

HIGHER SECANT VARIETIES OF $\mathbb{P}^n \times \mathbb{P}^m$ EMBEDDED IN BI-DEGREE $(1, d)$

ALESSANDRA BERNARDI, ENRICO CARLINI, AND MARIA VIRGINIA CATALISANO

ABSTRACT. Let $X_{(1,d)}^{(n,m)}$ denote the Segre-Veronese embedding of $\mathbb{P}^n \times \mathbb{P}^m$ via the sections of the sheaf $\mathcal{O}(1, d)$. We study the dimensions of higher secant varieties of $X_{(1,d)}^{(n,m)}$ and we prove that there is no defective s^{th} secant variety, except possibly for n values of s . Moreover when $\binom{m+d}{d}$ is a multiple of $(m+n+1)$, the s^{th} secant variety of $X_{(1,d)}^{(n,m)}$ has the expected dimension for every s .

INTRODUCTION

The s^{th} higher secant variety of a projective variety $X \subset \mathbb{P}^N$ is defined to be the Zariski closure of the union of the span of s points of X (see Definition 1.1). We will denote it with $\sigma_s(X)$.

Secant varieties have been intensively studied (see for example [AH], [BCS], [Gr], [Li], [St], [Za]). One of the first problems of interest is the computation of their dimensions. In fact, there is an expected dimension for $\sigma_s(X) \subset \mathbb{P}^N$, that is, the minimum between N and $s(\dim X) + s - 1$. There are well known examples where that dimension is not attained. For example, the variety of secant lines to the Veronese surface in \mathbb{P}^5 . A variety X is said to be $(s-1)$ -defective if there exists an integer $s \in \mathbb{N}$ such that the dimension of $\sigma_s(X)$ is less than the expected value. We would like to notice that only for Veronese varieties a complete list of all defective cases is given. This description is obtained using a result by J. Alexander and A. Hirschowitz [AH] recently reproposed with a shorter proof in [BO].

The interest around these varieties has been recently revived from many different areas of mathematics and applications when X is a variety parameterizing certain kind of tensors (for example Electrical Engineering; Antenna Array Processing [ACCF], [DM] and Telecommunications [Ch], [dLC]; Statistics, cumulant tensors [McC]; Data Analysis; Independent Component Analysis [Co1], [JS]; for other applications see also [Co2], [CR], [dLMV], [SBG] [GVL]).

Segre varieties parameterize completely decomposable tensors (i.e., projective classes of tensors in $\mathbb{P}(V_1 \otimes \cdots \otimes V_t)$ that can be written as $v_1 \otimes \cdots \otimes v_t$, with $v_i \in V_i$ and V_i vector spaces for $i = 1, \dots, t$). The s^{th} higher secant variety of a Segre variety is therefore the closure of the sets of tensors that can be written as a linear combination of s completely decomposable tensors. In particular, the higher secant varieties of the Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ are well known as they parameterize matrices of bounded rank (e.g., see [Hr]).

A Segre-Veronese variety can be described in two ways. Either as the embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_t}$ with the sections of the sheaf $\mathcal{O}(d_1, \dots, d_t)$ into \mathbb{P}^N , for certain $d_1, \dots, d_t \in \mathbb{N}$, with $N = \prod_{i=1}^t \binom{n_i+d_i}{d_i} - 1$, or as a section of a Segre variety. Consider

the Segre variety that naturally lives in $\mathbb{P}(V_1^{\otimes d_1} \otimes \cdots \otimes V_t^{\otimes d_t})$ with V_i vector spaces of dimensions $n_i + 1$ for $i = 1, \dots, t$. Then the Segre-Veronese variety is obtained intersecting that Segre variety with the projective subspaces $\mathbb{P}(S^{d_1} V_1 \otimes \cdots \otimes S^{d_t} V_t)$ of projective classes of partially symmetric tensors (where $S^{d_i} V_i \subset V_i^{\otimes d_i}$ is the subspace of completely symmetric tensors of $V_i^{\otimes d_i}$).

These two different ways of describing Segre-Veronese varieties allow us to translate problems about partially symmetric tensors into problems on forms of multi-degree (d_1, \dots, d_t) and viceversa. People working in Applied Mathematics are interested in many kind of tensors that they call “structured tensors”, which means tensors with a well defined structure. Partially symmetric tensors turns out to be useful in the Antenna Array processing. In this problem, one needs to give a precise localization of different antennas sets (whose positions are unknown) by using signals coming from receivers whose locations are known.

In this paper we will describe the s^{th} higher secant varieties of the embedding of $\mathbb{P}^n \times \mathbb{P}^m$ into \mathbb{P}^N ($N = (n+1)\binom{m+d}{d} - 1$), by the sections of the sheaf $\mathcal{O}(1, d)$, for almost all $s \in \mathbb{N}$ (see Theorem 2.3). We will follow the description of Segre-Veronese variety as the variety parameterizing forms of certain multi-degree.

One of the first instances of the study of the two factors Segre-Veronese varieties is the one of $\mathbb{P}^1 \times \mathbb{P}^2$ embedded in bi-degree $(1, 3)$ and appears in a paper by London [Lo], for a more recent approach see [DF] and [CaCh]. A first generalization for $\mathbb{P}^1 \times \mathbb{P}^2$ embedded in bi-degree $(1, d)$ is treated in [DF]. The general case for $\mathbb{P}^1 \times \mathbb{P}^2$ embedded in any bi-degree (d_1, d_2) is done in [BD]. In [ChCi] the case $\mathbb{P}^1 \times \mathbb{P}^n$ embedded in bi-degree $(d, 1)$ is treated.

In [CaCh] one can find the defective cases $\mathbb{P}^2 \times \mathbb{P}^3$ embedded in bi-degree $(1, 2)$, $\mathbb{P}^3 \times \mathbb{P}^4$ embedded in bi-degree $(1, 2)$ and $\mathbb{P}^2 \times \mathbb{P}^5$ embedded in bi-degree $(1, 2)$.

The paper [CGG] studies also the cases $\mathbb{P}^n \times \mathbb{P}^m$ with bi-degree $(n+1, 1)$; $\mathbb{P}^1 \times \mathbb{P}^1$ with bi-degree (d_1, d_2) and $\mathbb{P}^2 \times \mathbb{P}^2$ with bi-degree $(2, 2)$.

In [Ab] the cases $\mathbb{P}^1 \times \mathbb{P}^m$ in bi-degree $(2d+1, 2)$, $\mathbb{P}^1 \times \mathbb{P}^m$ in bi-degree $(2d, 2)$, and $\mathbb{P}^1 \times \mathbb{P}^m$ in bi-degree $(d, 3)$ can be found. A recent result on $\mathbb{P}^n \times \mathbb{P}^m$ in bi-degree $(1, 2)$ is in [AB], where the authors prove the existence of two functions $\underline{s}(n, m)$ and $\bar{s}(n, m)$ such that $\sigma_s(X_{(1,2)}^{(n,m)})$ has the expected dimension for $s \leq \underline{s}(n, m)$ and for $s \geq \bar{s}(n, m)$. In the same paper it is also shown that $X_{(1,2)}^{(1,m)}$ is never defective and all the defective cases for $X_{(1,2)}^{(2,m)}$ are described.

The varieties $\mathbb{P}^n \times \mathbb{P}^m$ embedded in bi-degree $(1, d)$ are related to the study of Grassmann defectivity ([DF]). More precisely, one can consider the Veronese variety X obtained by embedding \mathbb{P}^m in \mathbb{P}^N using the d -uple Veronese embedding ($N = \binom{m+d}{d}$). Then consider, in $\mathbb{G}(n, N)$, the $(n, s-1)$ -Grassmann secant variety of X , that is, the closure of the set of n -dimensional linear spaces contained in the linear span of s linearly independent points of X . The variety X is said to be $(n, s-1)$ -Grassmann defective if the $(n, s-1)$ -Grassmann secant variety of X does not have the expected dimension. It is shown in [DF], following Terracini’s ideas in [Te1], that X is $(n, s-1)$ -Grassmann defective if and only if the s^{th} higher secant varieties of the embedding of $\mathbb{P}^n \times \mathbb{P}^m$ into \mathbb{P}^N ($N = (n+1)\binom{m+d}{d} - 1$), by the sections of the sheaf $\mathcal{O}(1, d)$, is $(s-1)$ -defective. Hence, the result proved in this paper gives information about the Grassmann defectivity of Veronese varieties (see Remark 2.5).

In Proposition 2.1 we prove the non-defectivity of the Hilbert function of a subscheme of \mathbb{P}^{n+m} made of a d -uple \mathbb{P}^{n-1} , t projective subspaces of dimension n containing it, a simple \mathbb{P}^{m-1} and a number of double points that is an integer multiple of $n-1$. This result, together with Theorem 1.1 in [CGG] (see Theorem 1.4 in this paper), gives immediately the regularity of the higher secant varieties of the Segre-Veronese variety that we are looking for.

More precisely, we consider (see Section 2) the case of $\mathbb{P}^n \times \mathbb{P}^m$ embedded in bi-degree $(1, d)$ for $d \geq 3$. Our main result is Theorem 2.3 where we prove that the s^{th} higher secant variety of such Segre-Veronese varieties has the expected dimension for $s \leq s_1$ and for $s \geq s_2$, where

$$s_1 = \max \left\{ s \in \mathbb{N} \mid s \text{ is a multiple of } (n+1) \text{ and } s \leq \left\lfloor \frac{(n+1)\binom{m+d}{d}}{m+n+1} \right\rfloor \right\},$$

$$s_2 = \min \left\{ s \in \mathbb{N} \mid s \text{ is a multiple of } (n+1) \text{ and } s \geq \left\lceil \frac{(n+1)\binom{m+d}{d}}{m+n+1} \right\rceil \right\}.$$

1. PRELIMINARIES AND NOTATION

We will always work with projective spaces defined over an algebraically closed field K of characteristic 0. Let us recall the notion of higher secant varieties and some classical results which we will often use. For definitions and proofs we refer the reader to [CGG].

Definition 1.1. Let $X \subset \mathbb{P}^N$ be a projective variety. We define the s^{th} higher secant variety of X , denoted by $\sigma_s(X)$, as the Zariski closure of the union of all linear spaces spanned by s points of X , i.e.:

$$\sigma_s(X) := \overline{\bigcup_{P_1, \dots, P_s \in X} \langle P_1, \dots, P_s \rangle} \subset \mathbb{P}^N.$$

When $\sigma_s(X)$ does not have the expected dimension, that is $\min\{N, s(\dim X + 1) - 1\}$, X is said to be $(s-1)$ -defective, and the positive integer

$$\delta_{s-1}(X) = \min\{N, s(\dim X + 1) - 1\} - \dim \sigma_s(X)$$

is called the $(s-1)$ -defect of X .

The basic tool to compute the dimension of $\sigma_s(X)$ is Terracini's Lemma ([Te]):

Lemma 1.2 (Terracini's Lemma). *Let X be an irreducible variety in \mathbb{P}^N , and let P_1, \dots, P_s be s generic points on X . Then the tangent space to $\sigma_s(X)$ at a generic point in $\langle P_1, \dots, P_s \rangle$ is the linear span in \mathbb{P}^N of the tangent spaces T_{X, P_i} to X at P_i , $i = 1, \dots, s$, hence*

$$\dim \sigma_s(X) = \dim \langle T_{X, P_1}, \dots, T_{X, P_s} \rangle.$$

A consequence of Terracini's Lemma is the following corollary (see [CGG, Section 1] or [AB, Section 2] for a proof of it).

Corollary 1.3. *Let $X_{(1,d)}^{(n,m)} \subset \mathbb{P}^N$ be the Segre-Veronese variety image of the embedding of $\mathbb{P}^n \times \mathbb{P}^m$ by the sections of the sheaf $\mathcal{O}(1, d)$ into \mathbb{P}^N , with $N = (n+1)\binom{m+d}{d} - 1$. Then*

$$\dim \sigma_s \left(X_{(1,d)}^{(n,m)} \right) = N - \dim(I_Z)_{(1,d)} = H(Z, (1, d)) - 1,$$

where $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ is a set of s generic double points, I_Z is the multihomogeneous ideal of Z in $R = K[x_0, \dots, x_n, y_0, \dots, y_m]$, the multigraded coordinate ring of $\mathbb{P}^n \times \mathbb{P}^m$, and $H(Z, (1, d))$ is the multigraded Hilbert function of Z .

Now we recall the fundamental tool which allows us to convert certain questions about ideals of varieties in multiprojective space to questions about ideals in standard polynomial rings (for a more general statement see [CGG, Theorem 1.1])

Theorem 1.4. *Let $X_{(1,d)}^{(n,m)} \subset \mathbb{P}^N$ and $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ as in Corollary 1.3. Let $H_1, H_2 \subset \mathbb{P}^{n+m}$ be generic projective linear spaces of dimensions $n-1$ and $m-1$, respectively, and let $P_1, \dots, P_s \in \mathbb{P}^{n+m}$ be generic points. Denote by*

$$dH_1 + H_2 + 2P_1 + \dots + 2P_s \subset \mathbb{P}^{n+m}$$

the scheme defined by the ideal sheaf $\mathcal{I}_{H_1}^d \cap \mathcal{I}_{H_2} \cap \mathcal{I}_{P_1}^2 \cap \dots \cap \mathcal{I}_{P_s}^2 \subset \mathcal{O}_{\mathbb{P}^{n+m}}$. Then

$$\dim(I_Z)_{(1,d)} = \dim(I_{dH_1+H_2+2P_1+\dots+2P_s})_{d+1}$$

hence

$$\dim \sigma_s \left(X_{(1,d)}^{(n,m)} \right) = N - \dim(I_{dH_1+H_2+2P_1+\dots+2P_s})_{d+1}.$$

Since we will make use of Castelnuovo's inequality several times, we recall it here (for notation and proof we refer to [AH2], Section 2).

Lemma 1.5 (Castelnuovo's inequality). *Let $H \subset \mathbb{P}^N$ be a hyperplane, and let $X \subset \mathbb{P}^N$ be a scheme. We denote by $\text{Res}_H X$ the scheme defined by the ideal $(I_X : I_H)$ and we call it the residual scheme of X with respect to H , while the scheme $\text{Tr}_H X \subset H$ is the schematic intersection $X \cap H$, called the trace of X on H . Then*

$$\dim(I_{X, \mathbb{P}^N})_t \leq \dim(I_{\text{Res}_H X, \mathbb{P}^N})_{t-1} + \dim(I_{\text{Tr}_H X, H})_t.$$

2. SEGRE-VERONESE EMBEDDINGS OF $\mathbb{P}^n \times \mathbb{P}^m$

First we prove a relevant fact about subschemes of \mathbb{P}^{n+m} . Using this result and Theorem 1.4 we will obtain our main theorem.

Proposition 2.1. *Let $d \geq 3$, $n, m \geq 1$ and let $s = (n+1)q$ be an integer multiple of $n+1$. Let $P_1, \dots, P_s \in \mathbb{P}^{n+m}$ be generic points and let $H_1 \simeq \mathbb{P}^{n-1}, H_2 \simeq \mathbb{P}^{m-1}$ be generic linear spaces in \mathbb{P}^{n+m} . Let $W_1, \dots, W_t \subset \mathbb{P}^{n+m}$ be t generic linear spaces of dimension n containing H_1 . Now consider the scheme*

$$(1) \quad \mathbb{X} := dH_1 + H_2 + 2P_1 + \dots + 2P_s + W_1 + \dots + W_t$$

Then for any $q, t \in \mathbb{N}$ the dimension of the degree $d+1$ piece of the ideal $I_{\mathbb{X}}$ is the expected one, that is

$$\dim(I_{\mathbb{X}})_{d+1} = \max \left\{ (n+1) \binom{m+d}{d} - s(n+m+1) - t(n+1); 0 \right\}.$$

Proof. We will prove the proposition by induction on n .

A hypersurface defined by a form of $(I_{dH_1})_{d+1}$ cuts on $W_i \simeq \mathbb{P}^n$ a hypersurface which has H_1 as a fixed component of multiplicity d . It follows that

$$\dim(I_{dH_1, W_i})_{d+1} = \dim(I_{0, W_i})_1 = n+1.$$

Hence the expected number of conditions that a linear space W_i imposes to the forms of $(I_{\mathbb{X}})_{d+1}$ is at most $n + 1$. Moreover a double point imposes at most $n + m + 1$ conditions. So, since by Theorem 1.4 with $Z = \emptyset$ we get

$$\dim(I_{dH_1+H_2})_{d+1} = \dim R_{(1,d)} = (n+1) \binom{m+d}{d},$$

(where $R = K[x_0, \dots, x_n, y_0, \dots, y_m]$), we have

$$(2) \quad \dim(I_{\mathbb{X}})_{d+1} \geq (n+1) \binom{m+d}{d} - s(n+m+1) - t(n+1).$$

Now let $H \subset \mathbb{P}^{n+m}$ be a generic hyperplane containing H_2 and let $\tilde{\mathbb{X}}$ be the scheme obtained from \mathbb{X} by specializing the nq points P_1, \dots, P_{nq} on H , (P_{nq+1}, \dots, P_s) remain generic points, not lying on H .

Since by the semicontinuity of the Hilbert Function $\dim(I_{\tilde{\mathbb{X}}})_{d+1} \geq \dim(I_{\mathbb{X}})_{d+1}$, by (2) we have

$$(3) \quad \dim(I_{\tilde{\mathbb{X}}})_{d+1} \geq (n+1) \binom{m+d}{d} - s(n+m+1) - t(n+1).$$

Let $V_i = \langle H_1, P_i \rangle \simeq \mathbb{P}^n$. Since the linear spaces V_i are in the base locus of the hypersurfaces defined by the forms of $(I_{\tilde{\mathbb{X}}})_{d+1}$, we have

$$(4) \quad (I_{\tilde{\mathbb{X}}})_{d+1} = (I_{\tilde{\mathbb{X}}+V_1+\dots+V_s})_{d+1}.$$

Consider the residual scheme of $(\tilde{\mathbb{X}} + V_1 + \dots + V_s)$ with respect to H : $Res_H(\tilde{\mathbb{X}} + V_1 + \dots + V_s)$

$$\begin{aligned} &= dH_1 + W_1 + \dots + W_t + P_1 + \dots + P_{nq} + 2P_{nq+1} + \dots + 2P_s + V_1 + \dots + V_s \\ &= dH_1 + W_1 + \dots + W_t + 2P_{nq+1} + \dots + 2P_s + V_1 + \dots + V_s \subset \mathbb{P}^{n+m}. \end{aligned}$$

Any form of degree d in $I_{Res_H(\tilde{\mathbb{X}}+V_1+\dots+V_s)}$ represents a cone whose vertex contains H_1 . Hence if $\mathbb{Y} \subset \mathbb{P}^m$ is the scheme obtained by projecting $Res_H(\tilde{\mathbb{X}} + V_1 + \dots + V_s)$ from H_1 in a \mathbb{P}^m , we have:

$$(5) \quad \dim(I_{Res_H(\tilde{\mathbb{X}}+V_1+\dots+V_s)})_d = \dim(I_{\mathbb{Y}})_d.$$

Since the image by this projection of each W_i is a point, and for $1 \leq i \leq nq$ the image of $P_i + V_i$ is a simple point, and for $nq + 1 \leq i \leq s$ the image of $2P_i + V_i$ is a double point, we have that \mathbb{Y} is a scheme consisting of $t + nq$ generic points and q generic double points.

Now by the Alexander-Hirschowitz Theorem (see [A.H]), since $d > 2$ and $t + nq > 1$ we have that the dimension of the degree d part of the ideal of q double points plus $t + nq$ simple points is always as expected. So we get

$$(6) \quad \dim(I_{\mathbb{Y}})_d = \max \left\{ \binom{m+d}{d} - q(m+1) - t - nq ; 0 \right\}.$$

Now let $n = 1$. In this case we have: $s = 2q$,

$$(7) \quad \begin{aligned} &\dim(I_{Res_H(\tilde{\mathbb{X}}+V_1+\dots+V_s)})_d \\ &= \dim(I_{\mathbb{Y}})_d = \max \left\{ \binom{m+d}{d} - q(m+1) - t - q ; 0 \right\}, \end{aligned}$$

moreover H_1 is a point, $H_1 \cap H$ is the empty set, the W_i and the V_i are lines, and V_i is the line spanned by the points H_1 and P_i .

Set $W'_i = W_i \cap H$, $V'_i = V_i \cap H$. Note that for $1 \leq i \leq q$ we have $V'_i = P_i$. The trace on H of $\tilde{\mathbb{X}} + V_1 + \cdots + V_s$ is

$$\begin{aligned} & Tr_H(\tilde{\mathbb{X}} + V_1 + \cdots + V_s) \\ &= H_2 + 2P_1 + \cdots + 2P_q + W'_1 + \cdots + W'_t + V'_1 + \cdots + V'_{2q} \\ &= H_2 + 2P_1 + \cdots + 2P_q + W'_1 + \cdots + W'_t + V'_{q+1} + \cdots + V'_{2q} \subset H \simeq \mathbb{P}^m \end{aligned}$$

So $Tr_H(\tilde{\mathbb{X}} + V_1 + \cdots + V_s) \subset H$ is a scheme in \mathbb{P}^m union of $H_2 \simeq \mathbb{P}^{m-1}$, plus q generic double points and $t+q$ generic simple points. As above, by [AH], since $d > 2$ and $t+q \geq 1$ we get

$$\begin{aligned} & \dim(I_{Tr_H(\tilde{\mathbb{X}}+V_1+\cdots+V_s)})_{d+1} = \dim(I_{2P_1+\cdots+2P_q+W'_1+\cdots+W'_t+V'_{q+1}+\cdots+V'_{2q}})_d \\ (8) \quad &= \max \left\{ \binom{m+d}{d} - q(m+1) - t - q ; 0 \right\}. \end{aligned}$$

By Castelnuovo's inequality (see Lemma 1.5), (7) and (8) we get

$$(9) \quad \dim(I_{\tilde{\mathbb{X}}+V_1+\cdots+V_s})_{d+1} \leq \max \left\{ 2 \binom{m+d}{d} - 2q(m+1) - 2t - 2q ; 0 \right\}.$$

So by (3), (4) and (9) we have

$$\dim(I_{\tilde{\mathbb{X}}})_{d+1} = \max \left\{ 2 \binom{m+d}{d} - 2q(m+2) - 2t ; 0 \right\}.$$

From here, by (2) and by the semicontinuity of the Hilbert Function we get

$$\dim(I_{\tilde{\mathbb{X}}})_{d+1} = \max \left\{ 2 \binom{m+d}{d} - 2q(m+2) - 2t ; 0 \right\}$$

and the result is proved for $n = 1$.

Let $n > 1$.

Set: $H'_1 = H_1 \cap H$; $W'_i = W_i \cap H$; $V'_i = V_i \cap H$. With this notation the trace of $\tilde{\mathbb{X}} + V_1 + \cdots + V_s$ on H is:

$$\begin{aligned} & Tr_H(\tilde{\mathbb{X}} + V_1 + \cdots + V_s) = \\ &= dH'_1 + H_2 + 2P_1 + \cdots + 2P_{nq} + W'_1 + \cdots + W'_t + V'_1 + \cdots + V'_s \subset H \simeq \mathbb{P}^{n+m-1}. \end{aligned}$$

Analogously as above, observe that the linear spaces $V'_i = \langle H'_1, P_i \rangle \simeq \mathbb{P}^n$ are in the base locus for the hypersurfaces defined by the forms of $(I_{dH'_1+2P_i})_{d+1}$, hence the parts of degree $d+1$ of the ideals of $Tr_H(\tilde{\mathbb{X}} + V_1 + \cdots + V_s)$ and of $Tr_H(\tilde{\mathbb{X}} + V_{nq+1} + \cdots + V_s)$ are equal. So we have

$$(I_{Tr_H(\tilde{\mathbb{X}}+V_1+\cdots+V_s)})_{d+1} = (I_{Tr_H(\tilde{\mathbb{X}}+V_{nq+1}+\cdots+V_s)})_{d+1} = (I_{\mathbb{T}})_{d+1},$$

where

$$\mathbb{T} = dH'_1 + H_2 + 2P_1 + \cdots + 2P_{nq} + W'_1 + \cdots + W'_t + V'_{nq+1} + \cdots + V'_s \subset \mathbb{P}^{n+m-1},$$

that is, \mathbb{T} is union of the d -uple linear space $H'_1 \simeq \mathbb{P}^{n-2}$, the linear space $H_2 \simeq \mathbb{P}^{m-1}$, $t+q$ generic linear spaces through H'_1 , and nq double points. Hence by the inductive hypothesis we have

$$(10) \quad \dim(I_{\mathbb{T}})_{d+1} = \max \left\{ n \binom{m+d}{d} - nq(n+m) - (t+q)n ; 0 \right\}.$$

By (4), by Lemma 1.5, by (5), (6) and (10) we get

$$\begin{aligned} \dim(I_{\mathbb{X}})_{d+1} &\leq \max \left\{ \binom{m+d}{d} - q(m+1) - t - nq ; 0 \right\} + \\ &\quad + \max \left\{ n \binom{m+d}{d} - nq(n+m) - (t+q)n ; 0 \right\} \\ &= \max \left\{ (n+1) \binom{m+d}{d} - s(n+m+1) - t(n+1) ; 0 \right\}. \end{aligned}$$

Now the conclusion follows from (2) and the semicontinuity of the Hilbert Function and this ends the proof. \square

Corollary 2.2. *Let $d \geq 3$, $n, m \geq 1$ and let*

$$\begin{aligned} s_1 &:= \max \left\{ s \in \mathbb{N} \mid s \text{ is a multiple of } (n+1) \text{ and } s \leq \left\lfloor \frac{(n+1) \binom{m+d}{d}}{m+n+1} \right\rfloor \right\} \\ s_2 &:= \min \left\{ s \in \mathbb{N} \mid s \text{ is a multiple of } (n+1) \text{ and } s \geq \left\lceil \frac{(n+1) \binom{m+d}{d}}{m+n+1} \right\rceil \right\}. \end{aligned}$$

Let $P_1, \dots, P_s \in \mathbb{P}^{n+m}$ be generic points and $H_1 \simeq \mathbb{P}^{n-1}, H_2 \simeq \mathbb{P}^{m-1}$ be generic linear spaces in \mathbb{P}^{n+m} . Consider the scheme

$$\mathbb{X} := dH_1 + H_2 + 2P_1 + \dots + 2P_s.$$

Then for any $s \leq s_1$ and any $s \geq s_2$ the dimension of $(I_{\mathbb{X}})_{d+1}$ is the expected one, that is

$$\dim(I_{\mathbb{X}})_{d+1} = \begin{cases} (n+1) \binom{m+d}{d} - s(n+m+1) & \text{for } s \leq s_1 \\ 0 & \text{for } s \geq s_2 \end{cases}$$

Proof. By applying Proposition 2.1, with $t = 0$, to the scheme $\mathbb{X} = dH_1 + H_2 + 2P_1 + \dots + 2P_s$, we get that the dimension of $I(\mathbb{X})_{d+1}$ is the expected one for $s = (n+1)q$ and for any $q \in \mathbb{N}$. Hence if s_1 is the biggest integer multiple of $n+1$ such that $\dim(I_{\mathbb{X}})_{d+1} \neq 0$ we get that for that value of s the Hilbert function $H(I_{\mathbb{X}}, d+1)$ has the expected value. Now if for such s_1 we have that $(I_{\mathbb{X}})_{d+1}$ has the expected dimension than it has the expected dimension also for every $s \leq s_1$.

Now, if s_2 is the smallest integer multiple of $n+1$ such that $\dim(I_{\mathbb{X}})_{d+1} = 0$ then obviously such a dimension will be zero for all $s \geq s_2$. \square

Theorem 2.3. *Let $d \geq 3$, $n, m \geq 1$, $N = (n+1) \binom{m+d}{d} - 1$ and let s_1, s_2 be as in Corollary 2.2.*

Then the variety $\sigma_s(X_{(1,d)}^{(n,m)}) \subset \mathbb{P}^N$ has the expected dimension for any $s \leq s_1$ and any $s \geq s_2$, that is

$$\dim \sigma_s \left(X_{(1,d)}^{(n,m)} \right) = \begin{cases} s(n+m+1) - 1 & \text{for } s \leq s_1 \\ N & \text{for } s \geq s_2 \end{cases}.$$

Proof. Let $H_1, H_2 \subset \mathbb{P}^{m+n}$ be projective subspaces of dimensions $n-1$ and $m-1$ respectively and let $P_1, \dots, P_s \in \mathbb{P}^{n+m}$ be s generic points of \mathbb{P}^{n+m} . Define $\mathbb{X} \subset \mathbb{P}^{m+n}$ to be the scheme $\mathbb{X} := dH_1 + H_2 + 2P_1 + \dots + 2P_s$. Theorem 1.1 in [CGG] shows that $\dim \sigma_s \left(X_{(1,d)}^{(n,m)} \right)$ is the expected one if and only if $\dim(I_{\mathbb{X}})_{d+1}$ is the

expected one. Therefore the conclusion immediately follows from Theorem 1.4 and Corollary 2.2. \square

Remark 2.4. If $\binom{m+d}{d}$ is multiple of $(m+n+1)$, say $\binom{m+d}{d} = h(m+n+1)$, we get

$$\left\lfloor \frac{(n+1)\binom{m+d}{d}}{m+n+1} \right\rfloor = \left\lceil \frac{(n+1)\binom{m+d}{d}}{m+n+1} \right\rceil = h(n+1)$$

so $s_1 = s_2$. Hence in this case the variety $\sigma_s \left(X_{(1,d)}^{(n,m)} \right)$ has the expected dimension for any s .

If $\binom{m+d}{d}$ is not multiple of $(m+n+1)$, it is easy to show that $s_2 - s_1 = n$. Thus there are at most n values of s for which the s^{th} higher secant varieties of $X_{(1,d)}^{(n,m)}$ can be defective.

Remark 2.5. Theorem 2.3 has a straightforward interpretation in terms of Grassmann defectivity. More precisely, we see that the d -uple Veronese embedding of \mathbb{P}^m is not $(n, s-1)$ -Grassmann defective when $s \leq s_1$ or $s \geq s_2$.

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(A. Bernardi) CIRM–FBK c/o UNIVERSITÀ DEGLI STUDI DI TRENTO, VIA SOMMARIVE 14,
38123 POVO (TRENTO), ITALY
E-mail address: bernardi@fbk.eu

(E. Carlini) DIPARTIMENTO DI MATEMATICA, POLITECNICO DI TORINO, TURIN, ITALY
E-mail address: enrico.carlini@polito.it

(M.V.Catalisano) DIPTTEM - DIPARTIMENTO DI INGEGNERIA DELLA PRODUZIONE, TERMOEN-
ERGETICA E MODELLI MATEMATICI, UNIVERSITÀ DI GENOVA, PIAZZALE KENNEDY, PAD. D 16129
GENOVA, ITALY.
E-mail address: catalisano@diptem.unige.it