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Budan Tables of Real Univariate Polynomials

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Abstract

The Budan table of f collects the signs of the iterated derivative of f . We revisit the classical Budan-Fourier theorem for a univariate real polynomial f and establish a new connexity property of its Budan table. We use this property to characterize the virtual roots of f , (introduced by Gonzales-Vega, Lombardi, Mahé in 1998); they are continuous functions of the coefficients of f . We also consider a property (\mathcal{P}) of a polynomial f , which is generically satisfied, it eases the topological-combinatorial description and study of the Budan tables. A natural extension of the information collected by the virtual roots provides alternative representations of (\mathcal{P}) -polynomials; while an attached tree structure allows a finite stratification of the space of (\mathcal{P}) -polynomials. The paper is illustrated with examples and pictures computed with the computer algebra system Maple.

Keywords: Budan-Fourier theorem; virtual roots; roots of real univariate polynomial; Budan table; genericity; representation of univariate polynomials; Hermite-Birkhoff interpolation; stratification; incidence relations.

1 Introduction

Counting and localizing roots of univariate polynomials is a basic task in Algebra and Numerical analysis. Many known techniques and interesting historical notes can be found e.g. in the book [21] of Rahman and Schmeisser. Established two centuries ago the Budan-Fourier theorem connects the number of real roots of a polynomial in an interval with the sign variations of a sequence and inspired the invention of Sturm sequences. It also gave rise to efficient and certified subdivision algorithms.

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Subdivision methods exploit the ordered structure of the real numbers, and are widely applied for calculating good approximations of solutions of polynomial equations or intersections of surfaces in many applied sciences. The analysis of their complexity and efficiency relies on the the geometric and algebraic nature of the inputs. The dictionary between invariants readable on equations and features of varieties, in complex algebraic geometry, is ultimately based on the fact that a polynomial of degree n admits n roots. This is not the case for real roots. A natural strategy for studying properties of real algebraic varieties is to consider simultaneously roots of iterated derivatives of the input. A progress was achieved by Gonzales-Vega, Lombardi, Mahé in [11] when they introduced the concept of the n virtual roots of a degree n polynomial f (which depend continuously on the coefficients of f), to provide a good substitute to its n complex roots. Their motivation was an attempt to prove Pierce-Birkhoff conjecture on the representation of C^0 -polynomial spline functions. Virtual roots were then studied by Coste, Lajous, Lombardi, Roy in [9], and recently by Bembé in his PhD thesis [4].

The table containing the signs of all the derivatives of a polynomial f is called, in this paper, Budan table. The table is an infinite rectangle formed by positive and negative rectangular blocks, separated by segments corresponding to roots of derivatives. This structure was used by various mathematicians for separating and labeling the different real roots of a polynomial, see [7]. However up to now, authors considered properties of either rows or columns of a table, or their sign variations but not the table “itself”. Here, we adopt a 2D point of view, re-interpret known results and present new ones. We analyze the different admissible configurations of successive rows in such a table and deduce the 2D same-sign-connectivity of the positive (resp. negative) blocks components, then characterize the virtual roots using this data.

We define a simple property, we call (\mathcal{P}) , on the roots of the iterative derivatives of a monic polynomial f , which is generically satisfied. Then, for a (\mathcal{P}) -polynomial, i.e. assuming (\mathcal{P}) , we study and characterize the topological and combinatorial properties of the corresponding Budan table and virtual roots. We specialize results of Hermite-Birkhoff interpolation (see e.g. [19]) to our setting, to better analyze the relation between a (\mathcal{P}) -polynomial f and its virtual roots. We also introduce a finite discrete invariant, the Budan tree of a (\mathcal{P}) -polynomial f ; and initiate the study of the attached stratification by semi-algebraic sets of the space of (\mathcal{P}) -polynomials. Then we consider the general case, and relate it to the generic case through continuity properties.

The same-sign-connected block components inside the Budan table can be interpreted as plane surfaces delimited by discretized curves which can be approximated by continuous ones. This point of view was initiated in [5] and in [12] using derivatives with non-integers orders, called fractional derivatives. We plan to develop it further in a future work.

The paper is organized as follows. Section 2 provides basic definitions and

properties presented with our point of view, Section 3 introduces our generic assumption (\mathcal{P}), then presents our results in this generic case; they are illustrated with simple examples; the proofs are collected in Section 4. Section 5 examines admissible configurations for Budan tables and other invariants in the non generic case and establishes continuity results. Section 6 is devoted to some extensions and potential applications.

A partial and preliminary version of our approach was presented in the first part of our talk at the ISSAC'2011 conference, (and in the corresponding proceeding paper [5]).

Acknowledgments:

We thank D. Bembé for his collaboration in the preparation of our joint proceeding paper [5]. He has now other interests and did not participate to the elaboration of the present work. The author was partially supported by the European ITN Marie Curie network SAGA.

Notations:

We denote by \mathbb{R} be the field of real numbers, by $\mathbb{R}[x]$ the ring of real univariate polynomial, by $\mathbb{R}[x]_n$ the corresponding vector space when the degree is at most n ; it is equipped with the L_2 norm on the coefficients in the monomial basis. We denote by f a monic polynomial of degree n , $f := x^n + \sum_{i=0}^{n-1} a_i x^i$, the set of coefficients (a_i) belongs to \mathbb{R}^n . The affine space \mathbb{E}_n of monic polynomials of degree n is identified with \mathbb{R}^n . For an integer m , we denote by $f^{(m)}$ the m -th derivative of f , with $f^{(0)} := f$.

2 Basics on Budan tables

In this section, after the definition of a Budan table, we list classical results around Budan-Fourier theorem, the proofs are elementary and can be found e.g. in [21] chapter 10, section 1.

The simplest observation that we exploit in our approach is the following elementary lemma. It is true not only for polynomials but also for any derivable function. Rolle's theorem is a consequence of that lemma when the derivative of the input function is continuous.

Lemma 2.1 *Assume that a is a simple root of f and the derivative f' is positive in an interval $]a - h, a + h[$, $h > 0$, then f is negative in $]a - h, a[$ and positive in $]a, a + h[$.*

Definition 2.2 Let f be a monic univariate polynomial of degree n . The Budan table of f is the union of $n + 1$ infinite rectangles of height one $L_i :=$

$\mathbb{R} \times [i - 1/2, i + 1/2[$ for i from 0 to n , called rows.

For i from 0 to n , each row L_i is the union of a set of open rectangles (possibly infinite), separated by vertical segments. We color in black the rectangles corresponding to negative values of the $(n - i)$ -th derivative $f^{(n-i)}$ of f , and we color in gray the rectangles corresponding to positive values of $f^{(n-i)}$.

Remark 2.3 1. Since f is assumed monic, every infinite right rectangle of each row is gray.

2. Since $f^{(n)}$ is a positive constant, the row L_0 is a gray infinite rectangle.
3. The first (infinite) rectangle of each row L_i is alternatively gray or black, depending on the parity of i : it is gray if $n - i$ is even.

4. We are interested by the connected components of the union of the closures of the gray rectangles; and respectively for the black rectangles.

It is clear that there is a gray connected component containing the infinite right rectangles of all rows. The other connected components (gray or black) are said bounded on the right.

Using Taylor expansions, one can extend the result of the previous lemma to the case of multiple roots.

Proposition 2.4 (multiple roots) *Let f be a monic real polynomial of degree n . If b is a root of multiplicity k of f with $k \leq n$ then for sufficiently small positive h , denoting by s the sign of $f^{(k)}(b)$, the columns of the Budan table of f near b are shown in the left part of the following figure.*

If c is a root of multiplicity k of $f^{(m)}$ with $0 < m < m + k \leq n$ then, for sufficiently small positive h , denoting by s_1 the sign of $f^{(m-1)}(c)$, and by s_2 the sign of $f^{(m+k)}(c)$, the columns of the Budan table of f near c are shown in the right part of the following figure.

	$b-h$	b	$b+h$		$c-h$	c	$c+h$
$sgn(f)$	$(-1)^k s$	0	s		$sgn(f^{(m-1)})$	s_1	s_1
$sgn(f')$	$(-1)^{k+1} s$	0	s		$sgn(f^{(m)})$	$(-1)^k s_2$	0
...	...	0	s		0
$sgn(f^{(k-1)})$	$-s$	0	s		$sgn(f^{(m+k-1)})$	$-s_2$	0
$sgn(f^{(k)})$	s	s	s		$sgn(f^{(m+k)})$	s_2	s_2

The figure shows that when x moves in \mathbb{R} , the signs in the columns of a Budan table are continuous on the right.

A classical descriptor attached to a Budan table is the function $V_f(x)$ of the real indeterminate x with values in the set of integers \mathbf{N} , it counts the number of sign changes in the sequence formed by f and its derivatives evaluated at x .

Definition 2.5 For a sequence $(a_0, \dots, a_n) \in (\mathbb{R} \setminus \{0\})^{n+1}$ the number of sign changes $\mathbf{V}(a_0, \dots, a_n)$ is defined inductively in the following way:

$$\mathbf{V}(a_0) := 0;$$

$$\mathbf{V}(a_0, \dots, a_i) := \begin{cases} \mathbf{V}(a_0, \dots, a_{i-1}) & \text{if } a_{i-1}a_i > 0, \\ \mathbf{V}(a_0, \dots, a_{i-1}) + 1 & \text{if } a_{i-1}a_i < 0. \end{cases}$$

To determine the number of sign changes of a sequence $(a_0, \dots, a_n) \in \mathbb{R}^{n+1}$, delete the zeros in (a_0, \dots, a_n) and apply the previous rule. (\mathbf{V} of the empty sequence equals 0).

We notice that the function V_f is computed from the Budan table of f , but two different tables (of two polynomials f and g) may have the same function $V_f = V_g$. Therefore the Budan table is a finer invariant than V_f attached to the polynomial f . We will study other invariants attached to f .

Proposition 2.6 (Budan-Fourier theorem) *Let $f \in \mathbb{R}[X]$ be monic of degree n . Then,*

- $V_f(-\infty) = n, V_f(\infty) = 0$.
- Near a real root c of multiplicity k of f , which is not a root of another derivative of f , V_f decreases by k when x moves from $c - h$ to $c + h$, for sufficiently small positive h .
- Near a real root c of multiplicity k of $f^{(m)}$, which is not a root of another non successive derivative of f , the following happens:
If k is even, V_f decreases by k .
If k is odd, V_f decreases by the even integer $k + s_1s_2$, with the notations of Proposition 2.4.
- Near c , a real root of several non successive derivative of f , V_f decreases by the sum of the quantities corresponding to each of them.
- Near the other points of \mathbb{R} , V_f is constant.
The function V_f is decreasing on \mathbb{R} .
- For $a, b \in \mathbb{R}$ with $a < b$, the number of real roots of f in the interval $]a, b]$ counted with multiplicities is at most $V_f(b) - V_f(a)$. Moreover the defect is an even integer.

3 A generic case

In this section and the next one, we assume a condition (\mathcal{P}) , generically satisfied, We introduce and study several data, continuous or discrete, attached to f and its Budan table.

3.1 (\mathcal{P}) -polynomials

Definition 3.1 A polynomial g in $\mathbb{R}[x]$ satisfies condition (\mathcal{P}) if and only if: each derivative of g has simple roots, and all these roots are two by two distinct. A monic polynomial satisfying this condition will be called a (\mathcal{P}) -polynomial.

Obviously, the parities of the number of real roots and degree of a (\mathcal{P}) -polynomial are equal.

We now see that the property (\mathcal{P}) is generically satisfied.

Proposition 3.2 *Let f be a monic real polynomial of degree n . Let 1 and $(a_i), 0 \leq i \leq n-1$ be the ordered set of its coefficients in the monomial basis. For any positive ϵ , there exists g , another monic real polynomial of degree n , such that:*

- g satisfies condition (\mathcal{P}) .
- $|a_i - b_i| < \epsilon$, $0 \leq i \leq n-1$,
where 1 and $(b_i), 0 \leq i \leq n$ form the ordered set of the coefficients of g .
- the number of real roots (counted with multiplicities) of f and g are equal.

Moreover, the set \mathbb{H}_n of monic polynomials in \mathbb{E}_n , identified with \mathbb{R}^n , satisfying (\mathcal{P}) form a semi-algebraic set of \mathbb{E}_n .

3.2 Generic Budan tables

Now, we determine the features of the Budan table B of a generic monic polynomial f with degree n .

Definition 3.3 We say that a table B with $(n+1)$ rows L_i formed by rectangles of alternating colors, gray and black, separated by vertical segments, is a \mathcal{GB} table of degree n if it satisfies the following properties. The letters \mathcal{G} and \mathcal{B} stand for generic Budan.

- The row L_0 is a gray infinite rectangle. The infinite rightmost rectangle of each row is gray. The first (infinite) rectangle of each row L_i is alternatively gray or black, depending on the parity of i : it is gray if $n-i$ is even.
- If i is even (resp. odd) the number of rectangles on the row L_i is even (resp. odd).
- Let $(l+1)$ be the number of rectangles of the top row L_n , then $l \leq n$ and $n-l$ is an even number $2p$. There are $l+p+1$ same-color-connected components of B . Each non first rectangle of L_i , $i > 0$ is connected on the left to a rectangle of the same color of the row L_{i-1} . The l first rectangles of L_n are in separated same-color-connected components. The p other connected components, bounded on the right, are surrounded by connected components of the opposite color.



Figure 1: A Budan table of degree 6



Figure 2: A Budan table of of degree 10

- The previous item is true, replacing n by any m , $0 < m < n$, and B by the table formed by the lower $m + 1$ rows.

Theorem 3.4 Let f be a (\mathcal{P}) -polynomial of degree n , and let $m \leq n$ be the number or real (simple) roots of f . Then m and n have the same parity, $n = m + 2p$ and the Budan table of f is a \mathcal{GB} table of degree n .

Example 1 The following figures show two \mathcal{GB} tables: one of degree 6 with $m = 0, p = 3$ and the other of 10 with $m = 4, p = 3$.

Definition 3.5 Let f be a (\mathcal{P}) -polynomial of degree n . The x value of the rightest upper segment of a connected component (either gray or black) of the Budan table of f is called a virtual root of f .

Any real root (in the usual sense) of f is a virtual root. Let $m \leq n$ be the number or real (simple) roots of f , and let $n - m = 2p$.

There are p virtual non real roots of f , we say that there are of multiplicity two; each of them is a root of some derivative of f of positive order.

Hence f admits n virtual roots, counted with multiplicities.

Definition 3.6 We call augmented virtual root of f the pair (y, k) formed by a virtual root of f and the order of the derivative of f which vanishes at y , i.e. $f^{(k)}(y) = 0$.

Remark 3.7 The augmented virtual roots of f only depend on the Budan table B of f . Virtual roots of a \mathcal{GB} table are well defined.

Proposition 3.8 Let f be a (\mathcal{P}) -polynomial of degree n .
By Rolle's theorem between two successive roots $a < b$ of some derivative $f^{(m)}$

with $0 \leq m \leq n-2$, (or in \mathbb{R} if $f^{(m)}$ has no root), there is an odd number $2r+1$ of roots $(X_1 < \dots < X_{2r+1})$ of the next derivative $f^{(m+1)}$. Then the r roots with an even index (X_2, \dots, X_{2r}) are virtual non real roots of f .

Similarly if a is the smallest (resp. the largest) root of $f^{(m)}$, in the infinite interval $]-\infty, a[$ (resp. $]a, \infty[$) there is an even number of roots $2r$ of roots $(X_1 < \dots < X_{2r})$ of the next derivative $f^{(m+1)}$. Then the r roots with an odd index (X_1, \dots, X_{2r-1}) , (resp. the k roots with an even index (X_2, \dots, X_{2r})) are virtual non real roots of f .

For each augmented virtual non real root (y, k) of f , we have

$$f^{(k-1)}(y)f^{(k+1)}(y) > 0.$$

The following corollary explicits the relation between the virtual roots of f and f' .

Corollary 3.9 *With the same hypothesis and notations.*

The Budan table of the derivative f' of f is obtained by deleting the first row of the Budan table of f .

Denote by $(x_i); 1 \leq i \leq u$ the ordered set of real roots of f and by $(y_j); 1 \leq j \leq v$ the ordered set of virtual roots of f with order 1; call U the re-ordered union of these 2 sets. Then f' has $u + 2v - 1$ real roots: the $(y_j); 1 \leq j \leq v$ and a set $(z_r); 1 \leq r \leq u + v - 1$ of values in each of the $u + v - 1$ intervals limited by the successive elements of U . The virtual roots of order k of f' for $k > 0$ are the virtual roots of order $k + 1$ of f .

Hence, the virtual roots of f and f' satisfy the “classical” interlacing property.

A (\mathcal{P}) -polynomial f of degree n , depends on n real coefficients, it has “only” m real roots and p virtual non real roots, with $n = m + 2p$. In order to get the same number of parameter and constraints, we must attach to each of the p virtual non real roots another real value.

We choose to attach to each augmented virtual root (y, k) with $k > 0$ the value $f^{(k-1)}(y)$, and so define a balanced system \mathcal{S} of constraints.

Definition 3.10 We denote by $\mathcal{S}(f)$ the system of $n = m + 2p$ data formed by the m real roots $(x_i), 1 \leq i \leq m$, of f ; the p augmented virtual root $(y_j, k_j), 1 \leq j \leq p$ with $k > 0$ and the p corresponding values $w_j := f^{(k-1)}(y_j), 1 \leq j \leq p$.

Two natural questions arise.

Question 1 *Is it true that two different (\mathcal{P}) -polynomials f and g in \mathbb{H}_n , define different systems $\mathcal{S}(f) \neq \mathcal{S}(g)$?*

Question 2 *Given a system \mathcal{S} as above, does there exist a (\mathcal{P}) -polynomial f which satisfies these data ?*

The first question is a so-called homogeneous Hermite-Birkhoff interpolation problem, Hermite-Birkhoff interpolation, (HB) in short, is an old subject comprehensively reviewed till 1983 in the book [19]. In contrast with Lagrange or Hermite interpolation, it does not always have a unique solution. In this theory, an important tool is the incidence matrix.

3.3 Transposed incidence matrix

In HB theory, the list of constraints are written line by line, hence the rows of the incidence matrix correspond to real values where the functions are evaluated, and the columns correspond to degrees or orders of differentiation. Since in Budan tables the roles of rows and columns are reversed, we will consider here the transposed of the usual HB incidence matrix.

Let h be the difference between two (\mathcal{P})-polynomials f and g having same augmented virtual roots and set of values $w_j : 1 \leq j \leq p$, as above. h has degree at most $n - 1$, hence its coefficients form n unknowns. It should satisfy $n = m + 2p$ linear vanishing conditions: m corresponding to the m real roots of f and $2p$ corresponding to the p double virtual double roots of f . We order the virtual roots $y_1 < \dots < y_{m+p}$ of f .

Definition 3.11 The transposed incidence matrix E of a (\mathcal{P})-polynomial f only depends on the augmented virtual roots (y_i, k_i) of f . It is the $(n, m + p)$ -matrix $E = (e_{j,i})$ such that n of its entries are 1, and the others are 0 according to the following rule:

1. If $k_i=0$ then $e_{0,i} = 1$.
2. If $k_i > 0$ then $e_{k_i,i} = 1$ and $e_{k_i-1,i} = 1$.
3. Otherwise $e_{j,i} = 0$.

Example 2 For a (\mathcal{P})-polynomial of degree 5, having one real root y_2 and two augmented virtual roots $(y_1, 3)$ and $(y_3, 1)$, with $y_1 < y_2 < y_3$; the matrix E is:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In HB theory the following properties of the incidence matrix are extensively used (here we translated on the transposed matrix):

Definition 3.12 E satisfies Polya condition if and only if $M_j > j$ for all $j = 0 \dots n - 1$; where M_j denotes the sum of the entries in the first $j + 1$ rows of the transposed incidence matrix E .

A maximal sequence of 1 in a column of E is called even (resp. odd) if the

number of its elements is even (resp. odd).

E is said conservative, if and only if every odd sequence of E is non supported. A maximal sequence of 1 in a column i of E starting at (j, i) is non supported if at least one of the two following sub-matrices of E (which can be void) does not contain any 1:

- The upper left sub-matrix of E formed by the rows $0 \dots j - 1$ and the columns $1 \dots i - 1$
- The upper right sub-matrix of E formed by the rows $0 \dots j - 1$ and the columns $i + 1 \dots m + p$.

In [2] it is proved that a homogeneous HB problem whose incidence matrix is both conservative and satisfies Polya condition, has only the trivial solution. See also [18] for simpler proofs.

Proposition 3.13 *The previously defined matrix E (attached to $m + p$ augmented virtual roots) is conservative and satisfies Polya condition. The corresponding homogeneous HB problem has only the trivial solution.*

Hence the answer to the first question is “Yes”.

Proposition 3.14 *Given a system \mathcal{S} formed by pairs of real numbers and integers: $(x_i, 0)$ with $1 \leq i \leq m$, (y_i, k_i) with $1 \leq i \leq p$, $k_i \leq n$, the set of values w_i with $1 \leq i \leq p$, $n = m + p$. There exist a unique monic polynomial F satisfying the vanishing conditions :
 $F(x_i) = 0, 1 \leq j \leq m+$, $F^{(k_i)}(y_i) = 0, 1 \leq j \leq p$, $w_i := F^{(k_i-1)}(y_i), 1 \leq i \leq p$.*

Remark 3.15 • The (y_i, k_i) with $1 \leq i \leq m + p$, are not necessarily the augmented virtual roots of F , as illustrated by the following example 3.

- The invertible matrix of the linear transformation is a generalization of a Vandermonde matrix, see the following example 4.

Hence the answer to the second question is in general “No”.

Example 3 Consider the general degree 3 monic polynomial $F := x^3 + a_2x^2 + a_1x + a_0$. Require that $y_1 := 0$ is a root of F , $y_2 := 1$ is a root of F' , and the extra condition $F(1) = w$.

The 3 constraints are: $a_0 = 0, 3 + 2a_2 + a_1 = 0, 1 + a_2 + a_1 = w$.

The solution is: $a_0 = 0, a_1 = 2w + 1, a_2 = -w - 2$.

Then $y_2 := 1$ will be a virtual non real root of F , if and only if $f(1)f''(1) > 0$. This means $w(w - 2) < 0$, hence $0 < w < 2$.

Example 4 The generalized Vandermonde matrix corresponding to the example 2 is a $(5, 5)$ -matrix with entries monomials in (y_1, y_2, y_3) . Its determinant $D(y_1, y_2, y_3)$ is a polynomial of degree 5 which does not vanish if the indeterminate are real and are pairwise distinct.

$$D(y_1, y_2, y_3) = 12(y_1 - y_2)^2((y_1 + 2 - 2y_3)^2 + 2(y_1 - y_3)^2).$$

Fix a \mathcal{GB} table of degree n , through the set of its $m + p$ augmented virtual roots, it defines an incidence matrix E , hence with a set of values $(w_l) \in \mathbb{R}^p$ it defines a system of constraints \mathcal{S} .

Proposition 3.16 *The set W of values $(w_i) \in \mathbb{R}^p$ such that the system \mathcal{S} represents a (\mathcal{P}) -polynomial, form a semi algebraic set (possibly empty) in \mathbb{R}^p . More precisely, if we fix the signs σ of the $(w_i) \in \mathbb{R}^p$ such that the system \mathcal{S} represents a (\mathcal{P}) -polynomial, then the corresponding set W_σ is a (possibly empty) open convex polytop in \mathbb{R}^p .*

The following weaker form of Question 2 is also false:
Let B be a \mathcal{GB} table of degree n , does there exist a (\mathcal{P}) -polynomial f of degree n , which admits B as its Budan table?

The reason is that any arrangements of roots, compatible with Rolle's theorem, is not realizable as the roots of the derivatives of a polynomial. This topic has been extensively studied by V. Kostov, see [14], [15], [16]. However he proved in [17], that any sequence of numbers of roots of the derivatives, compatible with Rolle's theorem, is realizable.

In the next section, we will define an invariant weaker than a Budan table but stronger than this sequence.

3.4 Budan tree

In the (transposed) incidence matrix E of a (\mathcal{P}) -polynomial f , the list of the $m + p$ ordered virtual roots are replaced by the list of the first $m + p$ numbers. We will apply, in our setting, the same idea to get a discrete invariant.

In the Budan table of (\mathcal{P}) -polynomial f , we replace the list of the augmented virtual roots (y_i, k_i) , by the corresponding list of pairs of integers (i, k_i) . with $1 \leq i \leq m + p$ to form the nodes of a bi color tree with $n + 1$ leaves.

The edges are obtained by contracting the same-color-connected components. Figures 3 and 4 illustrate our construction with the trees corresponding to examples 1.

Definition 3.17 The Budan tree is a bi color tree constructed on a grid $[0..m + p + 1, 0..n]$ as follows.

We order the augmented virtual roots (y_i, k_i) of f , with $0 \leq k_i \leq n - 2$, $1 \leq i \leq m + p$, by the value y_i , and position the $m + p$ nodes of the tree at (i, k_i) . with $1 \leq i \leq m + p$. We position the $n + 1$ leaves at the points $(i, 0)$, $0 \leq i \leq n$ and color them alternatively in gray and black; $(n, 0)$ is gray. The root of the tree is positioned at $(0, m + p + 1)$. The root of the tree, the nodes and the leaves are connected by colored edges which are the contraction of the the same-color-connected components.of the Budan table of f .

We say that the Budan tree is decorated if the coordinates (i, k_i) , $1 \leq i \leq m + p$ of its nodes are given.

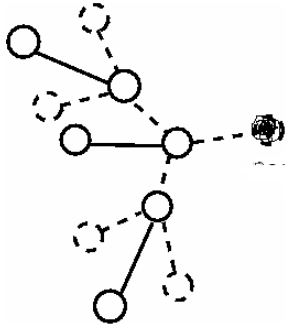


Figure 3: A Budan tree of degree 6

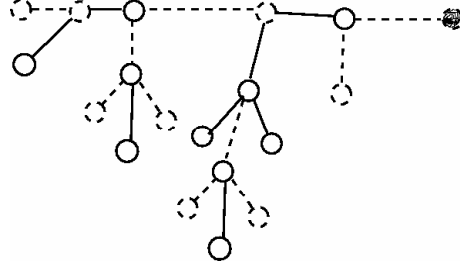


Figure 4: A Budan tree of degree 10

The obtained tree is rooted on the right of the picture and it is drawn (as usual) with short branches.

From a node $(i, 0)$ corresponding to a simple root of f start two branches of different colors. From a node (i, k_i) with $k_i > 0$ corresponding to a virtual non real root of f , start three branches of different colors: the two extremal ones have the same color as the one of its stem and the middle one is of opposite color.

Here is a weak form of Question 2.

Question 3 *Let B be a \mathcal{GB} table of degree n , does there exist a (\mathcal{P}) -polynomial f of degree n , which admits the same Budan tree than B ?*

Definition 3.18 Let $T(f)$ denote the decorated Budan tree associated to a (\mathcal{P}) -polynomial f . For a fixed decorated tree T corresponding to a \mathcal{GB} table of degree n , (we also denote by \mathcal{T} the non decorated tree), we consider the stratum Σ_T of \mathbb{H}_n formed by all the (\mathcal{P}) -polynomials f such that $T(f) = T$. We denote by Σ this stratification of \mathbb{H}_n .

We also consider a coarser stratification Θ whose strata $\Theta_{\mathcal{T}}$ are the union of all the strata Σ_T corresponding to the same non decorated tree \mathcal{T} .

Proposition 3.19 *Each stratum Σ_T of \mathbb{H}_n is a semi-algebraic set of \mathbb{E}_n .*

Question 4 *Are the strata Σ_T connected in \mathbb{H}_n ?*

Proposition 3.20 *The virtual roots of a the polynomials f in a stratum Σ_T of \mathbb{H}_n depend analytically of the coefficients of f .*

Proposition 3.21 *Each stratum $\Theta_{\mathcal{T}}$ of \mathbb{H}_n is a semi-algebraic set of \mathbb{E}_n .*

4 Proofs

This section provides the proofs of the statements of the previous section. In the proofs of each statement we keep the notations of that statement.

4.0.1 Proposition 3.2

It is precisely Lemma 10.2.2 of [21], its proof relies on a continuous representation of f by conjugated complex roots and Ostrowski continuity theorem, plus the use of discriminant.

4.0.2 Theorem 3.4

Items 1 and 2 are obvious. Let's prove item 3, item 4 will follow.

By Lemma 2.1, each (non first) rectangle of row L_i is connected to a rectangle of the same color on row L_{i-1} located on its left. So, by induction, any rectangle is same-color-connected to one of the $(n+1)$ first infinite rectangle on the left side of the table.

Each black connected component is bounded on its right; so it has a rightmost upper segment (call x its abscissa): if it is on L_n , then x is a real root of f , else if it is on L_i , $i < n$, it is surrounded below above and on the right by gray rectangles. In this last case, the black component stops at (x, i) and we also consider that the gray component above the segment coming from the left stops at $(x, i-1)$ since it is in an affluent of the gray component coming from the right which continues upward. Therefore two components (a black one and a gray one) stop at the abscissa x . The same reasoning works for the gray components bounded on the right, exchanging the roles of gray and black.

Let l denote the number of real roots of f and p denote the number of rightmost upper segment of a same-color-connected component bounded on the right. Since the gray component containing L_0 is unbounded, there are n tails moving upward from the left side. Hence $n = l + 2p$.

4.0.3 Proposition 3.8

Here is the proof of the first assertion, the second is very similar.

Assume without loss of generality that between a and b , $f^{(m)}$ is positive. Then by Lemma 2.1, on the left of x_1 , $f^{(m+1)}$ is also positive and changes sign at each of its successive root, so $f^{(m+1)}$ is positive at the right of X_2 . Between X_1 and X_2 , there are an odd number $2u+1$ of roots of $f^{(m+2)}$, denoted by Y_1, \dots, Y_{2u+1} . Then $f^{(m+2)}$ is positive at the right of Y_{2u+1} . Hence the rectangle $]X_1..X_2[$ on L_{n-m} is black and we deduce that it is surrounded (above, on its right and below) by gray rectangles.

4.0.4 Corollary 3.9

Apply Rolle's theorem.

4.0.5 Proposition 3.13

E is conservative because the only columns with one 1, have the 1 situated on the first row; the other columns have two 1.

Polya condition is clearly satisfied for the first two rows, because if a polynomial

f has no real root then its derivative has a at least one real root, hence f has at least a virtual root of order 1.

Let ρ_j denote the number of real roots of $f^{(j)}$, $j \geq 0$. Then by Proposition 3.8, the number λ_j , $j > 0$, of virtual roots of order j of f is: $\lambda_j = \frac{1}{2}(\rho_j - \rho_{j-1} + 1)$. Each of them gives rises to two 1 in E . The count of 1 in the first $j + 1$ rows of E gives: $M_j = j + \frac{1}{2}(\rho_j + \rho_{j-1} + 1)$. Hence $M_j > j$.

4.0.6 Proposition 3.14

Indeed the corresponding generalized Vandermonde matrix is invertible, by Proposition 3.13.

4.0.7 Proposition 3.16

Consider p other extra constraints $f^{(k_i+1)}(y_i) = z_i$, at the augmented non real virtual roots. The new system \mathcal{S}' is solvable if and only if the $2p$ parameters (z_i, w_i) satisfy a set of affine relations. And we also require for each i , $1 \leq i \leq p$ the non linear inequality $z_i \cdot w_i > 0$. The solution is an union of convex polytops whose vertices depend rationally on the y_i , then we consider its projection on the (w_i) space \mathbb{R}^p . If we fix the signs σ of the (w_i) , hence those of (z_i) , the solutions correspond to the intersection of an affine space with an hyper-octant in \mathbb{R}^{2p} , that we project on \mathbb{R}^p to get an open polytop.

4.0.8 Proposition 3.19

The proof of Proposition 3.16 gives the semi-algebraicity by elimination of the (w_i) .

4.0.9 Proposition 3.21

$\Theta_{\mathcal{T}}$ is an union of $\Sigma_{\mathcal{T}}$ which are disconnected.

5 General case

In this section we consider the general case where the condition (\mathcal{P}) is not necessarily satisfied. We generalize the definitions of virtual roots and of Budan tree, and prove some continuity results.

5.1 Multiplicities and augmented virtual roots

Definition 5.1 Let f be a monic real polynomial of degree n . The x value of the rightest upper segment of a connected component (either gray or black) of the Budan table of f is called a virtual root of f . Any real root (in the usual sense) of f is a virtual root of f . Any multiple real root (in the usual sense) of any derivative of f is also a virtual root of f . The virtual multiplicities are counted as follows:

- the multiplicities of events appearing along a same x -value are added,
- the multiplicity of a simple root of f counts 1,
- the multiplicity of a simple virtual non real root (i.e. it is not a multiple root of a derivative of f) counts 2,
- the multiplicity of a multiple root of f of order k counts k ,
- the multiplicity of a multiple virtual non real root which is a multiple root of order k of a derivative of f counts k if k is even, and otherwise $k + s_1 s_2$ with the notations of the second table of Proposition 2.4.

Budan theorem (Proposition 2.6) implies that f admits n virtual roots counted with multiplicities. Moreover the following result holds.

Proposition 5.2 *Let $f \in \mathbb{R}[X]$ be monic of degree n . Let $y_1 \leq \dots \leq y_n$ the ordered virtual roots of f , repeated according to their multiplicities, and $y_0 = -\infty$, $y_{n+1} = \infty$. Then we have for $1 \leq r \leq n + 1$, with $y_{r-1} \neq y_r$,*

$$x \in [y_{r-1}, y_r[\iff \mathbf{V}(f(x), f'(x), \dots, f^{(n)}(x)) = n + 1 - k$$

(resp. for $r = 1$ the interval $x \in] - \infty, y_1[$).

Theorem 5.3 ([11], see also [9]) *The virtual roots of a monic polynomial f depend continuously on the coefficients of f .*

Here is a quick proof: Let g_t tend to f in \mathbb{E}_n as t tends to 0. Let c be a virtual root of f of multiplicity k , there exists a sufficiently small positive h such that in the two intervals $[c - h, 0[$ and $]0, c + h]$ all derivatives $f^{(m)}$, $0 \leq m \leq n$ do not vanish, and a sufficiently small positive t_0 , such that for $t \leq t_0$, and $0 \leq m \leq n$ $f^{(m)}(c - h)$ and $g_t^{(m)}(c - h)$ have same sign, and similarly at $c + h$. Hence k virtual roots of g_t counted with multiplicities tend to c as t tends to 0. Since f and g_t have the same number n of virtual roots, we are done.

Definition 5.4 We call augmented virtual root of f a triple (y, k, r) formed by a virtual root of f , an order of differentiation $k \leq 0$ and a multiplicity $r \leq 1$ such that we have: $f^{(k)}(y) = 0, \dots, f^{(k+r-1)}(y) = 0$ but $f^{(k+r)}(y) \neq 0$; and if $k > 0$ $f^{(k-1)}(y) \neq 0$. Different augmented virtual root of f can correspond to the same virtual root of f , moreover they only depend on the Budan table of f .

Note that if f is a (\mathcal{P}) -polynomial then the last coordinates of each triple is 1, so it can be identified with a pair, as we did in the previous sections.

5.2 Budan trees and transposed incidence matrix

Definition 5.5 Let f be a monic polynomial with degree n , denote by u the number of ordered virtual roots of f and by (y_i, k_i, r_i) , $1 \leq i \leq u$ the augmented virtual roots of f .

We call Budan tree of f a tree T traced on a discrete grid $[0..n, 0..u+1]$ which consists of the contraction of the Budan table of f . The root of the tree T is at $(0, u+1)$, and the leaves of T are the $n+1$ points $(j, 0)$, $0 \leq j \leq n$. Each augmented virtual root (y_i, k_i, r_i) may define several nodes of T ; according to their multiplicities in Definition 5.1 and the following rule.

- If $k_i = 0, r_i = 1$, i.e y_i is a simple root, a node is positioned at $(0, i)$.
- If $k_i > 0, r_i = 1$, i.e y_i is a simple virtual non real root, a node is positioned at (k_i, i) .
- If $k_i = 0, r_i > 1$, i.e y_i is a multiple root of f of order r . If r_i is even, $r_i = 2q_i$ then q_i nodes are positioned at $(2j, i)$ with $0 \leq j \leq q_i - 1$. If r_i is odd, $r_i = 2q_i + 1$ then q_i nodes are positioned at $(2j+1, i)$ with $0 \leq j \leq q_i$;
- If $k_i > 0, r_i > 1$, i.e y_i is a multiple virtual non real root. If r_i is even, $r_i = 2q_i$ then q_i nodes are positioned at $(2j + k_i, i)$ with $0 \leq j \leq q_i - 1$; If r_i is odd, and with the notations of Proposition 2.4 $r_i + s_1 s_2 = 2q_i$ then q_i nodes are $(r_i + k_i - 2j, i)$ with $0 \leq j \leq q_i - 1$.

We say that the Budan tree is decorated if the coordinates (i, k_i) , of its nodes in the grid are given.

Denoting by m the number of nodes on the top line of the grid and by p the number of the other nodes; then by Proposition 2.4 $n = m + 2p$.

Example 5 We consider the monic polynomial $f := (x-1)^4 g/20$ of degree 7 such that $g := 20x^3 + 10x^2 + 4x + 1$ chosen such that we have at $x = 0$: $f(0) > 0$, $f'(0) = 0$, $f^{(2)}(0) = 0$, $f^{(3)}(0) = 0$, $f^{(4)}(0) > 0$. f also have a real root of order 4 at $x = 1$ and another simple real negative root.

Then the Budan table and Budan tree of f are shown in Figures 5 and 6.

Definition 5.6 With the same procedure than for the generic case we attach to a Budan table B a transposed incidence matrix $E = (e_{i,j})$. If B has degree n and l virtual roots, then E is a $(n+1, l)$ matrix with n entries equal to 1 and the others to 0, according to the following rule.

Let (i, k_i) be a node of the Budan tree then:

If $k_i = 0$ then $e_{i,0} = 1$. If $k_i > 0$ then $e_{i,k_i} = 1$ and $e_{i,k_i-1} = 1$. Otherwise $e_{i,j} = 0$.

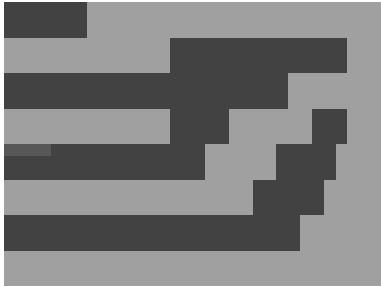


Figure 5: Budan table of example 5

Again we have:

Proposition 5.7 *The matrix E is conservative and satisfies Polya condition. The corresponding homogeneous HB problem has only the trivial solution.*

5.3 Deformations and continuity

Conjecture 1 *Let $f \in \mathbb{R}[X]$ be monic of degree n and T be its Budan tree, there exists a \mathcal{P} -polynomial g of degree n with the same Budan tree.*

To prove that we need to control the deformations of f at the multiple roots of f and its derivatives. The requirement is to deform f into F such that if x_0 is a root of $f(k)$ of multiplicity r it gives rise near-by x_0 to no root of $F(k)$ if r is even and to only one root of $F(k)$ if r is odd.

Example 6 For example 5, the deformation $F := x^7 + 10.5x^6 + 46.2x^5 + 110.x^4 + 154.x^3 + 126.x^2 + 55.6x + 10.4$ of f works. We constructed it by a sequence of deformations and integrations starting from the third derivative of f . The figures show the two families of graphs up to the third derivative.

Assuming this result, we could extend to \mathbb{E}_n the stratification Θ of \mathbb{H}_n . The study of incidence relations among the strata of Θ (or Σ) is an interesting question, but out of the scope of the present paper. The first tasks are the following.

Let f be a monic real polynomial of degree n with Budan tree T_1 . Let g_t be an infinitesimal deformation of f in \mathbb{H}_n , such that all g_t have the same Budan tree T_2 . By a refinement of theorem 5.3, when t tends to 0, the set of virtual roots of each $g_t^{(j)}$ tends to the set of virtual roots of $f^{(j)}$, counted with multiplicities. Some rectangles of the Budan table of g_t may move to become aligned or be squeezed to become segments. So one need to classify all the possible configurations.

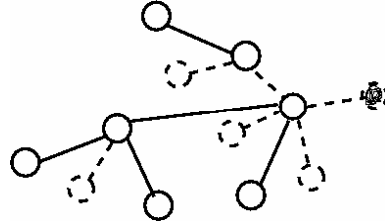


Figure 6: Budan tree of example 5

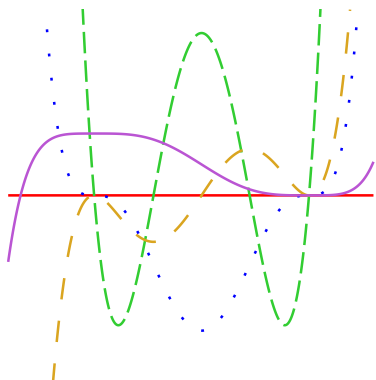


Figure 7: Graphs of the derivatives of example 5

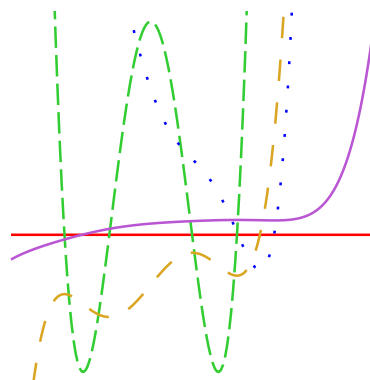


Figure 8: Graphs corresponding to the deformation

Question 5 *Are the strata $\Theta_{\mathcal{T}}$ connected in \mathbb{E}_n ?*

6 Conclusion

6.1 Extensions

In this paper, we have studied the same-sign-connected components of the Budan table of a univariate polynomial f , and extracted two types of information: the virtual roots which depend continuously on the coefficients of f and its Budan tree which is a finite discrete invariant attached to f . We related this construction to Hermite-Birkhoff interpolation, which has received a renewed interest in Computer algebra, see [3]. Then we defined a stratification of the space of polynomials and started its study. We left open several questions on the properties of the strata and their incidence relations, which are directions for further investigations and experimentation.

There are different possibilities of extensions of the previous approach together with directions for future research.

6.1.1 Truncated Budan table

One can truncate a Budan table, either by limiting it to an interval $[a, \infty[$ or to consider only the r top rows, then study its same-color connected components. For instance if the $(n - r)$ -th row, say after a , is gray, then we get a Budan-Fourier estimation on the number of real roots $\leq a$ just by considering the first r derivatives. This property could be used in a subdivision algorithm coupled e.g. with interval arithmetic.

6.1.2 Fewnomials

In [9], the Budan-Fourier theorem and the continuity property of the virtual roots, were generalized to the case of Fewnomials, with a modified set of differentiations depending on an infinite sequence of functions \mathbf{f} .

More precisely, with our notations, our Lemma 2.1 and Proposition 2.4 are valid in that setting. so many of our constructions, could also be extended to that setting. For the HB interpolation one should look, case by case, depending on the considered family \mathbf{f} .

6.1.3 “Circular” differentiation

Another extension could be to replace the input polynomial $f(x)$ of degree n by its homogenization $F(X, Y)$ in degree n , and then set $X = \cos(t)$, $Y = \sin(t)$ to get a trigonometric polynomial $G(t)$ depending on n coefficients, hence G takes bounded values in \mathbb{R} . In that case, the derivative $G'(t)$ of $G(t)$ with respect to t can be obtained similarly from another homogeneous polynomial.

Then we can define a periodic generalization of the Budan table, bounded on the columns and unbounded on the rows, but we can truncate it. There is again a generic case and one can study the corresponding “virtual roots”.

6.2 Two other directions of research

- Study the relationship between virtual roots in an interval and pairs of conjugate complex roots which lie in a sector close to this interval as initiated in Obreschkoff theorem, see [21], chapter 10.
- Investigate what happens if the coefficients set of the input polynomials are is not only generic but follows a Gaussian random distribution.

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