A general Hamilton-Jacobi framework for nonlinear state-constrained control problems

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Albert Altarovici[*]   Olivier Bokanowski[†]   Hasnaa Zidani[‡]

Abstract

The paper deals with deterministic optimal control problem with state constraints and non-linear dynamics. It is known for such a problem that the value function is in general discontinuous and its characterization by means of an HJ equation requires some controllability assumptions involving the dynamics and the set of state constraints. Here, we first adopt the viability point of view and look at the value function as its epigraph. Then, we prove that this epigraph can always be described by an auxiliary optimal control problem free of state constraints, and for which the value function is Lipschitz continuous and can be characterized, without any additional assumption, as the unique viscosity solution of a Hamilton-Jacobi equation. The idea introduced in this paper bypass the regularity issues on the value function of the constrained control problem and leads to a constructive way to compute its epigraph by a large panel of numerical schemes. Our approach can be extended to more general control problems. We study in this paper the extension to the infinite horizon problem as well as for the two-player game setting. Finally, an illustrative numerical example is given to show the relevance of the approach.

Keywords: State constraints, optimal control problems, nonlinear controlled systems, Hamilton-Jacobi equations, viscosity solutions.
Mathematics Subject Classification. 35B37, 49J15, 49Lxx, 49J45, 90C39

1 Introduction

This paper deals with the characterization of the value function of a deterministic optimal control problem with state constraints. For a given finite horizon $T > 0$, consider the dynamical system

$$\dot{y}(s) = f(s, y(s), \alpha(s)), \ \text{a.e.} \ s \in (t, T), \tag{1.1a}$$

$$y(t) = x. \tag{1.1b}$$

where $\alpha : [0, T] \to \mathcal{A}$ is a measurable function and $\mathcal{A}$ is a compact set of $\mathbb{R}^p$ $(p \geq 1)$, and $f : [0, T] \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}^d$ is a continuous function (see Section 2 for precise assumptions). The corresponding absolutely continuous solution will be denoted by $y = y_{t,x}^\alpha$.

For a given non empty and closed subset $\mathcal{K}$ of $\mathbb{R}^d$, we consider a control problem and its value function given by

$$\vartheta(t, x) := \min_{\alpha \in L^\infty((t,T),\mathcal{A})} \left\{ \int_t^T \ell(s, y_{t,x}^\alpha(s), \alpha(s)) ds + \varphi(y_{t,x}(T)) \ \middle| \ y_{t,x}^\alpha(\theta) \in \mathcal{K} \quad \forall \theta \in [t, T] \right\}, \tag{1.2}$$

with the usual convention that $\inf \emptyset = +\infty$, and where $\ell : [0, T] \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ are continuous functions (see Section 2 for precise assumptions). Without assuming any additional assumption, it may happen that the value function is discontinuous.

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In the case when $\mathcal{K} = \mathbb{R}^d$ and φ is Lipschitz continuous, under classical assumptions on f and ℓ, it is known that the value function ϑ is the unique continuous viscosity solution of a Hamilton-Jacobi equation [22, 8, 6]. This result is extended also to the lower semicontinuous (l.s.c.) setting in [12, 25].

When the control problem is in presence of state constraints ($\mathcal{K} \neq \mathbb{R}^d$) a state-space constrained HJB equation has been associated to the value function (1.2) in [37, 38]. In our setting, this HJB equation is in the form:

$$-u_t + \mathcal{H}(t, x, \nabla_x u) = 0 \quad \text{in } (0, T) \times \mathcal{K}, \tag{1.3a}$$

$$u(T, x) = \varphi(x) \quad \text{in } \mathcal{K}, \tag{1.3b}$$

where $\mathcal{H}(t, x, p) := \max_{a \in \mathcal{A}}(-f(t, x, a) \cdot p - \ell(t, x, a))$. In Soner's formulation, a function u is a viscosity solution of (1.3) if it is sub-solution in $(0, T) \times \overset{\circ}{\mathcal{K}}$ and a super-solution on $(0, T) \times \mathcal{K}$.

Even though it is easy to establish that the value function ϑ satisfies (1.3) in the constrained viscosity sense, it is more complicated to prove the uniqueness of the solution of (1.3). Actually, the uniqueness requires restrictive controllability assumptions on $\mathcal{K}$ and on the dynamics. The main difficulty comes from the fact that the state-space HJB equation may admit several solutions (in the constrained viscosity sense) if the behavior of the solution on the boundary is not taken into account, see [15, 30].

The most classical controllability assumption is called "inward pointing qualification condition (IPQ)". It was first introduced by Soner in [37, 38]. It asks that at each point of the boundary of $\mathcal{K}$ there exists a field of the system pointing inward $\mathcal{K}$. Under this assumption the value function is Lipschitz continuous in $\mathcal{K}$, and uniqueness can be established even though the constrained viscosity notion does not take into account the behavior of the solution at the boundary. From the viewpoint of the dynamical system, the inward condition ensures that all the trajectory hitting the boundary can be approximated by a sequence of trajectories that stay inside the interior of $\mathcal{K}$ [28]. We refer to [17] and [33, 34] for weaker inward pointing assumptions, and to [31, 32] for more properties and numerical approximation of continuous constrained viscosity solutions.

An other controllability assumption, called "outward pointing condition" (OPQ), has been considered in [26, 28]. This assumption states that each point on the boundary of $\mathcal{K}$ can be reached by a trajectory coming from the interior of $\mathcal{K}$. Under this assumption it is still possible to characterize the value function as the unique lower semi-continuous solution of an HJB equation.

However there are many control problems where the controllability assumptions are never satisfied.

On the other hand, using the viability tools [1, 4] and non-smooth analysis it is always possible to characterize the value function and more precisely its epigraph, see [5, 18, 19, 2] and the references therein.

In the present work we show that the epigraph of the value function can be described by means of a Lipschitz continuous function, which turns out to be the value function of an auxiliary control problem free of state constraints. Then standard viscosity theory can be applied to characterize the new value function as the unique continuous viscosity solution of a variational HJ inequation. Our approach is quite general and simple, and we believe that it can be easily extended to a large class of control problems with different type of state constraints. In the present paper, we will investigate also the case of infinite horizon control problem as well as the two-player game problem and we will show in each case how the auxiliary control problem should be defined.

The paper is organised as follows. The setting of the problem and the assumptions are made precise in section 2. Main results and proofs are presented in section 3. The extension of the results to an infinite horizon control problem as well as two-player games is done in sections 4&5. Finally, a numerical example is given in section 6.

2 Problem formulation

2.1 Statement of the state-constrained control problem

For a given non-empty compact subset $\mathcal{A}$ of $\mathbb{R}^p$ ($p \geq 1$) and a given $T \in \mathbb{R}$ ($T > 0$), we consider the set of admissible controls defined by:

$$\mathbb{A} := \left\{ \alpha : (0, T) \to \mathbb{R}^p \text{ measurable, } \alpha(t) \in \mathcal{A} \text{ a.e.} \right\}.$$

Consider the controlled system:

$$\dot{y}(s) = f(s, y(s), \alpha(s)), \text{ a.e. } s \in (t, T), \tag{2.1a}$$
$$y(t) = x, \tag{2.1b}$$

where $\alpha \in \mathbb{A}$, and $f : (0, T) \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}^d$ is continuous and is assumed to satisfy the following regularity and growth properties:

$$\textbf{(A1)} \begin{cases} (i) \ f : [0, T] \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}^d \text{ is continuous} \\ (ii) \text{ There exists } L \geq 0, \text{ for any } x, y \in \mathbb{R}^d, \text{ for all } a \in \mathcal{A}, \text{ and for all } t, s \in [0, T]: \\ \qquad |f(t, x, a) - f(s, y, a)| \leq L(|x - y| + |t - s|), \end{cases}$$

where $|\cdot|$ is a norm on $\mathbb{R}^d$. It is known that under assumption **(A1)**, for any $\alpha \in \mathbb{A}$ and for any $(t, x) \in (0, T) \times \mathbb{R}^d$, there exists a unique absolutely continuous trajectory $y = y_{t,x}^\alpha$ satisfying (2.1). The set of all feasible trajectories starting in x at time t will be denoted as:

$$S_{[t,T]}(x) := \left\{ y = y_{t,x}^\alpha, \ y \text{ satisfies (2.1) for some measurable } \alpha \in \mathbb{A} \right\}.$$

Under the assumption **(A1)**, for any $t \in [0, T]$ and $x \in \mathbb{R}^d$, $S_{[t,T]}(x)$ is a compact set in $W^{1,1}(t, T)$ for the topology of $C([0, T]; \mathbb{R}^d)$. Moreover, the set-valued application $x \rightsquigarrow S_{[t,T]}(x)$ is Lipschitz continuous from $\mathbb{R}^d$ in $C([0, T]; \mathbb{R}^d)$.

Let $\mathcal{K}$ be a non-empty closed set of $\mathbb{R}^d$ (no additional assumptions on $\mathcal{K}$ will be made). A trajectory $y \in S_{[t,T]}(x)$ will be said *admissible* (on the time interval (t, T)) if

$$y(s) \in \mathcal{K}, \quad \text{for all } s \in (t, T). \tag{2.2}$$

Now, consider a distributed cost function $\ell : [0, T] \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}$, satisfying:

$$\textbf{(A2)} \begin{cases} (i) \ \ell : [0, T] \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R} \text{ is continuous} \\ (ii) \text{ There exists } L \geq 0, \text{ for any } x, y \in \mathbb{R}^d, \text{ for any } a \in \mathcal{A} \text{ and for any } \forall t, s \in [0, T], \\ \qquad |\ell(t, x, a) - \ell(s, y, a)| \leq L(|x - y| + |t - s|), \end{cases}$$

and a given final cost function φ satisfying:

(A3) $\quad \varphi : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous.

The state-constrained Bolza problem is formulated as follows:

$$\vartheta(t, x) := \inf \left\{ \int_t^T \ell(s, y_{t,x}^\alpha(s), \alpha(s)) \, ds + \varphi(y_{t,x}^\alpha(T)) \ \middle| \ \alpha \in \mathbb{A}, \text{ and } y_{t,x}^\alpha(s) \in \mathcal{K}, \ \forall s \in [t, T] \right\}, \tag{2.3}$$

with the convention that $\inf \emptyset = +\infty$.

3

First, let us recall that the Bolza problem (2.3) can be set back to a Mayer problem. For this, we introduce the following "augmented" dynamics $\hat{f}$ defined by

$$\hat{f}(t,(x,z),a) := \begin{pmatrix} f(t,x,a) \\ -\ell(t,x,a) \end{pmatrix} \quad \forall (x,z) \in \mathbb{R}^d \times \mathbb{R}, \ \forall a \in \mathcal{A}, \ \forall t \in (0,T)$$

(the choice of $-\ell$ instead of ℓ is just for notational convenience). Introduce also, for $\alpha \in \mathbb{A}$, the augmented trajectory $\hat{y} = \hat{y}_{t,(x,z)}^{\alpha}$ solution of

$$\dot{\hat{y}}(s) = \hat{f}(s, \hat{y}(s), \alpha(s)), \ s \in (t,T), \tag{2.4a}$$

$$\hat{y}(t) = (x,z)^{\mathsf{T}}. \tag{2.4b}$$

In particular, for any $\hat{x} = (x,z) \in \mathbb{R}^d \times \mathbb{R}$, the solution of (2.4) is given by: $\hat{y}_{t,\hat{x}}^{\alpha}(s) = (y_{t,x}^{\alpha}(s), \zeta_{t,\hat{x}}^{\alpha}(s))$, where $\zeta_{t,\hat{x}}^{\alpha}(s) := z - \int_t^s \ell(\theta, y_{t,x}^{\alpha}(\theta), \alpha(\theta)) \, d\theta$ and $y_{t,x}^{\alpha}$ is the solution of (2.1). Define the set of corresponding trajectories:

$$\hat{S}_{[t,T]}(\hat{x}) := \left\{ \hat{y} = (y_{t,x}^{\alpha}, \zeta_{t,\hat{x}}^{\alpha}), \ \hat{y} \text{ satisfies } (2.4) \text{ for some } \alpha \in \mathbb{A} \right\},$$

for $\hat{x} = (x,z) \in \mathbb{R}^d \times \mathbb{R}$. The new control problem is then defined as follows:

$$\hat{\vartheta}(t,\hat{x}) := \inf_{\hat{y}=(y,\zeta) \in \hat{S}_{[t,T]}(\hat{x})} \left\{ \varphi(y(T)) - \zeta(T), \ y(s) \in \mathcal{K}, \ \forall s \in (t,T) \right\}. \tag{2.5}$$

It is clear that $\hat{\vartheta}(t,\hat{x}) = \vartheta(t,x) - z$ for any $\hat{x} := (x,z)$, and in particular $\vartheta(t,x) = \hat{\vartheta}(t,(x,0))$. In the sequel, for simplicity of the presentation, the following assumption will be also assumed:

(A4) For any $(t,x) \in [0,T] \times \mathbb{R}^d$, $\hat{f}(t,x,\mathcal{A})$ is a convex set.

Therefore, for any $\hat{x} \in \mathbb{R}^d \times \mathbb{R}$, $\hat{S}_{[t,T]}(\hat{x})$ is a compact set. Moreover, the value function ϑ (resp. $\hat{\vartheta}$) is l.s.c. in $[0,T] \times \mathbb{R}^d$ (resp. in $[0,T] \times \mathbb{R}^d \times \mathbb{R}$).

Remark 2.1. Notice that $\hat{S}_{[t,T]}(\hat{x})$ may not be a closed set if **(A4)** is not satisfied, and therefore the infimum value of $\hat{\vartheta}$ may not be achieved by an admissible trajectory. In this case, it would be natural to consider the closure of $\hat{S}_{[t,T]}(\hat{x})$ (for the topology induced by the $C^0([t,T])$ norm), see [13, 27]. For this, one should introduce $\hat{f}^{\sharp}$ the convexified set-valued dynamics

$$\hat{f}^{\sharp}(t,\hat{x}) := \overline{\text{co}(\hat{f}(t,\hat{x},\mathcal{A}))}, \quad \text{for } t \in [0,T], \ \hat{x} \in \mathbb{R}^d \times \mathbb{R}.$$

Here, since $\mathcal{A}$ is a compact set of $\mathbb{R}^d$ and $\hat{f}$ is continuous, by the caracthéodory theorem $\text{co}(\hat{f}(t,\hat{x},\mathcal{A}))$ is also a compact set and thus $\hat{f}^{\sharp}(t,\hat{x}) \equiv \text{co}(\hat{f}(t,\hat{x},\mathcal{A}))$. Under assumptions **(A1)-(A2)** the following differential inclusion admits absolutely continuous solutions in $[t,T]$ (see [3]):

$$\dot{y}(s) \in \hat{f}^{\sharp}(s,y(s)), \quad a.e. \ s \in (t,T). \tag{2.6a}$$

$$y(t) = x, \tag{2.6b}$$

Let us denote by $\hat{S}_{[t,T]}^{\sharp}(x,z)$ the set of all the solutions of (2.6). This set is precisely the closure of $\hat{S}_{[t,T]}^{\sharp}(x,z)$ for the topology of $C^0(0,T)$ (see for instance [27]):

$$\hat{S}_{[t,T]}^{\sharp}(\xi) \equiv \overline{\hat{S}_{[t,T]}(x,z)}^{C^0}. \tag{2.7}$$

Moreover, by Filippov's theorem, $\hat{S}_{[t,T]}^{\sharp}(x,z)$ is a compact set of $C^0([t,T])$. Now, define the relaxed control problem, and its associated value function $\vartheta^{\sharp}$, as follows:

$$\vartheta^{\sharp}(t,\hat{x}) := \min_{\hat{y}=(y,\zeta) \in \hat{S}_{[t,T]}^{\sharp}(\hat{x})} \left\{ \varphi(y(T)) - \zeta(T) \ \middle| \ y(s) \in \mathcal{K}, \ \forall s \in (t,T) \right\} \tag{2.8}$$

4

where $\hat{y}(s) = (y(s), \zeta(s))$ denotes the two components of a given trajectory of $\hat{S}^\sharp_{[t,T]}(x,z)$. In this case, the function $\vartheta^\sharp$ is l.s.c. and the minimum in (2.8) is achieved.

2.2 Some properties of the value function $\hat{\vartheta}$

In order to state the dynamic programming principle (DPP), two more notations will be used: the set of admissible trajectories starting from $\hat{x}$

$$\hat{S}^\mathcal{K}_{[t,T]}(\hat{x}) := \left\{ \hat{y} \in \hat{S}_{[t,T]}(\hat{x}), \hat{y}(s) \in \mathcal{K} \times \mathbb{R} \text{ for } s \in [t,T] \right\},$$

and the set of admissible *backward* trajectories arriving at $\hat{x}$

$$\hat{S}^{\mathcal{K},-}_{[\tau,t]}(\hat{x}) := \big\{ \hat{y} \in W^{1,1}(\tau, t), \ \hat{y} \text{ satisfies (2.4a) for some measurable } \alpha \in \mathbb{A} \text{ on } [\tau,t],$$
$$\hat{y}(t) = x \text{ and } \hat{y}(s) \in \mathcal{K} \times \mathbb{R} \text{ for } s \in [\tau,t] \big\}.$$

Lemma 2.2. *Assume* **(A1)-(A4)***. For any $\hat{x} \in \mathbb{R}^d \times \mathbb{R}$ and any $t \in [0,T[$, the following statements hold:*
(i) (Forward DPP) For any $h > 0$ such that $t + h \leq T$:

$$\hat{\vartheta}(t,\hat{x}) = \min_{\hat{y} \in \hat{S}^\mathcal{K}_{[t,t+h]}(\hat{x})} \hat{\vartheta}(t+h, \hat{y}(t+h))$$

with the convention $\hat{\vartheta}(t,\hat{x}) = +\infty$ whenever $\hat{S}^\mathcal{K}_{[t,t+h]}(\hat{x}) = \emptyset$.
(ii) (Backward DPP) For all $s \in [0,t]$ and $\hat{x} \in \mathbb{R}^d \times \mathbb{R}$, for every $\hat{y} \in \hat{S}^{\mathcal{K},-}_{[t-h,t]}(\hat{x})$:

$$\hat{\vartheta}(t,\hat{x}) \geq \hat{\vartheta}(t-h, \hat{y}(t-h)).$$

Since $\hat{S}_{[t,T]}(\hat{x})$ is a compact set, the forward DPP leads to the statement that $\hat{\vartheta}$ satisfies the *increasing principle*, i.e for any $(t,\hat{x}) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$:

$$\exists \hat{y} \in \hat{S}_{[t,T]}(\hat{x}), \quad \hat{\vartheta}(t,\hat{x}) \geq \hat{\vartheta}(t+s, \hat{y}(t+s)) \qquad \text{for any } s \in [t, T-t]. \tag{2.9}$$

This amounts to saying that the epigraph of $\hat{\vartheta}$ is weakly invariant with respect to the set-valued application:

$$(t,\hat{x}) \rightsquigarrow (1, \ \hat{f}(t,\hat{x},\mathcal{A}), \ 0)^\mathsf{T}.$$

Under assumption **(A1)-(A4)**, the mapping $(t,x) \rightsquigarrow \hat{f}(t,\hat{x},\mathcal{A})$ is upper semi-continuous and has nonempty convex compact images. Therefore, the characterization of the weak invariance principle by means of a HJB inequality is straightforward. More precisely, one can prove that (see [25, 9, 21, 15]):

$$u \text{ satisfies (2.9)} \quad \Longleftrightarrow \quad -\partial_t u(t,\hat{x}) + \sup_{a \in \mathcal{A}} (-\hat{f}(t,\hat{x},a) \cdot D_{\hat{x}} u(t,\hat{x})) \leq 0 \ \text{ in} [0,T] \times \mathcal{K} \times \mathbb{R}.$$

On the other hand, the backward DPP expresses the fact that $\hat{\vartheta}$ satisfies also the *decreasing principle* of the value function $\hat{\vartheta}$ along admissible backward trajectories $\hat{y} \in \hat{S}^{\mathcal{K},-}_{[t-s,t]}(\hat{x})$ for every $\hat{x} \in \mathcal{K} \times \mathbb{R}$ and every $t \in [0,T]$ and $s \in [0,t]$.

Let $T_\mathcal{K}(\hat{x})$ denote the tangent cone of $\mathcal{K}$ at $\hat{x}$. If we assume that $\mathcal{K}$ is smooth enough and has a nonempty interior, and if the vector-field $\hat{f}(t,\hat{x},\mathcal{A})$ is assumed to satisfy $\hat{f}(t,\hat{x},\mathcal{A}) \cap T_\mathcal{K}(\hat{x}) \neq \emptyset$, then from the decreasing principle one can conclude that $\mathcal{E}pi(\hat{\vartheta})$ is strongly invariant with respect to the application

$$(t,\hat{x}) \rightsquigarrow \begin{bmatrix} -1 \\ -(\hat{f}(t,\hat{x},\mathcal{A}) \cap T_\mathcal{K}(\hat{x})) \\ 0 \end{bmatrix}.$$

In general, although $\hat{f}$ is Lipschitz continuous, the set-valued application $(t,\hat{x}) \rightsquigarrow \hat{f}(t,\hat{x},\mathcal{A}) \cap T_{\mathcal{K}}(\hat{x})$ may have empty images and may not be Lipschitz continuous. Therefore, the characterization of the strong invariance principle by means of a HJB inequality is not clear (see [21, 15]). As mentioned in the introduction, several works have been done to investigate the characterization of strong invariance under additional controllability assumptions.

In the next section, we will follow a completely different reasoning and will prove that the epigraph of ϑ can be described using a value function of an auxiliary control problem without state-constraints. This description does not require any additional assumption on $\mathcal{K}$ neither on the regularity of ϑ.

3 Characterization of the epigraph

3.1 Auxiliary control problem. Main result

Here we shall focus on the characterization of the epigraph of $\vartheta(t,.)$

$$\mathcal{E}pi(\vartheta(t,.)) := \left\{ (x,z) \in \mathbb{R}^d \times \mathbb{R}, \ \vartheta(t,x) \leq z \right\}.$$

(This set also corresponds to $\{\hat{x} = (x,z) \in \mathbb{R}^d \times \mathbb{R}, \ \hat{\vartheta}(t,\hat{x}) \leq 0\}$.) For this, consider a "level set" function $g : \mathbb{R}^d \to \mathbb{R}$, Lipschitz continuous, that represents the set of constraints $\mathcal{K}$ in the following way:

$$\forall x \in \mathbb{R}^d, \quad g(x) \leq 0 \Leftrightarrow x \in \mathcal{K}. \tag{3.1}$$

Since $\mathcal{K}$ is closed, such a function g exists. Indeed, if we denote by d_K the signed distance to $\mathcal{K}$ (where $d_{\mathcal{K}}(x) := d(x,\mathcal{K})$ if $x \notin \mathcal{K}$ and $d_{\mathcal{K}}(x) = -d(x,\mathbb{R}^d\backslash\mathcal{K})$ otherwise), then the function $g \equiv d_{\mathcal{K}}(\cdot)$ is Lipschitz continuous and satisfies the statement (3.1). Therefore, for any $y \in S_{[t,T]}(x)$:

$$\left(y(\theta) \in \mathcal{K}, \ \forall \theta \in [t,T] \right) \quad \Leftrightarrow \quad \max_{\theta \in [t,T]} g\big(y(\theta)\big) \leq 0.$$

Now, introduce the auxiliary control problem and its associated value function w^g defined by:

$$w^g(t,x,z) := \inf_{\hat{y}=(y,\zeta)\in\hat{S}_{[t,T]}(x,z)} \left(\varphi(y(T)) - \zeta(T) \right) \bigvee \max_{\theta\in(t,T)} g(y(\theta)) \tag{3.2}$$

for $x \in \mathbb{R}^d$, $z \in \mathbb{R}$, $t \in [0,T]$, and where we have used, for convenience, the notation $a \vee b := \max(a,b)$.

In this auxiliary control problem, the term "$\max_{\theta\in[t,T]} g\big(y(\theta)\big)$" is an exact penalization of the state constraints. Here, we shall use the problem (3.2) to characterize the epigraph of the value function ϑ without requiring any additional assumption on $\mathcal{K}$ neither on the dynamics $\hat{f}$.

Theorem 3.1. *Assume that* **(A1)-(A4)** *hold and that $\mathcal{K}$ is closed and non-empty. Then for any $t \in [0,T]$ and $(x,z) \in \mathbb{R}^d \times \mathbb{R}$, the following holds:*
(i)

$$\vartheta(t,x) - z \leq 0 \quad \Longleftrightarrow \quad \hat{\vartheta}(t,(x,z)) \leq 0 \quad \Longleftrightarrow \quad w^g(t,x,z) \leq 0.$$

(ii) Moreover, the function ϑ is characterized by w^g through the following relation

$$\vartheta(t,x) = \min \left\{ z \in \mathbb{R}, \ w^g(t,x,z) \leq 0 \right\}. \tag{3.3}$$

Proof. (i) Let us assume that $\vartheta(t,x) \leq z$. There exists a sequence y_n of admissible trajectories of $S_{[t,T]}(x)$, such that

$$\lim_{n\to\infty} \int_t^T \ell(s,y_n(s),\alpha(s))\, ds + \varphi(y_n(T)) - z = \vartheta(t,x) - z \leq 0.$$

Since all the trajectories y_n are admissible, we have, for all $n \geq 0$, $\max_{\theta \in [t,T]} g(y_n(\theta)) \leq 0$. Hence

$$
\begin{aligned}
w^g(t,x,z) &\leq \liminf_{n \to \infty} \left(\int_t^T \ell(s, y_n(s), \alpha(s))\, ds + \varphi(y_n(T)) - z \right) \bigvee \max_{\theta \in [t,T]} g(y_n(\theta)) \\
&\leq 0.
\end{aligned}
$$

Conversely, let us assume that $w^g(t,x,z) \leq 0$. We know that $\hat{S}_{[t,T]}(\xi)$ is a compact set in $C^0([t,T])$, hence the infimum in $w^g(t,x,z)$ is reached and there exists a minimizer $y \in \hat{S}[t,T]((x,z))$. Moreover,

$$
0 \geq w^g(t,x,z) = \left(\int_t^T \ell(s, y(s), \alpha(s))\, ds + \varphi(y(T)) - z \right) \bigvee \max_{\theta \in [t,T]} g(y(\theta))
$$

Therefore, on one hand, $\max_{\theta \in [t,T]} g(y(\theta)) \leq 0$ and y satisfies the state constraints, and on the other hand,

$$
\vartheta(t,x) - z \leq \int_t^T \ell(s, y(s), \alpha(s))\, ds + \varphi(y(T)) - z \leq 0
$$

which is the desired result.

The proof of (ii) is a consequence of (i). $\qquad\square$

Remark 3.2. When the convexity assumption **(A4)** is not satisfied, the statements of the above theorem are valid for the l.s.c. value function $\vartheta^\sharp$ instead of $\hat{\vartheta}$. Indeed, let $\ell \equiv 0$, $f(t,x,\alpha) := (1,\alpha)^\mathsf{T}$ with $\alpha \in \mathcal{A} := \{\pm 1\}$, $\mathcal{K} := \{(x_1,x_2) \in \mathbb{R}^2,\ |x_2| \leq |x_1 - \frac{1}{2}|^2\}$, and $T = 1$. One can check that, for $\bar{x} = (0,0)^\mathsf{T}$, there exist no admissible trajectory starting in $\bar{x}$ and staying in $\mathcal{K}$ on $[0,T]$. Hence $\hat{\vartheta}(0, (\bar{x}, z)) = +\infty$ for all $z \in \mathbb{R}$. On the other hand, let $(y_n)_{n \geq 1}$ be the sequence of trajectories defined for $\theta \in (0,T)$ by

$$
y_n(\theta) = \begin{cases} (\frac{k}{n}, 0)^\mathsf{T} + (\theta - \frac{k}{n})(1,1)^\mathsf{T} & \text{if } \theta \in [\frac{k}{n}, \frac{k+1/2}{n}[, \\ (\frac{k+1/2}{n}, \frac{1/2}{n})^\mathsf{T} + (\theta - \frac{k+1/2}{n})(1,-1)^\mathsf{T} & \text{if } \theta \in [\frac{k+1/2}{n}, \frac{k+1}{n}[. \end{cases} \tag{3.4}
$$

Then y_n converges uniformly on $[0,1]$ toward the limit $y(t) = (t,0)^\mathsf{T}$, and $w^g(0,x,z)$ will have a finite negative value whenever $\varphi((T,0)^\mathsf{T}) < z$.

3.2 A particular case: backward reachable sets

Consider the problem of backward reachable sets to a closed target $\mathcal{C}$ of $\mathbb{R}^d$. It consists in characterizing, for every $t \in [0,T]$, the set of all initial positions from which it is possible to find an admissible trajectory that reaches the target at time T while lying in the set $\mathcal{K}$ on $[t,T]$:

$$
\mathcal{R}(t) := \left\{ x \in \mathbb{R}^d : \exists y \in S_{[t,T]}(x) \text{ such that } y(T) \in \mathcal{C}, \text{ and } y(s) \in \mathcal{K} \text{ on } \mathcal{K} \right\}.
$$

In that case we can consider $\ell \equiv 0$ and a Lipschitz continuous function $\phi : \mathbb{R}^d \to \mathbb{R}$ satisfying:

$$
\forall x \in \mathbb{R}^d, \qquad x \in \mathcal{C} \quad \Leftrightarrow \quad \phi(x) \leq 0.
$$

As in [14], we consider the control problem:

$$
v^g(t,x) = \inf_{y \in S_{[t,T]}(x)} \phi(y(T)) \bigvee \max_{\theta \in (t,T)} g(y(\theta))
$$

Therefore, under the same assumption **(A1)** and with a closed set $\mathcal{K}$, the backward reachable set is equivalent to:

$$
\mathcal{R}(t) \equiv \left\{ x \in \mathbb{R}^d \mid v^g(t,x) \leq 0 \right\},
$$

in other terms, the region where the function $v^g(t,\cdot)$ is negative represents the backward reachable set at time t, see [14]. In view of this result, theorem 3.1 can be seen as a generalisation to a more general setting

where the function φ is any Lipschitz continuous function. Actually, in the general case, the epigraph of the function φ should be considered as the target set and the epigraph of $\hat{\vartheta}(t, \cdot)$ represents the backward reachable set under the dynamics $\hat{f}$. This claim can be proved by using viability tools, see for instance [19, 2]. Moreover, taking this claim into account, one can compute the value function by using viability algorithm [5, 36]. Here, we use only viscosity arguments and give a description of the epigraph by means of a simple continous solution to a HJ inequation. This opens the way to use a very large pannel of numerical schemes (Semi-Lagrangian, ENO, WENO, Discontinuous Galerkin, ...).

3.3 Properties of the auxiliary value function w^g

The main feature of the auxiliary control problem (3.2) lies in the fact that this new problem is not under state constraints any more. Moreover, function w^g enjoys more regularity properties and can be characterized by a Hamilton-Jacobi equation without assuming any controllability assumption. Let us denote by $\mathcal{H}$ the Hamiltonian defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ by:

$$\mathcal{H}(t, x, z, p, q) := \max_{a \in \mathcal{A}} \left(-f(t, x, a) \cdot p + \ell(t, x, a) \cdot q \right). \tag{3.5}$$

Proposition 3.3. *Assume that* **(A1)-(A4)** *hold and let g be a Lipschitz continuous function satisfying* (3.1). *Then w^g is a locally Lipschitz continuous function on $[0, T] \times \mathbb{R}^d \times \mathbb{R}$. Furthermore, for any $t \in [0, T]$, and $h \geq 0$ such that $t + h \leq T$,*

$$w^g(t, x, z) := \min_{\hat{y} = (y, \zeta) \in \hat{S}_{[t, t+h]}(x, z)} \left\{ w^g \left(t + h, \, y(t + h), \, \zeta(t + h) \right) \bigvee \max_{\theta \in (t, t+h)} g(y(\theta)) \right\}. \tag{3.6}$$

Proof. The proof of the DPP (3.6) is classical and can be obtained by the same arguments as in [11, 10]. Consider $\hat{x} = (x, z), \hat{x}' = (x', z') \in \mathbb{R}^d \times \mathbb{R}$, $t \in [0, T]$, and $w_0(\hat{x}) := w^g(0, \hat{x}) \equiv \max(\varphi(x) - z, g(x))$. By using the definition of w^g and the simple inequalities:

$$\max(A, B) - \max(C, D) \leq \max(A - C, B - D), \quad \text{and} \quad \inf A_\alpha - \inf B_\alpha \leq \sup(A_\alpha - B_\alpha), \tag{3.7}$$

we get:

$$\begin{aligned}
|w^g(t, \hat{x}) - w^g(t, \hat{x}')| &\leq \sup_{\alpha \in \mathbb{A}} \max \left(\left| w_0(\hat{y}^\alpha_{t, \hat{x}}(T)) - w_0(\hat{y}^\alpha_{t, \hat{x}'}(T)) \right|, \max_{\theta \in (t, T)} \left| g(y^\alpha_{t, x}(\theta)) - g(y^\alpha_{t, x'}(\theta)) \right| \right), \\
&\leq \sup_{\alpha \in \mathbb{A}} \left(L_0 \left| \hat{y}^\alpha_{t, \hat{x}}(T) - \hat{y}^\alpha_{t, \hat{x}'}(T) \right|, \, L_g \max_{\theta \in (t, T)} \left| y^\alpha_{t, x}(\theta) - y^\alpha_{t, x'}(\theta) \right| \right)
\end{aligned}$$

where L_0 and L_g denotes respectively the Lipschitz constant of w_0 and g. By assumption **(A1)**, for $\theta \in (t, T)$, $\left| \hat{y}^\alpha_{t, \hat{x}}(\theta) - \hat{y}^\alpha_{t, \hat{x}'}(\theta) \right| \leq e^{\hat{L}(\theta - t)} |\hat{x} - \hat{x}'| \leq e^{\hat{L} T} |\hat{x} - \hat{x}'|$ (where $\hat{L}$ is the Lipschitz constant of $\hat{f}$). Then we conclude that:

$$|w^g(t, \hat{x}) - w^g(t, \hat{x}')| \leq \max(L_0, L_g) e^{\hat{L} T} |\hat{x} - \hat{x}'|. \tag{3.8}$$

On the other hand, let $\hat{x} = (x, z)$ be in $\mathbb{R}^d \times \mathbb{R}$, and let $t \geq 0$, $h \geq 0$. Using that $w^g(t, \hat{x}) \geq g(x)$, we deduce from the dynamic programming principle for w^g that

$$\begin{aligned}
|w^g(t + h, \hat{x}) - w^g(t, \hat{x})| &= \left| \inf_\alpha \max \left(w^g(t, \hat{y}^\alpha_{t, \hat{x}}(t + h)), \max_{\theta \in [t, t+h]} g(y^\alpha_{t, x}(\theta)) \right) - \max \left(w^g(t, \hat{x}), \, g(x) \right) \right| \\
&\leq \sup_\alpha \max \left(\left| w^g(t, \hat{y}^\alpha_{t, \hat{x}}(t + h)) - w^g(t, \hat{x})) \right|, \max_{\theta \in (t, t+h)} \left| g(y^\alpha_{t, x}(\theta)) - g(x) \right| \right) \\
&\leq \max \left(\max(L_0, L_g) e^{\hat{L} T} |\hat{y}^\alpha_{t, \hat{x}}(t + h) - \hat{x}|, \, L_g \max_{\theta \in (t, t+h)} |y^\alpha_{t, x}(\theta) - x| \right)
\end{aligned}$$

where we have used (3.8).

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Furthermore, denoting $C_f := \max\limits_{(s,a)\in(0,T)\times\mathcal{A}} |f(s,0,a)| < \infty$, we have $|f(s,x,a)| \le C_f + L|x|$. Hence by a Gronwall estimate, we have $|y_{t,x}^\alpha(\theta) - x| \le (C_f + L|x|)e^{Lh}h \le (C_f + L|x|)e^{LT}h$ for $\theta \in (t, t+h)$. We obtain in the same way the estimate: $|\hat{y}_{t,\hat{x}}^\alpha(\theta) - \hat{x}| \le (\hat{C}_f + \hat{L}|\hat{x}|)e^{\hat{L}T}h$ for every $\theta \in (t, t+h)$.

Therefore, we conclude that $|w_g(t', \hat{x}) - w_g(t, \hat{x})| \le C(1 + |\hat{x}|)|t' - t|$ for some constant $C > 0$. Combining all the inequalities above, we get:

$$|w_g(t', \hat{x}') - w_g(t, \hat{x})| \le C(1 + |\hat{x}|)\,(|t' - t| + |\hat{x}' - \hat{x}|),$$

for some constant $C \ge 0$. In particular the following linear growth holds: $|w_g(t, \hat{x})| \le C(1 + |\hat{x}|)$. $\qquad\square$

Moreover, the following proposition holds.

Proposition 3.4. *Assume that* **(A1)-(A4)** *are satisfied. Then the function $u = w^g$ is the unique continuous viscosity solution of the following HJB equation*

$$\min\left(-\partial_t u(t,x,z) + \mathcal{H}(t,x,z,\nabla_x u, \partial_z u), \quad u(t,x,z) - g(x) \right) = 0,$$
$$\forall t \in [0,T),\ x \in \mathbb{R}^d,\ z \in \mathbb{R}, \qquad (3.9\text{a})$$
$$u(T,x,z) = (\varphi(x) - z) \vee g(x), \quad x \in \mathbb{R}^d,\ z \in \mathbb{R}. \qquad (3.9\text{b})$$

Proof. The HJB equation can be derived from the DPP satisfied by w^g. For sake of completeness, we give here the main lines of the proof. We first show that w^g is a solution of (3.9). The fact that w^g satisfies the initial condition comes directly from the definition of w^g.

Let us check the super-solution property of w^g. From the DPP, we get that for any $\tau \ge 0$

$$w^g(t, (x,z)) \ge \min_{\hat{y}=(y,\zeta)\in\hat{S}_{[t,t+h]}(x,z)} w^g(t+h, y(t+h), \zeta(t+h)).$$

Hence, classical arguments in viscosity theory yield to:

$$-\partial_t w^g + \mathcal{H}(t,x,z,\nabla_x w^g, \partial_z w^g) \ge 0$$

in the viscosity sense. Moreover, by definition of w^g, for every $(t,x,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$, we have

$$w^g(t,x,z) \ge \min_{\hat{y}=(y,\zeta)\in\hat{S}_{[t,T]}(x,z)} \max_{\theta\in[t,T]} g(y(\theta)) \ge g(x).$$

Combining this two inequalities, we get

$$\min(\partial_t w^g + \mathcal{H}(t,x,z,\nabla_x w^g, \partial_z w^g), w^g - g) \ge 0$$

in the viscosity sense, i.e., w^g is a super-solution of (3.2).

It remains to prove that w^g is a sub-solution. Let $(x,z) \in \mathbb{R}^d \times \mathbb{R}$ and $t \in [0,T]$. If $w^g(t,x,z) \le g(x)$, then it is clear that w^g satisfies:

$$\min(-\partial_t w^g + \mathcal{H}(t,x,z,\nabla_x w^g, \partial_z w^g), w^g(t,x,z) - g(x)) \le 0.$$

Now, assume that $w^g(t,x,z) > g(x)$. By continuity of g and w^g, there exists some $\tau > 0$ such that for every $\hat{y} = (y,\zeta) \in \hat{S}_{[t,t+h]}(x,z)$, we have: $w^g(\theta, y(\theta), \zeta(\theta)) > g(y(\theta))$ for all $\theta \in [t, t+h]$ (since $y(\theta)$ will stay in a neighborhood of x). Hence, by using the DPP, we get that

$$w^g(t,x,z) = \min_{\hat{y}=(y,\zeta)\in\hat{S}_{[t,T]}(x,z)} w^g(t+h, y(t+h), \zeta(t+h)), \quad \text{for any } 0 \le h \le T - t.$$

Therefore, with classical arguments [6] one can obtain that $\partial_t w^g(t,x,z) + \mathcal{H}(t,x,z,\nabla_x w^g, \partial_z w^g) \le 0$ in the viscosity sense. Therefore, w^g is a viscosity sub-solution of (3.2).

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The fact that w^g is the unique continuous viscosity solution of (3.2) follows from the general comparison principle in appendix A, and the fact that there exists $C_0 > 0$ such that:

$$|\mathcal{H}(t,x,z,p,q) - \mathcal{H}(s,x',z',p,q)| \leq C_0(|p| + |q| + 1)\,(|t-s| + |x-x'| + |z-z'|), \quad (3.10a)$$

$$|\mathcal{H}(t,x,z,p,q) - \mathcal{H}(t,x,z,p',q')| \leq C_0(|x| + 1)(|p-p'| + |q-q'|), \quad (3.10b)$$

for every $(x,z), (x',z') \in \mathbb{R}^d \times \mathbb{R}$ and for every $(p,q), (p',q') \in \mathbb{R}^d \times \mathbb{R}$. $\qquad\square$

Remark 3.5. Notice that when $\mathcal{K}$ is a bounded set, then it is possible to modify f, ℓ and g outside $\mathcal{K}$ in order to get a bounded auxiliary value function. Indeed, if $\mathcal{K} \subset B(0,R)$ then by setting $h(x) := d(x, B(0,R))$, it is possible to consider $f_R(t,x,\alpha) := f(t,x,\alpha)(1 - h(x))_+$, $\ell_R(t,x,\alpha) := \ell(t,x,\alpha)(1 - h(x))_+$ and $g_R(x) = \max(g(x), R)$ (the functions f_R, ℓ_R and g_R coincide respectively with f, ℓ and g on $\mathcal{K}$). Moreover, f_R, ℓ_R and g_R are still Lipschitz continuous in (t,x), and compactly supported in $B(0, R+1)$. Therefore, the function w^{g_R} defined using (f_R, ℓ_R, g_R) satisfies $w^{g_R}(t,x,z) \leq 0 \Leftrightarrow w^g(t,x,z) \leq 0$, and it can be used for the characterization of $\vartheta(t,x)$.

Let us point out that the value function w^g depends on the choice of g. However, the set

$$\{(x,z),\ w^g(t,x,z) \leq 0\}$$

is independent of the choice of g.

Lemma 3.6. *If $\tilde{g}$ and g are Lipschitz continuous function satisfying (3.1), then for all $t \leq T$,*

$$\{(x,z),\ w^g(t,x,z) \leq 0\} \quad = \quad \{(x,z),\ w^{\tilde{g}}(t,x,z) \leq 0\}.$$

Remark 3.7 (Extension to terminal state constraints). Consider the case when the state variable is constrained to satisfy a final constraint $y(T) \in \mathcal{C}$, in addition to (2.2), where $\mathcal{C}$ is a closed subset of $\mathbb{R}^d$ (of course $\mathcal{C} \cap \mathcal{K}$ has to be considered nonempty, otherwise the problem becomes trivial). Then the control problem is defined by:

$$\vartheta(t,x) := \inf\left\{ \int_t^T \ell(s, y_{t,x}^\alpha(s), \alpha(s))\,ds + \varphi(y_{t,x}^\alpha(T)) \;\middle|\; \alpha \in \mathbb{A},\ y_{t,x}^\alpha(T) \in \mathcal{C} \text{ and } y_{t,x}^\alpha(s) \in \mathcal{K},\ \forall s \in [t,T] \right\}. \tag{3.11}$$

In this case, one should consider a Lipschitz continuous function $\psi : \mathbb{R}^d \to \mathbb{R}$ such that $\psi(x) \leq 0 \Longleftrightarrow x \in \mathcal{C}$ (since $\mathcal{C}$ is closed, such a Lipschitz function ψ exists). Hence the auxiliary control problem can be considered as:

$$w^g(t,x,z) := \inf_{\hat{y}=(y,\zeta)\in \hat{S}_{[t,T]}(x,z)} \left(\varphi(y(T)) - \zeta(T) \right) \bigvee \max_{\theta \in (t,T)} g(y(\theta)) \bigvee \psi(y(T)). \tag{3.12}$$

Here again, the description of the value function ϑ is given by:

$$\mathcal{E}pi(\vartheta(t,\cdot)) = \{(x,z) \in \mathbb{R}^d \times \mathbb{R} \mid w^g(t,x,z) \leq 0\},$$

and w^g satisfies the same HJB equation as in proposition 3.4 excepted the fact that the final condition at time $t = T$ is now: $w^g(T,x,z) = (\varphi(x) - z) \vee g(x) \vee \psi(x)$.

Before concluding this section, we would like to stress on the fact that considering the auxiliary function w^g allows to bypass all the regularity issues which arise when the control problem is in presence of state constraints. This constructive way to describe the value function ϑ involves only the classical framework of Lipschitz continuous viscosity solution, even though ϑ is only l.s.c. This approach opens the way for computing the epigraph of ϑ by a large panel of schemes for continuous viscosity solutions. One can criticize the fact that the definition of w^g involves one more state component, however it is important to keep in mind that only the level set $\{w^g(t,x,z) = 0\}$ is needed. Since w^g is Lipschitz continuous, one can use a local numerical scheme (as in [35]) to track this $0-$level set in $\mathbb{R}^{d+1}$ and reduce the numerical complexity to almost the same numerical complexity of solving an HJ equation in a d-dimensional space.

4 Infinite horizon problem with state constraints

Here we consider the case of an infinite horizon optimal control problem with state constraints (see [6] for standard results) and assume throughout this section that f, ℓ satisfies **(A1)-(A2)** and **(A4)**, and that these functions do not depend on the time variable:

$$f : \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}^d, \quad \ell : \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}.$$

Let $\mathbb{A}$ be the set of measurable controls $\alpha : (0, \infty) \to \mathcal{A}$. For any $\alpha \in \mathbb{A}$, consider the system obeying:

$$\dot{y}(s) = f(y(s), \alpha(s)) \quad \text{a.e. } s \in (0, \infty), \tag{4.1a}$$
$$y(0) = x, \tag{4.1b}$$

and let $y = y_x^\alpha$ denotes its solution. Let also $\mathcal{K}$ be a nonempty closed set of $\mathbb{R}^d$. Set L is assumed to be a majorant of the Lipschitz constants for f and ℓ with respect to the x variable, as stated in **(A1)-(A2)**, and set $\Lambda > L$. The infinite horizon control problem is:

$$\widetilde{\vartheta}(x) := \min_{\alpha \in \mathbb{A}} \left\{ \int_0^\infty e^{-\lambda s} \ell(y_x^\alpha(s), \alpha(s)) ds \ \middle| \ y_x^\alpha(\theta) \in \mathcal{K} \quad \forall \theta \in (0, \infty) \right\}. \tag{4.2}$$

We again focus on the characterization of the epigraph of $\widetilde{\vartheta}$:

$$\mathcal{E}pi(\widetilde{\vartheta}) := \left\{ (x, z) \in \mathbb{R}^d \times \mathbb{R}, \ \widetilde{\vartheta}(x) \leq z \right\}.$$

Let $g : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz continuous satisfying (3.1). We introduce an auxiliary control problem with value function $\widetilde{w}^g$ defined for $x \in \mathbb{R}^d$, $z \in \mathbb{R}$, and $\lambda > L$, by:

$$\widetilde{w}^g(x, z) := \min_{\alpha \in \mathbb{A}} \ \left(\int_0^\infty e^{-\lambda s} \ell(y_x^\alpha(s), \alpha(s)) \, ds - z \right) \bigvee \max_{\theta \in (0, \infty)} \left(e^{-\lambda \theta} g(y_x^\alpha(\theta)) \right). \tag{4.3}$$

Theorem 4.1. *Assume that **(A1)-(A2)** and **(A4)** hold, $\mathcal{K}$ is closed and non-empty, and $\lambda > L$.*
(i) For any $(x, z) \in \mathbb{R}^d \times \mathbb{R}$, the following holds:

$$\widetilde{\vartheta}(x) - z \leq 0 \quad \Longleftrightarrow \quad \widetilde{w}^g(x, z) \leq 0.$$

Moreover, the function $\widetilde{\vartheta}$ is characterized by $\widetilde{w}^g$ through the following relation

$$\widetilde{\vartheta}(x) = \min \left\{ z \in \mathbb{R}, \ \widetilde{w}^g(x, z) \leq 0 \right\}.$$

(ii) $\widetilde{w}^g$ is Lipschitz continuous on $\mathbb{R}^d \times \mathbb{R}$.
(iii) The function $u = \widetilde{w}^g$ is the unique Lipschitz continuous viscosity solution of

$$\min \left(\lambda u + \max_{a \in A}(-f(x, a) \cdot \nabla_x u + \ell(x, a) \partial_z u), \ u(x, z) - g(x) \right) = 0,$$
$$x \in \mathbb{R}^d, \ z \in \mathbb{R}. \tag{4.4}$$

The derivation of the HJ equation for $\widetilde{w}^g$ is based on the following DPP principle. We define $\zeta_{x,z}^\alpha(t) := e^{\lambda t} z - \int_0^t e^{\lambda(t-s)} \ell(y_x^\alpha(s), \alpha(s)) \, ds$. Equivalently, $\zeta(t) = \zeta_{x,z}^\alpha(t)$ is the absolutely continuous solution of

$$\dot{\zeta}(s) = \lambda \zeta(s) - \ell(y_x^\alpha(s), \alpha(s)) \quad \text{a.e. } s > 0, \tag{4.5a}$$
$$\zeta(0) = z. \tag{4.5b}$$

Lemma 4.2 (Dynamic programming principle). *For all $h \geq 0$ and $(x, z) \in \mathbb{R}^d \times \mathbb{R}$, we have*

$$\widetilde{w}^g(x, z) = \min \left\{ \left(e^{-\lambda h} \widetilde{w}^g(y_x^\alpha(h), \zeta_{(x,z)}^\alpha(h)) \right) \bigvee \max_{\theta \in (0,h)} \left(e^{-\lambda \theta} g(y_x^\alpha(\theta)) \right), \right.$$

$$\left. \alpha : (0, h) \to \mathcal{A} \ measurable, \ (y_x^\alpha, \zeta_{(x,z)}^\alpha) \ solution \ of \ (4.5) \right\} \qquad (4.6)$$

where the minimization is over all measurable controls $\alpha : (0, h) \to \mathcal{A}$.

Proof. The definition of $\widetilde{w}^g$ involves the maximum between two terms, and the DPP here seems to be new. Even though the proof is not difficult we prefer to present it for sake of completeness.

For any measurable control $\alpha : (0, \infty) \to \mathcal{A}$, we shall denote α_1 the restriction of α on $(0, h)$ and α_2 the measurable control of $\mathcal{A}$ such that $\alpha_2(t) = \alpha(t + h)$ a.e. $t \geq 0$. Using $y_x^\alpha(s + h) = y_{y_x^{\alpha_1}(h)}^{\alpha_2}(s)$, we obtain on the first hand:

$$\int_0^\infty e^{-\lambda s} \ell(y_x^\alpha(s), \alpha(s)) \, ds - z = \int_h^\infty e^{-\lambda s} \ell(y_x^\alpha(s), \alpha(s)) \, ds + \int_0^h e^{-\lambda s} \ell(y_x^\alpha(s), \alpha(s)) \, ds - z$$

$$= e^{-\lambda h} \left(\int_0^\infty e^{-\lambda s} \ell(y_x^\alpha(s + h), \alpha(s + h)) \, ds - \zeta_{x,z}^{\alpha_1}(h) \right)$$

$$= e^{-\lambda h} \left(\int_0^\infty e^{-\lambda s} \ell(y_{y_x^{\alpha_1}}^{\alpha_2}(s), \alpha_2(s)) \, ds - \zeta_{x,z}^{\alpha_1}(h) \right) \qquad (4.7)$$

and on the other hand:

$$\max_{\theta \in (0,\infty)} e^{-\lambda \theta} g(y_x^\alpha(\theta)) = \max_{\theta \in (0,\infty)} e^{-\lambda(\theta + h)} g(y_x^\alpha(\theta + h)) \bigvee \max_{\theta \in (0,h)} e^{-\lambda \theta} g(y_x^{\alpha_1}(\theta))$$

$$= \left(e^{-\lambda h} \max_{\theta \in (0,\infty)} e^{-\lambda \theta} g(y_{y_x^{\alpha_1}(h)}^{\alpha_2}(\theta)) \right) \bigvee \max_{\theta \in (0,h)} e^{-\lambda \theta} g(y_x^{\alpha_1}(\theta)). \qquad (4.8)$$

Combining (4.7) and (4.8), taking measurable controls $\alpha_1 : (0, h) \to \mathcal{A}$ and $\alpha_2 \in \mathbb{A}$, we obtain

$$\widetilde{w}^g(x, z) = \inf_{\alpha_1} \inf_{\alpha_2} \left(e^{-\lambda h} \int_0^\infty e^{-\lambda s} \ell(y_{y_x^{\alpha_1}}^{\alpha_2}(s), \alpha_2(s)) \, ds - \zeta_{x,z}^{\alpha_1}(h) \right) \bigvee \left(e^{-\lambda h} \max_{\theta \in (0,\infty)} e^{-\lambda \theta} g(y_{y_x^{\alpha_1}(h)}^{\alpha_2}(\theta)) \right)$$

$$\bigvee \max_{\theta \in (0,h)} e^{-\lambda \theta} g(y_x^{\alpha_1}(\theta))$$

$$= \inf_{\alpha_1} \left(e^{-\lambda h} \widetilde{w}^g(y_x^{\alpha_1}(h), \zeta_{x,z}^{\alpha_1}(h)) \right) \bigvee \max_{\theta \in (0,h)} e^{-\lambda \theta} g(y_x^{\alpha_1}(\theta))$$

which is the desired result. $\qquad \square$

5 Two-player games with state constraints

The approach presented in the previous sections can be extended to more general control problems lacking controllability assumptions. We consider here the case of two-player games [40, 24, 6, 39, 7, 20].

In addition to the player taking control values in $\mathcal{A}$, we consider $\mathcal{B}$ a non empty compact set of $\mathbb{R}^m$ ($m \geq 1$), and the set controls for the second player:

$$\mathbb{B} := \left\{ \beta : (0, T) \to \mathbb{R}^m \text{ measurable}, \ \beta(t) \in \mathcal{B} \text{ a.e.} \right\}.$$

We also consider a new dynamics f, distributed cost ℓ, and terminal cost φ such that:

$$(A_1') \quad \begin{cases} (i) \ \ (t,x,a,b) \in [0,T] \times \mathbb{R}^d \times \mathcal{A} \times \mathcal{B} \to f(t,x,a,b) \in \mathbb{R}^d \text{ is continuous} \\ (ii) \ \exists L \geq 0, \ \forall x,y \in \mathbb{R}^d, \ \forall (a,b) \in \mathcal{A} \times \mathcal{B}, \ \forall t,s \in [0,T], \\ \qquad |f(t,x,a,b) - f(s,y,a,b)| \leq L(|x-y| + |t-s|), \end{cases}$$

$$(A_2') \quad \begin{cases} (i) \ \ (t,x,a,b) \in [0,T] \times \mathbb{R}^d \times \mathcal{A} \times \mathcal{B} \to \ell(t,x,a,b) \in \mathbb{R}^d \text{ is continuous} \\ (ii) \ \exists L \geq 0, \ \forall x,y \in \mathbb{R}^d, \ \forall (a,b) \in \mathcal{A} \times \mathcal{B}, \ \forall t,s \in [0,T],, \\ \qquad |\ell(t,x,a,b) - \ell(s,y,a,b)| \leq L(|x-y| + |t-s|), \end{cases}$$

$(A_3') \quad \varphi : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous.

Again, we consider the "augmented" dynamics $\hat{f} : [0,T] \times \mathbb{R}^{d+1} \times \mathcal{A} \times \mathcal{B} \to \mathbb{R}^{d+1}$ defined as follows:

$$\hat{f}(t,(x,z),a,b) := \begin{pmatrix} f(t,x,a,b) \\ -\ell(t,x,a,b) \end{pmatrix}, \quad \forall (x,z) \in \mathbb{R}^d \times \mathbb{R}, \ a \in \mathcal{A}, \ b \in \mathcal{B}, \ t \in (0,T).$$

The following convexity assumption of $\hat{f}$ will be considered:

$$(A_4') \qquad \forall t, x, \beta, \quad \hat{f}(t,x,\mathcal{A},\beta) \text{ is a convex set.}$$

Notice that this includes the case when $f(t,x,a,b) = f_1(t,x,b) \cdot a + f_2(t,x,b)$ and $\mathcal{A}$ is a convex set.

For every $x \in \mathbb{R}^d$ and $(\alpha,\beta) \in \mathbb{A}_t \times \mathbb{B}$ we define the trajectory $y = y_{t,x}^{\alpha,\beta}$ as the solution of

$$\dot{y}(s) = f(s,y(s),\alpha(s),\beta(s)) \text{ for a.e. } s \in [t,T], \quad y(t) = x. \tag{5.1}$$

in the Caratéodory sense.

Following the formulation of Elliott and Kalton [24], we define the set of non-anticipative strategies $a \equiv a[\cdot]$ as follows:

$$\Gamma_t := \Bigg\{ a : \mathbb{B} \to \mathbb{A}, \ \forall (\beta, \tilde{\beta}) \in \mathbb{B}, \text{ and } \forall s \in [t,T],$$

$$\Big(\beta(\theta) = \tilde{\beta}(\theta). \ a.e. \ \theta \in [t,s] \Big) \Rightarrow \Big(a[\beta](\theta) = a[\tilde{\beta}](\theta), a.e. \text{ on } [t,s] \Big) \Bigg\}.$$

The value function for the first player is given by:

$$\overline{\vartheta}(t,x) \ := \tag{5.2}$$

$$\inf_{a \in \Gamma_t} \max_{\beta \in \mathbb{B}} \Bigg\{ \int_t^T \ell(s, y_{t,x}^{a[\beta],\beta}(s), \alpha(s), \beta(s)) \, ds + \varphi(y_{t,x}^{a[\beta],\beta}(T)) \ \Big| \ y_{t,x}^{\alpha}(s) \in \mathcal{K}, \ \forall s \in [t,T] \Bigg\}.$$

As in Section 3, we add a new state variable $z \in \mathbb{R}$, and we define the following auxiliary control problem:

$$\overline{w}^g(t,x,z) := \inf_{a \in \Gamma_t} \max_{\beta \in \mathbb{B}} \ \left(\varphi(y_{t,x}^{a[\beta],\beta}(T)) - \zeta_{t,(x,z)}^{a[\beta],\beta}(T) \right) \bigvee \max_{\theta \in [t,T]} g(y_{t,x}^{a[\beta],\beta}(\theta)). \tag{5.3}$$

where $\hat{f} = (f,-\ell)^\mathsf{T}$ and $\hat{y} = \hat{y}_{t,(x,z)}^{\alpha,\beta} \equiv (y_{t,x}^{\alpha,\beta}, \zeta_{t,(x,z)}^{\alpha,\beta})^\mathsf{T}$ is an absolutely continuous solution of

$$\dot{\hat{y}}(s) = \hat{f}(s, \hat{y}(s), \alpha(s), \beta(s)) \quad \text{a.e. } s \in (t,T), \tag{5.4a}$$

$$\hat{y}(t) = (x,z)^\mathsf{T}. \tag{5.4b}$$

In a similar way to the case of a one-player game, one can check easily that we have:

Lemma 5.1. *Let $(A'_1) - (\bar{A}'_4)$ hold.*
(i) $\bar{\vartheta}$ is a lower semi-continuous function.
(ii) The infimum in (5.3) is reached. In particular,

$$\overline{w}^g(t,x,z) \leq 0 \iff \exists a[\cdot] \in \Gamma_t, \ \forall \beta \in \mathbb{B},$$
$$\int_t^T \ell(s, y_{t,x}^{a[\beta],\beta}(s), \alpha(s), \beta(s)) \, ds + \varphi(y_{t,x}^{a[\beta],\beta}(T)) \leq z,$$
$$y_{t,x}^{a[\beta],\beta}(\theta) \in \mathcal{K}, \ \forall \theta \in [t,T].$$

Therefore the value $\overline{w}^g(t,x,z)$ is reached by some non-anticipative strategy, as soon as φ and g are continuous. By using similar arguments as in the previous sections, we get:

Theorem 5.2. *Assume $(A'_1) - (A'_4)$, and g Lipschitz continuous.*
(i) $\forall t \in [0,T], \ \forall (x,z) \in \mathbb{R}^d \times \mathbb{R}$,

$$\bar{\vartheta}(t,x) \leq z \iff \overline{w}^g(t,x,z) \leq 0.$$

(ii) The function $\overline{w}^g$ is locally Lipschitz continuous on $[0,T] \times \mathbb{R}^d \times \mathbb{R}$, more precisely there exists $C \geq 0$ such that

$$|\overline{w}^g(t',\hat{x}') - \overline{w}^g(t,\hat{x})| \leq C(1+|\hat{x}|) \left(|t'-t| + |\hat{x}'-\hat{x}|\right)$$

In particular, $\overline{w}^g$ has a linear growth : $|\overline{w}^g(t,\hat{x})| \leq C(1+|\hat{x}|)$ for some constant $C \geq 0$.
(iii) Furthermore, $u = \overline{w}^g$ is the unique continuous viscosity solution of the following HJ equation:

$$\min(-\partial_t u + \overline{\mathcal{H}}(t,(x,z),\nabla u), \ u - g(x)) = 0, \quad t \in (0,T), \ x \in \mathbb{R}^d, \ z \in \mathbb{R}, \tag{5.5a}$$
$$u(T,x,z) = \max(\varphi(x) - z, g(x)), \quad x \in \mathbb{R}^d. \tag{5.5b}$$

where $\overline{\mathcal{H}}(t,(x,z),(p,q)) := \max\limits_{\alpha \in \mathcal{A}} \min\limits_{\beta \in \mathcal{B}} -\hat{f}(t,x,\alpha,\beta) \cdot (p,q) \equiv \max\limits_{\alpha \in \mathcal{A}} \min\limits_{\beta \in \mathcal{B}} -f(t,x,\alpha,\beta) \cdot p + \ell(t,x,\alpha,\beta)q.$

Corollary 5.3. *Under $(A'_1) - (A'_4)$, the function $\bar{\vartheta}$ is characterized by means of the function $\overline{w}^g$ through the following relation*

$$\bar{\vartheta}(t,x) = \inf \left\{ z \in \mathbb{R}, \ \overline{w}^g(t,x,z) \leq 0 \right\}. \tag{5.6}$$

This gives again a characterization of $\bar{\vartheta}$ for two-player games with state constraints, by using a continuous viscosity approach and without any controllability assumption.

6 Numerical example

We consider the classical zermelo type problem. A boat with coordinates $y(t) = (y_1(t), y_2(t))$ navigates in a canal $\mathbb{R} \times [-2,2]$, starting from $y(0) = x = (x_1, x_2)$, and wants to reach an island $B = \bar{B}(0, r_0)$, $r_0 > 0$, with minimal fuel consumption. The dynamics is given by

$$\dot{y}_1 = v \cos(u) + c - a y_2^2, \tag{6.1a}$$
$$\dot{y}_2 = v \sin(u), \tag{6.1b}$$

where $u \in [0, 2\pi]$ is the first control (angle), $v \in [0, V_{max}]$ is a second control (the speed of the boat), and $c - a y_2^2$ is the current drift (along the x_1-axis). We shall choose the parameters $V_{max} = 1$ and $c = 2$, $a = 0.5$. The boundary $y_2 = \pm\sqrt{\frac{c}{a}} \equiv \pm 2$ (where the drift term $c - a y_1^2$ vanishes) corresponds to the coast. The evolution of the fuel mass $m(t)$ is given by

$$\dot{m} = -b\frac{v}{V_{\max}}, \tag{6.2}$$

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where $b = 1$ and $\frac{v}{V_{\max}}$ is a speed ratio. Therefore, the fuel consumption $\int_t^T |\dot{m}(s)|\,ds$ is proportional to

$$\int_t^T v(s)\,ds.$$

Because of the drift term (which can be greater than $V_{\max}$), the system is not controllable. Consider the set of constraints given by $\mathcal{K} := \{x \in \mathbb{R}^2,\ g(x) \leq 0\}$ where

$$g(x) := \max\left(r_a - \|x - a\|_\infty),\ r_b - \max(|x_1 - b_1|, \frac{1}{5}|x_2 - b_2|)\right) \tag{6.3}$$

and where $r_a = 0.4$, $a = (-0.5, 0.5)$ and $r_b = 0.2$, $b = (-1, -1.5)$. The target $\mathcal{C} \equiv B(c, r_0)$ with $r_0 = 0.25$ and $c = (1.5, 0)$ is represented by a function ψ defined by

$$\psi(x) := \|x - c\| - r_0.$$

The optimal control problem is to minimize, whenever this is possible, the following

$$\vartheta(t, x) = \inf\left\{\int_t^T v(s)\,ds,\ \alpha = (u, v) \in \mathbb{A},\ y_x^\alpha(T) \in \mathcal{C},\ \left(y_x^\alpha(\theta) \in \mathcal{K},\ \forall \theta \in [t, T]\right)\right\}$$

where $\mathbb{A}$ is the set of measurable controls $\alpha = (u, v) : (0, T) \to [0, 2\pi] \times [0, 1]$. Therefore we are in the situation of Section 3.7. We consider $\ell(t, x, (u, v)) := v$, a terminal cost $\varphi(x) \equiv 0$, and the PDE becomes

$$\min\left(-w_t + \max(0,\ \|\nabla_x w\| + w_z) - (c - ay_2^2)w_{y_1},\ w(t, x, z) - g(x)\right) = 0,$$

$$t \in (0, T),\ (x, z) \in \mathbb{R}^2 \times \mathbb{R}, \tag{6.4}$$

$$w(T, x, z) = \max(-z,\ \psi(x),\ g(x)),\quad (x, z) \in \mathbb{R}^2 \times \mathbb{R}. \tag{6.5}$$

This last HJ equation is then solved by a finite difference method (ENO scheme of second order in space, see for instance [14]). Results are shown in Figure 1, at time $t = 0$ with $T = 10$ (computations done with 70^3 grid point). The value is recovered finally using $\vartheta(t, x) = \min\left\{z \in \mathbb{R},\ w(t, x, z) \leq 0\right\}$.

We have considered a discretisation of 70^3 spatial mesh points of the domain $[-3, 2] \times [-2, 2] \times [Z_{min}, Z_{max}]$ where $Z_{min} = -0.1$ and $Z_{max} = 10$. The time interval is $[0, T] \equiv [0, 10]$. This example was solved by using the C++ HJB-solver "Binope-HJ" [16].

A   Comparison principle for HJ equations with obstacle terms

The aim of this section is to prove a comparison principle for the following HJ equation in presence of an obstacle term:

$$\min(-u_t + H(t, x, \nabla u), u - g(t, x)) = 0 \quad \text{on } (0, T) \times \mathbb{R}^d, \tag{A.1}$$

$$u(T, x) = u_0(x),\quad x \in \mathbb{R}^d \tag{A.2}$$

where $T > 0$ and $g \in C((0, T) \times \mathbb{R}^d)$, and $H : (0, T) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is continuous and assumed to satisfy:
   (**H1**) there exists $C \geq 0$ such that, for all (t, x) in $(0, T) \times \mathbb{R}^d$, $p, q \in \mathbb{R}^d$,

$$|H(t, x, p) - H(t, x, q)| \leq C(|x| + 1)|p - q|, \tag{A.3}$$

   (**H2**) for any $R > 0$, there exists a function $w_R : [0, \infty) \to [0, \infty)$, $\lim_{r \to 0+} w_R(r) = 0$ and

$$|H(t, x, p) - H(t, y, p)| \leq w_R((1 + |p|)|x - y|) \tag{A.4}$$

for every $(t, p) \in (0, T) \times \mathbb{R}^d$, $x, y \in B_R$, where B_R denotes the open ball centred at 0 and of radius R.
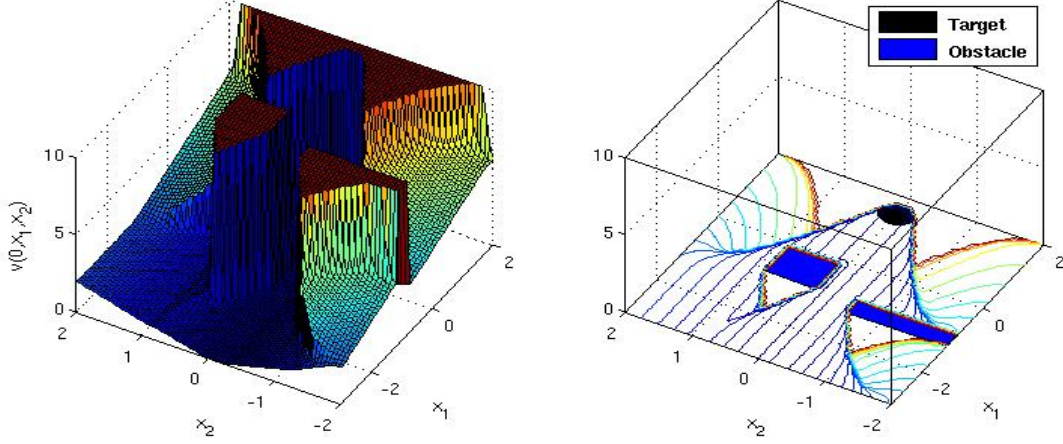
Figure 1: Left: values of $v(t,.)$ at time $t = 0$. Right: isovalues in logarithmic scale. Here the values such that $\vartheta(0, x, y) = 10$ correspond to a non-reachable zone

Theorem A.1. *Let u, v be two functions of $C([0,T] \times \mathbb{R}^d)$, and let g, h be in $C([0,T] \times \mathbb{R}^d)$. We assume that u (resp. v) is a subsolution (resp. supersolution) of* (A.1) *in $(0,T) \times \mathbb{R}^d$:*

$$\min(-u_t + H(t, x, \nabla u), u - g) \leq 0 \quad in \ (0,T) \times \mathbb{R}^d, \tag{A.5}$$

$$\min(-v_t + H(t, x, \nabla v), v - h) \geq 0 \quad in \ (0,T) \times \mathbb{R}^d. \tag{A.6}$$

We denote $u_T(x) := u(T, x)$ and $v_T(x) := v(T, x)$. Then for all $t \in [0, T]$,

$$\sup_{\mathbb{R}^d}(u(t,.) - v(t,.)) \leq \max\left(\sup_{\mathbb{R}^d}(u_T - v_T), \sup_{(t,T) \times \mathbb{R}^d}(g - h)\right). \tag{A.7}$$

Proof. The result without the obstacle term can be found for instance in Ishii [29]. It suffices to prove the result for $T > 0$ small enough, the result for any $T > 0$ can then be deduced by immediate recursion.

Assuming that $C > 0$, we take $T = 1/(2C)$ and $L = L(x_0) := 2C(|x_0| + 1)$, and we define the following cone

$$\mathcal{O}_{x_0} := \left\{(t, x) \in (0, T) \times \mathbb{R}^d, \ |x - x_0| < Lt\right\}.$$

(The case when $C = 0$ is trivial). We claim that for every $t_0 \in (0, T)$:

$$u(t_0, x_0) - v(t_0, x_0) \leq \max\left(\sup_{x_0 + B_{L(T-t_0)}}(u_T - v_T), \sup_{\mathcal{O}_{x_0}}(g - h)\right), \tag{A.8}$$

which concludes (A.7). Let us fix $t_0 \in (0, T)$ and prove our claim (A.8).

First, remark that for any $(t, x) \in \mathcal{O}_{x_0}$, the following holds: $C(|x| + 1) \leq C(|x_0| + 1) + C|x - x_0| \leq \frac{L}{2} + CLT \leq \frac{L}{2} + \frac{1}{2}L = L$, and thus

$$|H(t, x, p) - H(t, x, q)| \leq L|p - q| \qquad (t, x) \in \mathcal{O}_{x_0}. \tag{A.9}$$

We also define for any $(\bar{t}, \bar{x}) \in \Omega_{x_0}$ and $\tau \in (\bar{t}, T)$ the following cone:

$$\mathcal{O}_{\bar{t}, \bar{x}, \tau} := \left\{(t, x) \in (\bar{t}, \tau) \times \mathbb{R}^d, \ |x - x_0| < L(t - \bar{t})\right\}.$$

According to Crandall-Lions [23], and Ishii [29], the following Lemma holds.

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Lemma A.2. *If u, v belongs to $C(\overline{\mathcal{O}}_{\bar{t}, \bar{x}, t})$ with $t \in (\bar{t}, T)$, and are respectively viscosity solutions of*

$$-u_t + H(t, x, \nabla u) \leq 0 \quad in\ \mathcal{O}_{\bar{t}, \bar{x}, t}, \tag{A.10}$$
$$-v_t + H(t, x, \nabla v) \geq 0 \quad in\ \mathcal{O}_{\bar{t}, \bar{x}, t}, \tag{A.11}$$

then

$$u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{x}) \leq \sup_{\bar{x} + B_{L(t - \bar{t})}} (u(t, .) - v(t, .)).$$

Consider the set:

$$\Sigma := \{(t, x) \in \overline{\mathcal{O}}_{x_0}, \ u(t, x) \leq g(x)\},$$

and its complementary in $\mathcal{O}_{x_0}$:

$$\Omega := \mathcal{O}_{x_0} \backslash \Sigma.$$

Since u is a subsolution of (A.5), u is also a subsolution of $u_t + H(t, x, \nabla u) = 0$ on the open set Ω. Furthermore v being a supersolution of (A.6) on $\mathcal{O}_{x_0}$, it is also a supersolution of $v_t + H(t, x, \nabla v) = 0$ on the open set Ω.

On the other hand, from (A.6), it follows that $v(t, x) \geq g(x)$ everywhere. Hence

$$\forall (t, x) \in \Sigma, \quad u(t, x) - v(t, x) \leq g(t, x) - h(t, x) \leq \sup_{\mathcal{O}_{x_0}} (g - h). \tag{A.12}$$

Now, assume that

$$u(t_0, x_0) - v(t_0, x_0) > M := \max \left(\sup_{x_0 + B_{L(T - t_0)}} (u_0 - v_0), \sup_{\mathcal{O}_{x_0}} (g - h) \right). \tag{A.13}$$

Using the continuity of $u - v$ in (t_0, x_0), there exists a neighborhood $\Upsilon \subset \mathcal{O}_{x_0}$ of (t_0, x_0) satisfying:

$$u(t, x) - v(t, x) > M \geq \sup_{\mathcal{O}_{x_0}} (g - h) \quad \forall (x, t) \in \Upsilon.$$

Taking into account (A.12), it follows that Υ is necessarily included in Ω. Hence there exists $\tau > t_0$ such that the cone $\mathcal{O}_{t_0, x_0, \tau}$ is also included in Ω. Set

$$t_1 := \sup \left\{ \tau \in (t_0, T], \ \mathcal{O}_{t_0, x_0, \tau} \bigcap \Sigma = \emptyset \right\}.$$

($\mathcal{O}_{t_0, x_0, t_1}$ is the greatest cone $\mathcal{O}_{t_0, x_0, \tau}$ such that $\mathcal{O}_{t_0, x_0, \tau} \subset \Omega$.) By using Lemma A.2 in the cone $\mathcal{O}_{t_0, x_0, t_1}$, we obtain:

$$u(t_0, x_0) - v(t_0, x_0) \leq \sup_{x_0 + B_{L(t_1 - t_0)}} (u(t_1, .) - v(t_1, .)).$$

If $t_1 = T$, then $u(t_0, x_0) - v(t_0, x_0) \leq \sup_{x_0 + B_{LT}} (u_T - v_T) \leq M$, which contradicts (A.13). Hence $t_1 < T$. Let $x_1 \in x_0 + \overline{B}_{L(t_1 - t_0)}$ be a maximum of $u(t_1, \cdot) - v(t_1, \cdot)$ on $x_0 + \overline{B}_{L(t_1 - t_0)}$. Then,

$$M < u(t_0, x_0) - v(t_0, x_0) \leq u(t_1, x_1) - v(t_1, x_1).$$

We re-iterate the previous argument and obtain the existence of a t_2 in (t_1, T) corresponding to the greatest cone of the form $\mathcal{O}_{t_1, x_1, t_2}$ and satisfying $\mathcal{O}_{t_1, x_1, t_2} \subset \Omega$, and then the existence of a point x_2 in $x_1 + \overline{B}_{L(t_2 - t_1)} \subset x_0 + \overline{B}_{L(t_2 - t_0)}$ such that

$$M < u(t_1, x_1) - v(t_1, x_1) \leq u(t_2, x_2) - v(t_2, x_2),$$

and so on. Therefore we construct an increasing sequence of times (t_k), and a sequence of points (x_k) s.t. all the sequence (t_k, x_k) belongs to the cone $\mathcal{O}_{x_0}$. Because $\overline{\mathcal{O}}_{x_0}$ is a compact set, we can extract a convergent subsequence towards an element (t^*, x^*) of $\overline{\mathcal{O}}_{x_0}$. Moreover,

$$M < u(t^*, x^*) - v(t^*, x^*).$$

If $t^* = T$, we obtain a contradiction. Hence $t^* < T$. Now by continuity, we still must have $u(t,x) - v(t,x) > M$ in a neighborhood of (t^*, x^*), for instance in a tube $\mathcal{C}$ centered at (t^*, x^*) and defined by

$$\mathcal{C} :=]t^* - \tau_0, t^* + \tau_0[\times (x^* + B_{L\tau_0}),$$

for a $\tau_0 > 0$ sufficiently small. In particular, for any $x \in \mathcal{C}$, $(t,x) \notin \Gamma$. On the other hand, as soon as $t^* - t_k < \tau_0$, we have $\mathcal{O}_{t_k,x_k,t_{k+1}} \subset \mathcal{O}_{t_k,x_k,t^*} \subset \mathcal{C} \subset \mathbb{R}^d \backslash \Sigma \equiv \Omega$. This contradicts the fact that t_{k+1} is the maximal time τ such that $\mathcal{O}_{t_k,x_k,\tau} \subset \Omega$, and the proof of (A.8) is complete. $\qquad\square$

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