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Some defective secant varieties to osculating varieties of Veronese surfaces

A. Bernardi, M. V. Catalisano *

Abstract. We consider the k -osculating varieties $O_{k,d}$ to the Veronese d -uple embeddings of \mathbb{P}^2 . By studying the Hilbert function of certain zero-dimensional schemes $Y \subset \mathbb{P}^2$, we find the dimension of $O_{k,d}^s$, the $(s-1)^{th}$ secant varieties of $O_{k,d}$, for $3 \leq s \leq 6$ and $s = 9$, and we determine whether those secant varieties are defective or not.

0. Introduction.

The problem of determining the dimension of the *higher secant varieties* of a projective variety is a classical subject of study. In the present paper we are concerned with the $(s-1)^{th}$ higher secant varieties of $O_{k,V_{n,d}}$, where $O_{k,V_{n,d}}$ is the k -osculating variety to the Veronese embedding $V_{n,d}$ of \mathbb{P}^n into \mathbb{P}^N ($N = \binom{d+n}{n} - 1$) via the complete linear system R_d , where $R = K[x_0, \dots, x_n]$, and K is an algebraically closed field of characteristic zero.

This matter has been dealt with by several authors in the last few years (see [2], [3], [4], [5], [7]). We wish to mention E. Ballico and C. Fontanari. In [2] and [3] they study the higher secant varieties of $O_{k,V_{n,d}}$ for $n = 2$ and $k = 1, 2$, and they prove the following results:

0.1. Proposition. *For $k = 1$, the $(s-1)^{th}$ higher secant variety of the tangential variety to $V_{2,d}$ has the expected dimension, unless $s = 2$ and $d = 3$.*

0.2. Proposition. *For $k = 2$, the $(s-1)^{th}$ higher secant variety of 2-osculating variety to $V_{2,d}$ has the expected dimension, unless $s = 2$ and $d = 4$.*

In this note, for $n = 2$, for $3 \leq s \leq 6$ and $s = 9$, and for all k , we will determine the dimension of $O_{k,V_{2,d}}^s$. The methods for proving our results are similar to the ones used by Ballico and Fontanari. The basic idea is to use Terracini's Lemma (see [13]), then, via apolarity, the calculation of $\dim O_{k,V_{2,d}}^s$ is related to the evaluation of the Hilbert function of certain 0-dimensional schemes $Y \subset \mathbb{P}^2$ supported at s generic points (see [5]), and this is done using geometric constructions, Bezout's theorem, and the Horace Method ([11]).

In the first section we fix some notation, and describe the relationship between the higher secant varieties we want to study, and the 0-dimensional schemes Y .

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In the second section we relate the Hilbert function of Y to the Hilbert function $H(X, d)$ of a scheme X of s generic $(k + 1)$ -fat points (Lemma 2.2, Proposition 2.6 to 2.9, and Proposition 2.12), and in Theorem 2.13 we prove the main result of this paper, i.e., for $s \leq 6$ and $s = 9$:

$$\dim O_{k, V_{2, d}}^s = \min\{H(X, d) + 2s, N + 1\} - 1,$$

except when $s = 2$, $d = k + 2$. In this case

$$\dim O_{k, V_{2, k+2}}^2 = H(T, d) - 1 = N - 1,$$

where $H(T, d)$ is the Hilbert function of a scheme T of s generic $(k + 2)$ -fat points.

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1. Preliminaries and notation.

1.1. Definition. If $V \subset \mathbb{P}^N$ is an irreducible projective variety, an m -fat point on V is the $(m - 1)^{th}$ infinitesimal neighborhood of a smooth point P in V , and it will be denoted by mP (i.e., the scheme mP is defined by the ideal sheaf $\mathcal{I}_{P, V}^m \subset \mathcal{O}_V$). If $\dim V = n$, then, mP is a 0-dimensional scheme of length $\binom{m-1+n}{n}$. If X is the union of the $(m - 1)^{th}$ infinitesimal neighborhoods in V of s generic smooth points of V , we will say for short that X is union of s generic m -fat points on V .

1.2. Remark. In general it is a hard problem to determine the postulation for a union of m -fat points. If $V = \mathbb{P}^2$, there is a conjecture for the postulation of a generic union $X \subset \mathbb{P}^2$ of s m -fat points (e.g. see [10]): for $s \geq 10$ the conjecture says that X is regular in any degree d . This has been proved for $m \leq 20$ in [8], and, when s is a square, by L.Evain in [9]. For $s \leq 9$ all the defective cases are known (e.g., see [8] or [10]), more precisely, for any m and $s \leq 9$ the cases in which $X \subset \mathbb{P}^2$ is not regular are:

- i) $s = 2$, and $m \leq d \leq 2m - 2$;
- ii) $s = 3$, and $\frac{3m}{2} \leq d \leq 2m - 2$;
- iii) $s = 5$, and $2m \leq d \leq \frac{5m-2}{2}$;
- iv) $s = 6$, and $\frac{12m}{5} \leq d \leq \frac{5m-2}{2}$;
- v) $s = 7$, and $\frac{21m}{8} \leq d \leq \frac{8m-2}{3}$;
- vi) $s = 8$, and $\frac{48m}{17} \leq d \leq \frac{17m-2}{6}$.

Now we recall the notions of higher secant variety and k^{th} osculating variety.

1.3. Definition. Let $V \subset \mathbb{P}^N$ be a closed irreducible projective variety; the $(s - 1)^{th}$ higher secant variety of V is the closure of the union of all linear spaces spanned by s points of V , and it will be denoted by V^s . Let $\dim V = n$; the expected dimension for V^s is

$$(\dagger) \quad \text{expdim} V^s = \min \{sn + s - 1, N\}$$

where the number $sn + s - 1$ corresponds to ∞^{sn} choices of s points on V , plus ∞^{s-1} choices of a point on the \mathbb{P}^{s-1} spanned by the s points. When $\dim V^s < \min\{sn + s - 1, N\}$, the variety V^s is said to be *defective*, with *defect* $\delta = \min\{sn + s - 1, N\} - \dim V^s$.

1.4. Definition. Let $V \subset \mathbb{P}^N$ be a variety, and let $P \in V$ be a smooth point; we define the k^{th} *osculating space to V at P* , and we denote it by $O_{k,V,P}$, as the linear space defined by the vanishing of all linear forms L such that $L|_V$ vanishes to order $k+1$ on V at P . Let $V_0 \subset V$ be the dense set of the smooth points where $O_{k,V,P}$ has maximal dimension. The k^{th} *osculating variety to V* is defined as:

$$O_{k,V} = \overline{\bigcup_{P \in V_0} O_{k,V,P}}.$$

1.5. Notation. Set $R = K[x, y, z] = \bigoplus R_d$. Let $V_d \subset \mathbb{P}^N$, $N = \binom{d+2}{2} - 1$, denote the d -ple *Veronese embedding* of \mathbb{P}^2 , defined by the linear system R_d of all forms of a given degree d . Set $O_{k,d} = O_{k,V_d}$, so that the $(s-1)^{\text{th}}$ higher secant variety to the k^{th} osculating variety to the Veronese surface V_d will be denoted by $O_{k,d}^s$.

1.6. Remark. We have (see [5], Lemma 2.3) that the dimension of $O_{k,d}$ is always the expected one, that is

$$\dim O_{k,d} = \min \left\{ \binom{k+2}{2} + 1, \binom{d+2}{2} - 1 \right\}.$$

For $d \leq k$ we immediately get $O_{k,d} = \mathbb{P}^N$, hence for $d \leq k$ and for all s , we have $O_{k,d}^s = \mathbb{P}^N$.

Now we briefly recall how to associate to $O_{k,d}^s$, a zero dimensional scheme $Y \subset \mathbb{P}^2$ (see [5], Remark 2.2).

1.7. Remark. Let $\mathbb{P}^N = \mathbb{P}(R_d)$, and let $d \geq k+1$. A form $M \in R_d$ will denote, depending on the situation, a vector in R_d or a point in \mathbb{P}^N . We can view V_d as the image's closure of the map $(\mathbb{P}^2)^* \rightarrow \mathbb{P}^N$, where $L \mapsto L^d$, $L \in R_1$. Hence

$$V_d = \{L^d, \quad L \in R_1\}.$$

At the point $Q = L^d$ we have $O_{k,V_d,Q} = \{L^{d-k}F, \quad F \in R_k\}$ and $O_{k,d} = \bigcup_{Q \in V_d} O_{k,V_d,Q}$. So we have:

$$O_{k,d} = \{L^{d-k}F, \quad L \in R_1, \quad F \in R_k\}$$

hence

$$O_{k,d}^s = \{L_1^{d-k}F_1 + \dots + L_s^{d-k}F_s, \quad L_i \in R_1, \quad F_i \in R_k, \quad i = 1, \dots, s\}.$$

Let $P_i = L_i^{d-k}F_i$ be a generic point in $O_{k,d}$, and let $T_{O_{k,d},P_i}$ be the tangent space of $O_{k,d}$ at P_i . The affine cone over $T_{O_{k,d},P_i}$ is $W_i = \langle L_i^{d-k}R_k, L_i^{d-k-1}F_iR_1 \rangle$.

Terracini's Lemma (see [13]) says that the tangent space of $O_{k,d}^s$ at a generic point of $\langle P_1, \dots, P_s \rangle$, ($P_1, \dots, P_s \in O_{k,d}$), is the span of the tangent spaces of $O_{k,d}$ at P_i ($1 \leq i \leq s$); if $T_{O_{k,d},P_i} = \mathbb{P}(W_i)$, then

$$\dim O_{k,d}^s = \dim \langle T_{O_{k,d},P_1}, \dots, T_{O_{k,d},P_s} \rangle = \dim \langle W_1, \dots, W_s \rangle - 1$$

Now consider the orthogonal space $W_i^\perp \subset R_d$, ($1 \leq i \leq s$) via the apolarity action (for the definition of W_i^\perp see [5], Remark 2.5). It generates an ideal in R defining a scheme $Z_i(k, d) \subset \mathbb{P}^2$. Let Y be a generic union of s schemes $Z_i(k, d)$ in \mathbb{P}^2 , ($1 \leq i \leq s$). Since

$$\dim \langle W_1, \dots, W_s \rangle - 1 = N - \dim[\langle W_1, \dots, W_s \rangle]^\perp = N - \dim(W_1^\perp \cap \dots \cap W_s^\perp) = N - h^0(\mathbb{P}^2, \mathcal{I}_Y(d)),$$

we have (see also [5], Remark 2.5):

$$(\ddagger) \quad \dim O_{k,d}^s = N - h^0(\mathbb{P}^2, \mathcal{I}_Y(d)) = H(Y, d) - 1$$

where $H(Y, d)$ is the Hilbert function of Y in degree d .

Since for $d = k + 1$, $O_{k,d}^2 = \mathbb{P}^N$ (see [5], Proposition 3.4 C)), then for $s \geq 2$ we immediately get:

1.8. Proposition. *For $d = k + 1$ and $s \geq 2$, we have $O_{k,d}^s = \mathbb{P}^N$.*

For $d \geq k + 2$, the schemes $Z_i(k, d)$ are zero-dimensional, and do not depend on d , in fact we have the following lemma (see [5], Lemmata 2.6, 2.7, 2.8):

1.9. Lemma. *Let $Z(k, d) = Z_i(k, d)$ be one such scheme with support at P . For $d \geq k + 2$, we have:*

- i) $(k + 1)P \subset Z(k, d) \subset (k + 2)P$;
- ii) the length of $Z(k, d)$ is $l(Z) = \binom{k+2}{2} + 2$;
- iii) $Z(k, d) = Z(k, k + 2)$.

Henceforth for $d \geq k + 2$ we will denote $Z(k, d)$ by $Z(k)$, or Z , if k is obvious by the context.

From (\ddagger) and the lemma above it follows that for $d \geq k + 2$ in order to study the dimension of $O_{k,d}^s$, we only need to study the postulation of unions of generic schemes $Z(k)$.

1.10. Remark. Let $d \geq k + 2$. Recall that $Z(k)$ is defined by the ideal generated by $W^\perp \subset R_d$, where $W = \langle L^{d-k}R_k, L^{d-k-1}FR_1 \rangle$, with $L \in R_1$ and $F \in R_k$. Now we want to give a specialization of the scheme $Z(k)$: put $L = x$ and $F = y^k$; we get

$$W = \langle x^{d-k}R_k, x^{d-k-1}y^kR_1 \rangle$$

hence

$$W^\perp = \langle x^{d-k-1}y^{k-1}z^2, \dots, x^{d-k-1}yz^k, x^{d-k-1}z^{k+1}, x^{d-k-2}y^{k+2}, x^{d-k-2}y^{k+1}z, \dots, \\ x^{d-k-2}yz^{k+1}, x^{d-k-2}z^{k+2}, x^{d-k-3}y^{k+1}, x^{d-k-3}y^kz, \dots, x^{d-k-3}yz^k, x^{d-k-3}z^{k+1}, \dots, \\ xy^{d-1}, xy^{d-2}z, \dots, xyz^{d-2}, xz^{d-1}, y^d, y^{d-1}z, \dots, yz^{d-1}, z^d \rangle.$$

Let I be the ideal generated by W^\perp . By a direct computation, it is easy to show that the saturation of I is the ideal

$$(I)^{sat} = (y, z)^{k+1} \cap ((y, z)^{k+2} + (z^2))$$

that defines a scheme supported at a point of \mathbb{P}^2 , whose structure is given by the union of its k^{th} infinitesimal neighbourhood, with the intersection of its $(k + 1)^{th}$ infinitesimal neighbourhood with a double line.

1.11. Notation. We fix the following notation:

- i) let P_1, \dots, P_s be s generic points in \mathbb{P}^2 ;
- ii) let X be the union of s generic $(k + 1)$ -fat points in \mathbb{P}^2 , with support in P_1, \dots, P_s ;

- iii) let T be the union of s generic $(k+2)$ -fat points in \mathbb{P}^2 , with support in P_1, \dots, P_s ;
- iv) let Z_i be a 0-dimensional scheme in \mathbb{P}^2 , as defined in Remark 1.7, with support in P_i ;
- v) let $Y = Z_1 + \dots + Z_s$;
- vi) denote by $(k+1, k+2)P$ a 0-dimensional scheme whose defining ideal is $\wp^{k+1} \cap (\wp^{k+2} + l^2)$ where \wp is the homogeneous ideal in $R = K[x, y, z]$ of a point $P \in \mathbb{P}^2$, and l is the ideal of a generic line through P ; we call $(k+1, k+2)P$ a $(k+1, k+2)$ point;
- vii) let Z_i be a $(k+1, k+2)$ point with support in P_i . By Remark 1.10, the scheme Z_i is a specialization of Z_i ;
- viii) let $\mathcal{Y} = Z_1 + \dots + Z_s$ (so \mathcal{Y} is a specialization of the scheme Y). We have

$$\deg \mathcal{Y} = \deg Y = s \left(\binom{k+2}{2} + 2 \right) = \deg X + 2s;$$

ix) if $\mathcal{C} \subset \mathbb{P}^2$ is a curve, and Z is a zero-dimensional scheme, the scheme Z' defined by the ideal $(I_Z : I_{\mathcal{C}})$ is called the residual of Z with respect to \mathcal{C} , and denoted by $\text{Res}_{\mathcal{C}} Z$.

In the following lemma we determine the subscheme of a $(k+1, k+2)$ point with support in P , residual to a curve \mathcal{C} .

1.12. Lemma. *Let Z be a $(k+1, k+2)$ point, with support in P with defining ideal $\wp^{k+1} \cap (\wp^{k+2} + l^2)$, where \wp is the ideal of P , and $l = (L)$ is the ideal of a generic line through P . Let $\mathcal{C} \subset \mathbb{P}^2$ be a curve having at P a singularity of multiplicity m , and having L as tangent direction with multiplicity t . Then $\text{Res}_{\mathcal{C}}(Z)$ is defined by the ideal*

$$I_{\text{Res}_{\mathcal{C}}(Z)} = \wp^{\max\{k+1-m; 0\}} \cap (\wp^{\max\{k+2-m; 0\}} + l^{\max\{2-t; 0\}}).$$

$\text{Res}_{\mathcal{C}}(Z)$ is a fat point, or a $(k+1-m, k+2-m)$ point, except for $m < k+1$ and $t = 1$, more precisely:

$$\text{Res}_{\mathcal{C}}(Z) = \begin{cases} 0P & \text{for } m \geq k+2, \text{ or } m = k+1 \text{ and } t \geq 2 \\ 1P & \text{for } m = k+1 \text{ and } t \leq 1 \\ (k+1-m)P & \text{for } m < k+1 \text{ and } t \geq 2 \\ 2P & \text{for } m = k \text{ and } t = 0 \\ (k+1-m, k+2-m)P & \text{for } m < k \text{ and } t = 0 \end{cases}.$$

Proof. Without loss of generality, we assume that $\wp = (x, y)$, $L = x$, and, by abuse of notation, that x, y are affine coordinates.

Let $x^t f_1 + f_2 = 0$ be an equation defining the curve \mathcal{C} , where f_1 is a homogeneous polynomial of degree $m-t$, $f_1 \notin (x)$, and $f_2 \in (x, y)^{m+1}$. We have to prove that

$$((x, y)^{k+1} \cap ((x, y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x, y)^{\max\{k+1-m; 0\}} \cap ((x, y)^{\max\{k+2-m; 0\}} + (x^{\max\{2-t; 0\}})).$$

This is obvious for $m \geq k+2$, and for $m = k+1, t \geq 2$, since in these cases $\text{Res}_{\mathcal{C}}(Z)$ is the emptyset.

Let $m = k+1, t \leq 1$. The equality above becomes

$$((x, y)^{k+1} \cap ((x, y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x, y).$$

“ \subseteq ” : To prove this inclusion, let $g = a + h$, $a \in K$, $h \in (x, y)$. If $g(x^t f_1 + f_2) = (a + h)(x^t f_1 + f_2) \in ((x, y)^{k+2} + (x^2))$, since $f_2 \in (x, y)^{m+1}$, $hx^t f_1 \in (x, y)^{m+1}$, and $m+1 = k+2$, it follows that

$ax^t f_1 \in ((x, y)^{k+2} + (x^2))$. But f_1 is a homogeneous polynomial of degree $m - t$, $f_1 \notin (x)$, $t \leq 1$, so it easily follows that $a = 0$, hence $g \in (x, y)$. The reverse inclusion is obvious.

Since $I_{Res_C(\mathcal{Z})} = (x, y)$, we have $Res_C(\mathcal{Z}) = 1P$.

Now, let $m < k + 1$, $t \geq 2$. In this case we have to prove that

$$((x, y)^{k+1} \cap ((x, y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x, y)^{k+1-m}.$$

If $g(x^t f_1 + f_2) \in (x, y)^{k+1}$, it immediately follows that $g \in (x, y)^{k+1-m}$, and the reverse inclusion is obvious. Moreover, since $I_{Res_C(\mathcal{Z})} = (x, y)^{k+1-m}$, we have that $Res_C(\mathcal{Z}) = (k + 1 - m)P$.

Let $m \leq k$, $t \leq 1$. Now we have to prove that

$$((x, y)^{k+1} \cap ((x, y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x, y)^{k+1-m} \cap ((x, y)^{k+2-m} + (x^{2-t})).$$

“ \subseteq ” : As in the previous case, if $g(x^t f_1 + f_2) \in (x, y)^{k+1}$, it follows that $g \in (x, y)^{k+1-m}$, so we can write

$$g = xg_1 + ay^{k+1-m} + g_2,$$

where $g_1 \in (x, y)^{k-m}$ is homogeneous of degree $k - m$, $g_2 \in (x, y)^{k+2-m}$, $a \in K$. In order to prove that

$$g(x^t f_1 + f_2) = (xg_1 + ay^{k+1-m} + g_2)(x^t f_1 + f_2) \in ((x, y)^{k+2} + (x^2))$$

since $g_2 x^t f_1$, and $f_2 \in (x, y)^{k+2}$, it suffices to prove that

$$x^{t+1} g_1 f_1 + ax^t y^{k+1-m} f_1 \in ((x, y)^{k+2} + (x^2)).$$

Since $x^{t+1} g_1 f_1 + ax^t y^{k+1-m} f_1$ is homogeneous of degree $k+1$, and $f_1 \notin (x)$, we get that $x^{t+1} g_1 + ax^t y^{k+1-m} \in (x^2)$. For $t = 1$, this implies $a = 0$, so $g \in ((x, y)^{k+2-m} + (x))$. For $t = 0$ this implies $a = 0$, and $g_1 \in (x)$, so $g \in ((x, y)^{k+2-m} + (x^2))$.

“ \supseteq ” : This inclusion is obvious.

So we have proved that, for $m \leq k$ and $t \leq 1$:

$$I_{Res_C(\mathcal{Z})} = \begin{cases} (x, y)^{k+1-m} \cap ((x, y)^{k+2-m} + (x)) = (x, y)^{k+1-m} \cap (x, y^{k+2-m}) & \text{for } m \leq k \text{ and } t = 1 \\ (x, y) \cap ((x, y)^2 + (x^2)) = (x, y)^2 & \text{for } m = k \text{ and } t = 0, \\ (x, y)^{k+1-m} \cap ((x, y)^{k+2-m} + (x^2)) & \text{for } m < k \text{ and } t = 0 \end{cases},$$

hence for $m = k$ and $t = 0$ we have $Res_C(\mathcal{Z}) = 2P$, for $m < k$ and $t = 0$ we have $Res_C(\mathcal{Z}) = (k + 1 - m, k + 2 - m)P$, while for $m \leq k$ and $t = 1$, $Res_C(\mathcal{Z})$ is the union of the fat point $(k + 1 - m)P$ with the intersection of the line $\{x = 0\}$ with the fat point $(k + 2 - m)P$. \square

2. Osculating varieties to Veronese surface and some of their higher secant varieties.

In this section we will compute the dimension of $O_{k,d}^s$ for $3 \leq s \leq 6$ and $s = 9$.

2.1. Remark. We recall that for $d \leq k + 1$ and $s \geq 2$ (see Remark 1.6 and Proposition 1.8):

$$\dim O_{k,d}^s = N.$$

So we have to study the dimension of $O_{k,d}^s$ only for $d \geq k + 2$. Since, for $d \geq k + 2$ (see(†))

$$\dim O_{k,d}^s = H(Y, d) - 1,$$

then, if we know the postulation of Y , we are done.

We wish to notice that, by (†), the expected dimension for $O_{k,d}^s$ is

$$\text{expdim } O_{k,d}^s = \min\{sn + s - 1, N\},$$

where $n = \dim O_{k,d} = \min\left\{\binom{k+2}{2} + 1, \binom{d+2}{2} - 1\right\} = \min\left\{\binom{k+2}{2} + 1, N\right\} = \min\left\{\frac{\deg Y}{s} - 1, N\right\}$ (see Remark 1.6 and Lemma 1.9 *ii*). Hence it easily follows that

$$\text{expdim } O_{k,d}^s = \min\{\deg Y, N + 1\} - 1 = \text{exp } H(Y, d) - 1$$

where $\text{exp } H(Y, d)$ is the expected value for $H(Y, d)$.

In the next lemmata we show that the postulation of Y is strictly related with the postulation of the specialized scheme \mathcal{Y} , and of the scheme of fat points X .

2.2. Lemma. *If the Hilbert function of the specialized scheme \mathcal{Y} in degree d is*

$$H(\mathcal{Y}, d) = \min\{H(X, d) + 2s, N + 1\},$$

then

$$H(Y, d) = \min\{H(X, d) + 2s, N + 1\}.$$

Proof. It follows from the obvious inequalities: $H(\mathcal{Y}, d) \leq H(Y, d) \leq \min\{H(X, d) + 2s, N + 1\}$. \square

2.3. Lemma. *Let $s > 2$. Then:*

- i) for $k = 1$, $\mathcal{Y} = Y = (2, 3)P_1 + \dots + (2, 3)P_s$, and $H(\mathcal{Y}, d) = \min\{\deg Y, N + 1\}$;*
- ii) for $k = 2$, $\mathcal{Y} = (3, 4)P_1 + \dots + (3, 4)P_s$, and $H(\mathcal{Y}, d) = \min\{\deg Y, N + 1\}$.*

Proof. *i)* If $d = 2$ see [7], Proposition 3.3; for $d = 3$ see [7], Proposition 4.5; for $d \geq 4$ see [2], Theorem 1.

ii) follows from [3] Theorems 1 and 2. \square

2.4. Lemma. *i) If $H(\mathcal{Y}, d_0) = H(X, d_0) + 2s$, then for every $d \geq d_0$ we have*

$$H(\mathcal{Y}, d) = H(X, d) + 2s;$$

ii) if $(I_{\mathcal{Y}})_{d_0} = (0)$, then for every $d \leq d_0$ we have $(I_{\mathcal{Y}})_d = (0)$.

Proof. *i)* Since $X \subset \mathcal{Y}$ and $H(\mathcal{Y}, d_0) = H(X, d_0) + 2s$, then it easily follows that $\dim(I_X/I_{\mathcal{Y}})_{d_0} = 2s$. Therefore there are $2s$ forms $f_1, \dots, f_{2s} \in (I_X)_{d_0}$ linearly independent modulo $(I_{\mathcal{Y}})_{d_0}$. Let $\{l = 0\}$ be a line not through any of the points P_1, \dots, P_s . The forms $f_1 l^{d-d_0}, \dots, f_{2s} l^{d-d_0} \in (I_X)_d$ are linearly independent modulo $(I_{\mathcal{Y}})_d$, hence $\dim(I_X/I_{\mathcal{Y}})_d \geq 2s$, so we have $H(\mathcal{Y}, d) \geq H(X, d) + 2s$. Since obviously $H(\mathcal{Y}, d) \leq H(X, d) + 2s$, then the conclusion follows.

ii) Obvious. □

Now we will study the postulation of \mathcal{Y} for each s separately ($s = 3, 4, 5, 6, 9$), but first we wish to mention the case $s = 2$.

2.5. Proposition. *For $s = 2$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} \text{for } k = 1 : & \begin{cases} N + 1 & \text{if } d \leq 2 \\ H(T, d) = 9 < \exp H(\mathcal{Y}, d) & \text{if } d = 3 \\ H(X, d) + 4 = \deg Y & \text{if } d \geq 4 \end{cases} \\ \text{for } k = 2 : & \begin{cases} N + 1 & \text{if } d \leq 3 \\ H(T, d) = 14 < \exp H(\mathcal{Y}, d) & \text{if } d = 4 \\ H(X, d) + 4 = \deg Y & \text{if } d \geq 5 \end{cases} \\ \text{for } k \geq 3 : & \begin{cases} N + 1 & \text{if } d \leq k + 1 \\ H(T, d) = N < \exp H(\mathcal{Y}, d) & \text{if } d = k + 2 \\ H(X, d) + 4 < \exp H(\mathcal{Y}, d) & \text{if } k + 3 \leq d \leq 2k \\ H(X, d) + 4 = \deg Y & \text{if } d \geq 2k + 1 \end{cases} \end{cases}$$

Proof. The case $d \leq k + 1$ follows from Lemma 2.4 ii), and [5], Proposition 3.4, C).

For $d = k + 2$, observe that the line L through P_1 and P_2 is a component of multiplicity at least $2(k + 1) - d = k$ for the curves defined by the forms both of $(I_{\mathcal{Y}})_d$ and of $(I_T)_d$. Since $\text{Res}_{kL}\mathcal{Y} = \text{Res}_{kL}T = 2P_1 + 2P_2$ (see Lemma 1.12), we get

$$\dim(I_{\mathcal{Y}})_{k+2} = \dim(I_T)_{k+2} = \dim(I_{2P_1+2P_2})_2 = 1$$

(the only curve is the $(k + 2)$ -uple line through the two points). Thus $H(\mathcal{Y}, d) = H(T, d)$. Moreover, since T is not regular in degree $k + 2$ (see Remark 1.2), we get $H(\mathcal{Y}, d) < \exp H(\mathcal{Y}, d)$ (see [5], Corollary 3.5).

For $k = 1, 2$ and $d \geq k + 3$, see [5], Corollary 3.8. For $k \geq 3$, and $d \geq 2k + 1$ see [5], Proposition 3.9.

Now let $k \geq 3$, and $k + 3 \leq d \leq 2k$. For $d = k + 3$ the line L through P_1 and P_2 is a component of multiplicity at least $\nu = 2(k + 1) - d = k - 1$ for the curves defined by the forms of both $(I_{\mathcal{Y}})_d$, and $(I_X)_d$, hence from the case $k = 1$, $d = 4$, we get

$$\dim(I_{\mathcal{Y}})_{k+3} = \dim(I_{\mathcal{Y}'})_{k+3-(k-1)} = \dim(I_{\mathcal{Y}'})_4 = 15 - 10 = 5, \quad \dim(I_X)_{k+3} = \dim(I_{X'})_4 = 9,$$

where $\mathcal{Y}' = \text{Res}_{\nu L}\mathcal{Y} = (2, 3)P_1 + (2, 3)P_2$ (see Lemma 1.12), and $X' = \text{Res}_{\nu L}X = 2P_1 + 2P_2$.

It follows that $H(\mathcal{Y}, k + 3) = H(X, k + 3) + 4$. Hence by Lemma 2.4 i), for every $d \geq k + 3$ we have

$$H(\mathcal{Y}, d) = H(X, d) + 4.$$

Since two $(k + 1)$ -fat points impose independent conditions to curves of degree d if and only if $d \geq 2k + 1$ (see Remark 1.2), then, for $k + 3 \leq d \leq 2k$, we have $H(X, d) < \deg X$, thus

$$H(\mathcal{Y}, d) = H(X, d) + 4 < \deg X + 4 = \deg Y.$$

Moreover, since for $d = k + 3$, $\dim(I_{\mathcal{Y}})_{k+3} = 5$, then for $d \geq k + 3$, $\dim(I_{\mathcal{Y}})_d$ is positive, that is $H(\mathcal{Y}, d) < \binom{d+2}{2}$. It follows that if $k + 3 \leq d \leq 2k$, then $H(\mathcal{Y}, d) < \min \left\{ \deg \mathcal{Y}, \binom{d+2}{2} \right\} = \exp H(\mathcal{Y}, d)$.

(For $k \geq 3$, and $k + 3 \leq d \leq 2k$, see also [5], Proposition 3.10). □

2.6. Proposition. For $s = 3$ we have:

$$i) \quad H(\mathcal{Y}, d) = \begin{cases} N + 1 & \text{if } d \leq \lceil \frac{3(k+1)}{2} \rceil \\ H(X, d) + 6 < \deg Y & \text{if } \lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k \\ H(X, d) + 6 = \deg Y & \text{if } d \geq \max\{\lceil \frac{3(k+1)}{2} \rceil + 1; 2k + 1\} \end{cases} .$$

$$ii) \quad H(\mathcal{Y}, d) < \exp H(\mathcal{Y}, d) \quad \text{iff} \quad \begin{cases} \lceil \frac{3(k+1)}{2} \rceil + 2 \leq d \leq 2k & \text{if } k + 1 \text{ is even} \\ \lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k & \text{if } k + 1 \text{ is odd} \end{cases} .$$

Proof. *i)* In case $d \leq \lceil \frac{3(k+1)}{2} \rceil$, it suffices to prove that $(I_{\mathcal{Y}})_d = (0)$ for $d = \lceil \frac{3(k+1)}{2} \rceil$.

Let \mathcal{C} be the curve formed by the three lines P_1P_2, P_1P_3, P_2P_3 . For $d = \lceil \frac{3(k+1)}{2} \rceil$, the curve \mathcal{C} is a fixed component, of multiplicity at least

$$\nu = 2(k+1) - d = \begin{cases} \frac{k+1}{2} & \text{if } k+1 \text{ is even} \\ \frac{k}{2} & \text{if } k+1 \text{ is odd} \end{cases}$$

for the curves defined by the forms of $(I_{\mathcal{Y}})_d$, so we have (see Lemma 1.12)

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d-3\nu}$$

where

$$\mathcal{Y}' = \text{Res}_{\nu\mathcal{C}}\mathcal{Y} = \begin{cases} P_1 + P_2 + P_3 & \text{if } k+1 \text{ is even} \\ 2P_1 + 2P_2 + 2P_3 & \text{if } k+1 \text{ is odd} \end{cases}, \quad d - 3\nu = \begin{cases} 0 & \text{if } k+1 \text{ is even} \\ 2 & \text{if } k+1 \text{ is odd} \end{cases} .$$

It immediately follows that $(I_{\mathcal{Y}})_d = (0)$.

Now let $d \geq \lceil \frac{3(k+1)}{2} \rceil + 1$. In order to prove that $H(\mathcal{Y}, d) = H(X, d) + 6$, by Lemma 2.4 it suffices to prove that $H(\mathcal{Y}, d) = H(X, d) + 6$ for $d = \lceil \frac{3(k+1)}{2} \rceil + 1$.

Let $d = \lceil \frac{3(k+1)}{2} \rceil + 1$. The curve \mathcal{C} is a fixed component, with multiplicity at least

$$\nu = 2(k+1) - d = \begin{cases} \frac{k-1}{2} & \text{if } k+1 \text{ is even} \\ \frac{k-2}{2} & \text{if } k+1 \text{ is odd} \end{cases}$$

for the curves defined by the forms of both $(I_{\mathcal{Y}})_d$ and $(I_X)_d$, then we have

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d-3\nu}, \quad \dim(I_X)_d = \dim(I_{X'})_{d-3\nu}$$

where (see Lemma 1.12)

$$d - 3\nu = \begin{cases} 4 & \text{if } k+1 \text{ is even} \\ 6 & \text{if } k+1 \text{ is odd} \end{cases},$$

$$\mathcal{Y}' = \text{Res}_{\nu\mathcal{C}}\mathcal{Y} = \begin{cases} (2, 3)P_1 + (2, 3)P_2 + (2, 3)P_3 & \text{if } k+1 \text{ is even} \\ (3, 4)P_1 + (3, 4)P_2 + (3, 4)P_3 & \text{if } k+1 \text{ is odd} \end{cases},$$

$$X' = \begin{cases} 2P_1 + 2P_2 + 2P_3 & \text{if } k+1 \text{ is even} \\ 3P_1 + 3P_2 + 3P_3 & \text{if } k+1 \text{ is odd} \end{cases} .$$

Since it is well known that $\dim(I_{2P_1+2P_2+2P_3})_4 = 6$ and $\dim(I_{3P_1+3P_2+3P_3})_6 = 10$, we have

$$\dim(I_{X'})_{d-3\nu} = \begin{cases} 6 & \text{if } k+1 \text{ is even} \\ 10 & \text{if } k+1 \text{ is odd} \end{cases},$$

moreover, by Lemma 2.3 we get that

$$\dim(I_{Y'})_{d-3\nu} = \begin{cases} 0 & \text{if } k+1 \text{ is even} \\ 4 & \text{if } k+1 \text{ is odd} \end{cases}.$$

It follows that $\dim(I_X)_d - \dim(I_Y)_d = 6$, hence $H(\mathcal{Y}, d) - H(X, d) = 6$.

Since three $(k+1)$ -fat points impose independent conditions to curves of degree d if and only if $d \geq 2k+1$ (see Remark 1.2), then for $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k$ we have $H(X, d) < \deg X$, while if $d \geq \max\left\{\lceil \frac{3(k+1)}{2} \rceil + 1; 2k+1\right\}$, then $H(X, d) = \deg X$. Since $\deg Y = \deg X + 6$ we get:

$$H(\mathcal{Y}, d) = \begin{cases} H(X, d) + 6 < \deg Y & \text{if } \lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k \\ H(X, d) + 6 = \deg Y & \text{if } d \geq \max\left\{\lceil \frac{3(k+1)}{2} \rceil + 1; 2k+1\right\} \end{cases}.$$

ii) For $d \leq \lceil \frac{3(k+1)}{2} \rceil$, or $d \geq \max\left\{\lceil \frac{3(k+1)}{2} \rceil + 1; 2k+1\right\}$, from *i)* we have $H(\mathcal{Y}, d) = \exp H(\mathcal{Y}, d)$.

If $k+1$ is even, and $d = \lceil \frac{3(k+1)}{2} \rceil + 1$, then $\dim(I_Y)_d = 0$, hence $H(\mathcal{Y}, d) = \binom{d+2}{2}$, the expected one.

If $k+1$ is even, and $d = \lceil \frac{3(k+1)}{2} \rceil + 2$, from *i)*, since $\dim(I_X)_{d-1} = 6$ implies $\dim(I_X)_d > 6$, we have:

$$\dim(I_Y)_d = \binom{d+2}{2} - H(\mathcal{Y}, d) = \binom{d+2}{2} - H(X, d) - 6 = \dim(I_X)_d - 6 > 0.$$

Hence, if $k+1$ is even, for $d = \lceil \frac{3(k+1)}{2} \rceil + 2$, and so also for $d \geq \lceil \frac{3(k+1)}{2} \rceil + 2$, we have $\dim(I_Y)_d > 0$, that is $H(\mathcal{Y}, d) < \binom{d+2}{2}$. Since, by *i)*, if $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k$, then $H(\mathcal{Y}, d) < \deg Y$, it follows that for $\lceil \frac{3(k+1)}{2} \rceil + 2 \leq d \leq 2k$ we have $H(\mathcal{Y}, d) < \min\left\{\deg Y, \binom{d+2}{2}\right\} = \exp H(\mathcal{Y}, d)$.

If $k+1$ is odd and $d \geq \lceil \frac{3(k+1)}{2} \rceil + 1$, from the proof of *i)* we get $\dim(I_Y)_d > 0$, that is $H(\mathcal{Y}, d) < \binom{d+2}{2}$. Moreover, by *i)*, if $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k$, then $H(\mathcal{Y}, d) < \deg Y$, and the conclusion immediately follows. \square

2.7. Proposition. *For $s = 4$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} \text{for } k \leq 6: & \begin{cases} N+1 & \text{if } d \leq 2k+2 \\ H(X, d) + 8 = \deg Y & \text{if } d \geq 2k+3 \end{cases} \\ \text{for } k \geq 6: & \begin{cases} N+1 & \text{if } d \leq 2k+1 \\ H(X, d) + 8 = \deg Y & \text{if } d \geq 2k+2 \end{cases} \end{cases}.$$

Proof. If $d \leq 2k+1$, by Bezout Theorem, each element of $(I_Y)_d$ is divisible by every form defining an irreducible conic through P_1, \dots, P_4 , hence $(I_Y)_d = (0)$.

Let $d = 2k+2$. Recall that the ideal of the scheme \mathcal{Z}_i is $\wp_i^{k+1} \cap (\wp_i^{k+2} + l_i^2)$, where l_i defines a generic line L_i through P_i ($1 \leq i \leq 4$) such that $\deg(\mathcal{Y} \cap L_i) = k+2$. Let \mathcal{C}_i be the conic through P_1, \dots, P_4 , tangent in P_i to L_i . For the genericity of the L_i 's, the conics $\mathcal{C}_1, \dots, \mathcal{C}_4$ are irreducible and distinct. Bezout's Theorem implies that each conic \mathcal{C}_i is a component of any curve defined by the forms of $(I_Y)_d$. By Lemma 1.12 we can determine $I_{Res_{\mathcal{C}_1+\dots+\mathcal{C}_4}\mathcal{Y}}$, and it is an easy computation (which will be omitted) that the intersection multiplicities of the curves defined by the forms of $(I_{Res_{\mathcal{C}_1+\dots+\mathcal{C}_4}\mathcal{Y}})_{d-8}$ with a conic \mathcal{C}_i , is bigger than $2(d-8)$. Hence by Bezout's Theorem we get that each conic \mathcal{C}_i is a component with multiplicity at least 2 of any curve

defined by the forms of $(I_{\mathcal{Y}})_d$. So these curves have a component of degree 16. It follows that, if $(I_{\mathcal{Y}})_d \neq (0)$, then $d \geq 16$, that is $k \geq 7$. Thus, for $k \leq 6$, we have $(I_{\mathcal{Y}})_d = (0)$, that is $H(\mathcal{Y}, d) = N + 1$. Observe that for $k = 6$, we have $N + 1 = H(X, d) + 8 = \deg Y$, in fact in this case $d = 2k + 2 = 14$, $N + 1 = \binom{16}{2} = 120$, and, since four 7-fat points impose independent conditions to curves of degree 14 (see Remark 1.2), then $H(X, d) = 112$. If $k \geq 7$ we have

$$\dim(I_{\mathcal{Y}})_{2k+2} = \dim(I_{\mathcal{Y}'})_{2k+2-16},$$

where $\mathcal{Y}' = \text{Res}_{2\mathcal{C}_1 + \dots + 2\mathcal{C}_4} \mathcal{Y} = (k-7)P_1 + \dots + (k-7)P_4$ is a scheme of four $(k-7)$ -fat points (see Lemma 1.12). Since \mathcal{Y}' imposes independent conditions to curves of degree $2k-14$ (see Remark 1.2), then $H(\mathcal{Y}, 2k+2) = \binom{2k+4}{2} - \dim(I_{\mathcal{Y}})_{2k+2} = \binom{2k+4}{2} - \dim(I_{\mathcal{Y}'})_{2k-14} = \binom{2k+4}{2} - \binom{2k-12}{2} + 4\binom{k-6}{2} = 4\binom{k+2}{2} + 8 = H(X, 2k+2) + 8 = \deg Y$.

Now let $d \geq 2k+3$. It suffices to prove that $H(\mathcal{Y}, 2k+3) = H(X, 2k+3) + 8 = \deg Y$ (see Lemma 2.4 *i*)), so let $d = 2k+3$. By induction on k . For $k = 1$ see Lemma 2.3. Let $k \geq 2$. Let \mathcal{C} be an irreducible conic through P_1, \dots, P_4 , and let Q_1, Q_2, Q_3 be three points on \mathcal{C} . Let $\tilde{\mathcal{Y}} = \mathcal{Y} + Q_1 + Q_2 + Q_3$. By Bezout's Theorem, the conic \mathcal{C} is a fixed component for the curves of degree $2k+3$ through $\tilde{\mathcal{Y}}$, then

$$\dim(I_{\tilde{\mathcal{Y}}})_{2k+3} = \dim(I_{\tilde{\mathcal{Y}'}})_{2k+1} = \binom{2k+3}{2} - H(\tilde{\mathcal{Y}'}, 2k+1),$$

where $\tilde{\mathcal{Y}}' = \text{Res}_{\mathcal{C}} \tilde{\mathcal{Y}} = \text{Res}_{\mathcal{C}} \mathcal{Y} = \sum_{i=1}^4 (k, k+1)P_i$ (see Lemma 1.12). By the inductive hypothesis we have that $H(\tilde{\mathcal{Y}}', 2k+1) = \deg \tilde{\mathcal{Y}}' = 4\binom{k+1}{2} + 8$, hence

$$H(\tilde{\mathcal{Y}}, 2k+3) = \binom{2k+5}{2} - \binom{2k+3}{2} + 4\binom{k+1}{2} + 8 = \deg \mathcal{Y} + 3 = \deg \tilde{\mathcal{Y}}.$$

Hence $\tilde{\mathcal{Y}}$ imposes independent conditions to curves of degree $2k+3$. Since $\mathcal{Y} \subset \tilde{\mathcal{Y}}$, then also \mathcal{Y} imposes independent conditions to curves of degree $2k+3$, that is $H(\mathcal{Y}, 2k+3) = \deg \mathcal{Y} = \deg Y$. □

2.8. Proposition. *For $s = 5$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} N + 1 & \text{if } d \leq 2k + 3 \\ H(X, d) + 10 < \exp H(\mathcal{Y}, d) & \text{if } 2k + 4 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1 \\ H(X, d) + 10 = \deg Y & \text{if } d \geq \max \left\{ 2k + 4; \lfloor \frac{5(k+1)}{2} \rfloor \right\} \end{cases}.$$

Proof. Let $d \leq 2k+3$. If we prove that $(I_{\mathcal{Y}})_d = (0)$ for $d = 2k+3$ we are done. So let $d = 2k+3$. For $k = 1$ see Lemma 2.3. Let $k \geq 2$. Any curve defined by a nonzero element of $(I_X)_d$ has the conic \mathcal{C} through P_1, \dots, P_5 as a component of multiplicity at least $5(k+1) - 2d = k-1$, where X is the fat point subscheme of 5 points of multiplicity $k+1$, hence the same is true for \mathcal{Y} in place of X , since $X \subset \mathcal{Y}$, so we have:

$$\dim(I_{\mathcal{Y}})_{2k+3} = \dim(I_{\mathcal{Y}'})_{2k+3-2(k-1)} = \dim(I_{\mathcal{Y}'})_5,$$

where, by Lemma 1.12, $\mathcal{Y}' = \text{Res}_{(k-1)\mathcal{C}} \mathcal{Y} = (2, 3)P_1 + \dots + (2, 3)P_5$. Since, by Lemma 2.3 *i*), $\dim(I_{\mathcal{Y}'})_5 = 0$, then the conclusion follows.

Now let $d \geq 2k+4$. We have to prove that

$$H(\mathcal{Y}, d) = H(X, d) + 10.$$

By Lemma 2.4, it suffices to prove that $H(\mathcal{Y}, d) = H(X, d) + 10$ for $d = 2k + 4$, so let $d = 2k + 4$. For $k = 1, 2$ see Lemma 2.3. If $k = 3$ (hence $d = 10$), let Q be a point on the conic \mathcal{C} through P_1, \dots, P_5 . The scheme $\mathcal{Y} + Q$ imposes independent conditions to curves of degree 10. In fact, since the conic \mathcal{C} is a fixed locus for $(I_{\mathcal{Y}+Q})_{10}$, from the case $k = 2$ we get:

$$\dim(I_{\mathcal{Y}+Q})_{10} = \dim(I_{\mathcal{Y}'})_8 = \binom{8+2}{2} - 5(8) = 5 = \binom{10+2}{2} - 5(12) - 1 = \binom{10+2}{2} - \deg(\mathcal{Y} + Q),$$

where $\mathcal{Y}' = \text{Res}_{\mathcal{C}}(\mathcal{Y} + Q) = (3, 4)P_1 + \dots + (3, 4)P_5$ (see Lemma 1.12). Since $\mathcal{Y} + Q$ imposes independent conditions to curves of degree 10, then \mathcal{Y} and X also do the same. It follows that

$$H(\mathcal{Y}, 10) = \deg Y = \deg X + 10 = H(X, 10) + 10.$$

For $k \geq 4$, since \mathcal{C} is a fixed component with multiplicity at least $(k - 3)$ for curves defined both by $(I_{\mathcal{Y}})_{2k+4}$ and by $(I_X)_{2k+4}$, it follows that

$$\dim(I_{\mathcal{Y}})_{2k+4} = \dim(I_{\mathcal{Y}'})_{2k+4-2(k-3)} = \dim(I_{\mathcal{Y}'})_{10}, \quad \dim(I_X)_{2k+4} = \dim(I_{X'})_{10},$$

where (see Lemma 1.12)

$$\mathcal{Y}' = \text{Res}_{(k-3)\mathcal{C}}\mathcal{Y} = (4, 5)P_1 + \dots + (4, 5)P_5, \quad X' = \text{Res}_{(k-3)\mathcal{C}}4P_1 + \dots + 4P_5.$$

From the case $k = 3$ it follows that

$$\dim(I_{\mathcal{Y}})_{2k+4} = 6, \quad \dim(I_X)_{2k+4} = 16,$$

hence $H(\mathcal{Y}, d) = H(X, d) + 10$.

So we have proved that for $d \geq 2k + 4$

$$H(\mathcal{Y}, d) = H(X, d) + 10.$$

Now, since $\dim(I_{\mathcal{Y}})_{2k+4}$ is positive, then $H(\mathcal{Y}, d) < \binom{d+2}{2}$ for any $d \geq 2k + 4$. Moreover, since five generic $(k + 1)$ -fat points impose independent conditions to curves of degree d if and only if $d \geq \lfloor \frac{5(k+1)}{2} \rfloor$ (see Remark 1.2), then for $2k + 4 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$, we have $H(X, d) < \deg X$, hence

$$H(\mathcal{Y}, d) = H(X, d) + 10 < \min \left\{ \deg X + 10, \binom{d+2}{2} \right\} = \min \left\{ \deg Y, \binom{d+2}{2} \right\} = \exp H(\mathcal{Y}, d).$$

If $d \geq \max \left\{ 2k + 4; \lfloor \frac{5(k+1)}{2} \rfloor \right\}$, then $H(X, d) = \deg X$, so $H(\mathcal{Y}, d) = \deg Y$. □

2.9. Proposition. *For $s = 6$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} \text{for } k = 1 : & \begin{cases} N + 1 & \text{if } d \leq 6 \\ H(X, d) + 12 = \deg Y & \text{if } d \geq 7 \end{cases} \\ \text{for } k = 2 : & \begin{cases} N + 1 & \text{if } d \leq 8 \\ H(X, d) + 12 = \deg Y & \text{if } d \geq 9 \end{cases} \\ \text{for } k \geq 3 \\ k \equiv 2 \pmod{5} : & \begin{cases} N + 1 & \text{if } d \leq \lceil \frac{12(k+1)}{5} \rceil - 1 \\ H(X, d) + 12 < \exp H(\mathcal{Y}, d) & \text{if } \lceil \frac{12(k+1)}{5} \rceil \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1 \\ H(X, d) + 12 = \deg Y & \text{if } d \geq \max \left\{ \lceil \frac{12(k+1)}{5} \rceil; \lfloor \frac{5(k+1)}{2} \rfloor \right\} \end{cases} \\ \text{for } k \geq 3 \\ k \not\equiv 2 \pmod{5} : & \begin{cases} N + 1 & \text{if } d \leq \lceil \frac{12(k+1)}{5} \rceil \\ H(X, d) + 12 < \exp H(\mathcal{Y}, d) & \text{if } \lceil \frac{12(k+1)}{5} \rceil + 1 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1 \\ H(X, d) + 12 = \deg Y & \text{if } d \geq \max \left\{ \lceil \frac{12(k+1)}{5} \rceil + 1; \lfloor \frac{5(k+1)}{2} \rfloor \right\} \end{cases} \end{cases}.$$

Proof. We start by proving four particular cases, that we need later in the proof.

2.10. Lemma. *We have:*

- i)* $\dim(I_{(8,9)P_1+\dots+(8,9)P_6})_{20} = 3$;
- ii)* $\dim(I_{(6,7)P_1+\dots+(6,7)P_6})_{15} = 0$;
- iii)* $\dim(I_{(5,6)P_1+\dots+(5,6)P_6})_{13} = 3$;
- iv)* $\dim(I_{(4,5)P_1+\dots+(4,5)P_6})_{11} = 6$.

Proof. Though all of the above equalities can be checked using CoCoA (see [6]), we prove *i)* by specializing the scheme \mathcal{Y} . The proofs of *ii)*, *iii)*, and *iv)* by using a specialization of \mathcal{Y} are similar, and we left them to the reader.

i) Let $Q \in \mathbb{P}^2$ be a generic point, and let $\mathcal{F} = \{F = 0\}$ be a rational integral curve of degree 5 passing through Q , and having at each P_i , $1 \leq i \leq 6$, an ordinary singularity of multiplicity 2, (so $F \in (I_{2P_1+\dots+2P_6})_5$), and let $\{\tilde{l}_i = 0\}$ be one of the two distinct lines contained in the tangent space $T_{\mathcal{F},P_i}$ to \mathcal{F} at the point P_i . Recall that the defining ideal of $\mathcal{Y} = (8,9)P_1 + \dots + (8,9)P_6$ is

$$I_{\mathcal{Y}} = (\wp_1^8 \cap (\wp_1^9 + l_1^2)) \cap \dots \cap (\wp_6^8 \cap (\wp_6^9 + l_6^2)).$$

Specialize the scheme \mathcal{Y} putting $l_i = \tilde{l}_i$ for $i = 1, 2, 3, 4$, and let \mathcal{Y}^* be such specialization of \mathcal{Y} . Since the expected dimension of $(I_{\mathcal{Y}})_{20}$ is $\binom{20+2}{2} - \deg \mathcal{Y} = 231 - 228 = 3$, then if we prove that $\dim(I_{\mathcal{Y}^*})_{20} = 3$, we are done. It is easy to see that the curves defined by the forms of $(I_{\mathcal{Y}^*+Q})_{20}$ have the quintic \mathcal{F} as fixed component with multiplicity at least 2, hence

$$\dim(I_{\mathcal{Y}^*+Q})_{20} = \dim(I_{\mathcal{W}})_{10}$$

where $\mathcal{W} = \text{Res}_{2\mathcal{F}}(\mathcal{Y}^* + Q) = 4P_1 + 4P_2 + 4P_3 + 4P_4 + (4,5)P_5 + (4,5)P_6$. Now let \mathcal{W}^* be a specialization of \mathcal{W} obtained by putting $l_i = \tilde{l}_i$ for $i = 5, 6$. Since the quintic \mathcal{F} is as fixed component with multiplicity at least 2 for $(I_{\mathcal{W}^*+Q})_{10}$, and since $\text{Res}_{2\mathcal{F}}(\mathcal{W}^* + Q) = \emptyset$ (see Lemma 1.12) we have

$$\dim(I_{\mathcal{W}^*+Q})_{10} = \dim(I_{\text{Res}_{2\mathcal{F}}(\mathcal{W}^*+Q)})_0 = 1.$$

Thus for the specialized scheme \mathcal{W}^* we have $\dim(I_{\mathcal{W}^*})_{10} = 2 = \binom{10+2}{2} - \deg \mathcal{W}^*$. Then \mathcal{W}^* , and so \mathcal{W} also, imposes independent conditions to curves of degree 10. It follows that $\dim(I_{\mathcal{W}})_{10} = 2$. So $\dim(I_{\mathcal{Y}^*+Q})_{20} = 2$, hence $\dim(I_{\mathcal{Y}^*})_{20} = 3$, and we are done. \square

Now let $k+1 = 5q+r$, ($0 \leq r \leq 4$). Thus $k \equiv 2 \pmod{5}$ iff $r = 3$.

For $k = 1, 2$ see Lemma 2.3.

Let $k \geq 3$. Let \mathcal{C}_i be the conic through $P_1, \dots, \widehat{P}_i, \dots, P_6$, ($i = 1, \dots, 6$), and let $\mathcal{C} = \sum_{i=1}^6 \mathcal{C}_i$. Observe that if $2d < 5(k+1)$, then the curves defined by the forms of $(I_{\mathcal{Y}})_d$, and by the forms of $(I_{\mathcal{X}})_d$ have the six conics \mathcal{C}_i as fixed components with multiplicity at least $\nu = 5(k+1) - 2d$.

Then

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d-12\nu}, \quad \dim(I_{\mathcal{X}})_d = \dim(I_{\mathcal{X}'})_{d-12\nu},$$

where

$$\mathcal{Y}' = \text{Res}_{\nu\mathcal{C}}\mathcal{Y} = (k+1-5\nu, k+2-5\nu)P_1 + \dots + (k+1-5\nu, k+2-5\nu)P_6,$$

$$X' = \text{Res}_{\nu\mathcal{C}}X = (k+1-5\nu)P_1 + \cdots + (k+1-5\nu)P_6.$$

We split the proof in four cases.

Case 1): $k \equiv 2 \pmod{5}$, and $d \leq \lceil \frac{12(k+1)}{5} \rceil - 1 = 12q + 7$.

In this case it suffices to prove that $(I_{\mathcal{Y}})_d = (0)$ for $d = 12q + 7$. Since $2d = 2(12q + 7) < 5(k+1) = 5(5q+3)$, then the curves defined by the forms of $(I_{\mathcal{Y}})_d$ should have a fixed locus of degree $12\nu = 12q + 12$, and this is impossible, since $d = 12q + 7$. It follows that $(I_{\mathcal{Y}})_d = (0)$.

Case 2): $k \equiv 2 \pmod{5}$, and $d \geq \lceil \frac{12(k+1)}{5} \rceil = 12q + 8$.

First we will prove that

$$H(\mathcal{Y}, d) = H(X, d) + 12.$$

By Lemma 2.4, it suffices to prove that $H(\mathcal{Y}, d) = H(X, d) + 12$, for $d = 12q + 8$. Since $k \geq 3$, and $k+1 = 5q+3$, then we have $q \geq 1$. Let $q = 1$, so $d = 20$, $k+1 = 8$, $\mathcal{Y} = (8,9)P_1 + \cdots + (8,9)P_6$, and $X = 8P_1 + \cdots + 8P_6$. Since $\dim(I_{(8,9)P_1+\cdots+(8,9)P_6})_{20} = 3$ (see Lemma 2.10 *i*), and six 8-fat points impose independent conditions to curves of degree 20 (see Remark 1.2), we have $\dim(I_X)_{20} = 15$. It follows that $H(\mathcal{Y}, d) = H(X, d) + 12$. If $q > 1$, then $\nu\mathcal{C} = \sum_{i=1}^6 \nu\mathcal{C}_i$ is a fixed locus for $(I_{\mathcal{Y}})_d$ and $(I_X)_d$. Since $\nu = 5(k+1) - 2d = 5(5q+3) - 2(12q+8) = q-1$, we have $d - 12\nu = 12q + 8 - 12(q-1) = 20$, and $k+1-5\nu = 5q+3-5(q-1) = 8$. So

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{20} = 3, \quad \dim(I_X)_d = \dim(I_{X'})_{20} = 15,$$

where $\mathcal{Y}' = \text{Res}_{\nu\mathcal{C}}\mathcal{Y} = (8,9)P_1 + \cdots + (8,9)P_6$, $X' = \text{Res}_{\nu\mathcal{C}}X = 8P_1 + \cdots + 8P_6$.

So we have proved that $H(\mathcal{Y}, d) = H(X, d) + 12$.

Now, since for $d = 12q + 8$, $\dim(I_{\mathcal{Y}})_d$ is positive (and in fact it is equal to $\dim(I_{\mathcal{Y}'})_{20} = 3$), then $H(\mathcal{Y}, d) < \binom{d+2}{2}$ for any $d \geq 12q + 8$.

Since six generic $(k+1)$ -fat points impose independent conditions to curves of degree d if and only if $d \geq \lfloor \frac{5(k+1)}{2} \rfloor$ (see Remark 1.2), then for $12q + 8 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$, we have $H(X, d) < \deg X$, hence

$$H(\mathcal{Y}, d) = H(X, d) + 12 < \min \left\{ \deg X + 12, \binom{d+2}{2} \right\} = \min \left\{ \deg Y, \binom{d+2}{2} \right\} = \exp H(\mathcal{Y}, d).$$

While for $d \geq \max \left\{ 12q + 8; \lfloor \frac{5(k+1)}{2} \rfloor \right\}$, we have $H(X, d) = \deg X$, so $H(\mathcal{Y}, d) = H(X, d) + 12 = \deg X + 12 = \deg Y$. That is enough to finish the proof of this case.

Case 3): $k \not\equiv 2 \pmod{5}$, and $d \leq \lceil \frac{12(k+1)}{5} \rceil$.

By Lemma 2.4 we have only to prove that $H(\mathcal{Y}, d) = N + 1$ for $d = \lceil \frac{12(k+1)}{5} \rceil = 12q + \lceil \frac{12r}{5} \rceil$. Since $k \geq 3$, we have $k+1 = 5q+r \geq 4$, hence $q \geq \frac{4-r}{5}$. As above, let $\nu = 5(k+1) - 2d$, $\mathcal{Y}' = \text{Res}_{\nu\mathcal{C}}\mathcal{Y}$, and let $d' = d - 12\nu$. We have:

r	$k + 1$	d	ν	\mathcal{Y}'	d'
0	$5q$	$12q$	$q > 0$	$P_1 + \cdots + P_6$	0
1	$5q + 1$	$12q + 3$	$q - 1 \geq 0$	$(6,7)P_1 + \cdots + (6,7)P_6$	15
2	$5q + 2$	$12q + 5$	$q > 0$	$(2,3)P_1 + \cdots + (2,3)P_6$	5
4	$5q + 4$	$12q + 10$	$q \geq 0$	$(4,5)P_1 + \cdots + (4,5)P_6$	10

Since for $\nu = 0$, we have $\mathcal{Y}' = \mathcal{Y}$ and $d' = d$, then for every $\nu \geq 0$ we have:

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d'}.$$

Now we will prove that $\dim(I_{\mathcal{Y}'})_{d'} = 0$.

For $r = 0$ it is obvious. For $r = 2$ see Lemma 2.3. For $r = 1$ by Lemma 2.10 *ii*), we have $\dim(I_{(6,7)P_1+\dots+(6,7)P_6})_{15} = 0$. For $r = 4$, let $\mathcal{F} = \{F = 0\}$ be a rational integral curve of degree 5 having at each P_i ($1 \leq i \leq 6$) an ordinary singularity of multiplicity 2, ($F \in (I_{2P_1+\dots+2P_6})_5$). If there exists a form $G \neq 0$, $G \in (I_{(4,5)P_1+\dots+(4,5)P_6})_{10}$, then $FG \neq 0$ and $FG \in (I_{(6,7)P_1+\dots+(6,7)P_6})_{15}$, but this is impossible by the previous case $r = 1$.

Case 4): $k \not\equiv 2 \pmod{5}$, and $d \geq \lceil \frac{12(k+1)}{5} \rceil + 1$.

First we will prove that

$$H(\mathcal{Y}, d) = H(X, d) + 12.$$

By Lemma 2.4, it suffices to prove that $H(\mathcal{Y}, d) = H(X, d) + 12$ for $d = \lceil \frac{12(k+1)}{5} \rceil + 1 = 12q + \lceil \frac{12r}{5} \rceil + 1$.

As usual, let $\nu = 5(k+1) - 2d$, $\mathcal{Y}' = \text{Res}_{\nu\mathcal{C}}\mathcal{Y}$, $X' = \text{Res}_{\nu\mathcal{C}}X$, and $d' = d - 12\nu$. We have:

r	$k + 1$	d	ν	$k + 1 - 5\nu$	\mathcal{Y}'	X'	d'
0	$5q$	$12q + 1$	$q - 2$	10	$\sum_{i=1}^6 (10, 11)P_i$	$\sum_{i=1}^6 10P_i$	25
1	$5q + 1$	$12q + 4$	$q - 3$	16	$\sum_{i=1}^6 (16, 17)P_i$	$\sum_{i=1}^6 16P_i$	40
2	$5q + 2$	$12q + 6$	$q - 2$	12	$\sum_{i=1}^6 (12, 13)P_i$	$\sum_{i=1}^6 12P_i$	30
4	$5q + 4$	$12q + 11$	$q - 2$	14	$\sum_{i=1}^6 (14, 15)P_i$	$\sum_{i=1}^6 14P_i$	35

Since for $\nu = 0$, we have $\mathcal{Y}' = \mathcal{Y}$, $X' = X$, and $d' = d$, then for every $\nu \geq 0$ we have:

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d'}, \quad \dim(I_X)_d = \dim(I_{X'})_{d'}.$$

It follows that

$$H(\mathcal{Y}, d) - H(X, d) = H(\mathcal{Y}', d') - H(X', d').$$

Hence in case $\nu \geq 0$ we have only to prove that:

- (a) $H(\sum_{i=1}^6 (10, 11)P_i, 25) = H(\sum_{i=1}^6 10P_i, 25) + 12;$
- (b) $H(\sum_{i=1}^6 (12, 13)P_i, 30) = H(\sum_{i=1}^6 12P_i, 30) + 12;$
- (c) $H(\sum_{i=1}^6 (14, 15)P_i, 35) = H(\sum_{i=1}^6 14P_i, 35) + 12;$
- (d) $H(\sum_{i=1}^6 (16, 17)P_i, 40) = H(\sum_{i=1}^6 16P_i, 40) + 12;$

Now we need the following lemma:

2.11. Lemma. *Let:*

$$\mathcal{Y} = (m, m+1)P_1 + \dots + (m, m+1)P_6,$$

$$\tilde{\mathcal{Y}} = (m+2, m+3)P_1 + \dots + (m+2, m+3)P_6,$$

$$\tilde{X} = (m+2)P_1 + \dots + (m+2)P_6.$$

If the integer $\eta = 5(d+5) - 12(m+2) + 1 \geq 0$, and $H(\mathcal{Y}, d) = \deg \mathcal{Y}$, then

- i) $H(\tilde{\mathcal{Y}}, d+5) = \deg(\tilde{\mathcal{Y}})$, $H(\tilde{X}, d+5) = \deg(\tilde{X})$;
ii) $H(\tilde{\mathcal{Y}}, d+5) = H(\tilde{X}, d+5) + 12$.

Proof. i) Let \mathcal{F} be (as above) a rational curve of degree 5 having at each P_i ($1 \leq i \leq 6$), an ordinary singularity of multiplicity 2. Let $Q_1, \dots, Q_\eta \in \mathcal{F}$ be generic points. Since $5(d+5) < 6(2(m+2)) + \eta$, by Bezout Theorem \mathcal{F} is a fixed component for the curves defined by the forms of $(I_{\tilde{\mathcal{Y}}+Q_1+\dots+Q_\eta})_{d+5}$. It follows that

$$\dim(I_{\tilde{\mathcal{Y}}+Q_1+\dots+Q_\eta})_{d+5} = \dim(I_{\mathcal{Y}})_d.$$

Since $\binom{d+5+2}{2} - \deg(\tilde{\mathcal{Y}} + Q_1 + \dots + Q_\eta) = \frac{1}{2}(d+7)(d+6) - (\deg \mathcal{Y} + 6(m+2) + 6(m+1) + \eta) = \binom{d+2}{2} - \deg \mathcal{Y} = \binom{d+2}{2} - H(\mathcal{Y}, d) = \dim(I_{\mathcal{Y}})_d$, we have

$$\dim(I_{\tilde{\mathcal{Y}}+Q_1+\dots+Q_\eta})_{d+5} = \binom{d+5+2}{2} - \deg(\tilde{\mathcal{Y}} + Q_1 + \dots + Q_\eta)$$

hence $H(\tilde{\mathcal{Y}} + Q_1 + \dots + Q_\eta, d+5) = \deg(\tilde{\mathcal{Y}} + Q_1 + \dots + Q_\eta)$.

Since obviously $\tilde{X} \subset \tilde{\mathcal{Y}} \subset \tilde{\mathcal{Y}} + Q_1 + \dots + Q_\eta$, it follows that $H(\tilde{\mathcal{Y}}, d+5) = \deg(\tilde{\mathcal{Y}})$, and $H(\tilde{X}, d+5) = \deg(\tilde{X})$.

ii) Obvious. □

By *Case 2*) we know that $H(\sum_{i=1}^6 (8, 9)P_i, 20) = H(\sum_{i=1}^6 8P_i, 20) + 12 = \deg(\sum_{i=1}^6 (8, 9)P_i)$, so by Lemma 2.11 ii) we have (a) : $H(\sum_{i=1}^6 (10, 11)P_i, 25) = H(\sum_{i=1}^6 10P_i, 25) + 12$. Moreover, by Lemma 2.11 i), $H(\sum_{i=1}^6 (10, 11)P_i, 25) = \deg(\sum_{i=1}^6 (10, 11)P_i)$, hence by Lemma 2.11 ii) we get (b) : $H(\sum_{i=1}^6 (12, 13)P_i, 30) = H(\sum_{i=1}^6 12P_i, 30) + 12$. Analogously, by Lemma 2.11, we have that (b) \Rightarrow (c) \Rightarrow (d), so, for $\nu \geq 0$, we have proved that $H(\mathcal{Y}, d) = H(X, d) + 12$.

Now let $\nu < 0$. In this case, since $k+1 = 5q+r \geq 3$, we are left with the following cases:

r	q	$k+1$	Y	X	d
0	1	5	$\sum_{i=1}^6 (5, 6)P_i$	$\sum_{i=1}^6 5P_i$	13
1	1	6	$\sum_{i=1}^6 (6, 7)P_i$	$\sum_{i=1}^6 6P_i$	16
1	2	11	$\sum_{i=1}^6 (11, 12)P_i$	$\sum_{i=1}^6 11P_i$	28
2	1	7	$\sum_{i=1}^6 (7, 8)P_i$	$\sum_{i=1}^6 7P_i$	18
4	0	4	$\sum_{i=1}^6 (4, 5)P_i$	$\sum_{i=1}^6 4P_i$	11
4	1	9	$\sum_{i=1}^6 (9, 10)P_i$	$\sum_{i=1}^6 9P_i$	23

hence we have to prove that:

- (e) $H(\sum_{i=1}^6 (5, 6)P_i, 13) = H(\sum_{i=1}^6 5P_i, 13) + 12$;
(f) $H(\sum_{i=1}^6 (6, 7)P_i, 16) = H(\sum_{i=1}^6 6P_i, 16) + 12$;
(g) $H(\sum_{i=1}^6 (11, 12)P_i, 28) = H(\sum_{i=1}^6 11P_i, 28) + 12$;
(h) $H(\sum_{i=1}^6 (7, 8)P_i, 18) = H(\sum_{i=1}^6 7P_i, 18) + 12$;
(i) $H(\sum_{i=1}^6 (4, 5)P_i, 11) = H(\sum_{i=1}^6 4P_i, 11) + 12$;

$$(l) \quad H(\sum_{i=1}^6 (9, 10)P_i, 23) = H(\sum_{i=1}^6 9P_i, 23) + 12.$$

By Remark 1.2 and by Lemma 2.10 *iii*), and *iv*), it easily follows that (e) and (i) hold, moreover by Lemma 2.11 we have that (e) \Rightarrow (h) \Rightarrow (l) \Rightarrow (g), and (i) \Rightarrow (f), so we have proved that $H(\mathcal{Y}, d) = H(X, d) + 12$ also for $\nu < 0$.

Now observe that for $d = \lceil \frac{12(k+1)}{5} \rceil + 1$, $\dim(I_{\mathcal{Y}})_d$ is positive. In fact, as shown above, we have if $\nu \geq 0$:

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d'} = \binom{d'+2}{2} - \deg X' - 12 = \begin{cases} \binom{25+2}{2} - 6\binom{10+1}{2} - 12 = 9 & \text{for } r = 0 \\ \binom{40+2}{2} - 6\binom{16+1}{2} - 12 = 33 & \text{for } r = 1 \\ \binom{30+2}{2} - 6\binom{12+1}{2} - 12 = 16 & \text{for } r = 2 \\ \binom{35+2}{2} - 6\binom{14+1}{2} - 12 = 24 & \text{for } r = 4 \end{cases}$$

if $\nu < 0$:

$$\dim(I_{\mathcal{Y}})_d = \begin{cases} \binom{13+2}{2} - 6\binom{5+1}{2} - 12 = 3 & \text{for } r = 0, q = 1 \\ \binom{16+2}{2} - 6\binom{6+1}{2} - 12 = 15 & \text{for } r = 1, q = 1 \\ \binom{28+2}{2} - 6\binom{11+1}{2} - 12 = 27 & \text{for } r = 1, q = 2 \\ \binom{18+2}{2} - 6\binom{7+1}{2} - 12 = 10 & \text{for } r = 2, q = 1 \\ \binom{11+2}{2} - 6\binom{4+1}{2} - 12 = 6 & \text{for } r = 4, q = 0 \\ \binom{23+2}{2} - 6\binom{9+1}{2} - 12 = 18 & \text{for } r = 4, q = 1 \end{cases}.$$

Since $\dim(I_{\mathcal{Y}})_d > 0$ for $d = \lceil \frac{12(k+1)}{5} \rceil + 1$, we have that $\dim(I_{\mathcal{Y}})_d$ is positive for any $d \geq \lceil \frac{12(k+1)}{5} \rceil + 1$, and this means that $H(\mathcal{Y}, d) < \binom{d+2}{2}$ for any $d \geq \lceil \frac{12(k+1)}{5} \rceil + 1$. Moreover, since six generic $(k+1)$ -fat points impose independent conditions to curves of degree d if and only if $d \geq \lfloor \frac{5(k+1)}{2} \rfloor$ (see Remark 1.2), then for $\lceil \frac{12(k+1)}{5} \rceil + 1 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$, we have $H(X, d) < \deg X$, hence

$$H(\mathcal{Y}, d) = H(X, d) + 12 < \min \left\{ \deg X + 12, \binom{d+2}{2} \right\} = \min \left\{ \deg Y, \binom{d+2}{2} \right\} = \exp H(\mathcal{Y}, d).$$

While for $d \geq \max \left\{ \lceil \frac{12(k+1)}{5} \rceil + 1; \lfloor \frac{5(k+1)}{2} \rfloor \right\}$, we have $H(X, d) = \deg X$, so $H(\mathcal{Y}, d) = H(X, d) + 12 = \deg X + 12 = \deg Y$, and this finish the proof. \square

2.12. Proposition. *For $s = 9$ we have:*

$$H(\mathcal{Y}, d) = \begin{cases} k = 1 : \begin{cases} N + 1 & \text{if } d \leq 8 \\ H(X, d) + 18 = \deg Y & \text{if } d \geq 8 \end{cases} \\ k = 2 : \begin{cases} N + 1 & \text{if } d \leq 10 \\ H(X, d) + 18 = \deg Y & \text{if } d \geq 11 \end{cases} \\ k = 3 : \begin{cases} N + 1 & \text{if } d \leq 13 \\ H(X, d) + 18 = \deg Y & \text{if } d \geq 14 \end{cases} \\ k \geq 4 : \begin{cases} N + 1 & \text{if } d \leq 3k + 3 \\ H(X, d) + 18 = \deg Y & \text{if } d \geq 3k + 4 \end{cases} \end{cases}.$$

Proof. For $k = 1, 2$ the statement is known by [2] and [3].

Let $k = 3$, so $\mathcal{Y} = (4, 5)P_1 + \cdots + (4, 5)P_9$.

For $d = 13$, by CoCoA (see [6]), or by specializing the scheme \mathcal{Y} it is easy to check that $\dim(I_{\mathcal{Y}})_{13} = 0$, hence for $d \leq 13$ the conclusion follows from Lemma 2.4.

Now let C be the unique (smooth) cubic curve passing through the support of \mathcal{Y} , i.e., through P_1, \dots, P_9 . Consider the following exact sequence, where $\mathcal{Y}' = \text{Res}_C \mathcal{Y}$:

$$0 \rightarrow \mathcal{I}_{\mathcal{Y}'}(d-3) \rightarrow \mathcal{I}_{\mathcal{Y}}(d) \rightarrow \mathcal{I}_{\mathcal{Y} \cap C, C}(d) \rightarrow 0$$

We have that $\mathcal{I}_{\mathcal{Y} \cap C, C}(d) = \mathcal{O}_C(dH - \mathcal{Y} \cap C)$, where H is a line section of C , and $\deg(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 3d - 9(k+1)$.

Let $d = 14$. Since $k = 3$, we have $\deg(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 14 \cdot 3 - 4 \cdot 9 = 6$. It follows that $h^1(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 0$. Since $\mathcal{Y}' = (3, 4)P_1 + \cdots + (3, 4)P_9$, from the case $k = 2$ we get $h^1(\mathcal{I}_{\mathcal{Y}'}(d-3)) = h^1(\mathcal{I}_{\mathcal{Y}'}(11)) = 0$. So by the exact sequence above it follows that $h^1(\mathcal{I}_{(4,5)P_1+\dots+(4,5)P_9}(14)) = 0$, which implies $H(\mathcal{Y}, 14) = \deg \mathcal{Y}$. For $d > 14$ the conclusion follows from Lemma 2.4.

Let $k \geq 4$.

Now we proceed by induction on k . For $k = 4$, we have $\mathcal{Y} = (5, 6)P_1 + \cdots + (5, 6)P_9$, and $3k + 4 = 16$. By CoCoA (see [6]), or by specializing the scheme \mathcal{Y} it is easy to check that $\dim(I_{\mathcal{Y}})_{16} = 0$. So, since $N + 1 = \binom{16+2}{2} = 9 \cdot 17 = \deg \mathcal{Y}$, it follows that $H(\mathcal{Y}, 16) = N + 1 = \deg \mathcal{Y}$. Hence by Lemma 2.4 it follows that for $d \leq 16$ we have $H(\mathcal{Y}, d) = N + 1$, while for $d \geq 16$ we have $H(\mathcal{Y}, d) = \deg \mathcal{Y}$.

Now let $k > 4$. We have:

$$\mathcal{Y} = (k+1, k+2)P_1 + \cdots + (k+1, k+2)P_9 \quad \mathcal{Y}' = (k, k+1)P_1 + \cdots + (k, k+1)P_9.$$

Since obviously if $d \leq 3k + 3$, then $d - 3 \leq 3(k-1) + 3$, and if $d \geq 3k + 4$, then $d - 3 \geq 3(k-1) + 4$, by the induction hypothesis we have $H(\mathcal{Y}', d-3) = N' + 1$ for $d - 3 \leq 3(k-1) + 3$, ($N' = \binom{d-3+2}{2}$), and $H(\mathcal{Y}', d-3) = \deg \mathcal{Y}'$ for $d - 3 \geq 3(k-1) + 4$. That is:

$$h^0(\mathcal{I}_{\mathcal{Y}'}(d-3)) = 0 \quad \text{for } d - 3 \leq 3(k-1) + 3,$$

$$h^1(\mathcal{I}_{\mathcal{Y}'}(d-3)) = 0 \quad \text{for } d - 3 \geq 3(k-1) + 4.$$

Moreover, since $\deg(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 3d - 9(k+1) \leq 0$ for $d \leq 3k + 3$, and $\deg(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 3d - 9(k+1) \geq 3$ for $d \geq 3k + 4$, we have:

$$h^0(\mathcal{I}_{\mathcal{Y} \cap C, C}(d)) = 0 \quad \text{for } d \leq 3k + 3,$$

$$h^1(\mathcal{I}_{\mathcal{Y} \cap C, C}(d)) = 0 \quad \text{for } d \geq 3k + 4.$$

So whenever $d \leq 3k + 3$, we get $h^0(\mathcal{I}_{\mathcal{Y}'}(d-3)) = h^0(\mathcal{I}_{\mathcal{Y} \cap C, C}(d)) = 0$, which by the exact sequence above implies $h^0(\mathcal{I}_{\mathcal{Y}}(d)) = 0$.

When $d \geq 3k + 4$, we get $h^1(\mathcal{I}_{\mathcal{Y}'}(d-3)) = h^1(\mathcal{I}_{\mathcal{Y} \cap C, C}(d)) = 0$, so by the exact sequence above we have $h^1(\mathcal{I}_{\mathcal{Y}}(d)) = 0$, and we are done. \square

With all these partial results we have actually proved the main theorem of this paper:

2.13. Theorem. *For $s \leq 6$, or $s = 9$, then*

$$\dim O_{k,d}^s = \min \left\{ H(X, d) + 2s, \binom{d+2}{2} \right\} - 1$$

except when $s = 2$, $d = k + 2$ where $\dim O_{k,k+2}^2 = H(T, d) - 1 = \binom{d+2}{2} - 2$.

Proof. For $s = 1$, since $H(X, d) = \min \left\{ \binom{k+2}{2}, \binom{d+2}{2} \right\}$, then the result follows from Remark 1.6.

For $s = 2$ and $d = k + 2$, since $H(\mathcal{Y}, d) = H(T, d)$ (see Propositions 2.5), by the obvious inequalities $H(\mathcal{Y}, d) \leq H(Y, d) \leq H(T, d)$ we get

$$H(Y, d) = H(\mathcal{Y}, d) = H(T, d)$$

and the conclusion follows from Remark 1.7 (‡).

In the other cases by Lemma 2.2, and from Propositions 2.5 to 2.9, and Proposition 2.12, we have

$$H(Y, d) = H(\mathcal{Y}, d) = \min\{H(X, d) + 2s, N + 1\},$$

hence from Remark 1.7 (‡) we get the conclusion. □

2.14. Corollary. *Let $\delta = \min\{\deg Y - 1, N\} - \dim O_{k,d}^s$ be the defect of $O_{k,d}^s$. If $s \leq 6$, or $s = 9$, then $O_{k,d}^s$ is defective only in the following cases:*

- i) $s = 2$, $d = k + 2$, with defect: $\delta = 1$.
- ii) $s = 2$, $k \geq 3$, $k + 3 \leq d \leq 2k$, with defect: $\delta = \min \left\{ \binom{2(k+1)-d}{2}; (d-k)^2 - 4 \right\}$.
- iii) $s = 3$, $k \geq 7$, k odd, $\lceil \frac{3(k+1)}{2} \rceil + 2 \leq d \leq 2k$, with defect: $\delta = \min \left\{ 3 \binom{2(k+1)-d}{2}; \binom{2d-3k-1}{2} - 6 \right\}$.
- iv) $s = 3$, $k \geq 6$, k even, $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k$, with defect: $\delta = \min \left\{ 3 \binom{2(k+1)-d}{2}; \binom{2d-3k-1}{2} - 6 \right\}$.
- v) $s = 5$, $k \geq 5$, $2k + 4 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$, with defect $\delta = \min \left\{ \binom{5(k+1)-2d}{2}; 5 \binom{d-2k-1}{2} - 9 \right\}$.
- vi) $s = 6$, $k \equiv 2 \pmod{5}$, $k \geq 17$, $\lceil \frac{12(k+1)}{5} \rceil \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$, with defect:
 $\delta = \min \left\{ 6 \binom{5(k+1)-2d}{2}; \binom{5d-12k-10}{2} - 12 \right\}$.
- vii) $s = 6$, $k \not\equiv 2 \pmod{5}$, $k \geq \begin{cases} 19 & \text{if } k \text{ odd} \\ 24 & \text{if } k \text{ even} \end{cases}$, $\lceil \frac{12(k+1)}{5} \rceil + 1 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$, with defect:
 $\delta = \min \left\{ 6 \binom{5(k+1)-2d}{2}; \binom{5d-12k-10}{2} - 12 \right\}$.

Proof. First observe that: $k + 3 \leq 2k$ implies $k \geq 3$; if k is odd and $\lceil \frac{3(k+1)}{2} \rceil + 2 \leq 2k$, then $3(k+1) + 4 \leq 4k$, that is $k \geq 7$, while if k is even and $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq 2k$, then $k \geq 6$; from $2k + 4 \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$ we get $k \geq 5$; finally, for $k \equiv 2 \pmod{5}$, it is easy to compute that $\lceil \frac{12(k+1)}{5} \rceil \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$ implies $k \geq 17$, while for $k \not\equiv 2 \pmod{5}$, if $\lceil \frac{12(k+1)}{5} \rceil + 1 \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$, then $k \geq \begin{cases} 19 & \text{if } k \text{ odd} \\ 24 & \text{if } k \text{ even} \end{cases}$.

From what we have seen above, and by Remark 1.6, Propositions 2.5 to 2.9 and 2.12, we immediately get that $O_{k,d}^s$ is defective only in the cases *i*) to *vii*).

For $s = 2$ and $d = k + 2$, since $\dim O_{k,k+2}^2 = N - 1$, while the expected dimension is N , we have $\delta = 1$. In the other cases we know that $H(Y, d) = H(X, d) + 2s$, so we have

$$\begin{aligned}\delta &= \min\{\deg Y - 1, N\} - \dim O_{k,d}^s = \min\{\deg Y - 1, N\} - H(Y, d) + 1 \\ &= \min\{\deg Y - H(X, d) - 2s, N + 1 - H(X, d) - 2s\} = \min\{\deg X - H(X, d), \dim(I_X)_d - 2s\}.\end{aligned}$$

For $s = 2$, $k \geq 3$ and $k + 3 \leq d \leq 2k$, computing the dimension of $(I_X)_d$ by removing the line P_1P_2 $(2(k + 1) - d)$ times, we get:

$$\dim(I_X)_d = \dim(I_{X'})_{2(d-k-1)} = \binom{2(d-k-1)+2}{2} - 2\binom{d-k}{2} = (d-k)^2,$$

where $X' = (d-k-1)P_1 + (d-k-1)P_2$, hence

$$\begin{aligned}\deg X - H(X, d) &= 2\binom{k+2}{2} - \binom{d+2}{2} + (d-k)^2 = \binom{2(k+1)-d}{2}, \\ \delta &= \min\left\{\binom{2(k+1)-d}{2}; (d-k)^2 - 4\right\}.\end{aligned}$$

In cases *iii*) and *iv*), computing the dimension of $(I_X)_d$ by cutting off the three lines P_1P_2, P_1P_3, P_2P_3 , $2(k + 1) - d$ times each, we have:

$$\begin{aligned}\dim(I_X)_d &= \dim(I_{X'})_{d-3(2k+2-d)} = \dim(I_{X'})_{2(2d-3k-3)} \\ &= \binom{2(2d-3k-3)+2}{2} - 3\binom{2d-3k-2}{2} = \binom{2d-3k-1}{2},\end{aligned}$$

where $X' = \sum_{i=1}^3(k+1-2(2k+2-d))P_i = \sum_{i=1}^3(2d-3k-3)P_i$, and from here we easily get:

$$\begin{aligned}\deg X - H(X, d) &= 3\binom{k+2}{2} - \binom{d+2}{2} + \binom{2d-3k-1}{2} = 3\binom{2(k+1)-d}{2}, \\ \delta &= \min\left\{3\binom{2(k+1)-d}{2}; \binom{2d-3k-1}{2} - 6\right\}.\end{aligned}$$

For $s=5$, computing the dimension of $(I_X)_d$ (by cutting off the fixed conics), we get:

$$\begin{aligned}\dim(I_X)_d &= \dim(I_{X'})_{d-2(5k+5-2d)} = \dim(I_{X'})_{5(d-2k-2)} \\ &= \binom{5(d-2k-2)+2}{2} - 5\binom{2d-4k-3}{2} = 5\binom{d-2k-1}{2} + 1,\end{aligned}$$

where $X' = \sum_{i=1}^5(k+1-(5k+5-2d))P_i = \sum_{i=1}^5(2d-4k-4)P_i$, and from here we get:

$$\begin{aligned}\deg X - H(X, d) &= 5\binom{k+2}{2} - \binom{d+2}{2} + 5\binom{d-2k-1}{2} + 1 = \binom{5(k+1)-2d}{2}, \\ \delta &= \min\left\{\binom{5(k+1)-2d}{2}; 5\binom{d-2k-1}{2} - 9\right\}.\end{aligned}$$

Finally, for $s=6$, calculating the dimension of $(I_X)_d$ by removing every conic C_i (see the proof of Proposition 2.9) $(5(k+1) - 2d)$ times, we get

$$\begin{aligned} \dim(I_X)_d &= \dim(I_{X'})_{d-12(5k+5-2d)} = \dim(I_{X'})_{25d-60k-60} \\ &= \binom{25d-60k-60+2}{2} - 6 \binom{10d-24k-24+1}{2} = \binom{5d-12k-10}{2}, \end{aligned}$$

where $X' = \sum_{i=1}^6 (k+1 - 5(5k+5-2d))P_i = \sum_{i=1}^6 (10d-24k-24)P_i$, and from here we get:

$$\deg X - H(X, d) = 6 \binom{k+2}{2} - \binom{d+2}{2} + \binom{5d-12k-10}{2} = 6 \binom{5(k+1)-2d}{2},$$

$$\delta = \min \left\{ 6 \binom{5(k+1)-2d}{2}; \binom{5d-12k-10}{2} - 12 \right\}.$$

□

2.15. Remark. Some examples, some computations, and a lack of geometric reasons, lead us to conjecture that also for $s = 7$, and $s = 8$ we have

$$\dim O_{k,d}^s = \min \left\{ H(X, d) + 2s, \binom{d+2}{2} \right\} - 1.$$

Unfortunately, by methods similar to the ones utilized for $s \leq 6$, the proof splits into many cases, and becomes too long and tedious to justify including.

E.Ballico and C.Fontanari in [4] give partial results about the regularity of $O_{k,d}^s$ for $2 \leq s \leq 8$. Our Corollary 2.14, for $s \leq 6$ or $s = 9$, improves the results of [4] and gives a complete classification of all the defective cases.

2.16. Remark. We wish to notice that there are no defective cases for $s = 4$ or $s = 9$.

In case $s = 2$, $d = k + 2$ defectivity is forced by the defectivity of T , in fact, since $Y \subset T$ implies that $H(Y, k+2) \leq H(T, k+2)$, and since $H(T, k+2) = N < \exp H(Y, k+2) = N + 1$, it follows that $H(Y, k+2) < \exp H(Y, k+2)$. In the other cases defectivity of $O_{k,d}^s$ is forced by the defectivity of X .

2.17. Remark. In light of Remarks 2.15 and 2.16, and the results of L.Evain (see Remark 1.2), we like to conjecture that if s is a square, then $O_{k,d}^s$ is regular in any degree d .

Anyway by the results of L.Evain, and by [5], Lemma 3.1, we easily deduce a partial result about the regularity of $O_{k,d}^s$:

If s is a square, and $N + 1 \leq \deg X$ or $N + 1 \geq \deg T$, then $\dim O_{k,d}^s$ is as expected.

In fact if s is a square, by [9] we know that X and T have maximal Hilbert function. Hence if $N + 1 \leq \deg X$, then $\dim(I_X)_d = 0$, and if $N + 1 \geq \deg T$, then $H(T, d) = \deg T$. Since $X \subset Y \subset T$, it follows that if $\dim(I_X)_d = 0$, then $H(Y, d) = N + 1$, and if $H(T, d) = \deg T$, then $H(Y, d) = \deg Y$, and now the conclusion follows from Remark 2.1.

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