, a unique sequence $(f_n)_{n \in \mathbb{N}}$ of functions f_n in $\widehat{L}^2(\mathbb{R}^n)$ such that Φ can be decomposed as $\Phi = \sum_{n=0}^{+\infty} I_n(f_n)$, where $\widehat{L}^2(\mathbb{R}^n)$ denotes the set of all symmetric functions f in $L^2(\mathbb{R}^n)$ and $I_n(f)$ denotes the n -th multiple Wiener-Itô integral of f with the convention that $I_0(f_0) = f_0$ for constants f_0 . Moreover we have the equality $\mathbb{E}[\Phi^2] = \sum_{n=0}^{+\infty} n! \|f_n\|_{L^2(\mathbb{R}^n)}^2$ where \mathbb{E} denotes the expectation with respect to μ . For any $\Phi := \sum_{n=0}^{+\infty} I_n(f_n)$ satisfying the condition $\sum_{n=0}^{+\infty} n! \|A^{\otimes n} f_n\|_0^2 < +\infty$, define the element $\Gamma(A)(\Phi)$ of (L^2) by $\Gamma(A)(\Phi) := \sum_{n=0}^{+\infty} I_n(A^{\otimes n} f_n)$, where $A^{\otimes n}$ denotes the n th tensor power of the operator A (see [12] appendix E for more details about tensor products of operators). The operator $\Gamma(A)$ is densely defined on (L^2) . It is invertible and its inverse $\Gamma(A)^{-1}$ is bounded. Let $\|\varphi\|_0^2 := \|\varphi\|_{L^2}^2$ for any random variable φ in (L^2) and, for n in \mathbb{N} , let $\mathbb{D}\text{om}(\Gamma(A)^n)$ be the domain of the n -th iteration of $\Gamma(A)$. Define the family of norms $(\|\cdot\|_p)_{p \in \mathbb{Z}}$ by:

$$\|\Phi\|_p := \|\Gamma(A)^p \Phi\|_0 = \|\Gamma(A)^p \Phi\|_{(L^2)}, \quad \forall p \in \mathbb{Z}, \quad \forall \Phi \in (L^2) \cap \mathbb{D}\text{om}(\Gamma(A)^p).$$

For p in \mathbb{N} , define $(\mathcal{S}_p) := \{\Phi \in (L^2) : \Gamma(A)^p \Phi \text{ exists and belongs to } (L^2)\}$ and define (\mathcal{S}_{-p}) as the completion of the space (L^2) with respect to the norm $\| \cdot \|_{-p}$. As in [13], we let (\mathcal{S}) denote the projective limit of the sequence $((\mathcal{S}_p))_{p \in \mathbb{N}}$ and $(\mathcal{S})^*$ the inductive limit of the sequence $((\mathcal{S}_{-p}))_{p \in \mathbb{N}}$. The space (\mathcal{S}) is called the space of stochastic test functions and $(\mathcal{S})^*$ the space of Hida distributions. One can show that, for any p in \mathbb{N} , the dual space $(\mathcal{S}_p)^*$ of \mathcal{S}_p is (\mathcal{S}_{-p}) . Thus we will write (\mathcal{S}_{-p}) , in the sequel, to denote the space $(\mathcal{S}_p)^*$. Note also that $(\mathcal{S})^*$ is the dual space of (\mathcal{S}) . We will note $\ll \cdot, \cdot \gg$ the duality bracket between $(\mathcal{S})^*$ and (\mathcal{S}) . If ϕ, Φ belong to (L^2) then we have the equality $\ll \Phi, \phi \gg = \langle \Phi, \phi \rangle_{(L^2)} = \mathbb{E}[\Phi \phi]$. A function $\Phi : \mathbb{R} \rightarrow (\mathcal{S})^*$ is called a stochastic distribution process, or an $(\mathcal{S})^*$ -process, or a Hida process. Since we have defined a topology given by a family of norms on the space $(\mathcal{S})^*$ it is possible to define a derivative and an integral in $(\mathcal{S})^*$. A Hida process Φ is said to be differentiable at $t_0 \in \mathbb{R}$ if $\lim_{r \rightarrow 0} r^{-1} (\Phi(t_0 + r) - \Phi(t_0))$ exists in $(\mathcal{S})^*$.

For f in $L^2(\mathbb{R})$, we define the *Wick exponential* of $\langle \cdot, f \rangle$, noted $: e^{\langle \cdot, f \rangle} :$, as the (L^2) random variable equal to $e^{\langle \cdot, f \rangle - \frac{1}{2} \|f\|_0^2}$. The S -transform of an element Φ of $(\mathcal{S})^*$, noted $S(\Phi)$, is defined as the function from $\mathcal{S}(\mathbb{R})$ to \mathbb{R} given by $S(\Phi)(\eta) := \ll \Phi, : e^{\langle \cdot, \eta \rangle} : \gg$ for every η in $\mathcal{S}(\mathbb{R})$. Finally for every $(\Phi, \Psi) \in (\mathcal{S})^* \times (\mathcal{S})^*$, there exists a unique element of $(\mathcal{S})^*$, called the Wick product of Φ and Ψ and noted $\Phi \diamond \Psi$, such that $S(\Phi \diamond \Psi)(\eta) = S(\Phi)(\eta) S(\Psi)(\eta)$ for every η in $\mathcal{S}(\mathbb{R})$.

5.1.2 Fractional and multifractional White noise

We introduce two operators, denoted M_H and $\frac{\partial M_H}{\partial H}$, that will prove useful for the definition of the integral with respect to fBm and mBm.

Operators M_H and $\frac{\partial M_H}{\partial H}$

Let H be a fixed real in $(0, 1)$. Following [11] and references therein, define the operator M_H , specified in the Fourier domain, by $\widehat{M_H(u)}(y) := \frac{\sqrt{2\pi}}{c_H} |y|^{1/2-H} \widehat{u}(y)$ for every y in \mathbb{R}^* . This operator is well defined on the homogeneous Sobolev space of order $1/2 - H$, denoted $L_H^2(\mathbb{R})$ and defined by $L_H^2(\mathbb{R}) := \{u \in \mathcal{S}'(\mathbb{R}) : \widehat{u} = T_f; f \in L_{loc}^1(\mathbb{R}) \text{ and } \|u\|_H < +\infty\}$, where the norm $\| \cdot \|_H$ derives from the inner product $\langle \cdot, \cdot \rangle_H$ defined on $L_H^2(\mathbb{R})$ by $\langle u, v \rangle_H := \frac{1}{c_H^2} \int_{\mathbb{R}} |\xi|^{1-2H} \widehat{u}(\xi) \widehat{v}(\xi) d\xi$ and where c_H was given in the definition of the fractional field \mathbf{B}_1 .

The definition of the operator $\frac{\partial M_H}{\partial H}$ is quite similar. More precisely, define for every H in $(0, 1)$, the space $\Gamma_H(\mathbb{R}) := \{u \in \mathcal{S}'(\mathbb{R}) : \widehat{u} = T_f; f \in L_{loc}^1(\mathbb{R}) \text{ and } \|u\|_{\delta_H(\mathbb{R})} < +\infty\}$, where the norm $\| \cdot \|_{\delta_H(\mathbb{R})}$ derives from the inner product on $\Gamma_H(\mathbb{R})$ defined by $\langle u, v \rangle_{\delta_H} := \frac{1}{c_H^2} \int_{\mathbb{R}} (\beta_H + \ln |\xi|)^2 |\xi|^{1-2H} \widehat{u}(\xi) \widehat{v}(\xi) d\xi$. Following [14], define the operator $\frac{\partial M_H}{\partial H}$ from $(\Gamma_H(\mathbb{R}), \langle \cdot, \cdot \rangle_{\delta_H(\mathbb{R})})$ to $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle_{L^2(\mathbb{R})})$, in the Fourier domain, by: $\widehat{\frac{\partial M_H}{\partial H}(u)}(y) := -(\beta_H + \ln |y|) \frac{\sqrt{2\pi}}{c_H} |y|^{1/2-H} \widehat{u}(y)$, for every y in \mathbb{R}^* . The reader interested in the properties of M_H and $\frac{\partial M_H}{\partial H}$ may refer to [14, section 3.2].

Fractional and multifractional White noise

[14, (5.10)] states that:

$$\text{Almost surely, for every } t, B_t^h = \mathbf{B}_1(t, h(t)) = \langle \cdot, M_{h(t)}(\mathbf{1}_{[0,t]}) \rangle = \sum_{k=0}^{+\infty} \left(\int_0^t M_{h(t)}(e_k)(s) ds \right) \langle \cdot, e_k \rangle. \quad (5.1)$$

We now define the derivative in the sense of $(\mathcal{S})^*$ of mBm. Define the (\mathcal{S}^*) -valued process $W^h := (W_t^h)_{t \in [0,1]}$ by

$$W_t^h := \sum_{k=0}^{+\infty} \left[\frac{d}{dt} \left(\int_0^t M_{h(t)}(e_k)(s) ds \right) \right] \langle \cdot, e_k \rangle. \quad (5.2)$$

Theorem-Definition 5.1. [14, Theorem-definition 5.1] *The process W^h defined by (5.2) is an $(\mathcal{S})^*$ -process which verifies, in $(\mathcal{S})^*$, the following equality:*

$$W_t^h = \sum_{k=0}^{+\infty} M_{h(t)}(e_k)(t) \langle \cdot, e_k \rangle + h'(t) \sum_{k=0}^{+\infty} \left(\int_0^t \frac{\partial M_H}{\partial H}(e_k)(s) \Big|_{H=h(t)} ds \right) \langle \cdot, e_k \rangle. \quad (5.3)$$

Moreover the process B^h is $(\mathcal{S})^*$ -differentiable on $[0, 1]$ and verifies $\frac{dB^h}{dt}(t) = W_t^h$ in $(\mathcal{S})^*$.

When the function h_0 is constant and identically equal to H , we will write $W^H := (W_t^H)_{t \in [0, 1]}$ and call the $(\mathcal{S})^*$ -process W^H a fractional white noise. Note that (5.3) may be written as

$$W_t^h = W_t^{h(t)} + h'(t) \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)), \quad (5.4)$$

where $W_t^{h(t)}$ is nothing but $W_t^H|_{H=h(t)}$ and where the equality holds in $(\mathcal{S})^*$.

5.1.3 Bochner integral

Since the objects we are dealing with are no longer random variables in general, the Riemann or Lebesgue integrals are not relevant here. However, taking advantage of the fact that we are working with vector linear spaces, we may use Pettis or Bochner integrals. In the frame of the Wick-Itô integral, the space E , defined at the beginning of section 3.2 will be a space (\mathcal{S}_{-p}) for some integer p . The fact that we need a norm on \mathcal{H}_E suggests the use of Bochner integral. A nice survey of this topic may be found in [13, p.247]. We only recall here the definition and two basic results.

Definition 5.1 (Bochner integral [13], p.247). *Let I be a subset of $[0, 1]$ endowed with the Lebesgue measure. One says that $\Phi : I \rightarrow (\mathcal{S})^*$ is Bochner integrable of index p on I if it satisfies the two following conditions:*

1. Φ is weakly measurable on I , i.e $t \mapsto \langle \Phi_t, \varphi \rangle$ is measurable on I for every φ in (\mathcal{S}) .
2. There exists p in \mathbb{N} such that $\Phi_t \in (\mathcal{S}_{-p})$ for almost every t in I and $t \mapsto \|\Phi_t\|_{-p}$ belongs to $L^1(I, dt)$.

The Bochner-integral of Φ on I is denoted $\int_I \Phi_t dt$.

Proposition 5.2. *If $\Phi : I \rightarrow (\mathcal{S})^*$ is Bochner-integrable on I with index p then $\|\int_I \Phi_t dt\|_{-p} \leq \int_I \|\Phi_t\|_{-p} dt$.*

Theorem 5.3 ([13], theorem 13.5). *Let $\Phi := (\Phi_t)_{t \in [0, 1]}$ be an $(\mathcal{S})^*$ -valued process such that:*

- (i) $t \mapsto S(\Phi_t)(\eta)$ is measurable for every η in $\mathcal{S}(\mathbb{R})$.
- (ii) There exist p in \mathbb{N} , b in \mathbb{R}^+ and a function L in $L^1([0, 1], dt)$ such that, for a.e. t in $[0, 1]$, $|S(\Phi_t)(\eta)| \leq L(t) e^{b|\eta|^2}$, for every η in $\mathcal{S}(\mathbb{R})$.

Then Φ is Bochner integrable on $[0, 1]$ and $\int_0^1 \Phi(s) ds \in (\mathcal{S}_{-q})$ for every $q > p$ such that $2be^2 D(q-p) < 1$ where e denotes the base of the natural logarithm and where $D(r) := \frac{1}{2^{2r}} \sum_{n=1}^{+\infty} \frac{1}{n^{2r}}$ for r in $(1/2, +\infty)$.

5.2 Wick-Itô integral with respect to fBm

The fractional Wick-Itô integral with respect to fBm (or integral w.r.t fBm in the white noise) was introduced in [11] and extended in [5] using the Pettis integral. However, in order to use Theorem 3.4, we need that Y_s belongs to (\mathcal{S}_{-p}) for almost every real s . It then seems reasonable to assume that $(Y_s)_{s \in [0, 1]}$ is Bochner integrable on $[0, 1]$. For this reason, we now particularize the fractional Wick-Itô integral with respect to fBm of [11] and [5] in the framework of the Bochner integral.

Definition 5.4 (Wick-Itô integral w.r.t fBm in the Bochner sense). *Let $H \in (0, 1)$, I be a Borel subset of $[0, 1]$, $B^H := (B_t^H)_{t \in I}$ be a fractional Brownian motion of Hurst index H , and $Y := (Y_t)_{t \in I}$ be an $(\mathcal{S})^*$ -valued process verifying:*

- (i) there exists $p \in \mathbb{N}$ such that $Y_t \in (\mathcal{S}_{-p})$ for almost every $t \in I$,

(ii) the process $t \mapsto Y_t \diamond W_t^H$ is Bochner integrable on I .

Then, Y is said to be Bochner-integrable with respect to fBm on I and its integral is defined by:

$$\int_I Y_s d^\circ B_s^H := \int_I Y_s \diamond W_s^H ds. \quad (5.5)$$

Lemma 5.5. *Let $Y := (Y_t)_{t \in [0,1]}$ be an $(\mathcal{S})^*$ -valued process, Bochner integrable of index $p_0 \in \mathbb{N}$. Then Y is integrable on $[0, 1]$, with respect to fBm of any Hurst index H , in the Bochner sense. Moreover, for any H in $(0, 1)$, $\int_{[0,1]} Y_s d^\circ B_s^H$ belongs to (\mathcal{S}_{-r_0}) for every $r_0 \geq p_0 + 1$ if $p_0 \geq 2$ and for every $r_0 \geq p_0 + 2$ if $p_0 \in \{0, 1\}$.*

Proof: Fix $H \in (0, 1)$, $p_0 \geq 2$ and $r_0 \geq p_0 + 1$. The map $t \mapsto Y_t \diamond W_t^H$ is weakly measurable since $t \mapsto S(Y_t \diamond W_t^H)(\eta)$ is measurable for every $\eta \in \mathcal{S}(\mathbb{R})$. Using [13, remark 2 p.92], one obtains that, for almost every t in $[0, 1]$, $\|Y_t \diamond W_t^H\|_{-r_0} \leq \|Y_t\|_{-p_0} \|W_t^H\|_{-p_0} < +\infty$. Hence $Y_t \diamond W_t^H$ belongs to (\mathcal{S}_{-r_0}) . Since the map $t \mapsto \|W_t^H\|_{-r}$ is continuous for every integer $r \geq 2$ (see [14, proposition 5.9]), one also gets:

$$\int_0^1 \|Y_t \diamond W_t^H\|_{-r_0} dt \leq \left(\sup_{t \in [0,1]} \|W_t^H\|_{-p_0} \right) \int_0^1 \|Y_t\|_{-p_0} dt < +\infty.$$

This shows that $t \mapsto Y_t \diamond W_t^H$ is Bochner-integrable of index r_0 .

Let us now assume that $p_0 \in \{0, 1\}$. It is sufficient to check that theorem 5.3 applies. Condition (i) is obviously fulfilled. Moreover, using [13, p.79], we obtain that, for every (t, η) in $[0, 1] \times \mathcal{S}(\mathbb{R})$,

$$|S(Y_t \diamond W_t^H)(\eta)| \leq \|Y_t\|_{-p_0} e^{\frac{1}{2}|\eta|^2} \sup_{t \in [0,1]} \|W_t^H\|_{-2} =: L(t) e^{\frac{1}{2}|\eta|^2}.$$

Since Y is Bochner integrable of index p_0 , it is clear that L belongs to $L^1([0, 1], dt)$. Moreover, $e^2 D(r_0 - p_0) < 1$, for every $r_0 \geq p_0 + 2$. Theorem 5.3 then allows to conclude that $t \mapsto Y_t \diamond W_t^H$ is Bochner integrable of index r_0 . \square

The following lemma, the proof of which is obvious in view of hypothesis (\mathcal{H}_2) , will be useful in the proof of proposition 5.9 below.

Lemma 5.6. *For every p in \mathbb{N} , the map $(t, H) \mapsto \frac{\partial \mathbf{B}_1}{\partial H}(t, H)$ is continuous from $[0, 1]$ into $((\mathcal{S}_{-p}), \|\cdot\|_p)$. In particular, for every subset $[a, b]$ of $(0, 1)$, there exists a positive real κ such that:*

$$\forall p \in \mathbb{N}, \quad \sup_{(s, H) \in [0, t] \times [a, b]} \left\| \frac{\partial \mathbf{B}_1}{\partial H}(s, H) \right\|_{-p} \leq \kappa. \quad (5.6)$$

5.3 Stochastic integral with respect to mBm with approximating fBms

We construct in this section the Wick-Itô integral w.r.t. mBm using approximating fBms. In that view, we shall apply Theorem 3.4. Fix p_0 in \mathbb{N} and s_0 in \mathbb{N} such that $s_0 \geq \max\{p_0 + 1, 3\}$. Set $E := (\mathcal{S}_{-p_0})$ and $F := (\mathcal{S}_{-s_0})$. Define

$$\Lambda_E := \mathcal{H}_E = \left\{ (Y_t)_{t \in [0,1]} \in (\mathcal{S}_{-p_0})^{\mathbb{R}} : \int_{[0,1]} Y_t d^\circ B_t^\alpha \in (\mathcal{S}_{-s_0}), \forall \alpha \in h([0, 1]) \right\},$$

and equip \mathcal{H}_E with the norm $\|\cdot\|_{\Lambda_E} := \|\cdot\|_{\mathcal{H}_E}$ defined by $\|\Phi\|_{\mathcal{H}_E} := \int_0^1 \|\Phi_t\|_{-p_0} dt$. This norm fulfils condition (3.21). Indeed, let A_1, A_2, \dots, A_n be disjoint Borelians of $[0, 1]$. Then

$$\begin{aligned} \sum_{k=1}^n \|Y \mathbf{1}_{A_k}\|_{\mathcal{H}_E} &= \sum_{k=1}^n \int_0^1 \|Y \mathbf{1}_{A_k}(s)\|_{-p_0} ds \leq \sum_{k=1}^n \int_0^1 \mathbf{1}_{A_k}(s) \|Y_s\|_{-p_0} ds \\ &\leq \int_0^1 \left(\sum_{k=1}^n \mathbf{1}_{A_k}(s) \right) \|Y_s\|_{-p_0} ds \leq \|Y\|_{\mathcal{H}_E}. \end{aligned} \quad (5.7)$$

Lemma 5.7. *The Wick-Itô integral w.r.t. fBm verifies condition (3.23).*

Proof: Since $(\mathcal{S}_{-p_0}) \subset (\mathcal{S}_{-2})$ if p_0 belongs to $\{0; 1\}$, we assume from now that $p_0 \geq 2$ and that $s_0 \geq p_0 + 1$. Let $Y \in \mathcal{H}_E$. Lemma 5.5 entails that Y is integrable with respect to fBm in the Bochner sense, for any Hurst index α in $(0, 1)$ and that $\mathcal{I}(Y, \alpha) = \int_{[0,1]} Y_t d^\diamond B_t^\alpha$ belongs to (\mathcal{S}_{-s_0}) . Now for any (α, α') in $(0, 1)^2$, using the same arguments we used in the proof of lemma 5.5,

$$\begin{aligned} \|\mathcal{I}(Y, \alpha) - \mathcal{I}(Y, \alpha')\|_{-s_0} &= \left\| \int_{[0,1]} Y_t \diamond (W_t^\alpha - W_t^{\alpha'}) dt \right\|_{-s_0} \leq \int_0^1 \|Y_t\|_{-p_0} \|W_t^\alpha - W_t^{\alpha'}\|_{-p_0} dt \\ &\leq \left(\sup_{t \in [0,1]} \|W_t^\alpha - W_t^{\alpha'}\|_{-p_0} \right) \|Y\|_{\mathcal{H}_E}, \end{aligned}$$

thus

$$\sup_{\|Y\|_{\mathcal{H}_E} \leq 1} \|\mathcal{I}(Y, \alpha) - \mathcal{I}(Y, \alpha')\|_{-s_0} \leq \sup_{t \in [0,1]} \|W_t^\alpha - W_t^{\alpha'}\|_{-p_0}. \quad (5.8)$$

By definition, $\|W_t^\alpha - W_t^{\alpha'}\|_{-p_0}^2 = \sum_{k=0}^{+\infty} \frac{(M_\alpha(e_k)(t) - M_{\alpha'}(e_k)(t))^2}{(2k+2)^{2p_0}}$.

For every (t, k) in $[0, 1] \times \mathbb{N}$, the function $\alpha \mapsto M_\alpha(e_k)(t)$ is differentiable on $(0, 1)$ (this is Lemma 5.5 in [14]. Using point 1 of [14, lemma 5.6] and the mean value theorem, one obtains the following fact: for every $[a, b] \subset (0, 1)$, there exists a positive real ρ such that for all $(t, \alpha, \alpha', k) \in [0, 1] \times [a, b]^2 \times \mathbb{N}$:

$$|M_\alpha(e_k)(t) - M_{\alpha'}(e_k)(t)| \leq \rho (k+1)^{2/3} \ln(k+1) |\alpha - \alpha'|.$$

As a consequence,

$$\|W_t^\alpha - W_t^{\alpha'}\|_{-p_0}^2 \leq \rho^2 |\alpha - \alpha'|^2 \sum_{k=0}^{+\infty} \frac{(k+1)^{4/3} \ln^2(k+1)}{2^{2p_0} (k+1)^{2p_0}}.$$

With (5.8), one gets

$$\sup_{\|Y\|_{\mathcal{H}_E} \leq 1} \|\mathcal{I}(Y, \alpha) - \mathcal{I}(Y, \alpha')\|_{-s_0} \leq |\alpha - \alpha'| \gamma_{p_0} \quad (5.9)$$

where $\gamma_{p_0} := \rho \left(\sum_{k=1}^{+\infty} \frac{\ln^2 k}{k^{2(p_0-2/3)}} \right)^{1/2}$ is finite since $p_0 \geq 2$. □

A consequence of the previous lemma is that Theorem 3.4 applies, that is, $\lim_{n \rightarrow +\infty} \int_{[0,1]} Y_t d^\diamond B_t^{h_n}$ exists as an element of (\mathcal{S}_{-s_0}) .

Lemma 5.8. *For every process $Y := (Y_t)_{t \in [0,1]}$ Bochner-integrable on $[0, 1]$, the map $t \mapsto h'(t) Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t))$ is integrable.*

Proof: Fix $p_0 \geq 2$ and $s_0 \geq p_0 + 1$. Using the same arguments as in the proof of lemma 5.5 one easily prove that $t \mapsto h'(t) Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t))$ on $[0, 1]$ is weakly measurable. Lemma 5.6 entails that:

$$\sup_{(s, H) \in [0,1] \times h([0,1])} \left\| \frac{\partial \mathbf{B}_1}{\partial H}(s, H) \right\|_{-p_0} \leq \kappa,$$

for every p_0 . We hence get

$$\left\| h'(t) Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) \right\|_{-s_0} \leq \|Y_s\|_{-p_0} \left(\sup_{s \in [0,1]} |h'(s)| \right) \sup_{s \in [0,1]} \left\| \frac{\partial \mathbf{B}_1}{\partial H}(s, h(s)) \right\|_{-p_0} < +\infty.$$

Thus there exists $\delta \in \mathbb{R}_+$, such that $\int_0^1 \|h'(s) Y_s \diamond \frac{\partial \mathbf{B}_1}{\partial H}(s, h(s))\|_{-s_0} ds \leq \delta \int_0^1 \|Y_s\|_{-p_0} ds < +\infty$. As a consequence, $\int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) dt$ is well defined in the sense of Bochner. □

Corollary 5.9. *Let $Y \in \mathcal{H}_E$. Then*

$$\int_0^1 Y_t d^\diamond B_t^h := \lim_{n \rightarrow \infty} \int_0^1 Y_t d^\diamond B_t^{h_n} + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) dt, \quad (5.10)$$

where the limit and the equality hold in (\mathcal{S}_{-s_0}) , is well-defined and belongs to (\mathcal{S}_{-s_0}) .

5.4 A comparison between multifractional Wick-Itô integral and limiting fractional Wick-Itô integral

A multifractional Wick-Itô integral with respect to mBm was defined [14]. It is interesting to check whether it coincides with the one defined in Corollary 5.9. In that view, we need to adapt the definition of [14], which used Pettis integrals, to deal with Bochner integrals.

Definition 5.10 (Multifractional Wick-Itô integral in Bochner sense). *Let I be a Borelian connected subset of $[0, 1]$, $B^h := (B_t^h)_{t \in I}$ be a multifractional Brownian motion and $Y := (Y_t)_{t \in I}$ be a $(\mathcal{S})^*$ -valued process such that:*

- (i) *There exists $p \in \mathbb{N}$ such that $Y_t \in (\mathcal{S}_{-p})$ for almost every $t \in I$,*
- (ii) *the process $t \mapsto Y_t \diamond W_t^h$ is Bochner integrable on I .*

Y is then said to be integrable on I with respect to mBm in the Bochner sense or having a multifractional Wick-Itô integral. This integral is defined to be $\int_I Y_s \diamond W_s^h ds$.

Remark 5.11. *From the definition of $(W_t^h)_{t \in [0,1]}$ [14, proposition 5.9], and the proof of lemma 5.5, it is clear that every $(\mathcal{S})^*$ -valued process $Y := (Y_t)_{t \in I}$ which is Bochner integrable on I of index p_0 , is integrable on I with respect to mBm, in the Bochner sense. Moreover $\int_{[0,1]} Y_t dB_t^{\circ h}$ belongs to (\mathcal{S}_{-r_0}) , where r_0 was defined in lemma 5.5.*

In order to compare our two integrals with respect to mBm when they both exist, it seems natural to assume that $Y = (Y_t)_{t \in [0,1]}$ is a Bochner integrable process of index $p_0 \in \mathbb{N}$. The space E and the norm $\|\cdot\|_{\mathcal{H}_E}$ are defined as in the previous subsection.

Theorem 5.12. *Let $Y = (Y_t)_{t \in [0,1]}$ be a Bochner integrable process of index $p_0 \in \mathbb{N}$. Then Y is integrable with respect to mBm in both senses of Corollary 5.9 and Definition 5.10. Moreover $\int_I Y_s \diamond W_s^h ds$ and $\int_{[0,1]} Y_t d^\circ B_t^h$ are equal in (\mathcal{S}^*) .*

Proof: Since Y is a Bochner integrable process of index $p_0 \in \mathbb{N}$, Proposition 5.9 and Remark 5.11 entail that $\int_{[0,1]} Y_s d^\circ B_s^h$ exists in (\mathcal{S}_{-s_0}) and that $\int_I Y_s \diamond W_s^h ds$ exist in (\mathcal{S}_{-r_0}) , where s_0 has been defined just above proposition 5.9 and r_0 has been defined in lemma 5.5. Moreover, thanks to (5.10) and using (3.4) and (5.5), we may write, in (\mathcal{S}_{-s_0}) ,

$$\begin{aligned} \int_{[0,1]} Y_s d^\circ B_s^h &= \lim_{n \rightarrow \infty} \int_0^1 Y_t d^\circ B_t^{h_n} + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) dt \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) Y_t d^\circ B_t^{h(x_k^{(n)})} + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) dt \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) Y_t \diamond W_t^{h(x_k^{(n)})} dt + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) dt. \end{aligned} \quad (5.11)$$

Besides, equality (5.10) and (5.4) entail that, in (\mathcal{S}_{-r_0}) ,

$$\int_0^1 Y_t \diamond W_t^h dt = \int_0^1 Y_t \diamond W_t^{h(t)} dt + \int_0^1 h'(t) Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) dt. \quad (5.12)$$

Since $s_0 \geq r_0$ we have $(\mathcal{S}_{-r_0}) \subset (\mathcal{S}_{-s_0})$. Thus it remains to show that, in (\mathcal{S}_{-s_0}) ,

$$L(Y) := \lim_{n \rightarrow \infty} \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) Y_t \diamond W_t^{h(x_k^{(n)})} dt \text{ is equal to } M(Y) := \int_0^1 Y_t \diamond W_t^{h(t)} dt.$$

Since $L(Y)$ and $M(Y)$ both belong to (\mathcal{S}_{-s_0}) , it is sufficient to show that they have the same S -transform. Using [13, theorem 8.6], one has, for η in $\mathcal{S}(\mathbb{R})$,

$$S(L(Y))(\eta) = \lim_{n \rightarrow \infty} S \left(\sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) Y_t \diamond W_t^{h(x_k^{(n)})} dt \right) (\eta).$$

Using now (ii) of [14, theorem 5.12], one gets

$$\begin{aligned} S(L(Y))(\eta) &= \lim_{n \rightarrow +\infty} \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} S(Y_t)(\eta) S(W_t^{h(x_k^{(n)})})(\eta) dt \\ &= \lim_{n \rightarrow +\infty} \int_{[0,1]} S(Y_t)(\eta) \left(\sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) M_{h(x_k^{(n)})}(\eta)(t) \right) dt. \end{aligned} \quad (5.13)$$

The map $(t, H) \mapsto M_H(\eta)(t)$ is continuous (see [14, lemma 5.5]). Define $K_\eta := \sup_{(t, H) \in [0,1] \times h([0,1])} M_H(\eta)(t)$.

For every n in \mathbb{N} and t in $[0, 1]$, one has

$$\begin{aligned} \left| S(Y_t)(\eta) \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) M_{h(x_k^{(n)})}(\eta)(t) \right| &\leq \sup_{(t, H) \in [0,1] \times h([0,1])} M_H(\eta)(t) e^{\frac{1}{2}|\eta|_{p_0}^2} \|Y_t\|_{-p_0} \\ &\leq K_\eta e^{\frac{1}{2}|\eta|_{p_0}^2} \|Y_t\|_{-p_0}. \end{aligned} \quad (5.14)$$

The map $t \mapsto K_\eta e^{\frac{1}{2}|\eta|_{p_0}^2} \|Y_t\|_{-p_0}$ belongs to $L^1(\mathbb{R}, dt)$. In addition, for almost every t in $[0, 1]$,

$$\lim_{n \rightarrow +\infty} \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)}]}(t) M_{h(x_k^{(n)})}(\eta)(t) = M_{h(t)}(\eta)(t),$$

Thus, by dominated convergence, one gets, using (5.14),

$$S(L(Y))(\eta) = \int_{[0,1]} S(Y_t)(\eta) M_{h(t)}(\eta)(t) dt = \int_{[0,1]} S(Y_t)(\eta) S(W_t^{h(t)})(\eta) dt = S \left(\int_{[0,1]} Y_t \diamond W_t^{h(t)} dt \right) (\eta),$$

Since the map $S : \Phi \mapsto S(\Phi)$ from $(\mathcal{S})^*$ into itself is injective, one deduces that $L(Y) = \int_{[0,1]} Y_t \diamond W_t^{h(t)} dt$, and the proof is complete. \square

Appendix

A Proof of proposition 3.1 in the case of \mathbf{B}_1 and \mathbf{B}_2

It is sufficient to establish the proof for \mathbf{B}_1 . Almost surely, $\mathbf{B}_1(t, H) := \langle \cdot, M_H(\mathbb{1}_{[0,t]}) \rangle$ for every (t, H) in $\mathbb{R} \times (0, 1)$ (see [11, section 2] for example). The following result is established in [14]: the map $H \mapsto M_H(\mathbb{1}_{[0,t]})$ is C^1 from $(0, 1)$ to $L^2(\mathbb{R})$ for every t . The map is $H \mapsto \mathbf{B}_1(t, H)$ is C^1 for every real t and its derivative is such that, almost surely, $\frac{\partial \mathbf{B}_1}{\partial H}(t, H) = \langle \cdot, \frac{\partial M_H}{\partial H}(\mathbb{1}_{[0,t]}) \rangle$ for every (t, H) in $(0, 1)$. Note moreover that the previous equality also holds in $L^2(\Omega)$ and that the process $(\frac{\partial \mathbf{B}_1}{\partial H}(t, H))_{(t,H) \in \mathbb{R} \times (0,1)}$ is Gaussian and centred. Now, with the notations of [14], $\mathbb{E}[\frac{\partial \mathbf{B}_1}{\partial H}(t, H) \frac{\partial \mathbf{B}_1}{\partial H}(s, H')] = \langle \frac{\partial M_H}{\partial H}(\mathbb{1}_{[0,t]}), \frac{\partial M_{H'}}{\partial H}(\mathbb{1}_{[0,s]}) \rangle_{L^2(\mathbb{R})}$, for every $(t, H) \in \mathbb{R} \times (0, 1)$. Hence,

$$J := \mathbb{E} \left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t, H) - \frac{\partial \mathbf{B}_1}{\partial H}(s, H) \right)^2 \right] = \left\| \frac{\partial M_H}{\partial H}(\mathbb{1}_{[0,t]} - \mathbb{1}_{[0,s]}) \right\|_{L^2(\mathbb{R})}^2 = \frac{1}{c_H^2} \int_{\mathbb{R}} (\beta_H + \ln |y|)^2 |y|^{1-2H} \left| \frac{1 - e^{iy(t-s)}}{y^2} \right|^2 dy.$$

where $\beta_H := \frac{c_H}{c_H}$. Fix τ in $(0, c)$ and let $M := e^{\frac{\ln 2}{\tau}}$. Note that $M > 1$ and that $|y|^\tau \geq 2$ for every y such that $|y| \geq M$. One computes:

$$\begin{aligned} J &= \frac{1}{c_H^2} \int_{|y| > M} (\beta_H + \ln |y|)^2 |y|^{1-2H} \left| \frac{1 - e^{iy(t-s)}}{y^2} \right|^2 dy + \frac{1}{c_H^2} \int_{|y| \leq M} (\beta_H + \ln |y|)^2 |y|^{1-2H} \left| \frac{1 - e^{iy(t-s)}}{y^2} \right|^2 dy \\ &\leq \frac{1}{c_H^2} \int_{|y| > M} (\beta_H + \ln |y|)^2 |y|^{1-2H} |t-s|^2 \frac{|y|^{2\tau}}{|y|^2} dy + |t-s|^2 \frac{2^2}{c_H^2} \int_{|y| \leq M} \frac{(\beta_H + \ln |y|)^2}{|y|^{2H-1}} dy \\ &\leq |t-s|^2 \frac{4}{c_H^2} \left(\int_{|y| > M} \frac{(\beta_H + \ln |y|)^2}{|y|^{1+2(H-\tau)}} dy + \int_{|y| \leq M} \frac{(\beta_H + \ln |y|)^2}{|y|^{2H-1}} dy \right) =: |t-s|^2 Q(H). \end{aligned}$$

Since $\Delta_1 := \sup_{H \in [c,d]} Q(H) < +\infty$,

$$\mathbb{E} \left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t, H) - \frac{\partial \mathbf{B}_1}{\partial H}(s, H) \right)^2 \right] \leq \Delta_1 |t-s|^2. \quad (\text{A.1})$$

Besides, $\mathbb{E} \left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t, H) - \frac{\partial \mathbf{B}_1}{\partial H}(t, H') \right)^2 \right] = \int_{\mathbb{R}} |1 - e^{ity}|^2 (g_y(H) - g_y(H'))^2 dy$, where the map $g_y : (0, 1) \rightarrow \mathbb{R}$ is defined by $g_y(H) := \frac{(\beta_H + \ln |y|)}{c_H} |y|^{1/2-H}$ for every y in \mathbb{R}^* . It is easily seen that g_y is C^1 on $[c, d]$ for every y in \mathbb{R}^* . The mean value theorem applies and there exists a positive constant K , which only depends on $[c, d]$, such that

$$\mathbb{E} \left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t, H) - \frac{\partial \mathbf{B}_1}{\partial H}(t, H') \right)^2 \right] \leq |H - H'|^2 K \int_{\mathbb{R}} |1 - e^{ity}|^2 |\Phi(y)|^2 dy,$$

where $\Phi(y) := 1 + (|y|^{1/2-c} + |y|^{1/2-d}) (1 + (1 + \ln |y|) \ln |y|)$. Since $\Delta_2 := K \int_{\mathbb{R}} |1 - e^{ity}|^2 |\Phi(y)|^2 dy < +\infty$, we have proven that

$$\mathbb{E} \left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t, H) - \frac{\partial \mathbf{B}_1}{\partial H}(t, H') \right)^2 \right] \leq \Delta_2 |H - H'|^2. \quad (\text{A.2})$$

Setting $\Delta := 2(\Delta_1 + \Delta_2)$ and using (A.1) and (A.2), one obtains:

$$\begin{aligned} \mathbb{E} \left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t, H) - \frac{\partial \mathbf{B}_1}{\partial H}(s, H') \right)^2 \right] &\leq 2\mathbb{E} \left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t, H) - \frac{\partial \mathbf{B}_1}{\partial H}(t, H') \right)^2 \right] + 2\mathbb{E} \left[\left(\frac{\partial \mathbf{B}_1}{\partial H}(t, H') - \frac{\partial \mathbf{B}_1}{\partial H}(s, H') \right)^2 \right] \\ &\leq \Delta (|t-s|^2 + |H - H'|^2). \quad \square \end{aligned}$$

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