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Multifractional stochastic volatility

Sylvain CORLAY[†]

Joachim LÉBOVITS^{‡§}

Jacques LÉVY-VÉHEL[§]

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Abstract

The aim of this work is to advocate the use of multifractional Brownian motion (mBm) as a relevant model in financial mathematics. Multifractional Brownian motion is an extension of fractional Brownian motion where the Hurst parameter is allowed to vary in time. This enables the possibility to accommodate for non-stationary local regularity, and to decouple it from long-range dependence properties. While we believe that mBm is potentially useful in a variety of settings, we focus here on a multifractional stochastic volatility Hull & White model that is an extension of the model studied in [10]. Using the stochastic calculus with respect to mBm developed in [25], we solve the corresponding stochastic differential equations. Since the solutions are of course not explicit, we take advantage of recently developed numerical techniques, namely functional quantization-based cubature methods, to get accurate approximations. This allows us to test the behavior of our model (as well as the one in [10]) with respect to its parameters, and in particular its ability to explain the smile effect of implied volatility. An advantage of our model is that it is able to both fit smiles at different maturities, and to take into account volatility persistence in a more precise way than in [10].

Keywords: Hull & White model, functional quantization, vector quantization, Karhunen-Loève, Gaussian process, fractional Brownian motion, multifractional Brownian motion, white noise theory, S-transform, Wick-Itô integral, stochastic differential equations.

[†]Bloomberg L.P. Quantitative Finance Research, 731 Lexington avenue, New York, NY 10022, USA.

[‡]Laboratoire de Probabilités et Modèles Aléatoires, C.N.R.S. (UMR 7599), Université Pierre et Marie Curie (Paris VI), case 188, 4 place Jussieu, F-75252 Paris Cedex 5, France.

[§]Regularity team, I.N.R.I.A. Saclay, Parc Orsay Université, 4 rue Jacques Monod - Bat. P - 91893 Orsay Cedex, France.

Introduction

Volatility in financial markets is both of crucial importance and hard to model in an accurate way. It has been long known that a constant volatility as in the Black & Scholes model is not consistent with empirical findings, such as the smile effect. More basically, there is no reason to expect that instantaneous volatility should be constant. Popular models allowing for a varying volatility include ARCH models and their generalizations, stochastic volatility models and local volatility models. Local volatility models, in particular, enable the possibility to mimic in an exact way implied volatility surfaces. However, such models do not take into account another well documented fact: while stocks do not typically exhibit correlations, volatility does display long-range correlations (see, *e.g.* [1]). Stochastic volatility models, on contrast, are able to incorporate this feature, provided an adequate driving noise is used. In [10, 9], this is performed by using fractional integration. More precisely, the model considered in [10] for the dynamic of the price of a risky asset reads as follows:

$$\begin{cases} dS_t = \mu(t, S_t)dt + S_t\sigma_t dW_t, \\ d\ln(\sigma_t) = \theta(\mu - \ln(\sigma_t))dt + \gamma dB_t^H, \quad \sigma > 0, \end{cases} \quad (1)$$

where W is a Brownian motion and B_t^H is an independent fractional Brownian motion (fBm) under the historical probability. Fractional Brownian motion is an extension of Brownian motion, parameterized by a real H in $(0, 1)$, whose increments display long-range dependence for $H > 1/2$ and constant local Hölder regularity equal to H . Such a model is consistent with the slow decay in the correlations of volatility observed in practice. It also accounts for two features related to the measured smile effect: the volatility process is less persistent in the short term than a standard diffusion, while it is more persistent in the long run ([9, Page 3]). We verify this fact in the case of Model (1) numerically in Section 6. However, by the very nature of this model, the evolution in time of the smile is governed by the single parameter H .

In this work, we replace fBm appearing in (1) with a more general process called multifractional Brownian motion (mBm). Multifractional Brownian motion is an extension of fBm where the Hurst parameter H is replaced by a function h . This enables the possibility to accommodate for non-stationary local regularity, and to decouple it from long-range dependence properties. Indeed, there is no reason to believe that the regularity of the volatility should be constant. In addition, graphs of estimated historical volatility (see, *e.g.* [1]) seem to indicate that this process is highly irregular. Modeling this evolution with the help of an fBm would thus require to choose a “small” H , *i.e.* $H < 1/2$, which is not compatible with long-range dependence properties. In contrast, mBm has at each time t local regularity $h(t)$, and, no matter the value of h in $(0, 1)$, always display long-range dependence as long h is not constant. Moreover, we directly place ourselves in the risk-neutral setting. In addition, as we will show from numerical experiments, the model:

$$\begin{cases} dF_t = F_t\sigma_t dW_t, \\ d\ln(\sigma_t) = \theta(\mu - \ln(\sigma_t))dt + \gamma_h d^\circ B_t^h + \gamma_\sigma dW_t^\sigma, \quad \sigma_0 > 0, \\ d\langle W, W^\sigma \rangle_t = \rho dt, \end{cases} \quad (2)$$

where B_t^h is an mBm, yields shapes of the smile at maturity T that are governed by a weighted average of the values of the function h up to time T : thus, by adequately choosing h , one may mimic an implied volatility surface more faithfully than with a Hull & White model driven by a fBm.

While we believe that mBm is potentially useful in a variety of settings, we focus here on the multifractional stochastic volatility (2). However, we also briefly consider a multifractional SABR model (with $\beta = 1$) in Section 5.

In order to assess the relevance of our model, we compute numerically the smiles at different maturities. Since the solution cannot be written in an explicit form, we need to resort to approximations. In our case, this is made possible by recent advances in the theory of functional quantization of Gaussian processes.

Functional quantization of Gaussian processes has become an active field of research in recent years since the seminal article by Luschgy and Pagès [28]. As far as applications are concerned, cubature methods [33, 12] and variance reduction methods [13, 26] based on functional quantization have been proposed. However, as the numerical use of functional quantizers requires the evaluation of the Karhunen-Loève eigenfunctions, this method was restricted to processes for which a closed-form expression for this expansion is known, such

as Brownian motion. In [11], a numerical method was proposed to perform numerical quadratic functional quantization of more general Gaussian processes, which will be applied here to multifractional Brownian motion.

We show that we can handle a fast and accurate forward start option pricing in this model thanks to a functional quantization-based cubature method similar to the one proposed by Pagès and Printems in [33] and in [12]. This allows us to study the dependency of the smile dynamics on the functional parameter of the considered mBm.

The remaining of this paper is organized as follows. We recall in Section 1 basic facts about mBm. In Section 2, we explain how to perform functional quantization of mBm and investigate the rate of decay of the corresponding quantization error. Quantization-based cubature is addressed in Section 2.3. Fractional Brownian motion is not a semimartingale, and thus classical Itô integration theory does not apply. At the time [10] was written, no theory for integration with respect to fBm was available yet. Various approaches to stochastic integration with respect to fBm have been developed since. Among these, the one based on white noise theory is well fitted for an extension to mBm. In particular, it allows to deal with any $H \in (0, 1)$ and to obtain Itô formulas. A white noise-based stochastic integral with respect to mBm was developed in [25]. This is the theory we will use in order to define precisely our stochastic volatility models. This is recalled in Section 3, where we also show how to solve some stochastic differential equations (S.D.E.) in this setting. General remarks on the quantization of solutions of S.D.E. are presented in Section 3.4. A detailed treatment of the Hull & White and SABR models are proposed in Sections 4 and 5. Numerical experiments and conclusions are gathered in Section 6.

1 The multifractional Brownian motion

Fractional Brownian motion (fBm) is a centered Gaussian process with features that make it a useful model in various applications such as financial and teletraffic modeling, image analysis and synthesis, geophysics and more. These features include self-similarity, long-range dependence and the ability to match any prescribed constant local regularity. Fractional Brownian motion is a Gaussian process which depends on a parameter, usually denoted by H and called the Hurst exponent, that belongs to $(0, 1)$. Its covariance function R_H reads, for every (t, s) in \mathbb{R}^2 ,

$$R_H(t, s) := \frac{\gamma_H}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$

where γ_H is a positive constant. A normalized fBm is one for which $\gamma_H = 1$. Obviously, when $H = \frac{1}{2}$, fBm reduces to standard Brownian motion. While fBm is a useful model, the fact that most of its properties are governed by the single number H restricts its application in some situations. In particular, its Hölder exponent remains the same all along its trajectory. Thus, for instance, long-range dependent fBm, which require $H > \frac{1}{2}$ must have smoother paths than Brownian motion. Multifractional Brownian motion [36, 3] was introduced to overcome these limitations. The basic idea is to replace the real H by a function $t \mapsto h(t)$ ranging in $(0, 1)$.

In order to allow the local regularity varying along the sample paths, it is convenient to consider the notion of fractional Brownian field. Let (Ω, \mathcal{F}, P) be a fixed probability space. We call fractional Brownian field on $\mathbb{R} \times (0, 1)$ every Gaussian field, denoted by $(\mathbf{B}(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$, such that for every H in $(0, 1)$ the process $(B_t^H)_{t \in \mathbb{R}}$ defined by $B_t^H := \mathbf{B}(t, H)$ for every t in \mathbb{R} is a fractional Brownian motion of Hurst parameter H . For every deterministic continuous function $h : \mathbb{R} \rightarrow (0, 1)$ we call multifractional Brownian motion (mBm) of functional parameter h the Gaussian process $B^h := (B_t^h)_{t \in \mathbb{R}}$ defined by $B_t^h := \mathbf{B}(t, h(t))$ for every t in \mathbb{R} . We say that h is the *regularity function* of the mBm. We will say that the fractional field $(\mathbf{B}(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$ is normalized when, for all H in $(0, 1)$, the fractional Brownian motion $(B_t^H)_{t \in \mathbb{R}}$, resulting from the fractional field $(\mathbf{B}(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$, is a normalized fBm. In this case we will also say that the mBm $B^h := (B_t^h)_{t \in \mathbb{R}}$ is normalized.

This definition of the mBm generalizes all the definitions of multifractional Brownian motions given up to now (from [36] in 1995 to [38, Definition 1.1] in 2006). The main interest of choosing a normalized mBm lies in the fact that, for every fixed H in $(0, 1)$, when we take the function h identically equal to H , we obtain a normalized fBm of Hurst index H .

Definition 1 (Well-balanced fractional Brownian field). *The fractional Brownian field defined on $\mathbb{R} \times (0, 1)$ by $X(t, H) := \frac{1}{c_H} \int_{\mathbb{R}} \frac{e^{itu} - 1}{|u|^{H+1/2}} \widetilde{W}(du)$ where \widetilde{W} denotes a complex-valued Gaussian measure (cf. [38] for more details) is called the well-balanced fractional Brownian field. For any function $h : \mathbb{R} \rightarrow \mathbb{R}$, the well-balanced mBm of functional parameter h is the process $B^h := (B_t^h)_{t \in \mathbb{R}}$ defined by $B_t^h := X(t, h(t))$ for every t in \mathbb{R} .*

The proof of the following proposition can be found in [2].

Proposition 1.1 (Covariance function of the well-balanced mBm). *Consider a function $h : \mathbb{R} \rightarrow (0, 1)$ and a mBm $B^h := (B_t^h)_{t \in \mathbb{R}}$ of functional parameter h . Its covariance function denoted by R_h , is given, for every (s, t) in \mathbb{R}^2 by*

$$R_h(t, s) = \frac{c_{h(t,s)}^2}{c_{h(t)}c_{h(s)}} \left(\frac{1}{2} \left(|t|^{2h_{t,s}} + |s|^{2h_{t,s}} - |t-s|^{2h_{t,s}} \right) \right), \quad (3)$$

where $h_{t,s} := \frac{h(t)+h(s)}{2}$ for every (s, t) in \mathbb{R}^2 and $c_x := \left(\frac{2\pi}{\Gamma(2x+1)\sin(\pi x)} \right)^{\frac{1}{2}}$ for all x in $(0, 1)$.

In general, the increments of multifractional Brownian motion are neither independent nor stationary (in fact, one can prove that, as soon as $t \mapsto h(t)$ is not constant, the increments are not stationary). Since any mBm B^h is a fBm of Hurst index H when h is constant, equal to H , there is no likelihood of confusion if we denote by B^H the fractional Brownian motion of Hurst index H , what we will do in the sequel.

In the following, we will restrict to fractional Brownian fields which satisfy the Hölder regularity hypothesis (\mathcal{H}). In this situation, if the functional parameter h is Hölder continuous, the associated mBm B^h will have a continuous modification.

$$\forall T \in \mathbb{R}_+^*, \forall [c, d] \subset (0, 1), \exists (\Lambda, \delta) \in (\mathbb{R}_+^*)^2 \text{ such that} \quad (\mathcal{H})$$

$$\mathbb{E} [(B(t, H) - B(t, H'))^2] \leq \Lambda |H - H'|^\delta \text{ for every } (t, H, H') \text{ in } [0, T] \times [c, d]^2.$$

We show in Proposition 2.5 that this assumption is satisfied by the well-balanced fractional Brownian field. The other main properties of mBm are as follows: the pointwise Hölder exponent at any point t of $B^{(h)}$ is almost surely equal to $h(t) \wedge \beta_h(t)$, where $\beta_h(t)$ is the pointwise Hölder exponent of h at t [18, 20]. For a smooth h , one thus may control the local regularity of the paths by the value of h . In addition, the increments of mBm display long-range dependence for all non-constant function h (see [2]). Finally, when h is C^1 , mBm is tangent to fBm with exponent $h(u)$ in the neighborhood of any u in the following sense [19]: for every real u ,

$$\left\{ \frac{B_{u+rt}^h - B_u^h}{r^{h(u)}}; t \in [a, b] \right\} \xrightarrow{r \rightarrow 0^+} \text{law} \{ B_t^{h(u)}; t \in [a, b] \}.$$

These properties show that mBm is a more versatile model than fBm: in particular, it is able to mimic in a more faithful way local properties of financial records, Internet traffic and natural landscapes [7, 27, 15] by matching their local regularity. This is important *e.g.* for purposes of detection or real-time control. The price to pay is of course that one has to deal with the added complexity brought by having a functional parameter instead of a single number.

In general, the increments of multifractional Brownian motion are neither independent nor stationary. Since an mBm B^h is an fBm of Hurst index H when h is constant and equal to H , there is no risk of confusion by denoting B^H the fractional Brownian motion with Hurst index H .

2 Functional quantization of multifractional Brownian motion

2.1 Computation of the quantization

The quantization of a random variable X valued in a reflexive separable Banach space $(E, |\cdot|)$ consists in its approximation by a random variable Y that is measurable with respect to X and that takes finitely many values in E . The resulting error of the discretization is usually measured by the L^p norm of the difference

$|X - Y|$. If we settle on a fixed maximum cardinal N for $Y(\Omega)$, the minimization of the error reduces to the following optimization problem:

$$\min \left\{ \| |X - Y| \|_p, Y : \Omega \rightarrow E \text{ measurable with respect to } X, \text{card}(Y(\Omega)) \leq N \right\}. \quad (4)$$

As Y is supposed to be measurable with respect to X , there exists a Borel map $\text{Proj} : E \rightarrow E$ valued in a finite subset Γ of E such that $Y = \text{Proj}(X)$. The finite subset Γ is called the codebook. Hence if $\Gamma = \{\gamma_1, \dots, \gamma_N\}$, there exists a Borel partition $C = \{C_1, \dots, C_N\}$ of E such that $\text{Proj} = \sum_{i=1}^N \gamma_i \mathbf{1}_{C_i}$. In other words, Proj performs the process of mapping the continuous set $X(\Omega)$ to the finite set Γ . Let Proj_Γ denote a nearest neighbor projection on Γ . Clearly,

$$|X - \text{Proj}_\Gamma(X)| \leq |X - \text{Proj}(X)| \quad \text{so that} \quad \| |X - \text{Proj}_\Gamma(X)| \|_p \leq \| |X - \text{Proj}(X)| \|_p.$$

Hence, in order to minimize the quantization error, it is optimal to use a nearest neighbor projection on the codebook Γ . A solution of (4) is called an L^p -optimal quantizer of X . An elementary property of an L^2 -optimal quantizer is stationarity property $\mathbb{E}[X|Y] = Y$. We denote by $\mathcal{E}_{N,p}(X, |\cdot|)$ the minimal L^p quantization error for the random variable X and the norm $|\cdot|$.

$$\mathcal{E}_{N,p}(X, |\cdot|) = \min \left\{ \| |X - Y| \|_p, Y \text{ measurable with respect to } X \text{ and } |Y(\Omega)| \leq N \right\}$$

We now assume that X is a bi-measurable stochastic process on $[0, T]$ verifying $\int_0^T \mathbb{E}[|X_t|^2] dt < \infty$ so that it can be considered as a random variable valued in the Hilbert space $H = L^2([0, T])$. We assume that its covariance function Γ^X is continuous. In [28], it is shown that, in the centered Gaussian case, linear subspaces U of H spanned by N -stationary quantizers correspond to principal components of X , in other words, are spanned by eigenvectors of the covariance operator of X . Thus, the quadratic optimal quantization of Gaussian processes consists in using its Karhunen-Loève decomposition $(e_n^X, \lambda_n^X)_{n \geq 1}$.

To perform optimal quantization, the Karhunen-Loève expansion is first truncated at a fixed order m and then the \mathbb{R}^m -valued Gaussian vector constituted of the m first coordinates of the process on its Karhunen-Loève decomposition is quantized. To reach optimal quantization, we have to determine both the optimal rank of truncation $d^X(N)$ (the quantization dimension) and the optimal $d^X(N)$ -dimensional Gaussian quantizer corresponding to the first coordinates, $\bigotimes_{j=1}^{d^X(N)} \mathcal{N}(0, \lambda_j^X)$. We have the following representation of the quadratic distortion $\mathcal{E}_N(X) := \mathcal{E}_{N,2}(X, |\cdot|_{L^2([0,T])})$:

$$\mathcal{E}_N(X)^2 = \sum_{j \geq m+1} \lambda_j^X + \mathcal{E}_N \left(\bigotimes_{j=1}^m \mathcal{N}(0, \lambda_j^X) \right)^2.$$

From a numerical viewpoint, we are thus confronted on one hand with the finite-dimensional quantization of the Gaussian distribution $\bigotimes_{j=1}^m \mathcal{N}(0, \lambda_j^X)$ and, on the other hand, with the numerical evaluation of the first Karhunen-Loève eigenfunctions $(e_n^X)_{1 \leq n \leq d^X(N)}$. Concerning the first point, various numerical algorithms have been developed. Let us mention Lloyd's algorithm and the Competitive Learning Vector Quantization (CLVQ). A review of these methods is available in [32]. As concerns the evaluation of the first Karhunen-Loève eigenfunctions, closed-expressions are available for the standard Brownian motion, the standard Brownian bridge and the Ornstein-Uhlenbeck process. Other examples of explicit Karhunen-Loève expansions are available in [14] and [37]. In the general case, the so-called Nyström method for approximating the solution of the associated integral equation may be used. It reads

$$\int_0^T \Gamma^X(\cdot, s) e_k^X(s) ds = \lambda_k^X e_k^X, \quad k \geq 1, \quad (5)$$

where both the eigenvalues and the eigenvectors have to be determined. The Nyström method relies on the use of a quadrature scheme to approximate the integral, so that it comes to a matrix eigensystem. When

dealing with the midpoint quadrature rule, and for sufficiently regular kernels Γ^X , the error admits an asymptotic expansion in the form of the sum of even powers of the step size, for both the eigenvalues and the eigenfunctions. We take advantage of this asymptotic expansion by using Richardson-Romberg extrapolation methods. This method has been benchmarked with the Karhunen-Loève eigensystem of standard Brownian motion, Brownian bridge and the Ornstein-Uhlenbeck process in [11].

Instead of using an optimal quantization for the distribution $\bigotimes_{j=1}^{d^X(N)} \mathcal{N}(0, \lambda_j^X)$, another possibility is to use a product quantization, that is to use the Cartesian product of the optimal quadratic quantizers of the standard one-dimensional Gaussian distributions $\mathcal{N}(0, \lambda_j^X)_{1 \leq j \leq d^X(N)}$. In the case of independent marginals, this yields a stationary quantizer, *i.e.* a quantizer Y of X which satisfies $\mathbb{E}[X|Y] = Y$. This property, shared with optimal quantizers, results in a convergence rate of a higher order for the quantization-based cubature scheme, as explained in [33]. An advantage of this approach is that the one-dimensional Gaussian quantization is a fast procedure.

In [32], deterministic optimization methods (*e.g.* Newton-Raphson) are shown to converge rapidly to the unique optimal quantizer of the one-dimensional Gaussian distribution. Moreover, a sharply optimized database of quantizers of standard univariate and multivariate Gaussian distributions is available at www.quantize.maths-fi.com [34] for download. Still, we have to determine the quantization level for each dimension to obtain optimal product quantization. In this case, the minimization of the distortion becomes:

$$\left(\mathcal{E}_N^{\text{prod}}(X)\right)^2 := \min \left\{ \sum_{j=1}^d \mathcal{E}_{N_j}^2(\mathcal{N}(0, \lambda_j^X)) + \sum_{j \geq d+1} \lambda_j^X, N_1 \times \dots \times N_d \leq N, d \geq 1 \right\}. \quad (6)$$

A solution of (6) is called an optimal K-L product quantizer. This problem can be solved by the “blind optimization procedure”, which consists in computing the criterion for every possible decomposition $N_1 \times \dots \times N_d$ with $N_1 \geq \dots \geq N_d$. The result of this procedure can be stored for future use. Optimal decompositions for a wide range of values of N for both Brownian motion and Brownian bridge are available on the web site www.quantize.maths-fi.com [34]. Another fact on quadratic functional product quantization is that it is shown to be rate-optimal under certain assumptions on the K-L eigenvalues (see Theorem 2.1).

In Figures 1 and 2, we display quadratic product quantizers of the fractional Brownian motion and the well-balanced multifractional Brownian motion with different values for the Hurst index H and functional parameters h . We settled for Hölder continuous functions h , so that the associated multifractional Brownian motion has a continuous modification and thus a continuous covariance function. In this setting, we can apply the results mentioned earlier in this section. The optimal product decomposition depends on h . Here, we settled for a fixed product decomposition for simplicity sake.

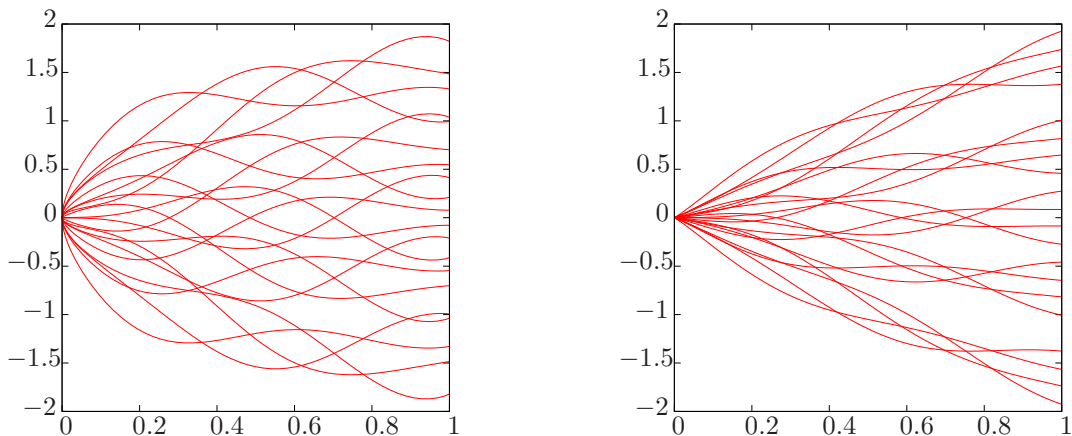


Figure 1: Quadratic $5 \times 2 \times 2$ -product quantizer of the fractional Brownian motion on $[0, 1]$ with Hurst index $H = 0.25$ on the left and with Hurst index $H = 0.75$ on the right.

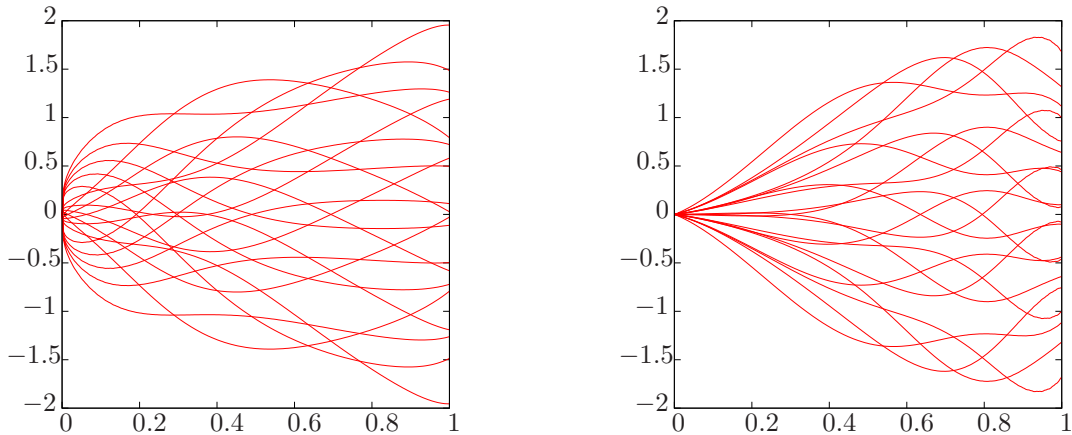


Figure 2: Quadratic $5 \times 2 \times 2$ -product quantizer of mBm on $[0, 1]$ with $h(t) := 0.1 + 0.8t$ on the left and with $h(t) := 0.9 - 0.8t$ on the right.

These graphs reflect, to some extent, the features of the quantized process, in particular its correlation and regularity properties.

In the case of fBm (Figure 1), when H increases, the rate of decay of the Karhunen-Loève eigenvalues also increases (and so does the pathwise Hölder regularity of the underlying process), so that even though we do not change the quantization dimension in this example, the contribution of higher-order Karhunen-Loève eigenvalues decreases. In Figure 1, we can see that curves of the functional quantizer localize around the quantization of the first Karhunen-Loève coordinate when $H = 0.75$, while this is not the case when $H = 0.25$.

Also, the distribution of the curves on the plane is related to the fact that the almost sure Hausdorff dimension of the paths of fBm is $2-H$: for small H , we expect the set of curves to be more space-filling than for large H , a feature that can be indeed be verified on the figure. In addition, the long-term correlation of fBm for $H > 1/2$, which results in paths typically having strong trends, translates here into curves which are roughly monotonous. Conversely, the negative correlations which characterizes the case $H < 1/2$ is reflected in the more oscillatory behavior of the curves in the left pane.

The case of mBm (Figure 2) makes even clearer the relation between the properties of the process and the optimal quantizer. In the right pane of Figure 2, the function h decreases linearly from 0.9 to 0.1. We see that, for small t , both the distribution of the curves and their trend look like the ones of fBm with large H . As t increases, the curves become more space-filling and oscillatory, in agreement with the fact that, for t close to 1, the corresponding mBm has larger local Hausdorff dimension. Similar remarks hold for the case where h is an increasing function (right pane of Figure 2).

Another way of interpreting these figures is to recall that mBm is tangent, at each t , to fBm with exponent $H = h(t)$. The behavior of the curves on Figure 2 is a translation of this fact in the quantization domain.

Finally, note that the shape of the convex envelopes in each of the four figures roughly matches the time evolution of the variances of the corresponding processes, *i.e.* $t^{0.25}$, $t^{0.75}$, $t^{0.1+0.8t}$ and $t^{0.9-0.8t}$.

2.2 Rate of decay of the quantization error for mBm

The rate of decay of the quadratic functional quantization error was first investigated in [28]. More precise results were then established in [29]. These results rely on assumptions on the asymptotic behavior of the Karhunen-Loève eigenvalues of the considered process. In Section 2.2.1, we recall the main result involving the rate of decay of these eigenvalues, leading to sharp rates of convergence for the quantization of fBm.

Unfortunately, such asymptotics for the Karhunen-Loève eigenvalues are not known at this time in the case of mBm. However, since the regularity of mBm is known, one may use another, less precise, type of results: these yield an upper estimate on the rate of decay of the quantization error [31]. This is explained

in Section 2.2.2 where we recall the main result of [31] and prove that mBm fulfills the conditions required to apply this result.

In the following, if $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ are two positive sequences. We write $x_n \underset{n \rightarrow \infty}{\sim} y_n$ (strong equivalence) if $x_n - y_n = o(y_n)$, which is equivalent to $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = 1$. Moreover, we write $x_n \underset{n \rightarrow \infty}{\lesssim} y_n$ (strong domination) if $\lim_{n \rightarrow \infty} \frac{x_n}{y_n} \leq 1$. Finally, we write $x_n \underset{n \rightarrow \infty}{\asymp} y_n$ if $x_n = O(y_n)$ and $y_n = O(x_n)$ as $n \rightarrow \infty$.

2.2.1 Sharp rates based on asymptotics of Karhunen-Loève eigenvalues

Definition 2 (Regularly varying function at infinity). *A measurable function $\phi : (s, \infty) \rightarrow (0, \infty)$, ($s > 0$) is regularly varying at infinity with index $b \in \mathbb{R}$ if for every $t > 0$, $\lim_{x \rightarrow \infty} \frac{\phi(tx)}{\phi(x)} = t^b$.*

Let X be a bi-measurable centered Gaussian process on $[0, T]$ with a continuous covariance function Γ^X and such that $\int_0^T \mathbb{E}[X_s^2] ds < \infty$. Denote by $(e_n^X, \lambda_n^X)_{n \geq 1}$ its Karhunen-Loève eigensystem.

Theorem 2.1 (Quadratic quantization error asymptotics [29]). *Assume that $\lambda_n^X \sim \phi(n)$ as $n \rightarrow \infty$, where $\phi : (s, \infty) \rightarrow (0, \infty)$ is a decreasing regularly varying function of index $-b < -1$ and $s > 0$. Set $\psi(x) := \frac{1}{x\phi(x)}$. Then*

$$\mathcal{E}_N(X) \sim \left(\left(\frac{b}{2} \right)^{b-1} \frac{b}{b-1} \right)^{1/2} \psi(\log(N))^{-1/2} \quad \text{as } N \rightarrow \infty.$$

Moreover, the optimal product quantization dimension $m^X(N)$ verifies $m^X(N) \sim \frac{2}{b} \log(N)$ as $N \rightarrow \infty$, and the optimal product quantization error $\mathcal{E}_N^{\text{prod}}(X)$ of level N satisfies

$$\mathcal{E}_N^{\text{prod}}(X) \lesssim \left(\left(\frac{b}{2} \right)^{b-1} \frac{b}{b-1} + C(1) \right)^{1/2} \psi(\log(N))^{-1/2} \quad \text{as } N \rightarrow \infty,$$

where $C(1)$ is a universal positive constant.

Though the optimal product quantization is not asymptotically optimal, it still provides a rate-optimal sequence of quantizers. The proof of the theorem is available in [29], where the case $b = 1$ is also handled. In this last case, the optimal product quantization does yield an asymptotically optimal quadratic quantization error.

The case of fractional Brownian motion

In [28], it is shown that the Karhunen-Loève eigenvalues of the fractional Brownian motion on $[0, T]$, $(\lambda_n^H)_{n \geq 1}$ verify

$$\lambda_n^H \sim \frac{\nu_H}{n^{2H+1}} \quad \text{as } n \rightarrow \infty,$$

where ν_H is a positive constant (This has also been proved by Bronski in [8]). Thus, the fractional Brownian motion satisfies the hypothesis of Theorem 2.1 and

$$\mathcal{E}_N(B^H) \sim \frac{K_H}{\log(N)^H} \quad \text{as } N \rightarrow \infty \text{ for some } K_H > 0, \quad \text{and} \quad \mathcal{E}_N^{\text{prod}}(B^H) \asymp \frac{1}{\log(N)^H} \quad \text{as } N \rightarrow \infty.$$

2.2.2 Mean regularity and domination of the functional quantization rate

In this case, we will need the notion of regular variation at 0:

Definition 3 (Regularly varying function at zero). *A measurable function $\phi : (0, s) \rightarrow (0, \infty)$, ($s > 0$) is regularly varying at 0 with index $b \geq 0$ if for every $t > 0$, $\lim_{x \rightarrow 0} \frac{\phi(tx)}{\phi(x)} = t^b$.*

This means that $x \mapsto \phi(1/x)$ is regularly varying at infinity with index $-b$.

Definition 4 (The ϕ -Lipschitz assumption). *Let X be a bi-measurable process on $[0, T]$. We say that X satisfies the ϕ -Lipschitz assumption for $\rho > 0$, which we denote by $(L_{\phi, \rho})$ if there is a nondecreasing function $\phi : \mathbb{R}_+ \rightarrow [0, \infty]$, continuous at 0 with $\phi(0) = 0$, such that*

$$(L_{\phi, \rho}) \equiv \begin{cases} \forall (s, t) \in [0, T]^2, \mathbb{E}[|X_t - X_s|^\rho] \leq (\phi(|t - s|))^\rho, & \text{if } \rho \geq 1 \\ \forall t \in [0, T], \forall h \in (0, T], \mathbb{E} \left[\sup_{t \leq s \leq (t+h) \wedge T} |X_s - X_t|^\rho \right] \leq (\phi(h))^\rho & \text{if } 0 < \rho < 1. \end{cases}$$

Remark 1 (On the ϕ -Lipschitz assumption). *The ϕ -Lipschitz assumption implies that $\mathbb{E}[|X|_{L^\rho([0, T])}^\rho] < \infty$ so that \mathbb{P} -almost surely, $t \mapsto X_t$ lies in $L^\rho([0, T])$.*

Theorem 2.2 (Mean regularity and quantization rate). *Let X be a bi-measurable process on $[0, T]$ such that $X_t \in L^\rho(\Omega)$ for every $t \in [0, T]$, $\rho > 0$. Assume that X satisfies the ϕ -Lipschitz assumption for some nondecreasing function $\phi : \mathbb{R}_+ \rightarrow [0, \infty]$, continuous at 0 with $\phi(0) = 0$, which we assume to be regularly varying with index $b \geq 0$ at 0. Then*

$$\forall (r, p) \in [0, \rho]^2, \quad \mathcal{E}_{N, r}(X, |\cdot|_{L^\rho([0, T])}) \leq C_{r, p} \begin{cases} \phi(1/\log(N)), & \text{if } b > 0, \\ \psi(1/\log(N)), & \text{if } b = 0, \end{cases}$$

with $\psi(x) := \left(\int_0^x \frac{(\phi(\xi))^{(r \wedge 1)}}{\xi} d\xi \right)^{\frac{1}{r \wedge 1}}$, assuming in addition that $\int_0^x \frac{(\phi(\xi))^{(r \wedge 1)}}{\xi} d\xi < \infty$ if $b = 0$.

In particular, if $\phi(u) = cu^b$, $b > 0$, then

$$\mathcal{E}_{N, r}(X, |\cdot|_{L^\rho([0, T])}) = O(\log(N)^{-b}).$$

The case of multifractional Brownian motion

Theorem 2.3 (L^2 -mean regularity of the multifractional Brownian motion). *Let $\beta > 0$ be fixed and $h : [0, T] \rightarrow (0, 1)$ be a β -Hölder continuous map¹. Let B^h be a mBm of functional parameter h associated with a fractional Brownian field $(\mathbf{B}(t, H))_{(t, H) \in \mathbb{R} \times (0, 1)}$ that satisfies the (\mathcal{H}) hypothesis. There exists a positive constant M such that*

$$\forall (s, t) \in [0, T]^2, \quad \mathbb{E} \left[(B_t^h - B_s^h)^2 \right] \leq M |t - s| \left(2 \inf_{u \in [0, T]} h(u) \wedge \beta \delta \right), \quad (7)$$

where δ is given in Hypothesis (\mathcal{H}) .

Proof: Let us show that there exists a positive constant M such that, for every (s, t) in $[0, T]^2$,

$$\mathbb{E} \left[(B_t^h - B_s^h)^2 \right] \leq M |t - s| \left(2 \inf_{u \in [0, T]} h(u) \wedge \beta \delta \right).$$

One can always assume that the fractional field $(\mathbf{B}(t, H))_{(t, H) \in [0, T] \times [c, d]}$ is normalized, what we do in the sequel. For every fixed (t, s) in $[0, T]^2$ we then have:

$$\begin{aligned} \mathbb{E} \left[(B_t^h - B_s^h)^2 \right] &\leq 2 \mathbb{E} \left[(\mathbf{B}(t, h(t)) - \mathbf{B}(s, h(t)))^2 \right] + 2 \mathbb{E} \left[(\mathbf{B}(s, h(t)) - \mathbf{B}(s, h(s)))^2 \right] \\ &\leq 2 \left(|t - s|^{2h(t)} + \Lambda |h(t) - h(s)|^\delta \right) \leq 2 \left(|t - s|^{2H_1} (1 + T^{2(H_2 - H_1)}) + \Lambda \eta^\beta |t - s|^{\beta \delta} \right) \\ &\leq 2 \left(1 + T^{2(H_2 - H_1)} \right) (1 + \Lambda \eta^\beta) \left(|t - s|^{2H_1} + |t - s|^{\beta \delta} \right) \leq M |t - s|^{2H_1 \wedge \beta \delta}, \end{aligned}$$

where $H_1 := \inf_{u \in [0, T]} h(u)$, $H_2 := \sup_{u \in [0, T]} h(u)$ and $M := 2 \left(1 + T^{2(H_2 - H_1)} \right) \left(1 + T^{2H_1 \vee \beta \delta - 2H_1 \wedge \beta \delta} \right) (1 + \Lambda \eta^\beta)$. □

¹ i.e. $\exists \eta \in \mathbb{R}_+^*$, $\forall (s, t) \in [0, T]^2$, $|h(s) - h(t)| \leq \eta |s - t|^\beta$.

Corollary 2.4 (Domination of the quantization error of the multifractional Brownian motion). *Let $T > 0$ and $\beta > 0$ are fixed. Let $h : [0, T] \rightarrow (0, 1)$ be a fixed β -Hölder continuous function such that $H_1 := \sup_{u \in [0, T]} h(u) > 0$. Let $B^h := (B_t^h)_{t \in [0, T]}$ be a mBm of functional parameter h associated with a fractional Brownian field that satisfies the (\mathcal{H}) hypothesis. Then we get, for every (r, p) in $(\mathbb{R}_+)^2$,*

$$\mathcal{E}_{N,r}(B^h, |\cdot|_{L^p([0, T])}) = O\left(\log(N)^{-(H_1 \wedge \frac{\beta\delta}{2})}\right).$$

Proof: Since B^h is a Gaussian process, Theorem 2.3 shows that B^h fulfills the ϕ -Lipschitz assumption for every integer ρ of the form $\rho := 2p$ where p is a positive integer and for the continuous function $\phi_\rho := \phi_{2p}$, $[0, +\infty]$ -valued and defined on \mathbb{R}_+ by $\phi_\rho(0) := 0$ and $\phi_\rho(x) := (\kappa_{\rho/2})^{1/2\rho} \sqrt{M} x^{H_1 \wedge \frac{\beta\delta}{2}}$ on \mathbb{R}_+^* , for every positive integer $\rho = 2p$, where we denoted, for every n in \mathbb{N} , κ_n the unique positive real such that $\mathbb{E}[Y^{2n}] = \kappa_n \mathbb{E}[Y^2]^n$ for every centered Gaussian random variable Y . Besides, it is clear that ϕ_ρ is regularly varying with index $H_1 \wedge \frac{\beta\delta}{2}$, which is positive. The result is then obvious in regard of Theorem 2.2. \square

Remark 2. *In fact one can easily extends Corollary 2.4 to every process $V^h := (V_t^h)_{t \in [0, T]}$, of the form $V_t^h := Z(t, h(t))$ where $Z := (Z(t, H))_{(t, H) \in \mathbb{R} \times [H_1, H_2]}$ is a Gaussian field such that one can find $(\Lambda, \gamma, \delta)$ in $(\mathbb{R}_+^*)^3$ verifying*

$$\forall (s, t, H, H') \in [0, T]^2 \times [H_1, H_2]^2, \quad \mathbb{E}\left[\left(Z(t, H) - Z(s, H')\right)^2\right] \leq \Lambda (|t - s|^\gamma + |H - H'|^\delta),$$

In this case, for every (r, p) in $(\mathbb{R}_+^)^2$, we get $\mathcal{E}_{N,r}(V^h, |\cdot|_{L^p([0, T])}) = O\left(\log(N)^{-(\frac{\gamma}{2} \wedge \frac{\beta\delta}{2})}\right)$.*

Remark 3 (Sharp rate). *We conjecture that the domination of the rate of decay of the quantization error for the mBm established in Corollary 2.4 is in fact a sharp rate and that we have, for every $(r, p) \in (\mathbb{R}_+^*)^2$,*

$$\mathcal{E}_{N,r}(B^h, |\cdot|_{L^p([0, T])}) \sim \frac{K_{h,r,p}}{\log(N)^{H_1 \wedge \beta}} \text{ as } N \rightarrow \infty \text{ for some } K_{h,r,p} > 0,$$

and that the quadratic optimal product quantization error $\mathcal{E}_N^{\text{prod}}(B^h)$ for the mBm satisfies

$$\mathcal{E}_N^{\text{prod}}(B^h) \asymp \frac{1}{\log(N)^{H_1 \wedge \beta}} \text{ as } N \rightarrow \infty,$$

which is consistent with the results of previous section on the fractional Brownian motion.

It is of a crucial importance to show that we can apply these results to the well-balanced fBm. This is the purpose of the following proposition.

Proposition 2.5. *The well-balanced fractional Brownian field (Def.1) satisfies assumption (\mathcal{H}) .*

Proof: Let $T > 0$ and $[c, d] \subset (0, 1)$ are fixed. For every (t, H, H') in $[0, T] \times [c, d]^2$, define $I_t^{H, H'} := \mathbb{E}\left[\left(\mathbf{B}(t, H) - \mathbf{B}(t, H')\right)^2\right]$. One gets:

$$I_t^{H, H'} = \int_{\mathbb{R}} \left| \frac{e^{it\xi} - 1}{c_H |\xi|^{H+1/2}} - \frac{e^{it\xi} - 1}{c_{H'} |\xi|^{H'+1/2}} \right|^2 du = \int_{\mathbb{R}} \left| \frac{e^{it\xi} - 1}{\xi} \right|^2 \left| \frac{1}{c_H} |\xi|^{1/2-H} - \frac{1}{c_{H'}} |\xi|^{1/2-H'} \right|^2 d\xi. \quad (8)$$

For every ξ in \mathbb{R}^* , the map $f_\xi : [c, d] \rightarrow \mathbb{R}_+$, defined by $f_\xi(H) := \frac{1}{c_H} |\xi|^{1/2-H}$ is C^1 since $H \mapsto c_H$ is C^1 on $(0, 1)$. There then exists a positive real D such that

$$\forall (\xi, H) \in \mathbb{R}^* \times [c, d], \quad |f'_\xi(H)| \leq D |\xi|^{1/2-H} (1 + |\ln(|\xi|)|) \leq D \left(|\xi|^{1/2-c} + |\xi|^{1/2-d} \right) (1 + |\ln(|\xi|)|).$$

We apply the mean-value theorem to Equation (8), which yields

$$\begin{aligned}
I_t^{H,H'} &\leq D^2 |H - H'|^2 \int_{\mathbb{R}} \frac{|e^{it\xi} - 1|^2}{|\xi|^2} \left(|\xi|^{1/2-c} + |\xi|^{1/2-d} \right)^2 (1 + |\ln(|\xi|)|)^2 d\xi \\
&\leq D^2 |H - H'|^2 \left(2^3 \int_{|\xi|>1} \frac{(1 + |\ln(|\xi|)|)^2}{|\xi|^{1+2c}} d\xi + (2t)^2 \int_{|\xi|\leq 1} |\xi|^{1-2d} (1 + |\ln(|\xi|)|)^2 d\xi \right) \\
&\leq (2^3 + T^2) D^2 \left(\int_{|\xi|>1} \frac{(1 + |\ln(|\xi|)|)^2}{|\xi|^{1+2c}} d\xi + \int_{|\xi|\leq 1} |\xi|^{1-2d} (1 + |\ln(|\xi|)|)^2 d\xi \right) |H - H'|^2.
\end{aligned}$$

Since the two integrals in the last line are finite, (\mathcal{H}) is verified with $\delta = 2$. \square

2.3 Quantization-based cubature

2.3.1 Basic formula and related inequalities in the case of Lipschitz continuous functionals

The idea of quantization-based cubature methods is to approximate the distribution of the random variable X by the distribution of a quantizer Y of X . As Y is a discrete random variable, we have $\mathbb{P}_Y = \sum_{i=1}^N p_i \delta_{y_i}$. Therefore, if $F : E \rightarrow \mathbb{R}$ is a Borel functional,

$$\mathbb{E}[F(Y)] = \sum_{i=1}^N p_i F(y_i). \quad (9)$$

Hence, the weighted discrete distribution $(y_i, p_i)_{1 \leq i \leq N}$ of Y allows us to compute the sum (9). We review some error bounds that can be derived when approximating $\mathbb{E}[F(X)]$ by (9). See [33] for more details.

1. If $X \in L^2$, Y a quantizer of X of size N and F is Lipschitz continuous, then

$$|\mathbb{E}[F(X)] - \mathbb{E}[F(Y)]| \leq [F]_{\text{Lip}} \|X - Y\|_2. \quad (10)$$

where $[F]_{\text{Lip}}$ is the Lipschitz constant of F . In particular, if $(Y_N)_{N \geq 1}$ is a sequence of quantizers such that $\lim_{N \rightarrow \infty} \|X - Y_N\|_2 = 0$, then the distribution $\sum_{i=1}^N p_i^N \delta_{x_i^N}$ of Y_N weakly converges to the distribution \mathbb{P}_X of X as $N \rightarrow \infty$.

2. If Y is a stationary quantizer of X , i.e. $Y = \mathbb{E}[X|Y]$, and F is differentiable with a Lipschitz continuous derivative DF , then

$$|\mathbb{E}[F(X)] - \mathbb{E}[F(Y)]| \leq [DF]_{\text{Lip}} \|X - Y\|_2^2, \quad (11)$$

where $[DF]_{\text{Lip}}$ is the Lipschitz constant of DF . If F is twice differentiable and D^2F is bounded, then we can replace $[DF]_{\text{Lip}}$ by $\frac{1}{2} \|D^2F\|_\infty$.

3. If F is a convex functional and Y is a stationary quantizer of X ,

$$\mathbb{E}[F(Y)] \leq \mathbb{E}[F(X)]. \quad (12)$$

2.3.2 Richardson-Romberg extrapolation

With respect to the quantization error

In the general setting of a non-uniform random variable X , a quadratic optimal N -quantizer Y_N of X and a C^1 functional with Lipschitz continuous derivative, Equation (11) does not provide a true asymptotic expansion which would allow us to use a Richardson-Romberg expansion, but it suggests to use a higher-order Taylor expansion of $F(X) - F(Y_N)$ to get one.

It follows from Taylor's formula that there exists a vector $\zeta \in [X, Y_N]$ such that

$$\begin{aligned} \mathbb{E}[F(X)] &= \mathbb{E}[F(Y_N)] + \underbrace{\mathbb{E}[\langle DF(Y_N), X - Y_N \rangle]}_{=\mathbb{E}[DF(Y_N) \cdot \mathbb{E}[X - Y_N | Y_N]] = 0 \text{ by stationarity.}} + \frac{1}{2} \mathbb{E}[D^2 F(Y_N)(X - Y_N)^{\otimes 2}] \\ &\quad + \frac{1}{6} \mathbb{E}[\zeta(X - Y_N)^{\otimes 3}] + o(\mathbb{E}[|X - Y_N|^3]) \\ &= \mathbb{E}[F(Y_N)] + \frac{1}{2} \mathbb{E}[D^2 F(Y_N)(X - Y_N)^{\otimes 2}] + O(\mathbb{E}[|X - Y_N|^3]). \end{aligned} \tag{13}$$

In [17], Graf *et al.* proved that the asymptotics of the L^s quantization error induced by a sequence of L^r -optimal quantizers remains rate-optimal in the case of probability distributions on \mathbb{R}^d , with $s < r + d$ for a class of distributions including the Gaussian distribution. In this case, this leads to $\mathbb{E}[|X - Y_N|^3] = O(\mathbb{E}[|X - Y_N|^2]^{\frac{3}{2}})$. This holds *e.g.* for the Brownian motion.

Unfortunately, no sharp equivalence between $\|X - Y_N\|_2^2$ and $\mathbb{E}[D^2 F(Y_N)(X - Y_N)^{\otimes 2}]$ has been established yet. Still, Equation (13) suggests to use a Richardson-Romberg extrapolation with respect to the quantization error $\mathbb{E}_N^2 := \|X - Y_N\|_2^2$. The two-steps extrapolation between $N = k$ and $N = l$ conducts to approximate $\mathbb{E}[F(X)]$ by the quantity

$$\frac{\mathbb{E}[F(Y_l)]\mathbb{E}_k^2 - \mathbb{E}[F(Y_k)]\mathbb{E}_l^2}{\mathbb{E}_k^2 - \mathbb{E}_l^2}. \tag{14}$$

Although this kind of Richardson-Romberg extrapolation has not received a full theoretical justification yet, it does dramatically increase the efficiency of quantization-based cubature formulas.

With respect to the quantization level

When the value of \mathbb{E}_k^2 is not known, we can also rely on an asymptotic expansion with respect to the quantization level.

Remark 4 (Romberg extrapolation with respect to the quantization level). *In Section 2.2, we have seen that under some assumptions on the eigenvalues of the convergence operator, the rate of convergence of optimal quantizers and K-L optimal product quantizers is $(\ln(N))^{-\alpha}$ for some $\alpha \in (0, 1)$. Replacing the distortion \mathbb{E}_N by its asymptotics $\frac{1}{\ln(N)^\alpha}$ as $N \rightarrow \infty$ in Equation (14) conducts to approximate $\mathbb{E}[F(X)]$ by the quantity*

$$\frac{\mathbb{E}[F(Y_l)](\ln l)^{2\alpha} - \mathbb{E}[F(Y_k)](\ln k)^{2\alpha}}{(\ln l)^{2\alpha} - (\ln k)^{2\alpha}}. \tag{15}$$

Multidimensional Richardson-Romberg extrapolation

In the situation where X^1 and X^2 are two independent random variables, in order to estimate the expectation $\mathbb{E}[F(X^1, X^2)]$ for some regular functional F , one can use a cubature based on a product quantization $(\widehat{X}^1, \widehat{X}^2)$ of (X^1, X^2) , and perform a multidimensional Richardson-Romberg extrapolation which comes to performing two Richardson-Romberg extrapolations as described already, one related to the quantization error of X^1 between quantization levels N_1 and M_1 , and one related to the quantization error of X^2 between quantization levels N_2 and M_2 . This conducts to approximate $\mathbb{E}[F(X^1, X^2)]$ by the quantity

$$\frac{\mathbb{E}_{M_1}^2 \mathbb{E}_{M_2}^2 F^{N_1, N_2} - \mathbb{E}_{N_1}^2 \mathbb{E}_{M_2}^2 F^{M_1, N_2} - \mathbb{E}_{M_1}^2 \mathbb{E}_{N_2}^2 F^{N_1, M_2} + \mathbb{E}_{N_1}^2 \mathbb{E}_{N_2}^2 F^{M_1, M_2}}{(\mathbb{E}_{M_1}^2 - \mathbb{E}_{N_1}^2)(\mathbb{E}_{M_2}^2 - \mathbb{E}_{N_2}^2)}, \tag{16}$$

where $F^{p,q}$ stands for the estimated expectation obtained with the quantization-based cubature and quantization levels of p and q for X^1 and X^2 respectively. In other words, $F^{p,q}$ is defined by

$$F^{p,q} := \mathbb{E}\left[F\left(\widehat{X}^{1p}, \widehat{X}^{2q}\right)\right]$$

where $\widehat{X}^{1p}, \widehat{X}^{2q}$ are quantizers of levels p and q for X^1 and X^2 respectively. In Equation (16), \mathbb{E}_{M_i} and \mathbb{E}_{N_i} denote the quadratic quantization error of level M_i and N_i for X_i .

3 Stochastic calculus with respect to mBm

From now on and until the end of the article we will only consider well-balanced multifractional Brownian motion defined in Section 1.

3.1 Some backgrounds on the white noise theory

Define the probability space as $\Omega := \mathcal{S}'(\mathbb{R})$ and let $\mathcal{F} := \mathbb{B}(\mathcal{S}'(\mathbb{R}))$ be the σ -algebra of Borel sets.

There exists a probability measure μ such that, for every f in $L^2(\mathbb{R})$, the map $\langle \cdot, f \rangle : \Omega \rightarrow \mathbb{R}$ defined by $\langle \cdot, f \rangle(\omega) = \langle \omega, f \rangle$ (where $\langle \omega, f \rangle$ is by definition $\omega(f)$, i.e the action of the distribution ω on the function f) is a centered Gaussian random variable with variance equal to $\|f\|_{L^2(\mathbb{R})}^2$ under μ . For every n in \mathbb{N} , define

$e_n(x) := (-1)^n \pi^{-1/4} (2^n n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n} (e^{-x^2})$ the n th Hermite function. Let $(|\cdot|_p)_{p \in \mathbb{Z}}$ be the family of

norms defined by $|f|_p^2 := \sum_{k=0}^{+\infty} (2k+2)^{2p} \langle f, e_k \rangle_{L^2(\mathbb{R})}^2$, for all (p, f) in $\mathbb{Z} \times L^2(\mathbb{R})$. The operator A defined on

$\mathcal{S}'(\mathbb{R})$ by $A := -\frac{d^2}{dx^2} + x^2 + 1$ admits the sequence $(e_n)_{n \in \mathbb{N}}$ as eigenfunctions and the sequence $(2n+2)_{n \in \mathbb{N}}$ as eigenvalues.

From now on we will denote as is customary (L^2) the space $L^2(\Omega, \mathcal{G}, \mu)$ where \mathcal{G} is the σ -field generated by $(\langle \cdot, f \rangle)_{f \in L^2(\mathbb{R})}$. For every random variable Φ of (L^2) there exists, according to the Wiener-Itô theorem, a

unique sequence $(f_n)_{n \in \mathbb{N}}$ of functions f_n in $\widehat{L}^2(\mathbb{R}^n)$ such that Φ can be decomposed as $\Phi = \sum_{n=0}^{+\infty} I_n(f_n)$, where

$\widehat{L}^2(\mathbb{R}^n)$ denotes the set of all symmetric functions f in $L^2(\mathbb{R}^n)$ and $I_n(f)$ denotes the n th multiple Wiener-Itô integral of f with the convention that $I_0(f_0) = f_0$ for constants f_0 . Moreover we have the equality $\mathbb{E}[\Phi^2] =$

$\sum_{n=0}^{+\infty} n! \|f_n\|_{L^2(\mathbb{R}^n)}^2$ where \mathbb{E} denotes the expectation with respect to μ . For any $\Phi := \sum_{n=0}^{+\infty} I_n(f_n)$ satisfying

the condition $\sum_{n=0}^{+\infty} n! |A^{\otimes n} f_n|_0^2 < +\infty$, define the element $\Gamma(A)(\Phi)$ of (L^2) by $\Gamma(A)(\Phi) := \sum_{n=0}^{+\infty} I_n(A^{\otimes n} f_n)$,

where $A^{\otimes n}$ denotes the n th tensor power of the operator A (see [22, Appendix E] for more details about tensor products of operators). The operator $\Gamma(A)$ is densely defined on (L^2) is called the second quantization operator of A . It is invertible and its inverse $\Gamma(A)^{-1}$ is bounded. Let us denote $\|\varphi\|_0^2 := \|\varphi\|_{(L^2)}^2$ for any random variable φ in (L^2) and, for n in \mathbb{N} , let $\text{Dom}(\Gamma(A)^n)$ be the domain of the n th iteration of $\Gamma(A)$. Define the family of norms $(\|\cdot\|_p)_{p \in \mathbb{Z}}$ by:

$$\|\Phi\|_p := \|\Gamma(A)^p \Phi\|_0 = \|\Gamma(A)^p \Phi\|_{(L^2)}, \quad \forall p \in \mathbb{Z}, \quad \forall \Phi \in (L^2) \cap \text{Dom}(\Gamma(A)^p).$$

For any p in \mathbb{N} , let $(\mathcal{S}_p) := \{\Phi \in (L^2) : \Gamma(A)^p \Phi \text{ exists and belongs to } (L^2)\}$ and define (\mathcal{S}_{-p}) as being the completion of the space (L^2) with respect to the norm $\|\cdot\|_{-p}$. As in [24], we let (\mathcal{S}) denote the projective limit of the sequence $((\mathcal{S}_p))_{p \in \mathbb{N}}$ and $(\mathcal{S})^*$ the inductive limit of the sequence $((\mathcal{S}_{-p}))_{p \in \mathbb{N}}$. The space (\mathcal{S}) is called the space of stochastic test functions and $(\mathcal{S})^*$ the space of Hida distributions. One can show that, for any p in \mathbb{N} , the dual space $(\mathcal{S}_p)^*$ of \mathcal{S}_p is (\mathcal{S}_{-p}) . Thus we will write (\mathcal{S}_{-p}) , in the sequel, to denote the space $(\mathcal{S}_p)^*$. Note also that $(\mathcal{S})^*$ is the dual space of (\mathcal{S}) . We will note $\langle\langle \cdot, \cdot \rangle\rangle$ the duality bracket between $(\mathcal{S})^*$ and (\mathcal{S}) . If Φ belongs to (L^2) then we have the equality $\langle\langle \Phi, \varphi \rangle\rangle = \langle \Phi, \varphi \rangle_{(L^2)} = \mathbb{E}[\Phi \varphi]$. Since we have defined a topology given by a family of norms on the space $(\mathcal{S})^*$ it is possible to define a derivative and an integral in $(\mathcal{S})^*$. A function $\Phi : \mathbb{R} \rightarrow (\mathcal{S})^*$ is called a stochastic distribution process, or an $(\mathcal{S})^*$ -process, or a Hida process.

The Hida process Φ is said to be differentiable at t_0 if $\lim_{r \rightarrow 0} r^{-1} (\Phi(t_0 + r) - \Phi(t_0))$ exists in $(\mathcal{S})^*$. Moreover we can also define an integral of an Hida process.

Theorem 3.1 (Integral in $(\mathcal{S})^*$). *Assume that $\Phi : \mathbb{R} \rightarrow (\mathcal{S})^*$ is weakly in $L^1(\mathbb{R}, dt)$, i.e. assume that for all φ in (\mathcal{S}) , the mapping $u \mapsto \langle\langle \Phi(u), \varphi \rangle\rangle$ from \mathbb{R} to \mathbb{R} belongs to $L^1(\mathbb{R}, dt)$. Then, there exists a unique element in $(\mathcal{S})^*$, denoted by $\int_{\mathbb{R}} \Phi(u) du$ such that*

$$\left\langle\left\langle \int_{\mathbb{R}} \Phi(u) du, \varphi \right\rangle\right\rangle = \int_{\mathbb{R}} \langle\langle \Phi(u), \varphi \rangle\rangle du \quad \text{for all } \varphi \text{ in } (\mathcal{S}). \quad (17)$$

We say that Φ is $(\mathcal{S})^*$ -integrable on \mathbb{R} or integrable on \mathbb{R} in the *Pettis* sense.

For every f in $L^2(\mathbb{R})$ we define the *Wick exponential* of $\langle \cdot, f \rangle$, noted $: e^{\langle \cdot, f \rangle} :$, as being the (L^2) random variable equal to $e^{\langle \cdot, f \rangle - \frac{1}{2} \|f\|_0^2}$. The S -transform of an element Φ of $(\mathcal{S})^*$, noted $S(\Phi)$, is defined as being the function from $\mathcal{S}(\mathbb{R})$ to \mathbb{R} given by $S(\Phi)(\eta) := \langle \langle \Phi, : e^{\langle \cdot, \eta \rangle} \rangle \rangle$ for every η in $\mathcal{S}(\mathbb{R})$. Finally for every $(\Phi, \Psi) \in (\mathcal{S})^* \times (\mathcal{S})^*$, there exists a unique element of $(\mathcal{S})^*$, called the Wick product of Φ and Ψ and noted $\Phi \diamond \Psi$, such that $S(\Phi \diamond \Psi)(\eta) = S(\Phi)(\eta) S(\Psi)(\eta)$; for every η in $\mathcal{S}(\mathbb{R})$.

Note that the map $S : \Phi \mapsto S(\Phi)$, from $(\mathcal{S})^*$ to $(\mathcal{S})^*$, is injective. Furthermore, let $\Phi : \mathbb{R} \rightarrow (\mathcal{S})^*$ be a fixed $(\mathcal{S})^*$ process. If Φ is $(\mathcal{S})^*$ -integrable over \mathbb{R} then for all η in $\mathcal{S}(\mathbb{R})$, $S(\int_{\mathbb{R}} \Phi(u) du)(\eta) = \int_{\mathbb{R}} S(\Phi(u))(\eta) du$. If Φ is $(\mathcal{S})^*$ -differentiable over \mathbb{R} then for all η in $\mathcal{S}(\mathbb{R})$, $S[\frac{d\Phi}{dt}(t)](\eta) = \frac{d}{dt} [S\Phi(t)](\eta)$.

For any Φ in $(\mathcal{S})^*$ and k in \mathbb{N}^* let $\Phi^{\diamond k}$ denote the element $\overbrace{\Phi \diamond \dots \diamond \Phi}^{k \text{ times}}$ of $(\mathcal{S})^*$. We can generalize the definition of \exp^\diamond to the case where Φ belongs to $(\mathcal{S})^*$. Indeed, for any Φ in $(\mathcal{S})^*$ such that the sum $\sum_{k=0}^{+\infty} \frac{\Phi^{\diamond k}}{k!}$ converges in $(\mathcal{S})^*$, define the element $\exp^\diamond \Phi$ of $(\mathcal{S})^*$ by setting $\exp^\diamond \Phi := \sum_{k=0}^{+\infty} \frac{\Phi^{\diamond k}}{k!}$. It is called Wick exponential of Φ .

For f in $L^2(\mathbb{R})$ and $\Phi := \langle \cdot, f \rangle$, it is easy to verify that $\exp^\diamond \Phi$ given previously exists and coincide with $: e^{\langle \cdot, f \rangle} :$ defined at the beginning of this section.

3.1.1 Fractional and multifractional White noise

Operators M_H and $\frac{\partial M_H}{\partial H}$.

Let H be a fixed real in $(0, 1)$. Following [25] and references therein, define the operator M_H , specified in the Fourier domain, by

$$\widehat{M_H(u)}(y) := \frac{\sqrt{2\pi}}{c_H} |y|^{1/2-H} \widehat{u}(y), \quad \forall y \in \mathbb{R}^*.$$

This operator is well defined on the homogeneous Sobolev space of order $1/2 - H$ noted $L_H^2(\mathbb{R})$ and defined by $L_H^2(\mathbb{R}) := \{u \in \mathcal{S}'(\mathbb{R}) : \widehat{u} = T_f; f \in L_{loc}^1(\mathbb{R}) \text{ and } \|u\|_H < +\infty\}$, where the norm $\|\cdot\|_H$ derives from the inner product $\langle \cdot, \cdot \rangle_H$ defined on $L_H^2(\mathbb{R})$ by: $\langle u, v \rangle_H := \frac{1}{c_H^2} \int_{\mathbb{R}} |\xi|^{1-2H} \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi$ and where c_H is defined right after Definition 1.1.

The definition of the operator $\frac{\partial M_H}{\partial H}$ is quite similar. Precisely, define for every H in $(0, 1)$ the space $\Gamma_H(\mathbb{R}) := \{u \in \mathcal{S}'(\mathbb{R}) : \widehat{u} = T_f; f \in L_{loc}^1(\mathbb{R}) \text{ and } \|u\|_{\delta_H(\mathbb{R})} < +\infty\}$, where the norm $\|\cdot\|_{\delta_H(\mathbb{R})}$ derives from the inner product $\langle \cdot, \cdot \rangle_{\delta_H}$ defined on $\Gamma_H(\mathbb{R})$ by $\langle u, v \rangle_{\delta_H} := \frac{1}{c_H^2} \int_{\mathbb{R}} (\beta_H + \ln |\xi|)^2 |\xi|^{1-2H} \widehat{u}(\xi) \overline{\widehat{v}(\xi)} d\xi$.

Following [25], define the operator $\frac{\partial M_H}{\partial H}$ from $(\Gamma_H(\mathbb{R}), \langle \cdot, \cdot \rangle_{\delta_H(\mathbb{R})})$ to $(L^2(\mathbb{R}), \langle \cdot, \cdot \rangle_{L^2(\mathbb{R})})$, in the Fourier domain by:

$$\widehat{\frac{\partial M_H}{\partial H}(u)}(y) := -(\beta_H + \ln |y|) \frac{\sqrt{2\pi}}{c_H} |y|^{1/2-H} \widehat{u}(y), \quad \forall y \in \mathbb{R}^*.$$

Fractional and multifractional White noise.

For any measurable function $h : \mathbb{R} \rightarrow (0, 1)$, it is easily seen that the process $B^h := (B_t^h)_{t \in \mathbb{R}}$ defined by

$$\forall (\omega, t) \in \Omega \times \mathbb{R}, \quad B_t^h := \sum_{k=0}^{+\infty} \left(\int_0^t M_{h(t)}(e_k)(s) ds \right) \langle \cdot, e_k \rangle$$

is a mBm. Assuming that h is differentiable, we define the $(\mathcal{S})^*$ -valued function $W^h := (W_t^h)_{t \in \mathbb{R}}$ by

$$W_t^h := \sum_{k=0}^{+\infty} \left[\frac{d}{dt} \left(\int_0^t M_{h(t)}(e_k)(s) ds \right) \right] \langle \cdot, e_k \rangle. \quad (18)$$

The following theorem states that, for all real t , the right-hand side of (18) does indeed belong to $(\mathcal{S})^*$ and is exactly the $(\mathcal{S})^*$ -derivative of B^h at point t .

Theorem-Definition 3.1 ([25, Theorem-definition 5.1]). *Let $h : \mathbb{R} \rightarrow (0, 1)$ be a C^1 deterministic function such that its derivative function h' is bounded. The process W^h defined by (18) is an $(\mathcal{S})^*$ -process which verifies, the following equality (in $(\mathcal{S})^*$):*

$$W_t^h = \sum_{k=0}^{+\infty} M_{h(t)}(e_k)(t) \langle \cdot, e_k \rangle + h'(t) \sum_{k=0}^{+\infty} \left(\int_0^t \frac{\partial M_H}{\partial H}(e_k)(s) \Big|_{H=h(t)} ds \right) \langle \cdot, e_k \rangle. \quad (19)$$

Moreover the process B^h is $(\mathcal{S})^*$ -differentiable on \mathbb{R} and verifies $\frac{dB^h}{dt}(t) = W_t^h$ in $(\mathcal{S})^*$.

Once again if the function h is constant, identically equal to H , we will denote by $W^H := (W_t^H)_{t \in \mathbb{R}}$ and call fractional white noise the process W^h .

3.2 Stochastic integral with respect to mBm

Following [25] we are now able to define the multifractional Wick-Itô integral. In the sequel we will always assume that h is a C^1 function on \mathbb{R} with bounded derivative.

Definition 5 (Multifractional Wick-Itô integral). *Let B^h be a normalized multifractional Brownian motion and $Y : \mathbb{R} \rightarrow (\mathcal{S})^*$ be a process such that the process $t \mapsto Y_t \diamond W_t^h$ is $(\mathcal{S})^*$ -integrable on \mathbb{R} . We say that the process Y is $d^\circ B^h$ -integrable on \mathbb{R} or integrable on \mathbb{R} with respect to mBm B^h . Moreover, we define the integral on \mathbb{R} of Y with respect to B^h by*

$$\int_{\mathbb{R}} Y_s d^\circ B_s^h := \int_{\mathbb{R}} Y_s \diamond W_s^h ds. \quad (20)$$

We also define $\int_I Y_s d^\circ B_s^h := \int_{\mathbb{R}} \mathbf{1}_I(s) Y_s d^\circ B_s^h$ for any interval I of \mathbb{R} .

When the function h is constant over \mathbb{R} , equal to H , the multifractional Wick-Itô integral coincides with the fractional Wick-Itô integral defined in [16], [6], [4] and [5]. In particular, when Y is adapted and when the function h is identically equal to $1/2$ our integral is nothing but the classical Itô integral with respect to Brownian motion. The multifractional Wick-Itô integral verifies the following properties:

Proposition 3.2. *Let B^h be an mBm and I be an interval of \mathbb{R} .*

- For all (a, b) in \mathbb{R}^2 such that $a < b$, we have $\int_a^b 1 d^\circ B_u^h = B_b^h - B_a^h$ almost surely.
- Let $X : I \rightarrow (\mathcal{S})^*$ be a $d^\circ B^h$ -integrable process over I . If $\int_I X_s d^\circ B_s^h$ belongs to (L^2) , we have $\mathbb{E}[\int_I X_s d^\circ B_s^h] = 0$.

An Itô formula has been derived for this stochastic integral with respect to the multifractional Brownian motion, see [25, Theorem 6.9].

Multifractional Wick-Itô integral of deterministic elements

In order to solve the differential equations driven by a mBm that we will need further in the article, it is necessary to know the exact nature of multifractional Wick-Itô integral of deterministic elements.

For H in $(0, 1)$ and f in $\mathcal{S}(\mathbb{R})$, define the function $g_f : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R}$ by $g_f(t, H) := \int_0^t M_H(f)(x) dx$ where M_H is the operator defined in at the beginning of Section 3.1.1. It has been shown that g_f belongs to $C^\infty(\mathbb{R} \times (0, 1), \mathbb{R})$ (cf. [25, Lemma 5.5]). The main result on the multifractional Wick-Itô integral of deterministic elements is the following:

Theorem 3.3. ([25, Theorem 5.25]) *Let $h : \mathbb{R} \rightarrow (0, 1)$ be a C^1 deterministic function and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a measurable deterministic function which belongs to $L^1_{\text{loc}}(\mathbb{R})$. Let $Z := (Z_t)_{t \in \mathbb{R}}$ be the process defined by $Z_t := \int_0^t f(s) d^\circ B_s^h$. Then Z is an $(\mathcal{S})^*$ -process which verifies the following equality in $(\mathcal{S})^*$*

$$\int_0^t f(s) d^\circ B_s^h = \sum_{k=0}^{+\infty} \left(\int_0^t f(s) \frac{d}{ds} [g_{e_k}(s, h(s))] ds \right) \langle \cdot, e_k \rangle. \quad (21)$$

Moreover Z is a (centered) Gaussian process if and only if $\sum_{k=0}^{+\infty} \left(\int_0^t f(s) \frac{d}{ds} [g_{e_k}(s, h(s))] ds \right)^2 < +\infty$, for all t in \mathbb{R} . In this case we have, for every t in \mathbb{R} ,

$$Z_t = \int_0^t f(s) d^\circ B_s^h \stackrel{\mathcal{L}}{\sim} \mathcal{N} \left(0, \sum_{k=0}^{+\infty} \left(\int_0^t f(s) \frac{d}{ds} [g_{e_k}(s, h(s))] ds \right)^2 \right). \quad (22)$$

In particular, Z is a Gaussian process when f belongs to $C^1(\mathbb{R}; \mathbb{R})$.

Deriving the quantity $\mathbb{E} [Z_t^2]$ in the previous theorem might be complicated using Equation (22). However, when f is a C^1 function, thanks to the Itô formula with respect to mBm [25, Theorem 6.9], we obtain the following integration-by-parts formula

$$\int_0^t f(s) d^\circ B_s^h \stackrel{(L^2)}{=} f(t) B_t^h - \int_0^t f'(s) B_s^h ds, \quad (23)$$

which leads to

$$\mathbb{E} [Z_t^2] = f(t)^2 t^{2h(t)} + \int_0^t \int_0^t f'(s) f'(u) R_h(s, u) ds du - 2f(t) \int_0^t f'(s) R_h(t, s) ds. \quad (24)$$

Remark 5. The integration-by-parts formula (23) allows to identify almost surely $\int_0^t f(s) d^\circ B_s^h$ with the quantity $I_t^f(B^h)$ where the map $I_t^f : C^0([0, t]; \mathbb{R}) \rightarrow \mathbb{R}$ is defined by

$$I_t^f : g \mapsto \left(f(t)g(t) - \int_0^t f'(s)g(s) ds \right). \quad (25)$$

3.3 Stochastic differential equations involving the mBm

For the rest of this section, we consider a fixed C^1 deterministic function $h : \mathbb{R} \rightarrow (0, 1)$. We also consider $(B_t)_{t \in \mathbb{R}}$ a standard Brownian motion, and $(B_t^h)_{t \in \mathbb{R}}$ a well-balanced multifractional Brownian motion, independent of B , of functional parameter h and $(\alpha_1, \alpha_2, \gamma_1, \gamma_2)$ in \mathbb{R}^4 .

3.3.1 Mixed multifractional Brownian S.D.E.

Let us consider, for some positive constants γ_1 and γ_2 , the following mixed multifractional stochastic differential equation,

$$\begin{cases} dX_t = X_t (\gamma_1 d^\circ B_t + \gamma_2 d^\circ B_t^h), \\ X_0 = x_0 \in \mathbb{R}. \end{cases} \quad (26)$$

Of course (26) is a shorthand notation for the equation

$$X_t = x_0 + \gamma_1 \int_0^t X_s d^\circ B_s + \gamma_2 \int_0^t X_s d^\circ B_s^h, \quad X_0 = x_0 \in \mathbb{R},$$

where the previous equality holds in $(\mathcal{S})^*$. A solution of this equation will be called *geometric mixed multifractional Brownian motion*. Rewriting the previous equation in terms of derivatives in $(\mathcal{S})^*$, we get:

$$\frac{dX_t}{dt} = X_t \diamond (\gamma_1 W_t^{1/2} + \gamma_2 W_t^h), \quad x_0 \in \mathbb{R}. \quad (27)$$

Theorem 3.4 (Geometric mixed multifractional Brownian motion). *The $(\mathcal{S})^*$ -process $(X_t)_{t \in [0, T]}$ defined by*

$$X_t := x_0 \exp^\diamond (\gamma_1 B_t + \gamma_2 B_t^h), \quad (28)$$

is the unique solution of (27) in $(\mathcal{S})^$.*

Proof: Applying the S -transform to both sides of Equation (27) and denoting by y_η the map $t \mapsto S(X_t)(\eta)$, for every η in $\mathcal{S}(\mathbb{R})$, we get:

$$y'_\eta(t) = y_\eta(t) \left(\gamma_1 M_{1/2}(\eta)(t) + \gamma_2 \frac{d}{dt} [g_\eta(t, h(t))] \right), \quad y_\eta(0) = x_0.$$

where the map $(t, H) \mapsto g_\eta(t, H)$ is defined on $\mathbb{R} \times (0, 1)$ by $g_\eta(t, H) := \int_0^t M_H(\eta)(x) dx$ (see [25, Lemma 5.5] for the proof that $t \mapsto g_\eta(t, h(t))$ is differentiable on \mathbb{R}). This equation admits a unique solution which verifies $y_\eta(t) = x_0 \exp \left\{ \gamma_1 \int_0^t M_{1/2}(\eta)(u) du + \gamma_2 \int_0^t \frac{d}{du} [g_\eta(u, h(u))] du \right\}$. Using (i) and (ii) of [25, Theorem 5.12] we hence get, for every η in $\mathcal{S}(\mathbb{R})$,

$$y_\eta(t) = x_0 \exp \left\{ \gamma_1 S(B_t)(\eta) + \gamma_2 S(B_t^h)(\eta) \right\} = S \left(x_0 \exp^\diamond \left\{ \gamma_1 B_t + \gamma_2 B_t^h \right\} \right) (\eta).$$

The injectivity of the S -transform allows us to conclude that $X_t = x_0 \exp^\diamond \left\{ \gamma_1 B_t + \gamma_2 B_t^h \right\}$ for every t in $[0, T]$. \square

Remark 6. (i) Using [22, Equality (3.16)], one sees that X is an (L^2) -valued process that may be represented as:

$$X_t = x_0 \exp \left\{ \gamma_1 B_t + \gamma_2 B_t^h - \frac{1}{2} \left(\gamma_1^2 t + \gamma_2^2 t^{2h(t)} \right) \right\}.$$

(ii) The theorem could also have been established by using [21, Theorem 3.1.2].

3.3.2 Mixed multifractional Ornstein-Uhlenbeck S.D.E.

Let us now consider the following mixed stochastic differential equation:

$$\begin{cases} dU_t = \theta(\mu - U_t)dt + (\alpha_1 d^\diamond B_t + \alpha_2 d^\diamond B_t^h) \\ U_0 = u_0 \in \mathbb{R}, \end{cases} \quad (29)$$

where $(B_t)_{t \in \mathbb{R}}$ and $(B_t^h)_{t \in \mathbb{R}}$ are independent, $\theta \geq 0$ and μ, α_1, α_2 belong to \mathbb{R} . A solution of this equation will be called a *mixed multifractional Ornstein-Uhlenbeck process*.

Theorem 3.5 (Mixed multifractional Ornstein-Uhlenbeck process). *The $L^2(\Omega)$ -valued process $(U_t)_{t \in \mathbb{R}}$ defined by*

$$U_t := u_0 e^{-\theta t} + \mu (1 - e^{-\theta t}) + \alpha_1 \int_0^t e^{\theta(s-t)} d^\diamond B_s + \alpha_2 \int_0^t e^{\theta(s-t)} d^\diamond B_s^h, \quad (30)$$

is the unique solution of the stochastic differential equation (29).

Proof: The proof that the process U defined by (30) is the unique solution of (29) is very similar to the one of Theorem 3.4. Indeed, setting $y_\eta(t) := S(U_t)(\eta)$ for every (t, η) in $\mathbb{R} \times \mathbb{R}$ and applying the S -transform to both sides of (29) we get, for every η in $\mathcal{S}(\mathbb{R})$, the ordinary differential equation

$$y'_\eta(t) = \theta(\mu - y_\eta(t)) + \alpha_1 M_{1/2}(\eta)(t) + \alpha_2 \frac{d}{dt} [g_\eta(t, h(t))], \quad y_\eta(0) = u_0. \quad (31)$$

It is clear that the unique solution of the previous ordinary differential equation is

$$y_\eta(t) = u_0 e^{-\theta t} + e^{-\theta t} \int_0^t e^{\theta s} \left(\theta \mu + \alpha_1 M_{1/2}(\eta)(s) + \alpha_2 \frac{d}{ds} [g_\eta(s, h(s))] \right) ds, \quad y_\eta(0) = u_0,$$

where $M_{1/2}$ is defined at the beginning of Section 3.1.1. The previous O.D.E. also reads $S(U_t)(\eta) = S \left(u_0 e^{-\theta t} + e^{-\theta t} \int_0^t e^{\theta s} \left(\theta \mu ds + \alpha_1 d^\diamond B_s + \alpha_2 d^\diamond B_s^h \right) \right) (\eta)$. Once again the injectivity of the S -transform allows us to conclude. \square

3.4 Quantization of solutions of S.D.E. driven by mBm

The quantization of a bi-measurable Gaussian process X on $[0, T]$ allows one to perform a quantization of the solution of a stochastic differential equation with respect to X , as soon as one can define the corresponding stochastic integral: indeed, in many cases, the solution of an S.D.E. can be seen as a functional of the underlying process. A quantizer of the solution can then be simply obtained by plugging the quantizer of the stochastic process into the so-defined functional.

In the one-dimensional setting, under rather general conditions on the diffusion coefficients, if X is continuous semimartingale, this functional is easily determined by using the Lamperti transform (see [30]). In this case, the corresponding quantizer of the stochastic differential equation is obtained by simply plugging the Gaussian quantizer in the S.D.E. written in the Stratonovich sense, leading to a finite set of ordinary differential equations. This leads to a simple and general constructive method to build a functional quantization of the solution of a S.D.E.. This approach based on the Lamperti transform cannot be generalized to the multidimensional setting though. In the case of continuous semimartingales, the multidimensional setting has been handled for the first time in [35] using the rough paths theory.

Unfortunately, no such result has been established in the case of a S.D.E. driven by a multifractional Brownian motion (or even by a fractional Brownian motion).

Still, in some situations, as when one has an explicit solution of the S.D.E. at ones disposal, we can directly plug the functional quantization of the Gaussian process in the solution. If the corresponding functional is regular enough, quantization-based cubatures can then be used. For example, the geometric mixed multifractional Brownian motion defined in Section 3.3.1 is a simple functional of the associated Brownian motion and multifractional Brownian motion (see Remark 6).

3.4.1 The case of a Wiener integral

Another favorable situation is the very simple case of a Wiener integral of the form $\int_0^t f(s) d^\circ B_s^h$, where f is a C^1 deterministic function. The integration-by-parts formula for mBm (23) reads $\int_0^t f(s) d^\circ B_s^h \stackrel{a.s.}{=} f(t)B_t^h - \int_0^t f'(s)B_s^h ds$. In other words, for $p \geq 1$, the stochastic process $t \mapsto \int_0^t f(s) d^\circ B_s^h$ seen as a random variable valued in $L^p(0, T)$ is the image of B^h by the map

$$\begin{aligned} J^f : L^p([0, T]) &\rightarrow L^p([0, T]) \\ g &\mapsto f(\cdot)g(\cdot) - \int_0^\cdot f'(s)g(s)ds. \end{aligned}$$

In other words we have (dt -almost everywhere) $J^f(g)(t) = I_t^f(g)$ where I_t^f has been defined in Remark 5.

Proposition 3.6 (L^p -regularity of the Wiener map). *For every $p \geq 1$, the so-defined map J^f is Lipschitz continuous in $L^p([0, T])$.*

Proof: It is straightforward that for $(g_1, g_2) \in L^p([0, T])^2$

$$\|J^f(g_1) - J^f(g_2)\|_p \leq \|f(g_1 - g_2)\|_p + \left\| \int_0^\cdot f'(s)(g_1(s) - g_2(s))ds \right\|_p \leq (\|f\|_\infty + \|f'\|_\infty T) \|g_1 - g_2\|_p.$$

□

In Appendix B, we prove that if the functional parameter h is C^1 , the Karhunen-Loève eigenfunctions of a well-balanced mBm B^h have bounded variations, and thus stationary quantizers of B^h have bounded variations as well (because it lies on a subspace of $L^2([0, T])$ spanned by a finite number of Karhunen-Loève eigenfunctions as mentioned already). In this setting, another integration by parts gives $I_t^f(\widehat{B}^h) = \int_0^t f(s) d\widehat{B}_s^h$ where $d\widehat{B}_s^h(\omega)$ stands for the signed measure associated with the function of bounded variation $s \mapsto \widehat{B}_s^h(\omega)$.

3.4.2 The case of a certain simple diffusions

Another case which can be handled in the in our setting is when one has an S.D.E. of the form

$$Y_t = y_0 + \int_0^t \beta(s, Y_s) ds + X_t, \quad (32)$$

where $\beta(s, y)$ is assumed to be Lipschitz continuous in y uniformly in s . This setting is addressed in [30, Pages 20-21]². Luschgy and Pagès considered the associated integral equation

$$y(t) = y_0 + \int_0^t \beta(s, y(s)) ds + g(t), \quad (33)$$

where $g \in L^p([0, T])$ is fixed. The existence and uniqueness in $L^p([0, T])$ of a solution for the integral equation (33) follows from the same approach used for ordinary differential equations. Then the solution of the associated S.D.E. (32) simply reads $U_t = \Psi_p^\beta(X)_t$, where $\Psi_p^\beta : L^p([0, T]) \rightarrow L^p([0, T])$ is the functional that maps $g \in L^p([0, T])$ to the unique solution in $L^p([0, T])$ of Equation (33). In [30], the map Ψ_p^β is showed to be Lipschitz continuous in $L^p([0, T])$. More precisely, we have

$$c([\beta]_{\text{Lip}}, T) \|g_1 - g_2\|_p^p \leq \|\Psi_p^\beta(g_1) - \Psi_p^\beta(g_2)\|_p^p \leq C([\beta]_{\text{Lip}}, T) \|g_1 - g_2\|_p^p,$$

with $c([\beta]_{\text{Lip}}, T) = \frac{1}{2^{p-1}(1-[\beta]_{\text{Lip}}^p T^p)}$ and $C([\beta]_{\text{Lip}}, T) = e^{2^{p-1}[\beta]_{\text{Lip}} T^{p-1}}$.

The case of the mixed multifractional Ornstein-Uhlenbeck process defined in Section 3.3.2 satisfies these conditions: we have $\beta(s, u) = \theta(\mu - u)$ and $X = \alpha_1 B + \alpha_2 B^h$.

4 Multifractional Hull & White stochastic volatility model

We assume that, under the risk-neutral measure, the forward price of a risky asset is the solution of the S.D.E.

$$\begin{cases} dF_t = F_t \sigma_t dW_t, \\ d \ln(\sigma_t) = \theta (\mu - \ln(\sigma_t)) dt + \gamma_h d^\circ B_t^h + \gamma_\sigma dW_t^\sigma, \quad \sigma_0 > 0, \end{cases} \quad (34)$$

where $\theta \geq 0$ and where W and W^σ are two standard Brownian motions and B^h is a well-balanced multifractional Brownian motion independent of W and W^σ with functional parameter h assumed to be continuously differentiable. We assume that W is decomposed into $\rho dW_t^\sigma + \sqrt{1 - \rho^2} dW_t^F$, where W^F is a Brownian motion independent of W^σ . Hence, S.D.E. (34) writes

$$\begin{cases} dF_t = F_t \sigma_t (\rho dW_t^\sigma + \sqrt{1 - \rho^2} dW_t^F) \\ d \ln(\sigma_t) = \theta (\mu - \ln(\sigma_t)) dt + \gamma_h d^\circ B_t^h + \gamma_\sigma dW_t^\sigma, \quad \sigma_0 > 0. \end{cases} \quad (35)$$

We denote respectively by \mathcal{F}^σ , \mathcal{F}^F and \mathcal{F}^h the natural filtrations of W^σ , W^F and B^h . We define the filtration $\mathcal{F}^{\sigma, h}$ by $\mathcal{F}_t^{\sigma, h} = \sigma(\mathcal{F}_t^\sigma, \mathcal{F}_t^h)$ and $\mathcal{F}^{F, \sigma, h}$ by $\mathcal{F}_t^{F, \sigma, h} = \sigma(\mathcal{F}_t^F, \mathcal{F}_t^\sigma, \mathcal{F}_t^h)$.

The unique solution of S.D.E. (34) reads

$$\begin{cases} F_t = F_0 \exp \left(\int_0^t \sigma_s dW_s - \frac{1}{2} \int_0^t \sigma_s^2 ds \right) \\ \sigma_s = \exp \left(\ln(\sigma_0) e^{-\theta s} + \mu (1 - e^{-\theta s}) + \gamma_\sigma \int_0^s e^{\theta(u-s)} dW_u^\sigma + \gamma_h \int_0^s e^{\theta(u-s)} d^\circ B_u^h \right). \end{cases} \quad (36)$$

In other words, $\ln(\sigma_t)$ is a mixed multifractional Ornstein-Uhlenbeck process. Note that, although the volatility process is not a semimartingale, the process $(F_t)_{t \in [0, T]}$ remains a (positive) $\mathcal{F}^{F, \sigma, h}$ -local martingale, and thus a super-martingale. By using the same method as Jourdain in [23], we can show that if $\rho = 0$ this local martingale is a real martingale. In the case where $\rho < 0$ the conclusion seems to be the same (based on numerically computations)

² Actually, in [30] the Lamperti transform is used to reduce a general Brownian diffusion to this case.

We now address the problem of the pricing of a forward start call option. The payoff of this option writes $\left(\frac{F_T}{F_\tau} - K\right)_+$ for some fixed maturity $\tau \in [0, T]$. The put case can be handled with the same method. Hence we are interested in the risk-neutral expectation $\mathbb{E} \left[\left(\frac{F_T}{F_\tau} - K\right)_+ \right]$.

We have the following decomposition

$$F_t = F_0 \underbrace{\exp \left(\rho \int_0^t \sigma_s dW_s^\sigma - \frac{\rho^2}{2} \int_0^t \sigma_s^2 ds \right)}_{\text{Measurable with respect to } \mathcal{F}_t^{\sigma, h}} \exp \left(\sqrt{1 - \rho^2} \int_0^t \sigma_s dW_s^F - \frac{1 - \rho^2}{2} \int_0^t \sigma_s^2 ds \right).$$

Preconditioning by $\mathcal{F}_T^{\sigma, h}$ yields

$$\begin{aligned} \mathbb{E} \left[\left(\frac{F_T}{F_\tau} - K\right)_+ \right] &= \mathbb{E} \left[\mathbb{E} \left[\left(\frac{F_T}{F_\tau} - K\right)_+ \middle| \mathcal{F}_T^{\sigma, h} \right] \right] \\ &= \mathbb{E} \left[\mathbb{E} \left[\left(F_{\tau, T} \exp \left(\sqrt{1 - \rho^2} \int_0^t \sigma_t dW_t^F - \frac{1 - \rho^2}{2} \int_0^t \sigma_s^2 ds \right) - K \right)_+ \middle| \mathcal{F}_T^{\sigma, h} \right] \right] \\ &= \mathbb{E} \left[\text{PrimeBS} \left(F_{\tau, T}, \underbrace{\left((1 - \rho^2)^{\frac{1}{2}} \int_\tau^T \sigma_s^2 ds \right)^{\frac{1}{2}}}_{=: I_{\tau, T}^\sigma}, T - \tau, K \right) \right], \end{aligned} \quad (37)$$

where $F_{\tau, T} := \exp \left(\rho \int_\tau^T \sigma_s dW_s^\sigma - \frac{\rho^2}{2} \int_\tau^T \sigma_s^2 ds \right)$ and where PrimeBS is the closed-form expression for the price of a Call option in the Black & Scholes model, detailed in Appendix A. The aim is to estimate the expectation (37) by a quantization-based cubature associated with the functional quantization of B^h and W^σ . We thus need to write the terms $F_{\tau, T}$ and $\int_\tau^T \sigma_s^2 ds$ as explicit functionals of the paths of W^σ and B^h in $L^2([0, T])$.

First, let us recall that σ is the exponential of a mixed multifractional Ornstein-Uhlenbeck process, which can be seen as an explicit functional of W^σ and B^h . More precisely we have

$$\sigma_t = \exp \left(\ln(\sigma_0) e^{-\theta t} + \mu (1 - e^{-\theta t}) + \gamma_\sigma e^{-\theta t} I_t^{e^{\theta \cdot}}(W^\sigma) + \gamma_h e^{-\theta t} I_t^{e^{\theta \cdot}}(B^h) \right). \quad (38)$$

Hence it is an explicit functional of the paths of W^σ and B^h , and so is $\int_\tau^T \sigma_s^2 ds$. We denote respectively by $(p_j^h)_{1 \leq j \leq N_1}$ and $(\chi_j^h)_{1 \leq j \leq N_1}$ the weights and paths of the quantizer \widehat{B}^h of B^h and respectively by $(p_j^\sigma)_{1 \leq j \leq N_2}$ and $(\chi_j^\sigma)_{1 \leq j \leq N_2}$ the weights and paths of the quantizer \widehat{W}^σ of W^σ . Conditionally on $B^h = \chi_i^h$, we have $I_{\tau, T}^\sigma = I_{\tau, T}^{\sigma^i}$ where

$$I_{\tau, T}^{\sigma^i} := \int_\tau^T \sigma_s^i dW_s^\sigma$$

and

$$\sigma_t^i = \exp \left(\ln(\sigma_0) e^{-\theta t} + \mu (1 - e^{-\theta t}) + \gamma_\sigma \int_0^t e^{\theta(s-t)} dW_s^\sigma + \gamma_h e^{-\theta t} I_t^g(\chi_i^h) \right).$$

Since we know, thanks to Appendix B, that χ_i^h has bounded variations, it is clear that σ^i is a semimartingale. Let us define $\langle \sigma^i, W^\sigma \rangle_{\tau, T} := \langle \sigma^i, W^\sigma \rangle_T - \langle \sigma^i, W^\sigma \rangle_\tau$, where $\langle \cdot, \cdot \rangle$ denotes the semimartingale bracket and let us denote by $\int_\tau^T \sigma_s^i \circ dW_s^\sigma$ the Stratonovich integral of σ^i . With these notations $I_{\tau, T}^{\sigma^i}$ reads

$$I_{\tau, T}^{\sigma^i} = \int_\tau^T \sigma_s^i \circ dW_s^\sigma - \frac{1}{2} \langle \sigma^i, W^\sigma \rangle_{\tau, T}.$$

Itô's formula yields

$$\begin{aligned} \int_\tau^T \sigma_t^i dW_t^\sigma &= \underbrace{\frac{\sigma_T^i - \sigma_\tau}{\gamma_\sigma} - \frac{1}{\gamma_\sigma} \int_\tau^T \sigma_t^i \theta (\mu - \ln(\sigma_t^i)) dt}_{= \int_\tau^T \sigma_t^i \circ dW_t^\sigma} - \frac{\gamma_h}{\gamma_\sigma} \int_\tau^T \sigma_t^i d\chi_i^h(t) - \underbrace{\frac{\gamma_\sigma}{2} \int_\tau^T \sigma_t^i dt}_{= \frac{1}{2} \langle \sigma^i, W^\sigma \rangle_{\tau, T}}. \end{aligned}$$

Moreover one has

$$\int_{\tau}^T \widehat{\sigma}_t^i d\widehat{W}_t^{\sigma} = \frac{\widehat{\sigma}_T^i - \widehat{\sigma}_{\tau}^i}{\gamma_{\sigma}} - \frac{1}{\gamma_{\sigma}} \int_{\tau}^T \widehat{\sigma}_t^i \theta \left(\mu - \ln \left(\widehat{\sigma}_t^i \right) \right) dt - \frac{\gamma_h}{\gamma_{\sigma}} \int_{\tau}^T \widehat{\sigma}_t^i d\chi_i^h(t).$$

One can hence approximate $\int_{\tau}^T \sigma_t^i \circ dW_t^{\sigma}$ by $\int_{\tau}^T \widehat{\sigma}_t^i d\widehat{W}_t^{\sigma}$ and $\int_{\tau}^T \sigma_t^i dt$ by $\int_{\tau}^T \widehat{\sigma}_t^i dt$. Thus we approximate $I_{\tau,T}^{\sigma^i}$ by $\widehat{I}_{\tau,T}^{\sigma^i} := \int_{\tau}^T \widehat{\sigma}_s^i d\widehat{W}_s^{\sigma} - \frac{\gamma_{\sigma}}{2} \int_{\tau}^T \widehat{\sigma}_s^i ds$.

In this setting, the cubature formula is completely explicit. We finally obtain the following approximation

$$\mathbb{E} \left[\left(\frac{F_T}{F_{\tau}} - K \right)_+ \right] \approx \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} p_i^h p_j^{\sigma} \text{PrimeBS} \left(F_{\tau,T}^{i,j}, \left((1 - \rho^2) \frac{1}{T - \tau} \int_{\tau}^T (\sigma^{i,j}(s))^2 ds \right)^{\frac{1}{2}}, T - \tau, K \right),$$

where

$$F_{\tau,T}^{i,j} = \exp \left(\rho \int_{\tau}^T \sigma^{i,j}(s) d\chi_j^{\sigma}(s) - \rho \gamma^{\sigma} \frac{1}{2} \int_{\tau}^T \sigma^{i,j}(s) ds - \frac{\rho^2}{2} \int_{\tau}^T (\sigma^{i,j}(s))^2 ds \right),$$

and

$$\sigma^{i,j}(t) := \exp \left(\ln(\sigma_0) e^{-\theta t} + \mu (1 - e^{-\theta t}) + \gamma_{\sigma} e^{-\theta t} I_t^{e^{\theta \cdot}}(\chi_j^{\sigma}) + \gamma_h e^{-\theta t} I_t^{e^{\theta \cdot}}(\chi_j^h) \right).$$

5 The multifractional SABR model

Let us consider $h \in C^1([0, T], (0, 1))$. We assume that, under the risk-neutral measure, the forward price of a risky asset is the solution of the S.D.E.

$$\begin{cases} dF_t = F_t \sigma_t dW_t, \\ d\sigma_t = \sigma_t \left(\gamma_h d^{\circ} B_t^h + \gamma_{\sigma} dW_t^{\sigma} \right), \quad \sigma_0 > 0, \end{cases} \quad (39)$$

where W and W^{σ} are two standard Brownian motions and B^h is a well-balanced multifractional Brownian motion independent of W and W^{σ} with functional parameter h . We assume that W is decomposed into $\rho dW_t^{\sigma} + \sqrt{1 - \rho^2} dW_t^F$, where W^F is a Brownian motion independent of W^{σ} . We use the same notations as in the previous section for \mathcal{F}^{σ} , \mathcal{F}^F , \mathcal{F}^h , $\mathcal{F}^{\sigma,h}$ and $\mathcal{F}^{F,\sigma,h}$. Hence, S.D.E. (39) writes

$$\begin{cases} dF_t = F_t \sigma_t \left(\rho dW_t^{\sigma} + \sqrt{1 - \rho^2} dW_t^F \right) \\ d\sigma_t = \sigma_t \left(\gamma_h d^{\circ} B_t^h + \gamma_{\sigma} dW_t^{\sigma} \right), \quad \sigma_0 > 0. \end{cases} \quad (40)$$

For the same reasons as in the case of the Hull & White model presented in Section 4, if $\rho \leq 0$ the so-defined process F is a $\mathcal{F}^{F,\sigma,h}$ -martingale. Following the same method as for the multifractional Hull & White model except that here, the solution of the stochastic differential equation related to the volatility process σ , established in Theorem 3.4 is

$$\sigma_t = \sigma_0 \exp^{\diamond} \left(\gamma_{\sigma} W_t^{\sigma} + \gamma_h B_t^h \right) = \sigma_0 \exp \left(\gamma_{\sigma} W_t^{\sigma} + \gamma_h B_t^h - \frac{1}{2} \left(\gamma_{\sigma}^2 t + \gamma_h^2 t^{2h(t)} \right) \right). \quad (41)$$

6 Numerical experiments

6.1 Variance reduction method for the quantization-based cubature

Numerical experiments carried out in [12] showed that in the case of vanilla options, the accuracy of the obtained implied volatility when using the *estimated forward* instead of the theoretical forward in the Black & Scholes formula is improved. We refer to [12] for more details. The counterpart of this method in the setting of forward start options is to plug the quantity

$$\mathcal{J}_{N_1, N_2} := \sum_{1 \leq i \leq N_1, 1 \leq j \leq N_2} p_i^h p_j^{\sigma} F_{\tau,T}^{i,j} \quad (42)$$

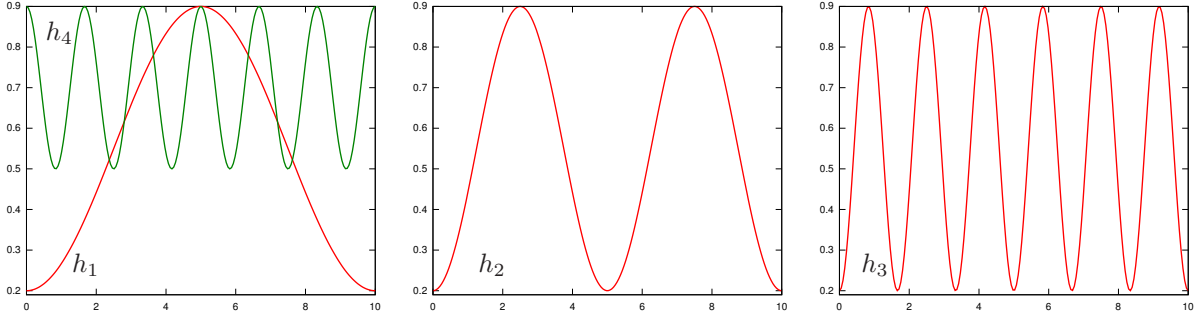


Figure 3: Left: functions h_1 and h_4 . Middle: function h_2 . Right: function h_3 .

instead of 1 in Formula (45). This also holds when using Richardson-Romberg extrapolation, case in which one uses the extrapolated value of \mathfrak{J}_{N_1, N_2} instead of 1 in Formula (45). We used this method to generate the numerical results presented below.

6.2 Numerical results

We present results on the multifractional Hull & White model. We have computed the price as a function of strike for different maturities: 1, 2.5, 5 and 10 years. Driving noises were chosen in the class of fBm and mBm. More precisely, we display results of our experiments with:

1. An fBm with $H = 0.2$.
2. An fBm with $H = 0.5$.
3. An fBm with $H = 0.75$.
4. An fBm with $H = 0.9$.
5. An mBm with $h = h_1 = 0.35 \sin\left(\frac{2\pi}{10}\left(t + \frac{15}{2}\right)\right) + 0.55$.
6. An mBm with $h = h_2 = 0.35 \sin\left(\frac{2\pi}{5}\left(t + \frac{15}{4}\right)\right) + 0.55$.
7. An mBm with $h = h_3 = 0.35 \sin\left(\frac{6\pi}{5}\left(t + \frac{5}{4}\right)\right) + 0.55$.
8. An mBm with $h = h_4 = -0.2 \sin\left(\frac{6\pi}{5}\left(t + \frac{5}{4}\right)\right) + 0.7$.

The four functions are plotted on Figure 6.2. The values for the other parameters are $\gamma_h = 0.3$, $\gamma_\sigma = \rho = 0$, $\theta = 0.3$, $\mu = \ln(0.2)$, $\sigma_0 = 0.2$ and $F_0 = 100$.

These particular h functions were chosen in order to evidence the fact that a weighted average of the past values of the regularity function governs the shape of the smile. The results displayed below provide an experimental justification to the claims made in the introduction. Indeed, one sees that, for the short maturity $t = 1$ year, in the fractional Hull & White model (*i.e.* with h constant), the smiles are more pronounced for small H and decrease as H increase, while the reverse is true for all maturities larger than one year (Figure 4). Thus, indeed, stronger correlations in the driving noise translate in this model into a slower decrease of the smile as maturities increase, as noted in [9]. However, with such an fBm-based model, an H larger than $1/2$ is needed to ensure long-range dependence and thus a more realistic evolution of the smile as compared to the Brownian case. As mentioned above, this is not compatible with empirical graphs of the volatility which show a very irregular behavior, and would require a small H . In addition, the local regularity of the volatility varies in time, calling for a varying H , *i.e.* an mBm.

Another aspect is that a fixed H , as in a modeling with fBm, does not allow to control independently the shape of the smiles at different maturities. This is possible with mBm, where the smile at maturity t depends on a weighted average of the values of h up to time t , as can be inferred from equalities (23) and

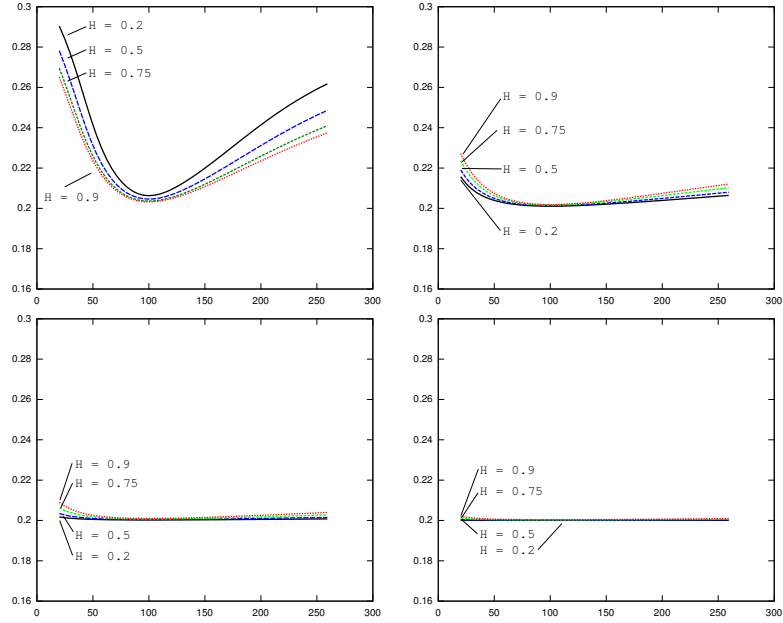


Figure 4: Comparisons of vanilla option volatility smiles for fBm with $H = 0.2, H = 0.5, H = 0.7$ and $H = 0.9$ at different maturities. Top left: $t = 1$. Top right: $t = 2.5$. Bottom left: $t = 5$. Bottom right: $t = 10$.

(36). This is apparent on Figure 5. We have compared fBms and mBms at various maturities t , where H and h are chosen such that $h(t) = H$, or, for the bottom right plot, $h_1(t) = h_4(t)$. One sees that the shape of the smile depends on a weighted average of past values of h . For instance, in the bottom left plot, the values of h before $t = 2.5$ are in average smaller than 0.9, resulting in a flatter smile. The fact that a *weighted* average must be considered is apparent on the bottom right plot: indeed, the smile is more pronounced for h_1 , although the average from 0 to 5 of this function is smaller than the one of h_4 . In contrast, the values in the immediate past of $t = 5$ are larger for h_1 than for h_4 , as may be checked on Figure 6.2. An adequate choice of h may thus allow one to better approximate a whole implied volatility surface. This topic will be addressed in a future work.

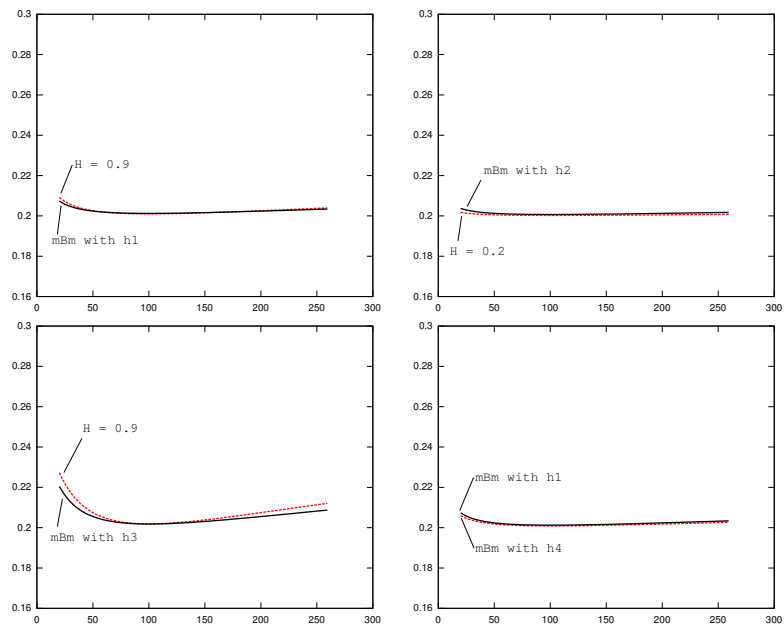


Figure 5: Comparisons of vanilla option volatility smiles for various fBm and mBm at several maturities. Top left: mBm with function h_1 and fBm with $H = 0.9$ at $t = 5$, where $h_1(t) = 0.9$. Top right: mBm with function h_2 and fBm with $H = 0.2$ at $t = 5$, where $h_2(t) = 0.2$. Bottom left: mBm with function h_3 and fBm with $H = 0.9$ at $t = 2.5$, where $h_3(t) = 0.9$. Bottom right: mBm with function h_1 and mBm with function h_4 at $t = 5$, where $h_1(t) = h_4(t) = 0.9$.

Appendices

A The implied forward start volatility

The forward start option price.

Let W be a standard Brownian motion on $[0, T]$ and $\tau \in (0, T)$. Let us consider the stochastic differential equation $dS_t = S_t \sigma_t dW_t$ (with $(\sigma_t)_{t \in [0, T]}$ a deterministic process) whose solution is a geometric Brownian motion $S_t = S_0 \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2} \int_0^t \sigma_s^2 ds\right)$. The forward start Call option price $\text{FSPrimeBS}(\sigma, \tau, T, K)$ is given by

$$\text{FSPrimeBS}(\sigma, \tau, T, K) = \mathbb{E} \left[\left(\frac{S_T}{S_\tau} - K \right)_+ \right] = \mathcal{N}(d_1) - K \mathcal{N}(d_2),$$

where $d_1 := \frac{\bar{\sigma}\sqrt{T-\tau}}{2} + \frac{\ln(K)}{\bar{\sigma}\sqrt{T-\tau}}$, $d_2 := d_1 - \frac{\bar{\sigma}\sqrt{T-\tau}}{2}$ and $\bar{\sigma}^2 := \frac{1}{T-\tau} \int_\tau^T \sigma_s^2 ds$. In other words, we have

$$\text{FSPrimeBS}(\sigma, \tau, T, K) = \text{PrimeBS}(1, \sigma, T - \tau, K), \quad (43)$$

where $(S_0, \text{Vol}, \text{Mat}, \text{Strike}) \mapsto \text{PrimeBS}(S_0, \text{Vol}, \text{Mat}, \text{Strike})$ is the closed-form expression for the vanilla Call option price in the Black & Scholes model.

The implied forward start volatility.

In the Black & Scholes model, where the asset price follows a geometric Brownian motion with a constant volatility, the forward start Call (or Put) option price is an increasing function of the volatility (if the strike is not zero). Conversely, for a given forward start Call (or Put) option price, the Black & Scholes implied volatility is the unique value of the volatility for which the Black & Scholes formula recovers the price; *i.e.* the implied forward start volatility associated with a given forward F_0 , a forward start date τ , maturity $T > \tau$, strike K , and option price P is defined by

$$P = \text{FSPrimeBS}(\text{ImpliedFSVolBS}(\tau, T, K, P), \tau, T, K). \quad (44)$$

Thanks to Equation (43), this yields to

$$\text{ImpliedFSVolBS}(\tau, T, K, P) = \text{ImpliedVolBS}(1, T - \tau, K, P), \quad (45)$$

where $\text{ImpliedVolBS}(\text{Fwd}, \text{maturity}, \text{Strike}, \text{Price})$ is the Black & Scholes implied volatility a certain forward, maturity, strike and option price.

B Variations of the Karhunen-Loève eigenfunctions of mBm

For every integer k in \mathbb{N} and C^1 deterministic function h , define the map $I_k : [0, T] \rightarrow \mathbb{R}$ by $I_k(t) := \int_0^T R_h(t, s) e_k^h(s) ds$, where R_h is the covariance function of a normalized mBm with functional parameter h (defined by (3)) and e_k^h is the k th Karhunen-Loève eigenfunction of mBm. Thanks to Equality (5), we know that I_k is nothing but $\lambda_k^h e_k^h$, where λ_k^h denotes the eigenvalue associated with e_k^h . The result is the following

Theorem B.1. *For every integer k , the map e_k^h has bounded variations on $[0, T]$.*

Proof: For every fixed (k, t) in $\mathbb{N} \times [0, T]$, we get

$$\begin{aligned} I_k(t) &= \int_0^T \frac{c_{h_t, s}^2}{c_{h(t)} c_{h(s)}} t^{2h_{t, s}} e_k^h(s) ds + \int_0^T \frac{c_{h_t, s}^2}{c_{h(t)} c_{h(s)}} s^{2h_{t, s}} e_k^h(s) ds - \int_0^T \frac{c_{h_t, s}^2}{c_{h(t)} c_{h(s)}} |t - s|^{2h_{t, s}} e_k^h(s) ds \\ &=: F_1(t) + F_2(t) - F_3(t). \end{aligned} \quad (46)$$

It is sufficient to show that F_i has bounded variations for every i in $\{1, 2, 3\}$. Since the case of F_1 , F_2 and F_3 is similar, we only treat here the case of F_1 . Let $(t_i)_{0 \leq i \leq N}$ be a sequence of elements of $[0, T]$ such that $0 = t_0 < t_1 < \dots < t_N = T$. For any i in $\{1, \dots, N\}$ we get,

$$\begin{aligned}
|F_1(t_i) - F_1(t_{i-1})| &\leq \overbrace{\sup_{s \in [0, T]} \left| \frac{e_k(s)}{c_h(s)} \right|}^{=: K_1} \int_0^T \left| \frac{c_{h_{t_i, s}}^2}{c_h(t_i)} t_i^{2h_{t_i, s}} - \frac{c_{h_{t_{i-1}, s}}^2}{c_h(t_{i-1})} t_{i-1}^{2h_{t_{i-1}, s}} \right| ds \\
&\leq K_1 \left(\underbrace{\int_0^T \frac{c_{h_{t_i, s}}^2}{c_h(t_i)} \left| t_i^{2h_{t_i, s}} - t_{i-1}^{2h_{t_{i-1}, s}} \right| ds}_{=: G_i} + \underbrace{\int_0^T \left| \frac{c_{h_{t_i, s}}^2}{c_h(t_i)} - \frac{c_{h_{t_{i-1}, s}}^2}{c_h(t_{i-1})} \right| t_{i-1}^{2h_{t_{i-1}, s}} ds}_{=: L_i} \right). \quad (47)
\end{aligned}$$

Since the map $(s, t) \mapsto \frac{c_{h_{t, s}}^2}{c_h(t)}$ is C^1 as soon as h is C^1 , we get, thanks to the mean-value theorem,

$$\left| \frac{c_{h_{t_i, s}}^2}{c_h(t_i)} - \frac{c_{h_{t_{i-1}, s}}^2}{c_h(t_{i-1})} \right| \leq \sup_{s \in [0, T]} |f'_s(t)| |t_i - t_{i-1}| =: K_2 |t_i - t_{i-1}|,$$

where $f'_s(t)$ denotes, for every s in $[0, T]$, the derivative, at point t , of the map $t \mapsto \frac{c_{h_{t, s}}^2}{c_h(t)}$. Setting $[H_1, H_2] := \left[\inf_{u \in [0, T]} h(u), \sup_{u \in [0, T]} h(u) \right]$, we hence have, for every integer i in $\{1, \dots, N\}$,

$$L_i \leq K_2 |t_i - t_{i-1}| \int_0^T t_{i-1}^{2h_{t_{i-1}, s}} ds \leq T (1 + K_2) |t_i - t_{i-1}| (e^{2H_1 T} + e^{2H_2 T}) =: K_3 |t_i - t_{i-1}|. \quad (48)$$

Besides, we get $G_i \leq \sup_{(t, s) \in [0, T]^2} \left| \frac{c_{h_{t, s}}^2}{c_h(t)} \right| \int_0^T \left| t_i^{2h_{t_i, s}} - t_{i-1}^{2h_{t_{i-1}, s}} \right| ds =: K_4 \int_0^T \left| t_i^{2h_{t_i, s}} - t_{i-1}^{2h_{t_{i-1}, s}} \right| ds$.

Now, writing

$$t_i^{2h_{t_i, s}} - t_{i-1}^{2h_{t_{i-1}, s}} = \underbrace{t_i^{2h_{t_i, s}} - t_{i-1}^{2h_{t_i, s}}}_{=: C_i(s)} - \underbrace{t_{i-1}^{2h_{t_i, s}} - t_{i-1}^{2h_{t_{i-1}, s}}}_{=: D_i(s)},$$

we easily get that

$$\forall s \in [0, T], \quad |C_i(s)| \leq 2H_2 \int_{t_{i-1}}^{t_i} (x^{2H_2-1} - x^{2H_1-1}) dx.$$

Moreover define the family of functions $(g_\alpha)_{\alpha \in \mathbb{R}_+^*}$, as being maps from \mathbb{R}_+ to \mathbb{R}_+ , such that $g_\alpha(x) := \alpha^x$ if $x > 0$ and by $g_\alpha(x) := 1$ if $x = 0$. Define the real $K_5 := \sup_{\alpha \in [0, T]} |\ln(\alpha)| (e^{2H_1 \ln(\alpha)} + e^{2H_2 \ln(\alpha)})$ and using the mean-value theorem with function g_α , we get

$$\forall s \in [0, T], \quad |D_i(s)| \leq 2^{-1} K_5 |2h_{t_i, s} - 2h_{t_{i-1}, s}| \leq K_5 \sup_{u \in [0, T]} |h'(u)| |t_i - t_{i-1}| =: K_6 |t_i - t_{i-1}|.$$

We hence have shown that

$$\forall i \in \{1; \dots; N\}, \quad G_i \leq \overbrace{(1+T)(1+2H_2)(1+K_4)(1+K_6)}^{=: K_7} \left(|t_i - t_{i-1}| + \int_{t_{i-1}}^{t_i} (x^{2H_2-1} - x^{2H_1-1}) dx \right). \quad (49)$$

Using (48) and (49) we finally obtain

$$\sum_{i=1}^N |F_1(t_i) - F_1(t_{i-1})| \leq 2K_7 \left(1 + \frac{1}{2H_1} \right) (T + T^{2H_1} + T^{2H_2}) < +\infty,$$

which ends the proof. □

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