Data completion problems solved as Nash games

A Habbal[1] **and M Kallel**[2]

[1] Laboratoire J.A.Dieudonné, Université de Nice & INRIA, France
[2] Laboratoire LAMSIN-ENIT & IPEIT, Université de Tunis, Tunisia

E-mail: `habbal@polytech.unice.fr` and `moez.kallel@ipeit.rnu.tn`

Abstract. We consider the Cauchy problem for an elliptic operator, formulated as a Nash game. The over specified Cauchy data are split among two players : the first player solves the elliptic equation with the Dirichlet part of the Cauchy data prescribed over the accessible boundary, and a variable Neumann condition (which we call first player's strategy) prescribed over the inaccessible part of the boundary. The second player makes use correspondingly of the Neumann part of the Cauchy data, with a variable Dirichlet condition prescribed over the inaccessible part of the boundary. The first player then minimizes the gap related to the non used Neumann part of the Cauchy data, and so does the second player with a corresponding Dirichlet gap. The two costs are coupled through a distributed field gaps. We prove that there exists always a unique Nash equilibrium, which turns out to be the reconstructed data when the Cauchy problem has a solution. We also prove that the completion algorithm is stable with respect to noise. Some numerical 2D and 3D experiments are provided to illustrate the efficiency and stability of our algorithm.

1. Introduction

We consider the following elliptic Cauchy problem :

$$\left\{\begin{array}{rcll} \nabla.(k\nabla u) & = & 0 & \text{in} \quad \Omega \\ u & = & f & \text{on} \quad \Gamma_c \\ k\nabla u.\nu & = & \Phi & \text{on} \quad \Gamma_c \end{array}\right. \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^d$ ($d = 2,\, 3$) with a sufficiently smooth boundary $\partial\Omega$ composed of two connected disjoint components Γ_c and Γ_i. The parameters k, f and Φ are given functions, ν is the unit outward normal vector on the boundary. The Dirichlet data f and the Neumann data Φ are the so-called Cauchy data, which are known on the accessible part Γ_c of the boundary $\partial\Omega$ and the unknown field u is the Cauchy solution.

The above Cauchy problem is also known as a data completion problem, where the data to be recovered, or missing data, are $u_{|\Gamma_i}$ and $k\nabla u.\nu_{|\Gamma_i}$, which are determined as soon as one knows u in the whole domain Ω. Cauchy problem is a prototype of Inverse Boundary Value Problems (IBVP), which model a wide field of applications ranging from medical imaging to detection and nondestructive testing, and addressing quasi exhaustively all the fields of physics, from electromagnetism to acoustics, fluid and structural mechanics (see e.g. [8, 9, 12, 16]).

Classically, IBVP are known to be ill-posed. For instance, the solution of Cauchy problem does not always exist for any pair of data (f, Φ), and if such a solution exists, it does not always depend continuously on the data (Hadamard's ill-posedness, see [20]). The Cauchy data (f, Φ) are called compatible (or consistent) if the corresponding Cauchy problem (1) has a solution (it is then unique thanks to classical continuation arguments). Ill-posedness in the sense of Hadamard makes classical numerical methods usually inappropriate because they are unstable, and there is a need for carefully stabilized dedicated computational methods, sometimes by regularizing (through reformulation of) the Cauchy problem itself. Readers may refer to a wide literature dealing with the efficient numerical solution of elliptic Cauchy problems, e.g. [3, 13, 14, 15, 17, 22] and, dealing with the ill-posedness for the Cauchy problem, [2, 10, 11] among many others.

Our purpose is to introduce an original method to solve the Cauchy problem, based on a game theory approach. We first recall in section 2 an optimal control formulation to solve the Cauchy problem used in [1]. We then show in section 3 that the control formulation naturally leads to a Nash game of static nature with complete information, which involves a Dirichlet gap and a Neumann gap costs. The existence and uniqueness of the Nash equilibrium is proved, and when the Cauchy solution exists, it turns out that the Nash equilibrium is exactly the pair of missing data of the Cauchy problem; afterwards, we end the section with a convergence result with respect to noisy data. Section 4 is devoted to sensitivity and implementation aspects, used to lead some numerical experiments. The numerical results are presented in section 5 to illustrate the efficiency of the present game-based approach. We end the paper by some concluding remarks.

The reader interested in some PDE's oriented applications of game theory may refer to [18, 19], and refer to [7, 23, 24] for a general introduction and proof of convergence of computational methods for Nash equilibria, and finally refer to [4, 5, 6] for a study of alternating algorithms which are in close link with our present approach.

2. An optimal control formulation of the Cauchy problem

We assume that the boundary $\partial\Omega$ and the data k, Φ and f are smooth enough, at least $\partial\Omega$ is of class C^2 and $(\Phi, f) \in L^2(\Gamma_c) \times H^1(\Gamma_c)$. In this case, the Cauchy solution u, if it exists, belongs to the space $H^{3/2}(\Omega)$.

In [1], the authors formulate the Cauchy problem (1) as an optimal control one. The setting is as follows :

For given $\eta \in L^2(\Gamma_i)$ and $\tau \in H^1(\Gamma_i)$, let us define $u_1(\eta)$ and $u_2(\tau)$ as the unique solutions in $H^1(\Omega)$ of the following elliptic boundary value problems :

$$(SP1)\begin{cases} \nabla.(k\nabla u_1) = 0 & \text{in } \Omega \\ u_1 = f & \text{on } \Gamma_c \\ k\nabla u_1.\nu = \eta & \text{on } \Gamma_i \end{cases} \quad (SP2)\begin{cases} \nabla.(k\nabla u_2) = 0 & \text{in } \Omega \\ u_2 = \tau & \text{on } \Gamma_i \\ k\nabla u_2.\nu = \Phi & \text{on } \Gamma_c \end{cases} \tag{2}$$

The optimization problem amounts to minimize, among all Neumann-Dirichlet pairs $(\eta,\tau) \in L^2(\Gamma_i) \times H^1(\Gamma_i)$, the following "Neumann-gap" cost :

$$\mathcal{J}_1(\eta,\tau) = \mathrm{J}_1(\eta,\tau,u_1(\eta),u_2(\tau)) \ = \ \frac{1}{2}\int_{\Gamma_c}(k\nabla u_1.\nu - \Phi)^2 d\Gamma_c + \frac{1}{2}\int_{\Gamma_i}(u_1 - u_2)^2 d\Gamma_i. \tag{3}$$

The authors in [1] proved that *when the Cauchy problem (1) has a solution*, then solving it is equivalent to solving the minimization problem

$$\min_{(\eta,\tau)\in L^2(\Gamma_i)\times H^1(\Gamma_i)} \mathcal{J}_1(\eta,\tau). \tag{4}$$

They also proved that the functional $\mathcal{J}_1$ is twice Fréchet differentiable and strictly convex.

The same conclusions above hold when a "Dirichlet-gap" cost is considered :

$$\mathcal{J}_2(\eta,\tau) = \mathrm{J}_2(\eta,\tau,u_1(\eta),u_2(\tau)) \ = \ \frac{1}{2}\int_{\Gamma_c}(u_2 - f)^2 d\Gamma_c \ + \ \frac{1}{2}\int_{\Gamma_i}(u_1 - u_2)^2 d\Gamma_i. \tag{5}$$

Let us remark that the choice of the functional spaces $L^2(\Gamma_i)$ and $H^1(\Gamma_i)$ as control spaces allows for the cost functions to be well defined, since we know from classical a priori estimates for elliptic boundary systems that u_1 and u_2 belong to the space $H^{3/2}(\Omega)$ (so that the normal derivatives $k\nabla u_1.\nu$ and $k\nabla u_2.\nu$ belong to $L^2(\Gamma_c)$, the natural space for the flux data Φ).

3. A Nash game formulation of the Cauchy problem

From the previous section, we remark that, formulated in the game theory language, the Neumann and Dirichlet controls η and τ do cooperate to minimize either the Neumann-gap or the Dirichlet-gap costs. These two controls could as well cooperatively minimize any convex combination of the two costs $\mathcal{J}_1$ and $\mathcal{J}_2$.

Now, the fields $u_1(\eta)$ and $u_2(\tau)$ are aiming at the fulfillment of a possibly antagonistic goals, namely minimizing the Neumann gap $\|k\nabla u_1.\nu - \Phi\|_{L^2(\Gamma_c)}$ and the Dirichlet gap $\|u_2 - f\|_{L^2(\Gamma_c)}$. This antagonism is intimately related to Hadamard's ill-posedness character of the Cauchy problem, and rises as soon as one requires that u_1 and u_2 coincide, which is exactly what the coupling term $\|u_1 - u_2\|_{L^2(\Gamma_i)}$ is for. Thus, one may think of an iterative process which minimizes in a smart fashion the three terms, namely Neumann-Dirichlet-Coupling terms.

Let us define the following two costs : for any $\eta \in L^2(\Gamma_i)$ and $\tau \in H^1(\Gamma_i)$,

$$J_1(\eta,\tau) = \ \frac{1}{2}\int_{\Gamma_c}(k\nabla u_1.\nu - \Phi)^2 d\Gamma_c \ + \ \frac{\alpha}{2}\int_{\Omega}(u_1 - u_2)^2 d\Omega \tag{6}$$

$$J_2(\eta,\tau) = \ \frac{1}{2}\int_{\Gamma_c}(u_2 - f)^2 d\Gamma_c \ + \ \frac{\alpha}{2}\int_{\Omega}(u_1 - u_2)^2 d\Omega \tag{7}$$

where the fields $u_1(\eta)$ and $u_2(\tau)$ are the unique solutions to (SP1) and (SP2), respectively. Differently from the definition of $\mathcal{J}_1, \mathcal{J}_2$ above, the coupling term is now distributed over the whole domain Ω and α is a given positive parameter (e.g. $\alpha = 1$).

One may consider a decomposition-like method where the variable η is used to minimize the Neumann gap + coupling term, in other words J_1, and τ is used to minimize the Dirichlet gap + coupling term, which defines J_2. Such a method fits into the area of mathematical games.

We shall say that there are two players, referred to as player 1 or Neumann-gap, and player 2 or Dirichlet-gap. Player 1 controls the strategy variable η, and player 2 controls the strategy variable τ. Each of the two players tries to minimize its own cost, namely J_1 for player 1, and J_2 for player 2. As classical, the fact that each player controls only his own strategy, while there is a strong dependance of each player's cost on the joint strategies (η, τ) justifies the use of the game theory framework (and terminology), a natural setting which may be used to formulate the negotiation between these two costs.

In order to be consistent with the initial formulation of the Cauchy problem, the relevant game theoretic framework to deal with is a static with complete information one. In this case, a commonly used solution concept (roughly speaking, in the game vocabulary, a rational and stable one) is the one of Nash Equilibria, defined as follows :

Definition 1 *A pair $(\eta_N, \tau_N) \in L^2(\Gamma_i) \times H^1(\Gamma_i)$ is a non-cooperative Nash equilibrium (NE) if*

$$\left\{ \begin{array}{ll} J_1(\eta_N, \tau_N) \leq J_1(\eta, \tau_N), & \forall \eta \in L^2(\Gamma_i), \\ \\ J_2(\eta_N, \tau_N) \leq J_2(\eta_N, \tau), & \forall \tau \in H^1(\Gamma_i). \end{array} \right. \tag{8}$$

It is of importance to notice that the present game has a separable structure. Indeed, the players criteria are formed of individual costs, the Neumann-gap depending only on η for player 1 and the Dirichlet-gap, depending only on τ for player 2, plus a *common* coupling cost which depends on both η and τ. Such game belongs to a family referred to as Inertial Nash Equilibration Processes in [5]. The game separable structure is crucial in our study, and we shall exploit it to prove that there exists a unique Nash equilibrium, which is shown to be the missing data when a Cauchy solution does exist. Based upon this structure of the criteria, we shall also establish a convergence result with respect to noisy data.

As a preliminary, let us remark that the field $u_1(\eta)$ is affine with respect to η, and so is the field $u_2(\tau)$ w.r.t. the variable τ. Thus, the functions J_1 and J_2 are quadratic. Following e.g. [1], it is an easy exercise to compute their second order differentials.

Let us consider the case of J_1, the one of J_2 follows the same steps. First, notice that we can set

$$u_1(\eta) = u_{1,0}(\eta) + u_{1,f}$$

where $u_{1,0}(\eta)$ solves the boundary value problem :

$$\left\{ \begin{array}{rcll} \nabla.(k\nabla u_{1,0}) & = & 0 & \text{in} \quad \Omega \\ u_{1,0} & = & 0 & \text{on} \quad \Gamma_c \\ k\nabla u_{1,0}.\nu & = & \eta & \text{on} \quad \Gamma_i \end{array} \right. \tag{9}$$

Then it is easy to compute the second order differential of J_1 w.r.t. η in any direction $\psi \in L^2(\Gamma_i)$ which reads :

$$(d^2 J_1(\eta, \tau).\psi, \psi) = \int_{\Gamma_c} (k\nabla u_{1,0}(\psi).\nu)^2 d\Gamma_c \; + \; \alpha \int_{\Omega} (u_{1,0}(\psi))^2 d\Omega.$$

It is immediate that if $(d^2 J_1(\eta,\tau).\psi,\psi) = 0$ then $u_{1,0}(\psi) = 0$, hence $\psi = 0$.

Indeed, strict convexity of J_1 and J_2 holds w.r.t. the pair (η,τ) as well. On the contrary, we shall see that partial ellipticity (or coerciveness) of the costs holds while it does not w.r.t. the pair (η,τ), precisely because of the coupling term.

Let us again focus on the case of J_1. If partial ellipticity fails, then there exists a sequence $(\psi_n) \subset L^2(\Gamma_i)$ such that

$$|\psi_n|_{L^2(\Gamma_i)} = 1 \quad \text{and} \quad (d^2 J_1(\eta,\tau).\psi_n,\psi_n) \to 0 \text{ when } n \to +\infty. \tag{10}$$

Thanks to the classical regularity results and a priori estimates for elliptic BVPs, one gets

$$\|u_{1,0}(\psi_n)\|_{H^{\frac{3}{2}}(\Omega)} \le |\psi_n|_{L^2(\Gamma_i)}$$

So, up to a subsequence, $(u_{1,0}(\psi_n))$ weakly converges in $H^{\frac{3}{2}}(\Omega)$. By invoking the Rellich-Kondrachov theorem, the sequence strongly converges to some $z \in H^1(\Omega)$. Since $(u_{1,0}(\psi_n))$ strongly converges to zero in $L^2(\Omega)$ thanks to (10), one has $z = 0$. Now, thanks to the continuity of the normal trace operator, one has $\psi_n \to k\nabla z.\nu$ (= 0) in $H^{-\frac{1}{2}}(\Gamma_i)$, which is in contradiction with $|\psi_n|_{L^2(\Gamma_i)} = 1$. We summarize the above preliminary results in the

Proposition 1 *The partial mapping $\eta \to J_1(\eta,\tau)$ (resp. $\tau \to J_2(\eta,\tau)$) is a quadratic strongly convex functional over $L^2(\Gamma_i)$ (resp. $H^1(\Gamma_i)$).*

It is important to notice that the partial ellipticity property of $\eta \to J_1(\eta,\tau)$ holds uniformly w.r.t. τ, and conversely for J_2. It allows us to bound the strategy spaces if necessary.

Following [5], let us introduce the functional $L(\eta,\tau)$ as follows :

$$L(\eta,\tau) = \frac{1}{2}\int_{\Gamma_c} (k\nabla u_1(\eta).\nu - \Phi)^2 d\Gamma_c + \frac{1}{2}\int_{\Gamma_c} (u_2(\tau) - f)^2 d\Gamma_c + \frac{\alpha}{2}\int_{\Omega} (u_1(\eta) - u_2(\tau))^2 d\Omega. \tag{11}$$

It is easy to check that L is strictly convex, and that any minimum of L is a Nash equilibrium and conversely, thanks to the separable structure of the present game (consider the necessary optimality conditions). We conclude that if a Nash equilibrium exists, then it is unique.

The existence of a Nash equilibrium $(\eta_N,\tau_N) \in L^2(\Gamma_i) \times H^1(\Gamma_i)$, *id est* a pair which fulfills (8), is obtained by a direct application of the Nash theorem : since strategy variables belong to Hilbert spaces, the uniform partial ellipticity of the costs allows for the choice of -large enough- closed bounded balls, which are then weakly compact convex sets, then continuity of the convex costs yields the weak lower semi-continuity over the so-defined balls.

Finally, remark that if a Cauchy solution u does exist, then by setting $\eta_C = k\nabla u.\nu_{|\Gamma_i}$ and $\tau_C = u_{|\Gamma_i}$, one has immediately $L(\eta_C,\tau_C) = 0$, since thanks to the uniqueness of the Cauchy solution, $u_1(\eta_C) = u_2(\tau_C) = u$. In this case, one has that (η_C,τ_C) is the minimum of the nonnegative functional L, and so it is also the Nash equilibrium (η_N,τ_N).

We summarize the above results in the following:

Proposition 2 *Consider the Nash game defined by (6)-(7)-(8).*

(i) There always exists a unique Nash equilibrium $(\eta_N,\tau_N) \in L^2(\Gamma_i) \times H^1(\Gamma_i)$. It is also the minimum of $L(\eta,\tau)$ given by (11).

(ii) When the Cauchy problem has a solution u*, then* $u_1(\eta_N) = u_2(\tau_N) = u$*, and* (η_N, τ_N) *are the missing data, namely* $\eta_N = k\nabla u.\nu_{|\Gamma_i}$ *and* $\tau_N = u_{|\Gamma_i}$*.*

Let us now consider the case of noisy data. We assume that there exists a pair of compatible data (f, Φ), and denote by u the corresponding Cauchy solution. We consider a family of not necessarily compatible data $(f^\delta, \Phi^\delta) \in H^1(\Gamma_c) \times L^2(\Gamma_c)$ such that :

$$\|f^\delta - f\|^2_{H^1(\Gamma_c)} + \|\Phi^\delta - \Phi\|^2_{L^2(\Gamma_c)} \leq \delta^2. \tag{12}$$

For given $(\eta, \tau) \in L^2(\Gamma_i) \times H^1(\Gamma_i)$, the fields $u_1^\delta(\eta)$ and $u_2^\delta(\tau)$ are solution of the respective problems :

$$\left\{ \begin{array}{rcll} \nabla.(k\nabla u_1^\delta) & = & 0 & \text{in } \Omega \\ u_1^\delta & = & f^\delta & \text{on } \Gamma_c \\ k\nabla u_1^\delta.\nu & = & \eta & \text{on } \Gamma_i \end{array} \right. \qquad \left\{ \begin{array}{rcll} \nabla.(k\nabla u_2^\delta) & = & 0 & \text{in } \Omega \\ u_2^\delta & = & \tau & \text{on } \Gamma_i \\ k\nabla u_2^\delta.\nu & = & \Phi^\delta & \text{on } \Gamma_c \end{array} \right. \tag{13}$$

We define the associated cost functionals :

$$J_1^\delta(\eta, \tau) = \frac{1}{2} \int_{\Gamma_c} (k\nabla u_1^\delta.\nu - \Phi^\delta)^2 d\Gamma_c + \frac{\alpha}{2} \int_\Omega (u_1^\delta - u_2^\delta)^2 d\Omega \tag{14}$$

$$J_2^\delta(\eta, \tau) = \frac{1}{2} \int_{\Gamma_c} (u_2^\delta - f^\delta)^2 d\Gamma_c + \frac{\alpha}{2} \int_\Omega (u_1^\delta - u_2^\delta)^2 d\Omega \tag{15}$$

The functions J_1^δ and J_2^δ have the properties declined in Proposition-1, so there exists a unique corresponding Nash equilibrium $(\eta_N^\delta, \tau_N^\delta) \in L^2(\Gamma_i) \times H^1(\Gamma_i)$.

One may ask if, when $\delta \to 0$, the Nash equilibrium $(\eta_N^\delta, \tau_N^\delta)$ does converge to the missing data $(k\nabla u.\nu_{|\Gamma_i}, u_{|\Gamma_i})$ or, in other words, do the fields $u_1^\delta(\eta_N^\delta)$ and $u_2^\delta(\tau_N^\delta)$ converge to the Cauchy solution u?

Let us again use the notation $\eta_C = k\nabla u.\nu_{|\Gamma_i}$ and $\tau_C = u_{|\Gamma_i}$ the Cauchy missing data, and introduce the auxiliary functions $z_1^\delta = u_1^\delta(\eta_C) - u$ and $z_2^\delta = u_2^\delta(\tau_C) - u$. Then, using the continuity of the trace operator from $H^{\frac{3}{2}}(\Omega)$ onto $H^1(\partial\Omega)$, and the continuity of the normal trace operator from $H^{\frac{3}{2}}(\Omega)$ onto $L^2(\partial\Omega)$, it is easy to show that (obvious notation is used to define L^δ from its definition in (11)) :

$$L^\delta(\eta_C, \tau_C) \leq (1+\alpha)(\|z_1^\delta\|^2_{H^{\frac{3}{2}}(\Omega)} + \|z_2^\delta\|^2_{H^{\frac{3}{2}}(\Omega)})$$

Using classical a priori estimates (z_1^δ fulfills the elliptic equation with non-homogeneous Dirichlet condition, and z_2^δ a one with non-homogeneous Neumann condition), one gets :

$$L^\delta(\eta_C, \tau_C) \leq (1+\alpha) \left(\|f^\delta - f\|^2_{H^1(\Gamma_c)} + \|\Phi^\delta - \Phi\|^2_{L^2(\Gamma_c)} \right) \leq (1+\alpha)\delta^2. \tag{16}$$

We have previously seen that the Nash equilibrium $(\eta_N^\delta, \tau_N^\delta)$ is also the unique minimum of L^δ, thus we have :

$$J_1^\delta(\eta_N^\delta, \tau_N^\delta) \leq L^\delta(\eta_N^\delta, \tau_N^\delta) \leq L^\delta(\eta_C, \tau_C) \leq (1+\alpha)\delta^2 \tag{17}$$

and $J_2^\delta(\eta_N^\delta, \tau_N^\delta) \leq (1+\alpha)\delta^2$ as well. Now, since the mapping $\eta \to J_1^\delta(\eta, \tau)$ is coercive, uniformly in τ (and in δ), and since the same corresponding property holds for $\tau \to J_2^\delta(\eta, \tau)$, we obtain that the sequence (η_N^δ) is uniformly bounded in $L^2(\Gamma_i)$ and (τ_N^δ) is uniformly bounded in $H^1(\Gamma_i)$. Up to a subsequence, we have from one part that η_N^δ converges weakly to some $\eta_N^0 \in L^2(\Gamma_i)$, strongly in the $H^{-\frac{1}{2}}(\Gamma_i)$ topology. From other part, the sequence τ_N^δ converges weakly to some $\tau_N^0 \in H^1(\Gamma_i)$, strongly in the $H^{\frac{1}{2}}(\Gamma_i)$ topology. Since the sequence f^δ strongly converges to f in $H^1(\Gamma_c)$ and Φ^δ strongly converges to Φ in $L^2(\Gamma_c)$, we conclude that the sequences $u_1^\delta(\eta_N^\delta)$ and $u_2^\delta(\tau_N^\delta)$, which are the solutions to equations (13), strongly converge in $H^1(\Omega)$ to respectively $\overline{u}_1(\eta_N^0)$ and $\overline{u}_2(\tau_N^0)$ which are the unique solutions to the respective equations:

$$\left\{\begin{array}{rcll} \nabla.(k\nabla\overline{u}_1) & = & 0 & \text{in} \quad \Omega \\ \overline{u}_1 & = & f & \text{on} \quad \Gamma_c \\ k\nabla\overline{u}_1.\nu & = & \eta_N^0 & \text{on} \quad \Gamma_i \end{array}\right. \qquad \left\{\begin{array}{rcll} \nabla.(k\nabla\overline{u}_2) & = & 0 & \text{in} \quad \Omega \\ \overline{u}_2 & = & \tau_N^0 & \text{on} \quad \Gamma_i \\ k\nabla\overline{u}_2.\nu & = & \Phi & \text{on} \quad \Gamma_c \end{array}\right. \tag{18}$$

Taking $\delta \to 0$ in (17) yields that $(k\nabla u_1^\delta.\nu - \Phi^\delta)$ strongly converges to 0 in $L^2(\Gamma_c)$, which means that $(k\nabla u_1^\delta.\nu)$ strongly converges to Φ in $L^2(\Gamma_c)$ so *a fortiori* in $H^{-\frac{1}{2}}(\Gamma_c)$. Hence, $k\nabla\overline{u}_1.\nu = \Phi$ over Γ_c. Thanks to the uniqueness of the Cauchy problem, we conclude that $\overline{u}_1 = u$. The same reasoning applied to the coupling term in J_1^δ would directly yield $\overline{u}_1 = \overline{u}_2$ and thanks to the equations (18), the two fields are equal to the Cauchy solution u as well.

Finally, we have proved that the sequence (η_N^δ) strongly converges in $L^2(\Gamma_i)$ to $\eta_N^0 = \eta_C = k\nabla u.\nu_{|\Gamma_i}$ and that the sequence (τ_N^δ) strongly converges in $H^1(\Gamma_i)$ to $\tau_N^0 = \tau_C = u_{|\Gamma_i}$.

Proposition 3 *Assume there exists a unique Cauchy solution $u \in H^1(\Omega)$ for a given compatible pair of data $(f,\Phi) \in H^1(\Gamma_c) \times L^2(\Gamma_c)$. Let $(f^\delta,\Phi^\delta) \in H^1(\Gamma_c) \times L^2(\Gamma_c)$ be any sequence of noisy data such that*

$$\|f^\delta - f\|^2_{H^1(\Gamma_c)} + \|\Phi^\delta - \Phi\|^2_{L^2(\Gamma_c)} \leq \delta^2.$$

Then, the Nash game corresponding to the costs J_1^δ and J_2^δ defined by (14)-(15) has a unique Nash equilibrium $(\eta_N^\delta,\tau_N^\delta) \in L^2(\Gamma_i) \times H^1(\Gamma_i)$ which strongly converges, as $\delta \to 0$, to the Cauchy missing data $(k\nabla u.\nu_{|\Gamma_i}, u_{|\Gamma_i})$.

Moreover, the solutions to the equations (13), respectively $u_1^\delta(\eta_N^\delta)$ and $u_2^\delta(\tau_N^\delta)$, strongly converge in $H^1(\Omega)$ to the Cauchy solution u.

4. Numerical procedure

From the computational viewpoint, in [5] the authors propose an alternating minimization algorithm to compute the Nash Equilibrium by means of the following iterative process :

Let (η^0,τ^0) be a given initial state ;

$$\left\{\begin{array}{l} \eta^{(k+1)} = \text{argmin}_\eta\{J_1(\eta,\tau^{(k)}) + \frac{\beta}{2}\int_{\Gamma_i}(\eta-\eta^{(k)})^2 d\Gamma_i\}, \\ \tau^{(k+1)} = \text{argmin}_\tau\{J_2(\eta^{(k+1)},\tau) + \frac{\beta}{2}\int_{\Gamma_i}(\tau-\tau^{(k)})^2 d\Gamma_i\}, \end{array}\right. \tag{19}$$

where β is a given positive parameter (e.g. $\beta = 1$).

In the cited reference, the convergence of the alternating algorithm above is proved, under suitable assumptions which also hold in our case, see Proposition 1.

Our algorithm is written as follows :

Set $k = 0$. Starting from an initial guess $S^{(0)} = (\eta^{(0)}, \tau^{(0)})$:

Step 1: Compute $\overline{\eta}^{(k)}$, which solves $\min_\eta J_1(\eta, \tau^{(k)})$;

Step 2: Compute $\overline{\tau}^{(k)}$, which solves $\min_\tau J_2(\eta^{(k+1)}, \tau)$;

Step 3: Set $S^{(k+1)} = (\eta^{(k+1)}, \tau^{(k+1)}) = t\,(\eta^{(k)}, \tau^{(k)}) + (1-t)\,(\overline{\eta}^{(k)}, \overline{\tau}^{(k)})$, $0 < t < 1$.

Redo (*Step 1*) until the sequence $S^{(k)}$ converges. As a stopping criterion we choose a classical one, the first k such that,

$$\|S^{(k+1)} - S^{(k)}\| \leq \epsilon,$$

where ϵ is given small enough parameter.

It is easy to show that the above procedure is equivalent to the algorithm (19) as soon as one uses a fixed step gradient method to solve the partial optimization problems in *Step 1* and *Step 2* above. For numerical problem, we used the discetized version which was proved to converge in [24].

To this end, the gradients may be efficiently computed by means of an adjoint state method. Let us define the following Lagrangian:

$$\begin{aligned}\mathcal{L}(\eta,\tau,\tau^*,u_1,u_2,\lambda_1,\lambda_2) = \;& \frac{\alpha}{2}\int_\Omega (u_1-u_2)^2\,d\Omega + \frac{1}{2}\int_{\Gamma_c}(k\nabla u_1.\nu - \Phi)^2\,d\Gamma_c \\ &+ \int_\Omega k\nabla u_1.\nabla\lambda_1\,d\Omega - \int_{\Gamma_i}\eta\lambda_1\,d\Gamma_i \\ &+ \int_\Omega k\nabla u_2.\nabla\lambda_2\,d\Omega - \int_{\Gamma_c}\Phi\lambda_2\,d\Gamma_c \\ &+ \int_{\Gamma_i}(u_2-\tau)\tau^*\,d\Gamma_i\end{aligned} \tag{20}$$

where $(\eta,\tau) \in L^2(\Gamma_i) \times H^1(\Gamma_i)$, $(u_1,u_2,\lambda_1,\lambda_2) \in H^{\frac{3}{2}}(\Omega) \times H^{\frac{3}{2}}(\Omega) \times W_1 \times W_2$ and $\tau^* \in L^2(\Gamma_i)$, where the latter two spaces are given by :

$$W_1 = \{v \in H^{\frac{3}{2}}(\Omega) \text{ such that } v|_{\Gamma_c} = 0\} \quad \text{and} \quad W_2 = \{v \in H^{\frac{3}{2}}(\Omega) \text{ such that } v|_{\Gamma_i} = 0\}.$$

The Lagrangian is used to compute the gradients $\nabla_\eta J_1$ and $\nabla_\tau J_2$:

Proposition 4 *We have the following two partial derivatives:*

$$\left\{\begin{aligned}&\frac{\partial J_1}{\partial\eta}(\eta,\tau)\xi = -\int_{\Gamma_i}\lambda_1\xi\,d\Gamma_i, \quad \textit{for all } \xi \in L^2(\Gamma_i)\\ &\\ &\textit{where } \lambda_1 \in W_1 \textit{ solves the adjoint problem:}\\ &\\ &\int_\Omega k\nabla\lambda_1.\nabla\gamma\,d\Omega = -\alpha\int_\Omega(u_1-u_2)\gamma\,d\Omega - \int_{\Gamma_c}(k\nabla u_1.\nu-\Phi)(k\nabla\gamma.\nu)\,d\Gamma_c, \quad \gamma\in W_1\end{aligned}\right. \tag{21}$$

and

$$\left\{\begin{array}{l} \dfrac{\partial J_2}{\partial \tau}(\eta,\tau)h = \displaystyle\int_{\Gamma_i} (k\nabla\lambda_2.\nu)h\, d\Gamma_i, \quad h \in H^1(\Gamma_i) \\ \\ \textit{where } \lambda_2 \in W_2 \textit{ solves the adjoint problem:} \\ \\ \left\{\begin{array}{lll} \nabla.k\nabla\lambda_2 = \alpha\,(u_1 - u_2) & \textit{in} & \Omega \\ k\nabla\lambda_2.n = f - u_2 & \textit{on} & \Gamma_c \\ \lambda_2 = 0 & \textit{on} & \Gamma_i \end{array}\right. \end{array}\right. \tag{22}$$

5. Numerical results

The computational methodology used to illustrate the efficiency of the present approach is classical. All experiments are performed on a Personal Computer and all the partial differential equations are numerically solved using *FreeFem++* [21], a Finite Element based free software. In order to obtain accurate approximations of the normal derivatives of u_1 and of λ_2, the dual Raviart-Thomas mixed finite elements are used.

We consider a domain Ω defined as the open bounded set delimited by two concentric circles in 2D test-cases, or two concentric spheres in 3D test-cases. The inner boundary plays the role of Γ_i, where the trace and normal derivative are missing, and the outer one plays the one of Γ_c where the latter information is over specified.

We then consider explicit well-known analytical solutions, generically denoted by u, which are harmonic inside the domain Ω, and set the trace and normal derivative of u over Γ_c as being the measured data $f = u_{|\Gamma_c}$ and $\Phi = (k\nabla u.\nu)_{|\Gamma_c}$. We sometimes refer to these data as temperature and flux (with obvious interpretation).

In order to test the robustness of the proposed method we add a white noise to the temperature f and the heat flux Φ as follows:

$$f^\sigma = u + \sigma w_1 \quad \text{and} \quad \Phi^\sigma = k\nabla u.\nu + \sigma w_2, \quad \text{on} \quad \Gamma_c, \tag{23}$$

where σ denotes the noise level relative to $\|.\|_{L^2(\Gamma_c)}$ of u and $k\nabla u.\nu$ respectively, whereas (w_1, w_2) are normelly distributed random functions.

Our algorithm performs a denoising task on the noisy prescribed Cauchy data f^σ and Φ^σ. For instance, let us denote by $(\tau_N^\sigma, \eta_N^\sigma)$ the Nash equilibrium associated to the latter noisy over specified data. Then, the pair of optimal solutions $(u_2(\tau_N)|_{\Gamma_c}, (k\nabla\tilde{u_1}.\nu)|_{\Gamma_c})$ may be viewed as regularized Cauchy data obtained from the noisy Cauchy data, where $\tilde{u_1}$ is the solution of

$$\left\{\begin{array}{lll} \Delta\tilde{u_1} = 0 & \text{in} & \Omega, \\ \tilde{u_1} = u_2(\tau_N) & \text{on} & \Gamma_c, \\ k\nabla\tilde{u_1}.\nu = \eta_N & \text{on} & \Gamma_i. \end{array}\right. \tag{24}$$

We present numerical results which illustrate the stability of our method with respect to noisy data, as well as an example of the noise deblurring property underlined above. The presented graphics are related to the profiles over Γ_i of the Dirichlet and Neumann missing data and to the fields u_1 and u_2. We also provide convergence relative errors history, related to the fields and to the missing profiles, as a function of iterations and for different noise levels. We also present a stationarity history for the Nash overall computation iterations.

The computation of the Nash equilibrium is performed as described in section 4, where the partial optimization tasks of Step-1 and Step-2 use a fixed line-search gradient method. Here, it is sufficient to carry out only few iterations in the optimization process in each step. A second formulation would be obtained, if we assume that the players compute not sequentially, but in parallel with step-1, in the Step-2 we compute $\overline{\tau}^{(k)}$ which solves $\min_\tau J_2(\eta^{(k)}, \tau)$.

An arbitrary initial guess such as $S^{(0)} = (\eta^{(0)}, \tau^{(0)}) = (0, 0)$ is chosen to start-up the algorithm, the physical function k takes the constant value 1 in Ω, and the parameters α (weight of the coupling term in the costs) and t (relaxation parameter in the computation of Nash equilibrium) are set to $\alpha = 1$ and $t = 0.25$.

5.1. Two 2D test-cases

We consider an annular domain Ω with circular boundary components Γ_i and Γ_c, both centered at $(0, 0)$ and with radii $R_i = 0.6$ and $R_c = 1$, respectively.

Test-case A. The first 2D experiment is related to a smooth case. The -artificial- Cauchy data f and Φ are defined as respectively the trace and normal derivative, over the circle Γ_c, of the harmonic function :

$$u(x, y) = e^x cos(y).$$

In figure-1, the missing data τ_N and η_N are presented at convergence of the algorithm (19) dedicated to the computation of the Nash equilibrium, for different noise levels σ. The obtained Dirichlet as well as Neumann profiles show remarkable stability with respect to noise.

Test-case B. The second 2D experiment is related to the singular function :

$$u(x, y) = Re(\frac{1}{z - a}), \quad \text{where } z = x + iy.$$

In this case, the singularity source, located at $a = (0.5, 0)$, is in the vicinity of the circle Γ_i, and reconstruction of the solution over this boundary is a numerically challenging task, particularly in the case of noisy data.

The results in figure-2 show again the stability of our method. The profile shape is well captured including the localization of the singularity peak, whose magnitude is however underestimated for the trace as well as for the normal derivative.

Finally, the denoising effect, through computing the Nash equilibrium and solving of equation (24), is actually observed in figure-3 for a noise level of 5%.

5.2. Two 3D test-cases

As for the 2D case, we consider a thick spherical shell domain Ω with boundary components Γ_i and Γ_c, which are two spheres both centered at $(0, 0, 0)$ and with radii given by respectively $R_i = 0.6$ and $R_c = 1$.

Again two functions, denoted by u, are selected to play the role of exact solutions to the Cauchy problem for the Laplace operator. To this end, the Cauchy data f and Φ are defined as respectively the trace and normal derivative of the involved functions over the sphere Γ_c.

Test-case C. The first function is radial, so it is of constant trace over each of the spherical components of the boundary:

$$u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}. \tag{25}$$

The function u given by (25) is the solution of $\Delta u = \delta_0$, where δ_0 is the Dirac distribution at the origin $(0, 0, 0)$, a point source that is not in Ω, so u is harmonic and smooth enough inside Ω.

In figure-4, level set slices are shown for the fields u_1 and u_2 at convergence for noise free and 5% noisy data. The overall Nash algorithm (19) converged in 150 iterations. More detailed issues related to the convergence are presented in figure-5. Relative L^2-errors behave well in the noise free and in the noisy cases. The relative errors on reconstructed fields decrease as well as do the ones relative to the missing data (converged Nash strategies, which we recall are respectively the trace of u_1 and the normal trace of u_2 over the sphere Γ_i).

The sensitivity of the reconstructed fields and missing data to the noise level σ is shown in figure-6. Interestingly, boundary missing data are much less sensitive than the domain distributed fields. Both of them exhibit a satisfactory stable behavior w.r.t. the noise magnitude.

Test-case D. The second function is given by

$$u(x, y, z) = \frac{1}{\sqrt{(x + 0.2)^2 + y^2 + z^2}}. \tag{26}$$

The function u given by (26) is the non-radial solution of $\Delta u = \delta_{X_0}$, where the source term is now $X_0 = (-0.2, 0, 0)$.

In this experiment we put 5% of noise in (f, Φ) according to (23). The obtained results are illustrated in figure-7, where the reconstructed fields u_1 and u_2 are presented at convergence. The non radial missing data are presented in figure-8 and relative L^2-errors on the reconstructed fields and boundary data are shown in figure-9.

Even though the error curves do monotonically decrease towards zero for the two test-cases C and D, the stagnation that appears quite early (around iterations 80-120 in the present case) suggests that multilevel or hierarchical optimization approaches should be designed to speed-up the convergence of the Nash algorithm.

6. Conclusion

Let us conclude the paper with some short remarks. First of all, we have used the simplest class of games to model the completion problem, namely the class of static games with complete information. This simple game formulation yields interesting results like the existence and uniqueness of a Nash equilibrium even when the Cauchy data are not compatible, and also the fact that the Nash equilibrium is the missing data when the Cauchy problem has a solution. Investigation of more sophisticated classes of games such as dynamical games with incomplete information may lead to new efficient data completion algorithms. It is also interesting to notice that solving the data completion problem with our method makes use of the standard computational tools, be it finite element or optimization codes. The numerical experiments presented for different test-cases prove that our method exhibits remarkable numerical stability with respect to noisy Cauchy data.

References

[1] Aboulaïch R, Ben Abda A and Kallel M 2008 Missing boundary data reconstruction via an approximate optimal control *Inverse Problems and Imaging* **2** 411–426

[2] Alessandrini G, Rondi L, Rosset E and Vessella S 2009 The stability for the Cauchy problem for elliptic equations *Inverse Problems* **25** 123004 (47pp)

[3] Andrieux S, Baranger T and Ben Abda A 2006 Solving Cauchy problems by minimizing an energy-like functional *Inverse Problems* **22** 115–133

[4] Attouch H, Redont P and Soubeyran A 2007 A new class of alternating proximal minization algorithms with costs-to-move *SIAM J. Optim.* **18** 1061–1081

[5] Attouch H, Bolte J, Redont P and Soubeyran A 2008 Alternating proximal algorithms for weakly coupled convex minimization problems. Applications to dynamical games and PDE's *J. Convex Anal.* **15** 485–506

[6] Attouch H and Soueycatt M 2009 Augmented Lagrangian and proximal alternating direction methods of multipliers in Hilbert spaces. Applications to games, PDE's and control *Pacific J. Optim* **5** 17–37

[7] Basar T 1987 Relaxation techniques and asynchronous algorithms for on-line computation of non-cooperative equilibria *Journal of Economic Dynamics and Control* **11** 531–549

[8] Ben Abda A, Kallel M, Leblond J and Marmorat J-P 2002 Line-segment cracks recovery from incomplete boundary data *Inverse Problems* **18** 1057–1077

[9] Ben Abda A, Ben Hassen F, Leblond J and Mahjoub M 2009 Sources recovery from boundary data: a model related to electroencephalography, *Mathematical and Computer Modelling* **49** 2213–2223

[10] Ben Belgacem F 2007 Why is the Cauchy problem severely ill-posed? *Inverse Problems* **23** 823–36

[11] Bourgeois L 2010 About stability and regularization of ill-posed elliptic Cauchy problems: the case of $C^{1,1}$ domains *ESAIM: M2AN* **44** 715–35

[12] Chaabane S, Jaoua M and Leblond J 2003 Parameter identification for Laplace equation and approximation in analytic classes *J. Inv. Ill-Posed Problems* **11** 35-57

[13] Cao H and Pereverzv S V 2007 The balancing principle for the regularization of elliptic Cauchy problems *Inverse Problems* **23** 1943–1961

[14] Chakib A and Nachaoui A 2006 Convergence analysis for finite element approximation to an inverse Cauchy problem *Inverse Problems* **22** 1191–1206

[15] Cimetière A, Delvare F, Jaoua M and Pons F 2001 Solution of the Cauchy problem using iterated Tikhonov regularization *Inverse Problems* **17** 553–570

[16] Cimetière A, Delvare F, Jaoua M, Kallel M and Pons F 2002 Recovery of cracks from incomplete boundary data *Inverse Problems in Engineering* **10** 377–392

[17] Dinh Nho Hao and Lesnic D 2000 The Cauchy problem for Laplace's equation via the conjugate gradient method *IMA Journal of Applied Mathematics* **65** 199–217

[18] Habbal A 2005 A topology Nash game for tumoral antiangiogenesis *Structural and Multidisciplinary Optimization* **30** 404–412

[19] Habbal A, Petersson J and Thellner M 2004 Multidisciplinary topology optimization solved as a Nash game *Int. J. Numer. Meth. Engng* **61** 949–963

[20] Hadamard J 1953 *Lectures on Cauchy's Problem in Linear Partial Differential Equation*, Dover, New york USA

[21] Hecht F, Le Hyaric A and Pironneau O *Freefem++*, Univ. Pierre et Marie Curie, Paris, software available at http://www.freefem.org/ff++/

[22] Kozlov V A, Maz'ya V G and Fomin A V 1991 An iterative method for solving the Cauchy problems for elliptic equations *Comput. Math. Phys.* **31** 45–52

[23] Li S and Basar T 1987 Distributed algorithms for the computation of noncooperative equilibria *Automatica* **23** 523–533

[24] Uryas'ev S and Rubinstein R Y 1994 On relaxation algorithms in computation of noncooperative equilibria *IEEE Transactions on Automatic Control* **39** 1263–1267

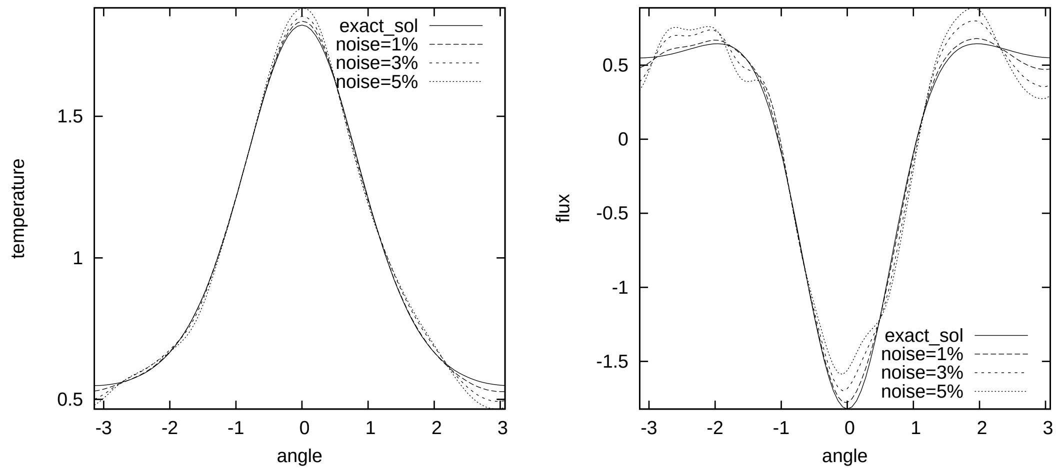

Figure 1. Test-case A. Reconstructed smooth Dirichlet (τ_N, left) and Neumann (η_N, right) data over Γ_i. The profiles are presented at convergence and for various amounts of noise level $\sigma \in \{1\%, 3\%, 5\%\}$. The corresponding traces of the exact solution are also plotted. The Finite Element computations are performed with 1529 nodes and 2788 triangles.

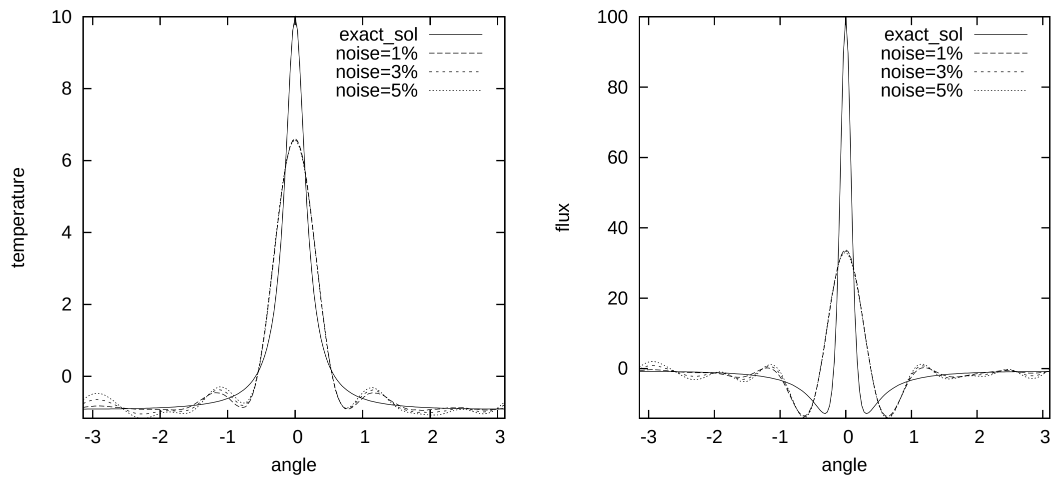

Figure 2. Test-case B. Reconstructed singular Dirichlet (τ_N, left) and Neumann (η_N, right) data over Γ_i. The profiles are presented at convergence and for various amounts of noise level $\sigma \in \{1\%, 3\%, 5\%\}$. The corresponding traces of the exact solution are also plotted. The Finite Element computations are performed with 5405 nodes and 10310 triangles.

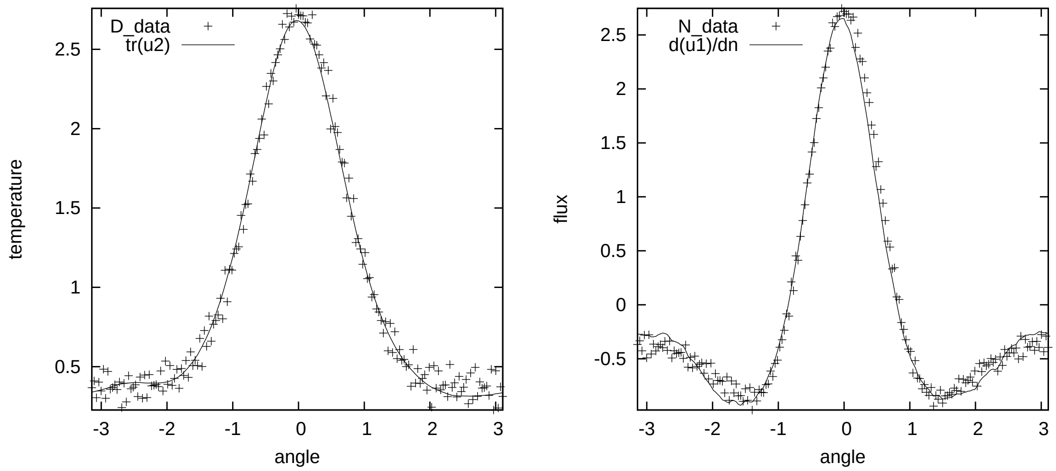

Figure 3. Test-case A. Regularization of noisy Cauchy data (noise level is $\sigma = 5\%$). At convergence : (left) the smoothed profile $u_2(\tau_N)|_{\Gamma_c}$ ($-$ line) is compared to the random f^σ ($+$ dots) ; (right) the smoothed flux profile $k\nabla\tilde{u_1}.\nu|_{\Gamma_c}$ ($-$ line) is compared to Φ^σ ($+$ dots).

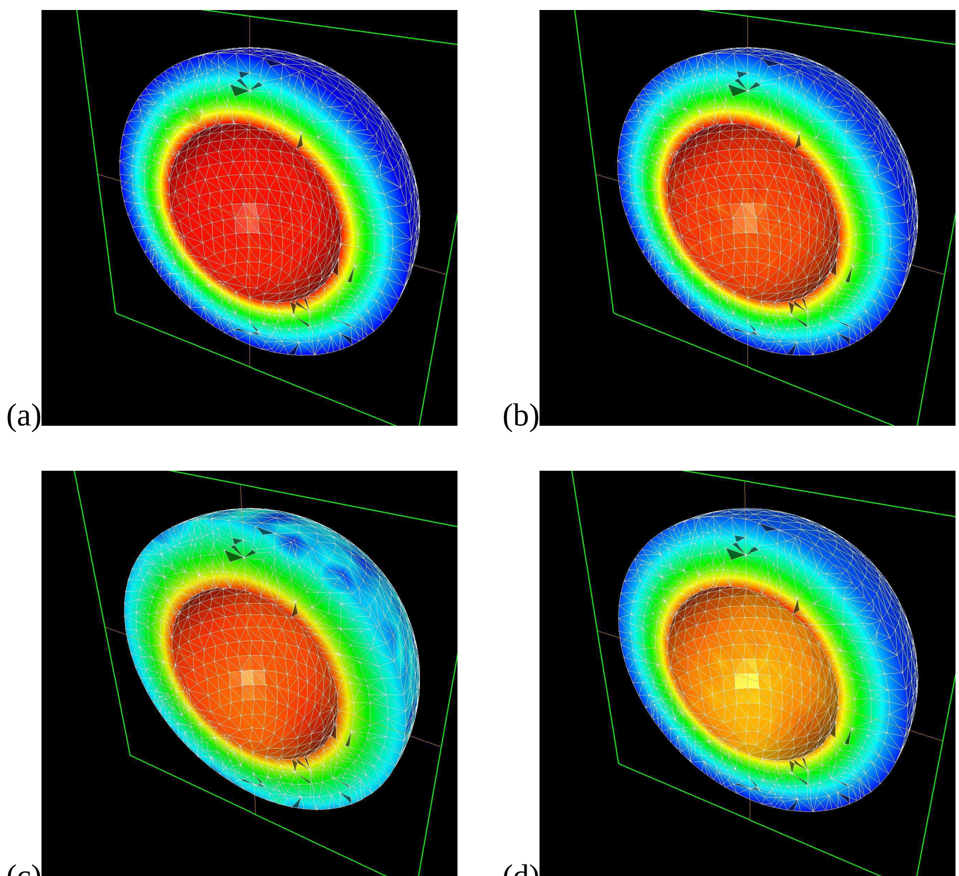

Figure 4. Test-case C. Top row : noise level is 0%. At convergence, we plot a plane slice of the level sets of (a) u_1 and (b) u_2. Bottom row : noise level is 5%. At convergence, are plotted the level sets of (c) u_1 and (d) u_2. The Finite Element computations are performed with 4740 nodes and 22795 tetrahedral elements.

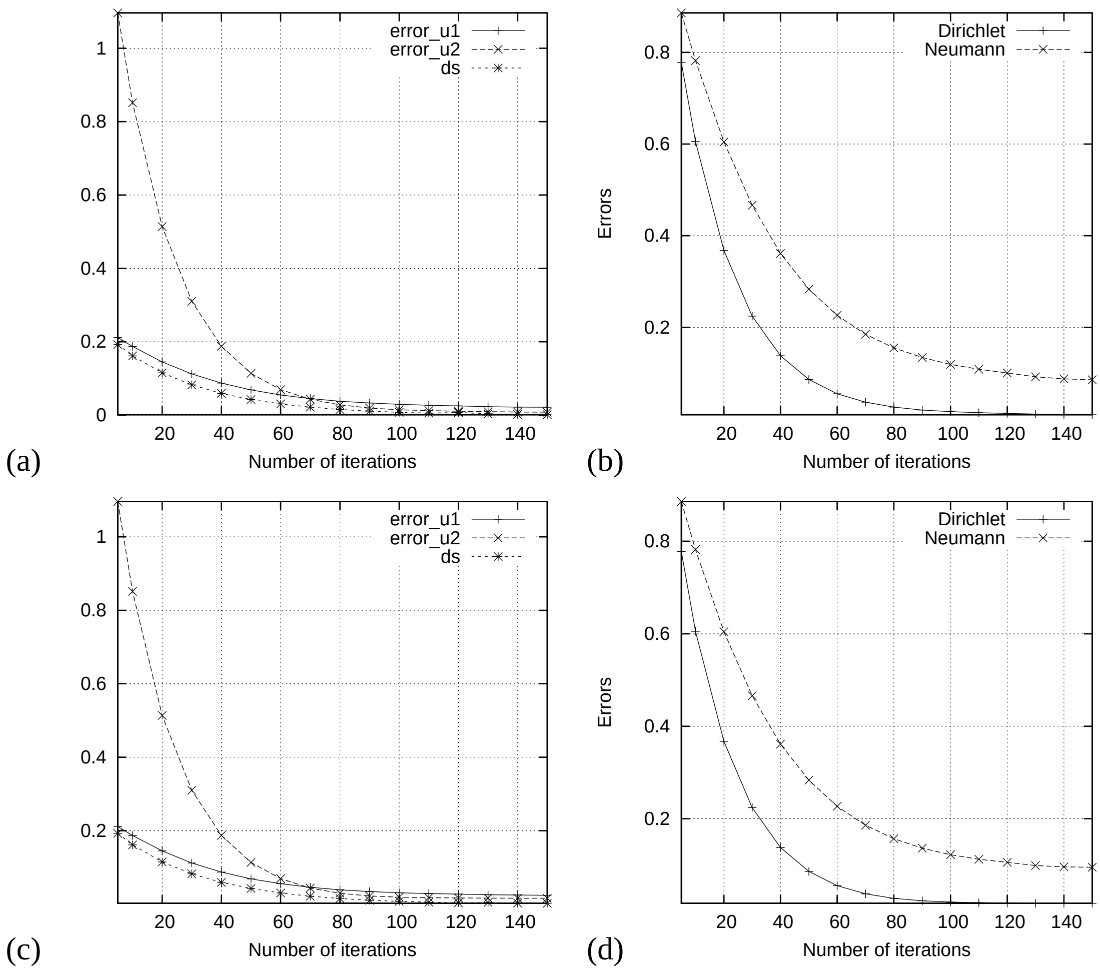

Figure 5. **Test-case C.** Top row : Relative L^2-errors are presented as a function of overall Nash iterations k, for a noise level $\sigma = 0\%$. (a) reconstructed fields : $\|u_i^{(k)} - u\|/\|u\|$, $i = 1, 2$ and Nash strategies $ds = \|S^{(k)} - S^{(k-1)}\|$; (b) missing Dirichlet data : $\|\tau^{(k)} - u_{|\Gamma_i}\|/\|u_{|\Gamma_i}\|$ and Neumann data $\|\eta^{(k)} - \frac{\partial u}{\partial \nu}_{|\Gamma_i}\|/\|\frac{\partial u}{\partial \nu}_{|\Gamma_i}\|$. Bottom row : The corresponding relative errors for a noise level $\sigma = 5\%$ are plotted in (c)–(d).

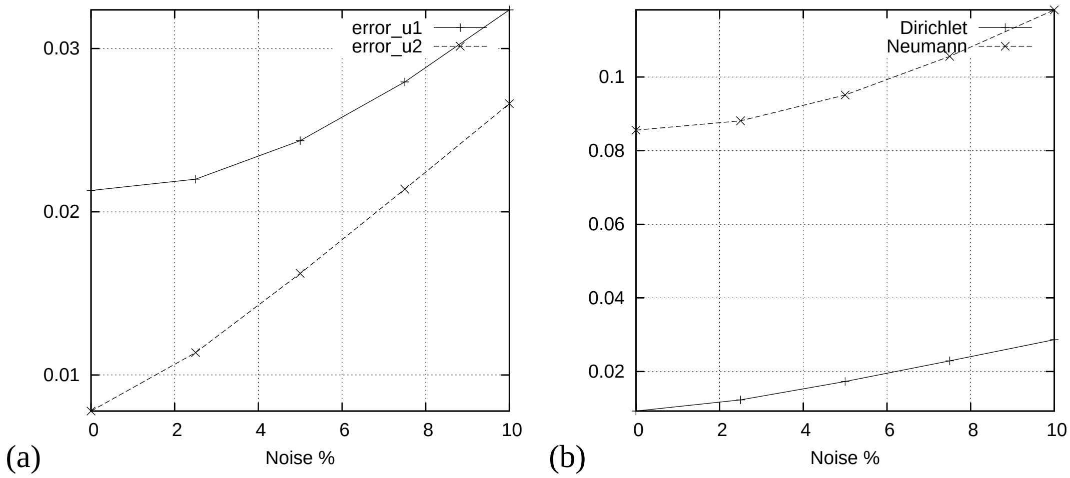

Figure 6. Test-case C. Sensitivity of the reconstructed fields to noisy Cauchy data (f^σ, Φ^σ). L^2-errors are presented as a function of the noise level σ : (a) reconstructed fields : $\|u_i - u\|/\|u\|$, $i = 1, 2$; (b) missing data : Dirichlet $\|\tau_N - u_{|\Gamma_i}\|/\|u_{|\Gamma_i}\|$ and Neumann $\|\eta_N - \frac{\partial u}{\partial \nu}_{|\Gamma_i}\|/\|\frac{\partial u}{\partial \nu}_{|\Gamma_i}\|$.

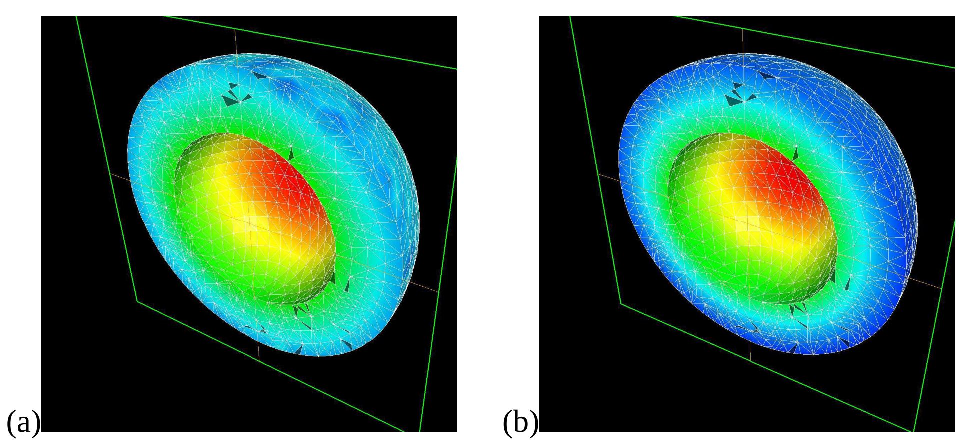

Figure 7. Test-case D. Non radial case. Noise level is 5%. The level sets of the reconstructed fields (a) u_1 and (b) u_2 are presented at convergence. The Finite Element computations are performed with 4740 nodes and 22795 tetrahedral elements.

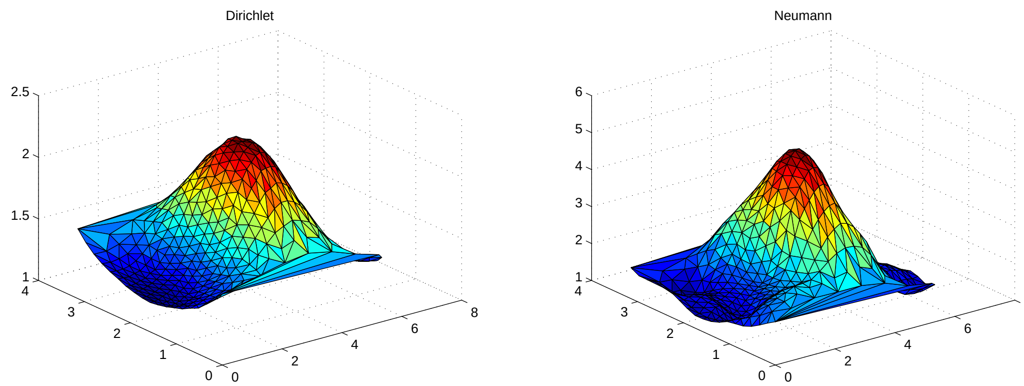

Figure 8. Test-case D. Reconstructed non radial Dirichlet (τ_N, left) and Neumann (η_N, right) data over Γ_i. The profiles are presented at convergence and for a noise level $\sigma = 5\%$.

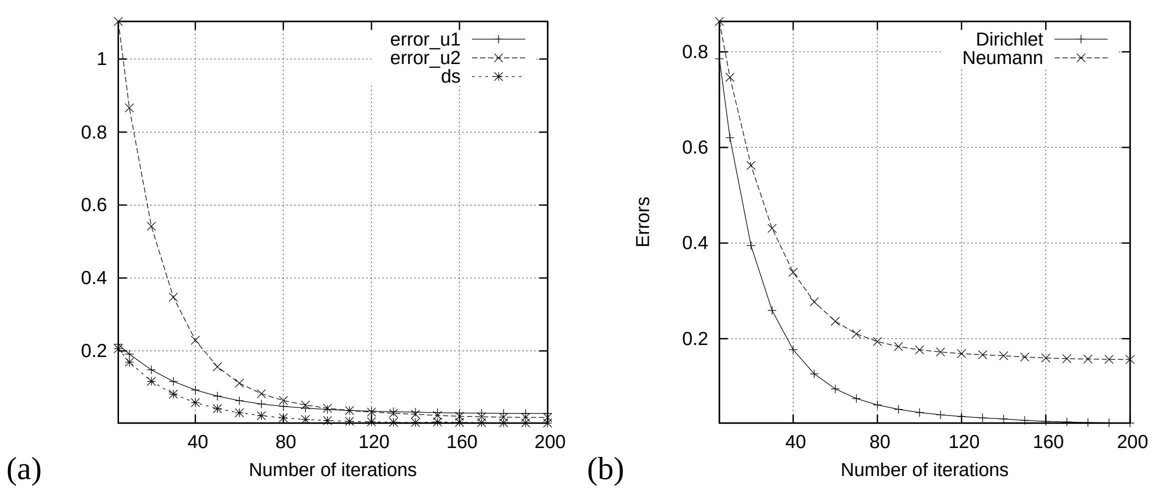

Figure 9. Test-case D. Relative L^2-errors are presented for the non radial case as a function of the overall Nash iterations. Noise level is 5%. (a) reconstructed fields error : $\|u_i^{(k)} - u\|/\|u\|$, $i = 1, 2$ and Nash strategies $ds = \|S^{(k)} - S^{(k-1)}\|$; (b) missing Dirichlet data : $\|\tau^{(k)} - u_{|\Gamma_i}\|/\|u_{|\Gamma_i}\|$ and Neumann data $\|\eta^{(k)} - \frac{\partial u}{\partial \nu}_{|\Gamma_i}\|/\|\frac{\partial u}{\partial \nu}_{|\Gamma_i}\|$.