Unilateral Altruism in Network Routing Games with Atomic Players

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Abstract— We study a routing game in which one of the players unilaterally acts altruistically by taking into consideration the latency cost of other players as well as his own. By not playing selfishly, a player can not only improve the other players' equilibrium utility but also improve his own equilibrium utility. To quantify the effect, we define a metric called the Value of Unilateral Altruism (VoU) to be the ratio of the equilibrium utility of the altruistic user to the equilibrium utility he would have received in Nash equilibrium if he were selfish. We show by example that the VoU, in a game with nonlinear latency functions and atomic players, can be arbitrarily large. Since the Nash equilibrium social welfare of this example is arbitrarily far from social optimum, this example also has a Price of Anarchy (PoA) that is unbounded. The example is driven by there being a small number of players since the same example with non-atomic players yields a Nash equilibrium that is fully efficient.

I. Introduction

The topic of network routing games has been studied extensively in recent years (e.g. [1]–[3]). Some of these investigations consider network games for which there are so many users that the traffic of each user can be considered negligible – the players for such games are said to be non atomic. Other work considers routing games for which each player is the source of a significant amount of the network's traffic. These players are said to be atomic. The analysis of atomic routing games can be more complex than that of non atomic games since the routing choice of a single player can affect the delays seen by others.

The Price of Anarchy (PoA) [4] is one popular measure of the efficiency of equilibria for such games. It is defined as the ratio of the maximum social welfare that could be achieved if a social planner selected the strategies of all players versus the lowest social welfare achieved in Nash equilibrium. The literature shows that in many cases the PoA can be arbitrarily large, although there are several important classes of examples in which the PoA can be bounded. (For example, Roughgarden [5], shows that routing games with non-atomic users and affine latency functions have a PoA of no more than 4/3.)

Naturally if players were not selfish, but instead all were striving to maximize social welfare, the equilibrium welfare could be much more efficient. However, an interesting question is what would happen if just one player chose not to be selfish and instead takes actions that consider the welfare of other players – in other words if he behaves altruistically. In this work we investigate the effects of unilateral altruism. We find by way of example that the effect of unilateral altruism can be arbitrarily large.

A selfish user chooses a path (or a set of paths) that minimizes his total cost without considering that their selection may degrade the performance of the other users in the system.

However in practice players may not play like this. Experimental studies [6]–[8] have shown that even in simple games in a controlled environment, players do not always act selfishly, for example they can show reciprocity, act out of spite, or be altruistic. Several explanations have been considered for such behaviour of players. Fehr [9], [10] argues that players act out of a perception of fairness and thus they consider a utility function that captures the costs of other players. Models of altruism and spite are discussed in [7], [8], [11], [12] etc. Most of the related models are discussed by Sobel in [6]. The fact that the social welfare observed in experimental trials is often better than that predicted by finding the Nash equilibrium of a model with selfish players is one reason to consider a model in which players act out of altruism. Since the players often interact over a longer period than that modeled in a one-shot game, players have a reason to play more cooperatively.

In this work, we focus on altruistic behaviour with atomic users. We consider a simple model where the perceived cost of a user is a linear combination of the other user's cost, as we proposed in [13]. We show that unilaterally altruism drastically increases the altruistic player's equilibrium utility. It is obvious that if all players are altruistic then the equilibrium utility can be improved, however it is surprising to observe a large improvement by a single player's altruistic action. However, it needs to be emphasized that the improvement only happens if the other players change their strategies in accordance with the belief that the altruistic player will truly be altruistic. In some sense, the altruistic player has to "credibly commit" to being altruistic.

In section II we present the system model and key definitions. We present an example to motivate the problem by showing very high price of anarchy in section III. Section IV analyzes a two user selfish game. In section V, we introduce and analyze the altruistic routing game. In section VI, we extend our model to a more general topology with n nodes and investigate the impact of altruism. Finally, we summarize with concluding remarks in section VII.

II. Basic Definitions and System Model

We consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{L})$, where a set of nodes $\mathcal{V}$ are connected by set of directed links $\mathcal{L} \subseteq \mathcal{V} \times \mathcal{V}$. We denote $v_s \in \mathcal{S} \subseteq \mathcal{V}, v_d \in \mathcal{D} \subseteq \mathcal{V}$ distinct nodes that are called source nodes and destination nodes, respectively. A route $R \in \mathcal{R}$ is a directed path with distinct nodes that connects a source and a destination. For every link $l \in \mathcal{L}$ and a route R, we write $l \in r$ whenever l is a part of route R.

A set $\mathcal{I} = \{1, 2, ..., I\}$ of users share the network $(\mathcal{V}, \mathcal{L})$. We consider that each source node v_s is associated with a user i, which ships a throughput demand of average data rate r^i to a common destination associated with node v_d. User i splits its fixed demand r^i among the paths (routes) connecting the

source to the destination, so as to optimize some individual performance objective. We assume that there exist at least one route from v_s to v_d.

Let x_l^i denote the expected flow that user i sends on link l. The vector of flows $\mathbf{x}^i = (x_l^i)_{l \in \mathcal{L}}$ that user i sends onto each path is called the routing strategy of user i. The set of strategies of user i that satisfy the user's demand and preserve its flow at all nodes is called the strategy space of user i and is denoted by $\mathbf{X}^i$, that is:

$$\mathbf{X}^i = \{\mathbf{x}^i \in \mathbb{R}^{|\mathcal{L}|}; \sum_{l \in \text{Out}(v)} x_l^i = \sum_{l \in \text{In}(v)} x_l^i + r_v^i, v \in \mathcal{V}\},$$

where $r_s^i = r^i, r_d^i = -r^i$ and $r_v^i = 0$ for $v \neq v_s, v_d$ and $\text{Out}(v)$ and $\text{In}(v)$ the set of out-going and in-coming links, respectively. The system flow configuration $\mathbf{x} = (x^1, ..., x^I)$ is called a *routing strategy profile* and takes values in the product strategy space $\mathbf{X} = \otimes_{i \in \mathcal{I}} \mathbf{X}^i$.

Every link $l \in \mathcal{L}$ is associated with a separable latency cost function $T_l(x_l) : \mathcal{R} \to \mathcal{R}$ which is assumed to be piecewise differentiable, convex, increasing, and $T_l(x_l) \geq 0$ for every $x_l > 0$. The queueing network interpretation is that $T_l(x_l)$ is the delay incurred by traffic crossing link l, or alternatively it can be read as the cost per unit of traffic when the load in link l is x_l.

Latency function: We consider the following latency function types in this paper:

A : Affine: $T_l(x_l) = ax_l + b$, where $a, b \geq 0$.
E : Elbow function: We define the function with offset o in parametric form as $T_l(x_l) = \max\left(o, \left(\frac{L}{\delta}\right)(x - r) + L\right)$. Unless stated otherwise, we use $o = 0, 0 < \delta \ll 1$, refer to Fig. 2.

The **E** type latency function posses several features that makes in mathematically tractable, e.g., piece-wise linearity, convexity. It turns out that a steep slope of elbow function causes users to react for even small flow changes which plays a key role in the latter sections. This is why, we designed the elbow function in a way so that the per-unit cost of traffic can increase very rapidly. This is also not unlike the delay in most queueing models such as $M/M/1$ delay function.

Load Balancing (LB) Network: We introduced a network topology called the "load balancing network" [13], where two users located one at each of the two nodes, called source nodes, deliver their demand to a common destination node as pictured in Fig. 1(a).

In this paper, we associate **E** type latency with l_1, l_2 while we associate **A** type latency with l_{12}, l_{21}. One of the reasons of such modeling is that, typically source-destination links are standard queueing network links with limited capacity (approximated by **E** type). The interconnecting/cross links are meant for balancing skewed loads on the local links. Thus, it is of high capacity but are expensive by provision (approximated by type **A**).

User Cost: Let $J_i(\mathbf{x})$ be the cost of user i under routing strategy profile $\mathbf{x}$. The total cost to user i is the sum of the separable link costs, given as

$$J_i(\mathbf{x}) = \sum_l x_l^i T_l(x_l), \text{ for each } i. \quad (1)$$

The objective of each user i is to find an admissible routing strategy $\mathbf{x}^i \in \mathbf{X}^i$ so as to minimize his cost. Later we consider

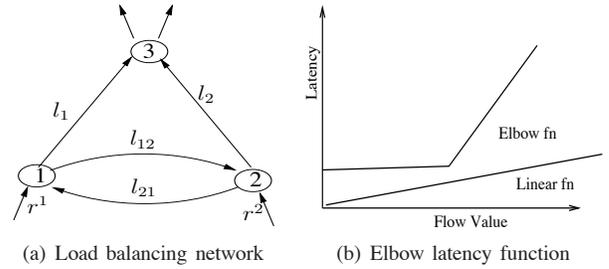(a) Load balancing network   (b) Elbow latency function

Fig. 1. Network topology and Latency function

altruistic users, where the objective function takes into account of other user's cost also.

A. Nash equilibrium

Each user in this framework minimizes his own cost functions which leads to the concept of Nash equilibrium. The minimization problem here depends on the routing decision of other users, i.e., their routing strategy

$$\mathbf{x}^{-i} = (\mathbf{x}^1, ..., \mathbf{x}^{i-1}, \mathbf{x}^{i+1}, ...\mathbf{x}^I),$$

Definition 2.1: A vector $\tilde{\mathbf{x}}^i$, $i = 1, 2, ..., I$ is called a Nash equilibrium if for each user i, $\tilde{\mathbf{x}}^i$ minimizes the cost function given that other users' routing decisions are $\tilde{\mathbf{x}}^j$, $j \neq i$. In other words,

$$J_i(\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2, ..., \tilde{\mathbf{x}}^I) = \min_{\mathbf{x}^i \in \mathbf{X}^i} J_i(\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2, ..., \mathbf{x}^i, ..., \tilde{\mathbf{x}}^I),$$
$$i = 1, 2, ..., I,$$

where $\mathbf{X}^i$ is the routing strategy space of user i.

Social Optimum: A centralized or coordinated solution is referred to as social optimum[1], which can be expressed as:

$$\mathbf{x}^{OPT} = \arg\min_{\mathbf{x} \in \mathbf{X}} \sum_i J_i(\mathbf{x}).$$

Price of Anarchy (PoA): A well known way to quantify the equilibrium inefficiency of a non-cooperative game (in the sense of [4]). The Price of Anarchy is defined ([2], [4]) as the ratio of worst case cost at Nash equilibrium when users are selfish to the cost at social optimum. Therefore,

$$PoA = \sup_{\mathbf{x} \in \mathbf{x}^{SE}} \frac{\sum_i J_i(\mathbf{x})}{\sum_i J_i(\mathbf{x}^{OPT})} \quad . \quad (2)$$

where $\mathbf{x}^{SE}$ denotes the set of equilibrium flow configurations when users are "selfish".

III. MOTIVATING EXAMPLE

In this section we illustrate the inefficiency of non-cooperative equilibrium with the help of a simple example and point out the key reason for the inefficiency to motivate the detailed investigation. We also note that the inefficiency appears only for the atomic users.

Consider a LB network (in Fig. 1(a)) with two **selfish users**, u_1 and u_2, at source nodes 1 and 2 respectively, having identical demand rates $r_1 = r_2 = r$, which is to be dispatched to the destination node 3. Since, link l_1 is the most direct way for user 1 to reach the destination, we refer to link l_1 his "local link". Similarly, we refer to link l_2 as user 2's local link. We refer to links l_{12} and l_{21} as "cross" links.

Example: Let the latency function be as shown in Fig. 2, which is expressed as:

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[1]Existence of minima is due to compact strategy set and piece-wise continuous cost function. It is unique because of potential function structure.
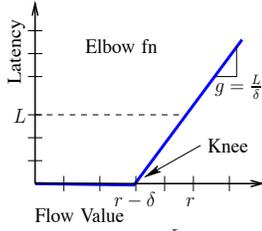
Fig. 2. Link latency function: $L_2(x) = \frac{L}{\delta}(x - r) + L$, for $r - \delta \leq x$ and the parameters are: $r = 1, L = 0.1, \delta = 10^{-3}, g = \frac{L}{\delta} = 100$.

$$T_l(x) = \begin{cases} \max(0, L_2(x)), & \text{for } l = l_1, l_2, \\ c, & \text{for } l = l_{12}, l_{21}. \end{cases}$$

We can easily observe that the social optimum is achieved when users use only local links. This is so because any positive flow on cross links add additional latency while retaining the total latency (due to flow) on links l_1 and l_2 remains unchanged. However, when users are selfish, each user optimize its own utility. We illustrate by way of iteratively finding the best responses of the two selfish players why they end up putting a large fraction of their traffic on the cross links, which is the cause of inefficiency.

Let the initial flow be $\{r, r\}$, and the corresponding initial cost of users are $\{L, L\} = \{0.1, 0.1\}$. Let u_1 pushes δ amount of flow to l_2 to bring his local link flow below the knee (see Fig. 2), which results in $T_1(.) = 0$. On the other hand, the total flow in l_2 raised to $r + \delta$ which results in $T_2(.) = L + \frac{L}{\delta}\delta = 2L$. The updated cost of u_1 is $J_1 = \delta c + \delta \cdot 2L = 0.12$, which is much lower than previous cost 0.1. However, the cost of u_2 has increased from 0.1 to $(1 - 10^{-3}) \approx 0.2$, almost doubled. In turn u_2 will reciprocate by pushing 2δ unit of flow to l_1 to bring his local link latency below the knee, which reduces his cost drastically to $2\delta \cdot (2L + c) \approx 0.003$, but raises the cost of u_1 to $(1 - \delta)2L + \delta c \approx 0.2$. Due to this raise in cost, again u_1 will push 2δ amount of flow to l_2 in order to bring his local link latency below the knee. This process will keep repeating till the flow configuration reaches to $x^{SE} \approx \{0.5, 0.5\}$, after which none of the users benefit by deviating even a small amount of traffic from their local link. In other words, x^{SE} is the Nash equilibrium. The total cost incurred by any user at x^{SE} is $\approx 0.5 \cdot (0.1 + 1) = 0.55$, which is much higher ($\approx 50$ times) than the cost at social optimum ($= 0.1$), i.e., $PoA \approx 50$.

We demonstrated by a simple LB Network example that selfish routing with atomic users can be significantly inefficient. In the following, we show that it can be even worse and can be arbitrarily large for atomic users. We also show that this is not the case with non-atomic users, suggesting that it is the ability of a user to chase the latency of the links with his strategy that drives the example.

IV. Two User Symmetric Game

In this section we establish results showing the inefficiency of selfish routing games by analyzing the two user LB network game. In the later sections, we construct and analyze the model in which users are altruistic, which exposes our key findings.

We consider a sequence of LB network examples indexed by $m \geq 1$. Consider a LB Network of three nodes as shown in Fig. 1(a) with the **E** type and **A** type latency functions as follows:

$$T_l(x) = \begin{cases} \max\left(0, \left(\frac{L}{\delta_m}\right)(x - r) + L\right), & \text{for } l_1, l_2, \\ c_m, & \text{for } l_{12}, l_{21}. \end{cases} \quad (3)$$

where $c_m > L > \delta_m > 0$ and $c_m < r\frac{L}{\delta_m}$ for all m . The sequence δ_m decreases and tends to zero such that the gradient $\frac{L}{\delta_m} \to \infty$ as $m \to \infty$. Similarly, c_m is an increasing sequence such that latency of cross link can be made very high for sufficiently large m. One example of such sequence can be $\delta_m = \delta^m$ and $c_m = c^m$. It turns out to be important that the gradient $\frac{L}{\delta_m}$ be larger than the cross-link latency c_m, but the cross-link latency c_m be bigger than the latency of the local links when the flow is r, which is L.

Using (1), the cost of users are given by,
$$J_i(\mathbf{x}) = x_i^i T_i(x_i) + \bar{x}_i^i(T_{ij}(\bar{x}_i^i) + T_j(x_j)), \quad \text{for } i, j = 1, 2.$$

where $\bar{x}_i^i = r^i - x_i^i$. We often denote $T_{l_i}(.)$ by $T_i(.)$.

A. Selfish Routing Game with Atomic Users

Lemma 4.1: Consider two atomic users of identical demands on a load balancing network (Fig. 1(a)) at node 1 and node 2 with the associated latency given by (3). In the two user selfish routing game, there exists a unique Nash equilibrium, given by

$$\mathbf{x}^{SE} = \{x_1^{1^*}, x_2^{2^*}\} = \{r/2 + \zeta_m, r/2 + \zeta_m\},$$
$$J_i^{SE} = rL + (r/2 - \zeta_m)c_m, \text{ for } i = 1, 2.$$

where $\zeta_m = \frac{1}{2}\left(\frac{c_m}{L/\delta_m}\right)$ is the ratio of unit latency of cross links to the marginal latency of local links. By abuse of notation, above we denoted $\mathbf{x}^{SE}$ as the equilibrium flow.

Proof: The proof of Lemma 4.1 uses techniques of Non Zero Sum Games (NZSG) [14]. We solve the game by explicitly computing the intersecting point of the best responses. The Nash equilibrium for the selfish game corresponds to the solution to the following minimization problem

$$\mathcal{P}_2: \quad \mathbf{x}^{i^*} = \arg\min_{\mathbf{x}^i \in \mathbf{X}^i} J_i(\mathbf{x}^i, \mathbf{x}^{-i}), \, \forall \, i = 1, 2.$$

The cost of user 1 and 2 in the above equation can be expressed as follows:
$$J_1(\mathbf{x}) = x_1^1 T_1(x_1) + \bar{x}_1^1(T_{12}(\bar{x}_1^1) + T_2(x_2)),$$
$$J_2(\mathbf{x}) = x_2^2 T_2(x_2) + \bar{x}_2^2(T_{21}(\bar{x}_2^2) + T_1(x_1)). \quad (4)$$

where $\bar{x}_i^i = r - x_i^i$ for $i = 1, 2$.

The compactness of the strategy set $\mathbf{X}$ with the concavity and continuity of the objective function $J_i(.)$ imply the existence of a Nash equilibrium (see [14, Thm. 4.4]). A formal proof of existence is provided in [13]. We provide the proof of unique Nash equilibrium for a general n node LB network (n user) in Lemma 6.1. Therefore, we conclude that there exist unique equilibrium for the problem $\mathcal{P}_1$, as it is a special case of Lemma 6.1 with $n = 2$. Therefore, if we can compute a Nash equilibrium, then that is the only Nash equilibrium. We can observe that a symmetric Nash equilibrium exist due the symmetric structure of $\mathcal{P}_1$. In the following, we compute the symmetric Nash equilibrium.

1) Computation of symmetric Nash equilibrium: We compute the best response for each player to compute the equilibrium. The best responses can be given by
$$BR_1(x_2^2) = \arg\min_{0 \leq x^{1^*} \leq r_1} J_1(x^1, x^2) = \max\{\min\{0, a^*\}, r\},$$
$$BR_2(x_1^1) = \arg\min_{0 \leq x^{2^*} \leq r_2} J_2(x^1, x^2) = \max\{\min\{0, b^*\}, r\}.$$

where (with some abuse of notation[2]) we denote the *internal solutions* by a^* and b^*. By the *internal solution*, we mean $0 <$

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[2] a^* is represented in short for $a^*(x_2^2)$ and b^* in short for $b^*(x_1^1)$.

a^*(or b^*) $< r$ when $a^* \Rightarrow \frac{\partial J_1(\mathbf{x})}{x_1^1} = 0$ and $b^* \Rightarrow \frac{\partial J_2(\mathbf{x})}{x_2^2} = 0$. Equating the above and doing some simple calculus we obtain,

$$a^*(x_2^2) = \frac{F(x_2) + (c_m + rg(x_2))}{G(x)},$$

$$b^*(x_1^1) = \frac{F(x_1) + (c_m + rg(x_1))}{G(x)}$$

where, $F(x_2) = T_2(x_2) - T_1(x_1) = -F(x_1)$, $g_1(x_1) = \frac{\partial T_{l_1}(x_1)}{\partial x_1}$, $g_2(x_2) = \frac{\partial T_2(x_2)}{\partial x_2}$, and $G(x) = g_1(x_1) + g_2(x_2)$. The gradient of $T_1(x)$ at r is $g_1(r) = L/\delta_m$.

Exploiting the symmetry we note $x_1 = x_2 = r$, $g(x_1) = g(x_2) = g(r)$, and $F(x_1) = F(x_2) = 0$ at equilibrium. Therefore, the equilibrium flow can be given by

$$x_1^{1^*} = x_2^{2^*} = \frac{c_m}{2g(r)} + \frac{rg(r)}{2g(r)} = \frac{r}{2} + \zeta_m \qquad (5)$$

where $\zeta_m = \frac{1}{2}\left(\frac{c_m}{L/\delta_m}\right)$. The Nash equilibrium flow for selfish users is expressed as $\mathbf{x}^{SE} = \{\frac{r}{2} + \zeta_m, \frac{r}{2} + \zeta_m\}$. Note that $\zeta_m \geq 0$ for large enough m.

2) Using (4), the equilibrium cost can be given by

$$J_1^{SE} = \left(\frac{r}{2} + \zeta_m\right) T_1(x_1) + \left(\frac{r}{2} - \zeta_m\right)\left(T_{12}\left(\frac{r}{2} - \zeta_m\right)\right.$$
$$\left. + T_2(x_2)\right) = rL + \left(\frac{r}{2} - \zeta_m\right) c_m.$$

This concludes the proof. ◇

Proposition 4.1: The **Price of Anarchy** of two atomic users on LB network (Fig. 1(a)) with the associated elbow latency function of (3) and with identical demands is given by

$$PoA = \left[1 + \frac{(\frac{r}{2} - \zeta_m)c_m}{rL}\right].$$

The PoA can be arbitrarily large for large enough m.

Proof: Recall the definition of price of anarchy from (2)

$$PoA = \sup_{\mathbf{x} \in \mathbf{x}^{SE}} \frac{\sum_i J_i(\mathbf{x})}{\sum_i J_i(\mathbf{x}^{OPT})} \quad.$$

One can easily observe that at social optimum (OPT) only local links are used. Therefore, the user cost J_i^{OPT} at social optima is given as

$$J_1^{OPT} = rT_1(r) = rL.$$

Therefore, the price of anarchy for any m can be given by

$$PoA = \frac{rL + \left(\frac{r}{2} - \zeta_m\right) c_m}{rL} = \left[1 + \frac{(\frac{r}{2} - \zeta_m)c_m}{rL}\right].$$

Note that c_m increases and ζ_m decreases with m. Hence, PoA can be arbitrary large for large enough m. ◇

B. Selfish Routing Game with Non-Atomic Users

We consider large number of selfish users at each nodes. The demand of each user is assumed very small such that choice of a single user has negligible impact on the system performance. The latency observed by a user is the sum of the link latency that a user traverse to reach to the destination node. Each user would choose a route so as to minimize its latency at equilibrium.

Proposition 4.2: The *Price of Anarchy* for the routing game on the LB network with non-atomic selfish users and associated latency given by (3) and with identical demands is equal to 1 for any m.

The bounded PoA is achieved because the user's use only their local links. This is because the latency observed by a user via a cross link is larger as compared to the case when he chose his local link. However, such direct intuition is not possible in case of atomic users. The proof of Lemma 4.2 is based on variational inequality at Wardrop equilibrium in the similar lines of [2] (refer appendix A).

Interestingly, the social welfare for selfish routing games with atomic users is unbounded whereas in similar settings it corresponds to social optimum for non-atomic users.

V. ALTRUISTIC ROUTING GAME

In this section, we consider routing games when users are not entirely selfish, but rather altruistic. Each user minimizes the perceived cost, a weighted sum of his own cost and other users' cost (we proposed this model in [13]). The weight coefficient is called *degree of Cooperation* (DoC).

Definition 5.1: Let α_k^i be the *degree of cooperation* of user i with user k. The perceived cost is a convex combination of the users' cost from the set $\mathcal{I}$ as follows:

$$\hat{J}_i(\mathbf{x}, \vec{\alpha}_i) = \sum_{k \in \mathcal{I}} \alpha_k^i J_k(\mathbf{x}); \quad \sum_k \alpha_k^i = 1, i = 1, ...|\mathcal{I}|.$$

We can express the DoC vector of user i as $\vec{\alpha^i} = (\alpha_1^i, \cdots, \alpha_{|\mathcal{I}|}^i)$. Based on the DoC, we can view the following properties for user i,

- Non-cooperative: if $\alpha_k^i = 0$, for all $k \neq i$. Denote the DoC of a selfish user by $\vec{e}_i = \{0, \cdots, e_i^i, \cdots, 0\}$, where $e_i^i = 1$.
- Altruistic: User i is fully cooperative with all users and does not care for his benefits, i.e., $\alpha_i^i = 0$.
- Equally-cooperative: if $\alpha_j^i = \frac{1}{|\mathcal{I}|}$, $\forall j \in \mathcal{I}$, user i is equally cooperative with each user.

Note that, although a user cooperates with others, he minimizes his own perceived cost. Therefore, this formulation allows us to analyze the game using the conventional techniques of non cooperative game theory, e.g., Nash equilibrium, etc.

A. Unilateral Altruism

We call user i **unilaterally altruistic**, if he cooperates with other users irrespective of whether the other users are cooperative or non cooperative with user i. More precisely, let all users $j \neq i$ be selfish and user i be unilaterally altruistic with degree of cooperation vector $\vec{\alpha}_i$, then $\vec{\alpha} = \{\vec{e}_1, \cdots, \vec{\alpha}_i, \cdots, \vec{e}_{|\mathcal{I}|}\}$. The perceived cost by user i and user $j \neq i$ can be expressed as

$$\hat{J}_i(\mathbf{x}, \vec{\alpha}_i) = \sum_{k \in \mathcal{I}} \alpha_k^i J_k(\mathbf{x}), \quad \hat{J}_j(\mathbf{x}, \vec{e}_j) = J_j(\mathbf{x}). \qquad (6)$$

Let $\mathbf{x}^{SE}$ denote the set of Nash equilibrium flow vectors for the selfish routing game. Then, the best selfish Nash equilibrium flow can be given by $\underline{\mathbf{x}}^{SE} = \arg\min_{\mathbf{x} \in \mathbf{x}^{SE}} \hat{J}_i(\mathbf{x}, \vec{e}_i)$. Similarly, let $\mathbf{x}_{\vec{\alpha}_i}^{AL}$ be the set of Nash equilibrium flows[3] (using (6)) when user i is unilaterally altruistic with DoC $\vec{\alpha}_i$ and all the other users are selfish. Recall that, an altruistic player truly commits his strategy to be altruistic such that the other players can change their strategy in accordance to this belief.

Definition 5.2 (VoU): We quantify the benefit of unilateral altruism by the term **Value of Unilateral Altruism (VoU)**, which is defined as the ratio of the best selfish equilibrium cost

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[3]Existence of equilibrium with unilateral altruism is guaranteed since the strategy set is compact and continuous, and the cost function is concave [14].

to his best equilibrium cost when he is unilaterally altruistic. Loosely speaking, VoU measures the extent of benefit that a selfish user can achieve by becoming altruistic. Mathematically, we can express

$$VoU(i) = \frac{J_i(\underline{\mathbf{x}}^{SE})}{\inf_{(\vec{\alpha}_i, \mathbf{x}_{\vec{\alpha}_i}^{AL})} J_i(\mathbf{x}_{\vec{\alpha}_i}^{AL})} \quad . \qquad (7)$$

Notice that the equilibrium is obtained based on minimization of perceived cost but the VoU is related to the actual cost incurred by the user. By VoU we identify the bound on the improvement that a user may achieve by becoming altruistic (if altruism is beneficial for the user). Note that, when there exists unique equilibrium with an altruistic user, the VoU corresponds to the $\vec{\alpha}_i$ that minimizes the equilibrium cost. However, when there exist multiple equilibria, the VoU corresponds to the best equilibria at its corresponding $\vec{\alpha}_i$. By definition, we have $\underline{\mathbf{x}}^{SE} \subseteq \mathbf{x}_{\vec{\alpha}_i}^{AL}$, therefore the feasible values of $VoU(i)$ must be ≥ 1. $VoU(i) = 1$ indicates, either no improvement or degradation with altruism as compared to the selfish situation. The more interesting is the case of $VoU(i) > 1$, which corresponds to the situation when a unilaterally altruistic user improves his cost as compared to he being selfish. This is non-intuitive, as an altruistic user is generally expected to incur more cost. In the following, we illustrate that one can achieve drastic improvement by being unilaterally altruistic.

Proposition 5.1: Suppose user 1 is altruistic in a two user routing game on a LB Network (Fig. 1(a)) with the associated latency given by (3) and with identical demand. The "Value of unilateral altruism" of user 1, $VoU(1)$, is arbitrarily large for sufficiently large m.

The following steps establishes the unboundedness of $VoU(1)$. The selfish users incur unbounded cost at equilibrium (unique), see Lemma 4.1. Therefore, a bounded equilibrium cost of user 1 when he is unilaterally altruistic implies the proposition. However, there can be multiple equilibria with altruism but due to unbounded $J_1^{SE}(.)$, the $VoU(1)$ still remains unbounded even if there exist other equilibria since VoU was defined using the infimum across multiple equilibria. Using these arguments we show $VoU(1)$ is unbounded. In fact, we show that $(x_1^{1^*}, x_2^{2^*}) = (r, r - \delta)$ is an equilibrium with unilateral altruism for a set of $\alpha \in \Gamma$. The equilibrium cost is unbounded.

Proof: In an altruistic routing game, users minimize their perceived cost ($\hat{J}_i(.)$). It is interesting to note that although the perceived cost is minimized non-cooperatively, the effect of altruism appears due the coupling through the cost. Therefore, the Nash equilibrium of this two user game can be expressed by the following equivalent problem:

$$\mathcal{P}_2 : \quad x^i = \arg\min_{\mathbf{x} \in \mathbf{X}} \hat{J}_i(\mathbf{x}), \, \forall \, i = 1, 2.$$

where the perceived cost of users (when user 1 is altruistic) can be expressed as
$$\hat{J}_1(\mathbf{x}) = (1 - \beta)J_1(\mathbf{x}) + \beta J_2(\mathbf{x}),$$
$$\hat{J}_2(\mathbf{x}) = J_2(\mathbf{x}).$$

Existence of equilibrium is direct from the compact strategy set $\mathbf{X}$ and concave and continuity of the cost function $\hat{J}_i(\mathbf{x})$ [14].

In this altruistic equilibrium, we are interested in an equilibrium which can improve the user's cost as the selfish equilibrium cost is very large. It turns out that the cost becomes

unbounded or large only when a significant amount of flow is routed thorough cross link, which is the main reason of inefficiency. Therefore, a bounded cost can be achieved only when one can restrict almost all the flow to one's local link. In this spirit an altruistic game can achieve the equilibrium flow to be $(x_1^1, x_2^2) = (r, r - \delta)$ for at least some β, because it can result in bounded cost.

We call $x_i^i = r$ as "load taker" strategy because user i does not reciprocate (send back flow) in response to any flow sent to his local link, i.e., he takes all the load. Similarly, $x_i^i = r - \delta$ is the so called "delta load taker" strategy as it only pushes enough flow to the other link so that his local link gets to the knee, provided the other player does not push back. We aim to show that $(x_1^1, x_2^2) = (\text{load taker}, \text{delta load taker}) = (r, r - \delta)$ is an equilibrium profile when m is sufficiently large.. We user δ and δ_m alternatively in the proof for ease of notation.

Computation of $BR_2(x_1^1)$: The cost of u_2 for a load taker strategy of user 1 is given by (let denote $p = x_2^2$ and $\bar{p} = r - x_2^2$),

$$J_2(p) = \begin{cases} \bar{p}[T_{12}(\bar{p}) + T_1(r + \bar{p})] & \text{if } p \leq r - \delta, \\ pT_2(p) + \bar{p}[T_{12}(\bar{p}) + T_1(r + \bar{p})] & \text{if } p > r - \delta. \end{cases} \quad (8)$$

Notice that the $J_2(p)$ has discontinuity only at $x_2^2 = r - \delta$. The marginal cost can be given as

$$\frac{\partial J_2(p)}{\partial p} = \begin{cases} A(p) < 0 & \text{if } p \leq r - \delta, \\ B_m(p) > 0 & \text{if } p > r - \delta, \end{cases}$$

Using (8), we have $A(p) = -\bar{p}\frac{L}{\delta m} - [d_m + \frac{L}{\delta m}(\bar{p}) + L] < 0$, and $B_m(p) = \frac{(2p - r)L}{\delta_m} - d_m > (r - 2\delta_m)\frac{L}{\delta_m} - d_m$. We note that $B_m(p) > 0$ for sufficiently large m. Hence, $BR_2(r) = r - \delta$.

Computation of $BR_1(x_2^2)$: The cost of u_1 for a delta load taker strategy of u_2 is given by (let denote $x_1^1 = p$ and $x_2^2 = \bar{\delta} = r - \delta$),

$$\begin{aligned} \hat{J}_1(p) &= (1 - \beta)J_1(\mathbf{x}) + \beta J_2(\mathbf{x}) \\ &= (1 - \beta)[pT_1(p + \delta) + \bar{p}(T_{12}(\bar{p}) + T_2(\bar{p} + \bar{\delta}))] \\ &\quad + \beta[\bar{\delta}T_2(\bar{p} + \bar{\delta}) + \delta(T_{21}(\delta) + T_1(p + \delta))]. \end{aligned}$$

Noting that $\hat{J}_1(p)$ is strictly convex in $p \in [0, r]$, if $\frac{\partial \hat{J}_1}{\partial p} < 0$ at $p = r$ then $\frac{\partial \hat{J}_1}{\partial p} < 0$ for $p \in [0, r]$. Therefore, we examine if $\frac{\partial \hat{J}_1}{\partial p} < 0$ holds true at $p = r$. The derivative of $\hat{J}_1(p)$ is

$$\begin{aligned} \frac{\partial \hat{J}_1(p)}{\partial p} &= (1 - \beta)[p(g_1(p + \delta) + g_2(\bar{p} + \bar{\delta})) + T_1(p + \delta) - T_2(\bar{p} + \bar{\delta}) \\ &\quad - (rg_2(\bar{p} + \bar{\delta}) + T_{12}(\bar{p}))] + \beta[\delta g_1(p + \delta)) - (\bar{\delta})g_2(\bar{p} + \bar{\delta})]. \end{aligned}$$

At $p = r^-$ (less than and close to r), we have $g_1(p + \delta) = g_2(\bar{\delta} + \bar{p}) = \frac{L}{\delta_m}$, $T_1(p + \delta) = L + (p + \delta - r)\frac{L}{\delta_m}$, and $T_2(\bar{\delta} + \bar{p}) = L + (r - \delta - p)\frac{L}{\delta_m}$. Plugging the values in (9), we obtain

$$\frac{\partial \hat{J}_2(p)}{\partial p} < 0, \text{ for } \beta \in \Gamma$$

where the set Γ turns out to be a non empty set, given by $(\frac{1}{2}, 1]$. Therefore, we can state that $BR_1(r - \delta) = r$, for $\beta \in \Gamma$.

Therefore, together it constitute the Nash equilibrium. Denoting the equilibrium flow by $\tilde{\mathbf{x}}^{AL}$, and cost of user 1 as $J_1(\tilde{\mathbf{x}}^{AL})$, we have
$$\tilde{\mathbf{x}}^{AL} = (r, r - \delta), \text{ for } \beta \in \Gamma,$$
$$J_1(\tilde{\mathbf{x}}^{AL}) = rT_1(r + \delta) = 2rL.$$

The set of equilibria is denoted by $\mathbf{x}^{AL}$ when there exist multiple equilibria. Denote $\underline{\mathbf{x}}^{AL} \in \mathbf{x}^{AL}$ such that

$$J_1(\underline{\mathbf{x}}^{AL}) < J_1(\mathbf{x}), \text{for any } \mathbf{x} \in \mathbf{x}^{AL}. \qquad (9)$$

Computation of VoU: Using (5) and (9), and the definition 5.2, we can express

$$VoU(1) = \frac{J_i(\mathbf{\underline{x}}^{SE})}{\inf_{(\vec{\alpha}_i, \mathbf{x}_{\vec{\alpha}_i}^{AL})} J_i(\mathbf{x}_{\vec{\alpha}_i}^{AL})} > \frac{J_1(\mathbf{x}^{SE})}{J_1(\tilde{\mathbf{x}}^{AL})}$$
$$= \frac{rL + (r/2 - \zeta_m)c_m}{2rL}.$$

The last inequality is from (11). The quantity $\frac{rL+(r/2-\zeta_m)c_m}{2rL}$ can be arbitrarily large for large m. This concludes the proof. $\diamond$

Note that by showing unilateral altruism, not only the altruistic user is able to improve his cost by keeping it bounded, rather the other selfish user also enjoy his improved cost which also remains bounded. This way, even the selfish user also get drastic benefit due to altruistic user's action.

VI. VoU FOR N PLAYERS

From the discussion of previous sections, one might think that the benefit of altruism (the unbounded VoU) is restricted to a two user scenario and it may disappear for a larger number of users. In this spirit we investigate for n user game in this section. We are able to construct a LB type network of $n+1$ nodes, shown in Figs. 1(a), on which the property of unbounded VoU is retained when a user is unilaterally altruistic. Note that one could trivially construct an example with 2 users with 1 unit of traffic and n-2 users with a tiny amount of traffic, and end up with an example that is almost the same as our 2 user example. However, in this section, we will construct an example in which we have N players, all of which have non-negligible traffic.

Consider that n users (a user at each node) have identical demands to be dispatched to a common destination node (marked as $n+1^{th}$ node). We compute the selfish routing game performance and subsequently study the performance with an altruistic user. It is intuitive that if all users turn altruistic then the efficiency of selfish equilibrium can be improved. In the earlier sections we studied the game with two users. However, it is more complex to study for n users. We chose a symmetric network, which is more tractable. We find that even in case of n users, the benefit of a single user's altruism is significant. We formally establish our findings with the help from the following lemma and proposition.
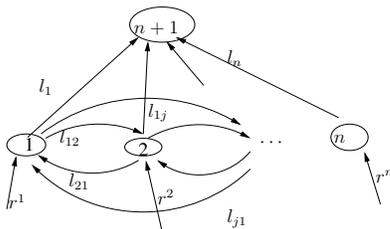

Fig. 3. A LB type network with n source nodes and a destination node.

Lemma 6.1: The PoA for the n player selfish routing game on a LB network (Fig. 3) with the latency function as in (3), and with identical demands is arbitrarily large for sufficiently large m.

The proof follows by first transforming the given network into a two node network with parallel links, then invoking the uniqueness result from [1, Theorem 1]. Further, exploiting the symmetric structure we obtain the NEP for selfish users and

the social optimum. Using these we prove that PoA can be arbitrarily large (refer to appendix B).

Proposition 6.1: Consider a user (say user $i = 1$) being unilaterally altruistic while other n users are selfish in a LB network (Fig. 3), where all other users are selfish users with identical demands and latency as in (3). The "Value of unilateral altruism" of the altruistic user, $VoU(i)$, is arbitrarily large for sufficiently large m.

Proof: The proof follows the similar techniques as that in proposition 5.1 and using lemma 6.1. The equivalent problem for the n user altruistic routing game considered can be expressed as (similar as in proposition 5.1)

$$\mathcal{P}_n: \quad x^i = \arg\min_{\mathbf{x} \in \mathbf{X}} \hat{J}_i(\mathbf{x}), \forall\ i = 1, \cdots, n.$$

where the perceived cost of users (when user i is altruistic) can be given by

$$\hat{J}_i(\mathbf{x}) = \sum_{(k=1)}^{n} \alpha_k^i J_k(\mathbf{x}).$$

Existence of an equilibrium is direct from the compact strategy set $\mathbf{X}$ and concave and continuity of the cost function $\hat{J}_i(\mathbf{x})$ [14]. Following the approach of proposition 5.1, we show that the load taker strategy of altruistic user and delta load taker strategy of other users constitutes a Nash equilibrium. Since this equilibrium yields a bounded cost for altruistic user, VoU is be arbitrarily large for large m.

Let index altruistic user by i and selfish users by $j(\neq i) = 1, \cdots, n$. We compute the best response for both type users.

Computation of $BR_j(x_i^i = r)$: The cost of a selfish user (u_j) for a load taker strategy of user i is given by (let denote $p = x_j^i$, $\bar{p} = r - x_j^i = x_j^j$, and $x_j^k = q$, $k \neq i, j$),

$$J_j(\bar{p}) = \bar{p}[T_j(\bar{p}) + (n-2)qT_k(\bar{p}) + (n-2)qT_{jk}(q) \\ + pT_i(r + (n-1)p) + pT_{ji}(p)]$$

Since the latency function is discontinuous, the cost function can be re-expressed as follows

$$J_j(\bar{p}) = \begin{cases} p[T_i(r+(n-1)p) + c_m] + (n-2)qc_m & \text{if } \bar{p} \geq r-\delta, \\ \bar{p}T_j(\bar{p}) + (n-2)qT_k(\bar{p}) + (n-2)qc_m & \qquad\qquad (10) \\ \quad + p[T_i(r+(n-1)p) + T_{ji}(p)] & \text{if } p > r-\delta. \end{cases}$$

Noting the discontinuity only at $x_j^j = r - \delta$, the marginal cost can be given as

$$\frac{\partial J_j(\bar{p})}{\partial \bar{p}} = \begin{cases} A^1(\bar{p}) < 0 & \text{if } \bar{p} \geq r-\delta, \\ B_m^1(\bar{p}) > 0 & \text{if } \bar{p} < r-\delta, \end{cases}$$

Using (10), we have $A^1(\bar{p}) = -p\frac{L}{\delta^m} - pT_i(r+(n-1)p) < 0$, and $B_m^1(\bar{p}) = \frac{(2\bar{p}-r)L}{\delta_m} + (n-2)\frac{L}{\delta_m} - c_m - (T_i(r+(n-1)p) - T_j(\bar{p}))$. We note that $B_m^1(p) > 0$ for sufficiently large m. Hence, the **best response is a delta load taker** strategy, i.e., $BR_j(r) = r - \delta$.

Computation of $BR_i(x_j^j = r-\delta)$: The cost of the altruistic user (u_i) for a delta load taker strategy of u_j is given by (let denote $x_i^i = p$ and $x_j^j = r - \delta = \bar{\delta}$),

$$\hat{J}_i(p) = (1-\beta)J_i(\mathbf{x}) + (n-1)\frac{\beta}{(n-1)}J_j(\mathbf{x})$$

where, $J_i(\mathbf{x}) = [pT_i(p+(n-1)\delta) + (n-1)\frac{\bar{p}}{n-1}(T_{ij}(\frac{\bar{p}}{n-1}) + T_j(\frac{\bar{p}}{n-1}+\bar{\delta}))]$ and $J_j(\mathbf{x}) = [\bar{\delta}T_j(\frac{\bar{p}}{n-1}+\bar{\delta}) + \delta(T_{ji}(\delta) + T_i(p+\delta(n-1)))]$. Noting that $\hat{J}_i(p)$ is convex in $p \in [0, r]$, if $\frac{\partial \hat{J}_i}{\partial p} < 0$ at $p = r^-$ (less than and close to r) then $\frac{\partial \hat{J}_i}{\partial p} < 0$ for $p \in [0, r]$.

Therefore, below we examine if $\frac{\partial \hat{J}_i}{\partial p} < 0$ holds true at $p = r^-$. The derivative of $\hat{J}_i(p)$ is

$$\frac{\partial \hat{J}_i(p)}{\partial p} = (1 - \beta)[p(g_i(p + (n-1)\delta) + g_j(\frac{\bar{p}}{n-1} + \bar{\delta}))$$
$$+ T_i(p + (n-1)\delta) - T_j(\frac{\bar{p}}{n-1} + \bar{\delta}) - (rg_j(\frac{\bar{p}}{n-1} + \bar{\delta})$$
$$+ T_{ij}(\frac{\bar{p}}{n-1}))] + \beta[\delta g_i(p + \delta(n-1))) - (\bar{\delta})g_j(\frac{\bar{p}}{n-1} + \bar{\delta})].$$

Plugging the values of $g(.)$ and $T(.)$ in (11), we obtain

$$\frac{\partial \hat{J}_i(p)}{\partial p} < 0, \quad \text{for } \beta \in \Gamma$$

where the set $\Gamma = (\frac{1}{n}, 1]$. Hence, we can state that $BR_i(r - \delta) = r$, for $\beta \in \Gamma$.

Therefore, together it constitute the Nash equilibrium. Denoting the equilibrium flow by $\tilde{\mathbf{x}}^{AL}$, and cost of user i as $J_i(\tilde{\mathbf{x}}^{AL})$, we have
$$\tilde{\mathbf{x}}^{AL} = (r, r - \delta), \quad \text{for } \beta \in \Gamma,$$
$$J_i(\tilde{\mathbf{x}}^{AL}) = rT_i(r + \delta) = rL.$$

The set of equilibria is denoted by $\mathbf{x}^{AL}$ when there exist multiple equilibria. Denote $\underline{\mathbf{x}}^{AL} \in \mathbf{x}^{AL}$ such that

$$J_i(\underline{\mathbf{x}}^{AL}) < J_i(\mathbf{x}), \text{for any } \mathbf{x} \in \mathbf{x}^{AL}. \tag{11}$$

Computation of VoU: Using (5) and (11), and the definition 5.2, we can express
$$VoU(i) = \frac{J_i(\underline{\mathbf{x}}^{SE})}{\inf_{(\vec{\alpha}_i, \mathbf{x}_{\vec{\alpha}_i}^{AL})} J_i(\mathbf{x}_{\vec{\alpha}_i}^{AL})} > \frac{J_1(\mathbf{x}^{SE})}{J_i(\tilde{\mathbf{x}}^{AL})}$$
$$= \frac{rL + (r/2 - \zeta_m)c_m}{2rL}.$$

The last inequality is from (11). The quantity $\frac{rL+(r/2-\zeta_m)c_m}{2rL}$ can be arbitrarily large for large m. $\diamond$

VII. CONCLUDING REMARKS AND PERSPECTIVE

Our investigation in this work reveals the benefits of unilateral altruism in a network routing game. We showed with the help of a simple example with two users that if one user becomes altruistic, his equilibrium utility drastically improves. An improvement by unilateral altruism is itself non-intuitive. Moreover, the improvement by altruism is drastic, which is quite surprising. To quantify this we proposed a metric, "Value of Unilateral Altruism". We also showed that the benefit of unilateral altruism is possible when there are n players each of whom have a non-negligible amount of traffic.

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APPENDIX

A. Proof of Lemma 4.2

Proof: Consider non-atomic type users at source node 1 and source node 2 with a total demand rate of r each to be dispatched to the destination node 3.

Let $p \in \mathcal{P}$ denote a path that a user can take from a source node to destination node and $\mathbf{T}_p(.)$ denote the cumulative latency observed for the path. The concept of road traffic wardrop equilibrium, originally introduced by Wardrop [15], which is the selfish equilibrium for non-atomic users in network traffic, is defined in terms of variational inequality as

$$\mathbf{T}_p(\mathbf{x}) - A \geq 0; \ (\mathbf{T}_p(\mathbf{x}) - A)x_p^i = 0, \quad \text{for } p \in \mathcal{P}, \tag{12}$$

where $A = \min_{p \in \mathcal{P}} \mathbf{T}_p(\mathbf{x})$.

The Wardrop equilibrium falls into the category of potential games with an infinite number of users and can be expressed by a single convex optimization [16]. Hence, there exist a unique equilibrium.

There are two possible paths from source node 1 to destination node: a) through l_1; and, b) through $l_3 \to l_2$. Path Similarly, from source node 2 following are the feasible paths: c) through l_2; and, d) through $l_4 \to l_1$. Using (12), we can easily see that at equilibrium the set $\mathcal{P} = \{a, c\}$, i.e., only local links are used. Therefore, the equilibrium flow is given by $\mathbf{x}^* = \{x_{1*}, x_{2*}\} = \{r, r\}$ and the latency cost experienced by a user is $T_1(.) = L$ and total cost of users from a source node equals $\sum_{k=1}^r T_1(.) = rL$.

Clearly, the user's cost is bounded and equal to the social optimum at Wardrop equilibrium. $\diamond$

B. Proof of Lemma 6.1

Proof: In order to compute PoA, we need to compute the worst equilibrium cost for selfish user and the cost at social optimum. To do so, We go in the following steps: i) We show that there exist unique Nash equilibrium for selfish users; ii) Since there can be only one equilibrium, we exploit the symmetric property and compute the Nash equilibrium. Obviously, this can be used for worst equilibrium cost of PoA computation; iii) Finally, we note that the social optimum is unique, the PoA follows by directly comparing the ration of selfish equilibrium and social optimum.

We allowed arbitrary number of node and a user associated with each user, as shown in Figs. 3. The Nash equilibrium for the selfish game corresponds to the solution to the following minimization problem

$$\mathcal{P}_1: \quad x^{i^*} = \arg\min_{\mathbf{x}^i \in \mathbf{X}^i} J_i(\mathbf{x}^i, \mathbf{x}^{-i}), \ \forall \ i = 1, \cdots, n.$$

The cost of user a i in the above equation can be given as:

$$J_i(\mathbf{x}) = x_i^i T_i(x_i) + \bar{x}_i^i (T_{ij}(\bar{x}_i^i) + T_j(x_j)),$$
$$\text{for } i \neq j = 1, \cdots, n.$$

where $\bar{x}_i^i = r^i - x_i^i$. We show the existence of unique Nash equilibrium by transforming our problem into parallel link network problem with arbitrary number of users, for which it known to exist unique Nash equilibrium (see [1, Theorem 1.]).

Consider a network consisting of two nodes: a and b, and of I parallel directed links, all from node a to node b. There are i users, $i = 1, \cdots, I$, having node a as the source node and node b as the destination node. The total demand rate for user i is r^i.

Let x_l^i be the rate at which user i sends over the link l, ($l = 1, \cdots, I$, which satisfy the positive flow constraint, $x_l^i \geq 0$, $\forall l$ and $\sum_l x_l^i = r^i$. User i determines $(x_1^i, \cdots, x_I^i)$ so as to minimize his cost $\overline{J}_i(\overline{\mathbf{x}})$ where $\overline{\mathbf{x}} = (x_l^i, l \in (1, \cdots, I), i \in \mathcal{I})$. Define $x_l = \sum_i x_l^i$.

We assume that $\overline{J}(\overline{\mathbf{x}})$ is the sum of the link cost functions: $\overline{J}_i(\overline{\mathbf{x}}) = \sum_l \overline{J}_i^l(\overline{\mathbf{x}})$, where $\overline{J}_i^p(\overline{\mathbf{x}})$ are expressed in terms of the cost $J_i^l(\mathbf{x})$, defined in the original problem, as follows. For any l and any i,
$$\overline{J}_i^i(\overline{\mathbf{x}}) = J_i^i(x_i^i, x_i)$$
$$\overline{J}_i^j(\overline{\mathbf{x}}) = J_i^{ij}(x_i^i, x_j) + J_i^j(x_j^i, x_j), \ \forall j \neq i.$$

recall that J_i^i represent cost of user i in its local link i and the corresponding flow is x_i^i.

We note that the cost function of the original problem satisfies the following assumptions (as that in [1]):

G1. $J_i(\mathbf{x})$ is the sum of the link costs incurred in each link, i.e.,
$$J_i(\mathbf{x}) = \sum_l J_i^l(\mathbf{x}).$$

G2. $J_i^l(\mathbf{x})$ is continuous (piecewise continuous with our elbow latency function, which suffice the condition) functions whose range is the nonnegative quadrant and their image is $[0, \infty]$.

G3. J_i^l are convex functions in the rate sent by user i over link l.

G4. Whenever finite, J_i^l is continuously differentiable in the flow sent by user i to link l. The marginal cost is denoted by $K_i^l(\mathbf{x}) = \frac{\partial J_i^l(\mathbf{x})}{\partial x_i^l}$.

G5. If not all users have finite cost and one user user has infinite cost then it can change its own flow o make this cost finite. This ensures that at any equilibrium users have finite cost if it is feasible.

The following assumption is also satisfied by $J_i(\mathbf{x})$, which is very important for uniqueness ([1]):

Π_1 $K_i^l(.)$ is function of two arguments: i) the total flow on the link l, x_l and ii) the flow of user i on link l, x_i^l. $K_i^l(x_i^l, x_i)$ is strictly increasing in both of its arguments.

1) Uniqueness of Nash equilibrium: If the cost of original problem with LB network satisfy the assumption (Π_1), it follows that the costs for the new routing problem also do satisfy the assumption (Π_1). The new routing problem has a unique Nash equilibrium under the assumptions (Π_1) (see Theorem 1 in [1]). By identifying the decision variable $\overline{\mathbf{x}}$ in the new routing

problem with the decision variable $\mathbf{x}$ of the original problem with LB network, we see that the minimization problem faced by each user is same in both the cases, and therefore we conclude that user optimum in our problem is also unique. Note, the although we considered a symmetric network, the proof holds good for non symmetric LB network also.

2) Computation of equilibrium: Since all the users have identical demands and facing identical network conditions, the users will have identical strategy at equilibrium. Such equilibrium are often called as symmetric equilibrium. Let x_i^i be the equilibrium strategy of user i, the flow in his local link, and x_i^j is the flow in a cross link to user j.

Let us compute the best response of user i for the strategy of user $j \neq i$. Let x_i^i be user i's best response for the strategy of user j be x_j^j. Note that we consider that all the other user's have identical strategy(x_j^j). Therefore, the best response of user i is given by
$$BR_i(x_j^j, n) = \arg\min_{x_i \in \mathbf{X}} J_i(\mathbf{x}).$$

The cost of user i is given by,
$$J_i(x_i^i, x_j^j) = x_i[T_i(x_i^j + (r - x_j^j))] + \sum_{j \neq i} \frac{\bar{x}_i^i}{n-1}\left[c_m\left(\frac{\bar{x}_i^i}{n-1}\right)\right.$$
$$\left. + T_j\left(x_j^j + \frac{(r - x_i^i)}{n-1}\right) + \frac{(n-2)(r - x_k^k)}{n-1}\right].$$

where $k \neq i, j$. The derivative of $J_i(.)$ is (denote $p = x_i^i$)
$$\frac{\partial J_i(p, x_j^j)}{\partial p} = pg_i(p + r - x_j^j) + T_i(p + r - x_j^j) - \bar{p}\Big[g_j$$
$$\left(x_j^j + \frac{(r - p)}{n-1} + \frac{r - x_k^k}{n-1}\right)\Big] - \left[c_m + T_j\left(x_j^j + \frac{(n-2)\bar{x}_k^k + \bar{p}}{n-1}\right)\right],$$
$$= p[g_i(x_i) + g_j(x_j)] + [T_i(x_i) - T_j(x_j)] - (rg_j(x_j) + c_m).$$

Due to symmetric structure, we note that $x_i = x_{(j \neq i)} = r$. Thus, we have $T_i(x_i) - T_j(x_j) = 0$ and $g_i(x_i) = g_j(x_j) = g$. Therefore,
$$BR_i(x_j^j) = \frac{c_m + rg}{2g} = \frac{1}{2}\left[r + \frac{c_m}{L/\delta_m}\right] = r/2 + \zeta_m.$$

Since all the users are symmetric, the equilibrium strategy of any user is same. Therefore, the equilibrium strategy and the equilibrium cost of a user i can be given by
$$x_i^{i^{SE}} = r/2 + \zeta_m, \ J_i^{SE} = rL + \left(\frac{r}{2} - \zeta_m\right)(c_m).$$

One can directly observe that at social optimum (OPT) only local links are used. Therefore, the user cost J_i^{OPT} at social optima is given as
$$J_i^{OPT} = rT_i(r) = rL, \ \forall \ i.$$

3) Computation of PoA: Therefore the price of anarchy can be expressed as
$$PoA = \sup_m \frac{\sum_i rL + (\frac{r}{2} - \zeta_m)(c_m)}{\sum_i rL}$$
$$= 1 + \sup_m \frac{(\frac{r}{2} - \zeta_m)(c_m)}{\sum_i rL}.$$

Evidently, the PoA can be arbitrarily large for sufficiently large m. This concludes the proof. $\diamond$