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Study of propagation of acoustic waves in junction of thin slots

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Abstract: In this document, we analyze via the theory of matched asymptotics the propagation of a time domain acoustic wave in a junction of thin slots. This allows us to propose Improved Kirchhoff conditions for the 1D limit problem. These conditions are analyzed and validated numerically.

Key-words: Matched asymptotics, wave equation, time domain, thin slots, Kirchhoff conditions

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Étude de la propagation des ondes acoustiques dans une jonction de fentes minces

Résumé : Dans cet article, nous utilisons la théorie des développements asymptotiques raccordés pour analyser la propagation d'ondes acoustiques à travers une jonction de fentes minces. ceci nous permet de proposer des conditions de Kirchhoff améliorées pour le problème limite 1D. Ces conditions sont analysées et validées numériquement.

Mots-clés : Méthodes asymptotiques, équation des ondes, temporel, fentes minces, conditions de Kirchhoff

Introduction

One can consider time harmonic wave propagation and time domain propagation in thin domains that are junctions of thin slots whose thickness ε is small with respect to the wave length λ and converge, when ε tends to 0, to a 1-dimensional graph. *A contrario*, one can imagine that we start from a 1D graph of straight lines that we thicken symmetrically so that we obtain a ε-thickness 2D graph (an example is given by the figure 1).
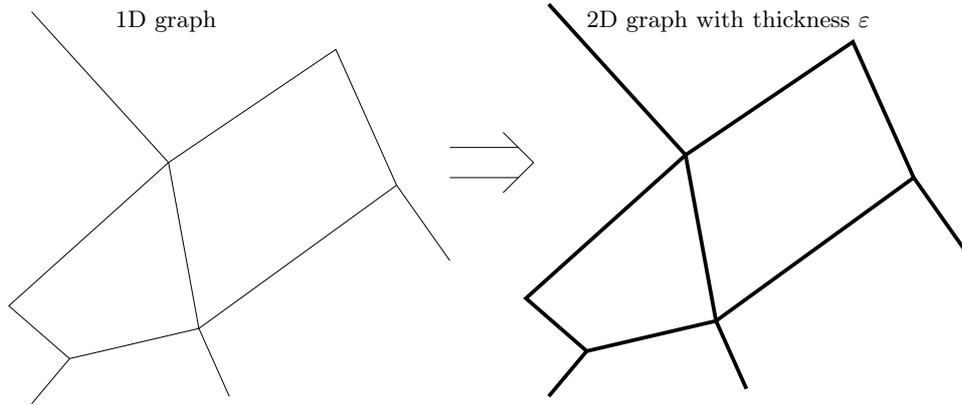

Figure 1: Configuration of the 1D graph (left) and the ε-thickness 2D graph (right)

On this 2D graph (that will be call Ω^ε in the following), one wants to solve the following problem: find $u^\varepsilon \in C^0(\mathbb{R}^*_+, H^1(\Omega^\varepsilon)) \cap C^1(\mathbb{R}^*_+, L^2(\Omega^\varepsilon))$ satisfying

$$\begin{cases} \dfrac{\partial^2 u^\varepsilon}{\partial t^2}(t, \mathbf{x}) - \Delta u^\varepsilon(t, \mathbf{x}) &= 0 \quad \text{for } (t, \mathbf{x}) \in \mathbb{R}^*_+ \times \Omega^\varepsilon \\[2mm] \dfrac{\partial u^\varepsilon}{\partial \vec{n}}(t, \mathbf{x}) &= 0 \quad \text{for } (t, \mathbf{x}) \in \mathbb{R}^*_+ \times \partial\Omega^\varepsilon \\[2mm] u^\varepsilon(0, \mathbf{x}) &= f^\varepsilon \quad \text{for } \mathbf{x} \in \Omega^\varepsilon \\[2mm] \dfrac{\partial u^\varepsilon}{\partial t}(0, \mathbf{x}) &= g^\varepsilon \quad \text{for } \mathbf{x} \in \Omega^\varepsilon \end{cases} \tag{1}$$

with $(f^\varepsilon, g^\varepsilon) \in H^1(\Omega^\varepsilon) \times L^2(\Omega^\varepsilon)$ suth that we assume, for ε small enough, that these functions vanish on the different junctions, and do not depend on the crosswise variable on the slots.

The limit model is known for very long time but its justification seems to be very recent, with the works of Jacob Rubinstein, Michelle Schatzman [7, 8] and Peter Kuchment [6]. In this limit model, the solution only depends of the topology of the graph and satisfy an one-dimensional time domain acoustic equation. Moreover, this solution is continuous at each node of the graph, and satisfy the Kirchhoff laws. To be more precise, the sum of the outgoing normal derivatives (with appropriate constants) is equal to zero.

For studying the behaviour of u^ε for small ε, we will use the method of matched asymptotics expansions. This is a well-known method that has been developed

at the beginning of the 70's, initially to analyse boundary layer phenomena. This approach has been developed quasi-independently (one does not find a lot of cross citations in the publications) by two mathematical schools, from two rather different points of view (see also [2, 10] for more references). In this article, the use of the matched asymptotics expansions is based on a recent work of Patrick Joly and Sébastien Tordeux [5]. As it is classical with this type of problem, the work can be naturally divided into three steps. The first one is related to obtaining the formal asymptotic expansion and constitutes the most algebraic and calculator part. The next two cover two different aspects of the mathematical justification and make use of various techniques for the analysis of PDE's.

- **Step 1:** Derive formal expansion. One starts from an ansatz, i.e. a form of *a priori* behaviour for the asymptotic expansion that is injected in the equations of the initial (ε dependant) problem, to deduce a series of (ε independent) elementary problem that are supposed to characterize the terms of the asymptotic expansion.

- **Step 2:** Show that the various terms of the asymptotic expansions are well defined, i.e. that the above elementary problems are well posed. This is not necessary straightforward since these problems are quite often not standard.

- **Step 3:** Justify the asymptotic expansion, namely establish error estimates between the true solution and truncated asymptotic expansions.

A very particular case has been treated in [3]: in this article, we consider the case of two slots of same width being connected by one junction. The outline of this paper is the following:

- in section 1, we explain the model problem we consider, and we claim the main results of this article,

- in section 2, we use the technic of matched asymptotics to give the development of the exact solution of the whole domain. One can remark that, for giving our approximated model, we only need the first terms of our development, but one can also see that the approximated model we need is the first order of a generalized family of approximated problems,

- in section 3, we justify the expansions of section 2, and we prove that we can build some approximate function which differs from the exact solution with an error as small as we want,

- finally in section 4, we explain how we build one 1D problem (in space) whose solution differs from the exact solution with an arror as small as we want.

1 Model problem and main results

1.1 The model problem

In this section we introduce the geometry and the equations of our problem. We consider a domain made of the junction of N straight slots (see figure 2). More precisely, we consider the union of N thin rectangles and a junction zone. The ith rectangle has length L_i and thickness $c_i\varepsilon$, where ε is the small parameter in the analysis. A geometrical characteristic of this domain is the relative width of the rectangles, given by the numbers c_i. For the analysis, we consider in fact a family of such thin domains denoted Ω^ε with varying ε. We make the choice (this has an influence on the asymptotic analysis) that each rectangle of Ω^ε expands symmetrically from a fixed segment S_i whose one of the two vertices does not depend on i (see again figure 2).
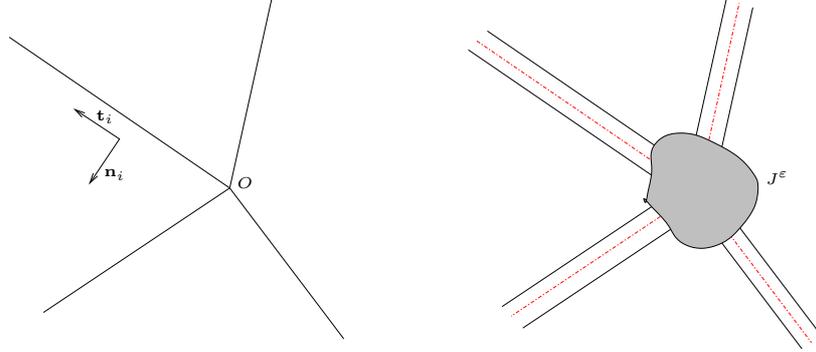

Figure 2: Configuration of the domain for the general case

Analytically, we have

$$\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \ldots \cup \Omega_N^\varepsilon \cup J^\varepsilon \tag{2}$$

with $J^\varepsilon = \varepsilon J$ where J does not depend on ε, and

$$\begin{cases} \Omega_i^\varepsilon = \left\{ \mathbf{x} \in \mathbb{R}^2 \quad / \quad 0 < \mathbf{x} \cdot \mathbf{t}_i < L_i, \quad -\frac{c_i\varepsilon}{2} < \mathbf{x} \cdot \mathbf{n}_i < \frac{c_i\varepsilon}{2} \right\} \\ \mathbf{t}_i \text{ and } \mathbf{n}_i \text{ are given by figure 2} \end{cases} \tag{3}$$

The problem we consider is: find $u^\varepsilon \in \mathrm{C}^0(\mathbb{R}_+^*, \mathrm{H}^1(\Omega^\varepsilon) \cap \mathrm{C}^1(\mathbb{R}_+^*, \mathrm{L}^2(\Omega^\varepsilon))$ such that (we recall here the wave system (1) we want to solve)

$$\begin{cases} \dfrac{\partial^2 u^\varepsilon}{\partial t^2}(t, \mathbf{x}) - \Delta u^\varepsilon(t, \mathbf{x}) &=& 0 \quad \text{for } (t, \mathbf{x}) \in \mathbb{R}_+^* \times \Omega^\varepsilon \\[2mm] \dfrac{\partial u^\varepsilon}{\partial \vec{n}}(t, \mathbf{x}) &=& 0 \quad \text{for } (t, \mathbf{x}) \in \mathbb{R}_+^* \times \partial\Omega^\varepsilon \\[2mm] u^\varepsilon(0, \mathbf{x}) &=& f^\varepsilon \quad \text{for } \mathbf{x} \in \Omega^\varepsilon \\[2mm] \dfrac{\partial u^\varepsilon}{\partial t}(0, \mathbf{x}) &=& g^\varepsilon \quad \text{for } \mathbf{x} \in \Omega^\varepsilon \end{cases}$$

For this system, we can define the associate energy

$$\mathcal{E}^\varepsilon(t, u^\varepsilon) = \frac{1}{2\varepsilon} \int_{\Omega^\varepsilon} |\nabla u^\varepsilon(t, \mathbf{x})|^2 + \left| \frac{\partial u^\varepsilon}{\partial t}(t, \mathbf{x}) \right|^2 d\mathbf{x} \qquad (4)$$

and we have (thanks to the Neumann boundary condition)

$$\mathcal{E}^\varepsilon(t, u^\varepsilon) = \mathcal{E}^\varepsilon(0, u^\varepsilon) = \frac{1}{2\varepsilon} \int_{\Omega^\varepsilon} |\nabla f^\varepsilon(\mathbf{x})|^2 + |g^\varepsilon(\mathbf{x})|^2 d\mathbf{x} \qquad (5)$$

In this energy, we add the ε^{-1} term to get the initial energy independant of ε when f^ε and g^ε satisfy somme good properties (namely hypothesis 1.2 detailed below).

Remark 1.1. In the following, we assume that $L_i = +\infty$. In fact, for t small enough, solution obtained on a finite domain with homogeneous boundary conditions is the same as solution obtained on a semi-finite domain. But effects of the junction does not depend on the choice of L_i, neither on conditions eventually put at $s_i = L_i$.

1.2   The 1D limit problem

When ε tends to 0, the domain Ω^ε "degenerates" into a "1D domain", namely the union of the n segments S_i (left part of the figure 2). Intuitively, one expects, assuming f^ε and g^ε have good properties (with respect to ε), that the solution u^ε "converges" to a "1D function" (in space), namely a function of time and the arclength s_i along S_i, solution of a "1D problem". It remains to give a more precise mathematical meaning to such a statement. To describe the "limit problem" inside the slots, we will use local normalized tangential and normal coordinates $(s_i, \widehat{\nu}_i)$, that express that Ω_i^ε is isomorphic to the rectangle

$$\widehat{\Omega}_i =]0, +\infty[\times \left] -\frac{c_i}{2}, \frac{c_i}{2} \right[$$

through the map

$$\mathbf{x} \mapsto (s_i, \widehat{\nu}_i) = (\mathbf{x} \cdot \mathbf{t}_i, \mathbf{x} \cdot \mathbf{n}_i / \varepsilon) \quad \text{from } \Omega_i^\varepsilon \text{ into } \widehat{\Omega}_i$$

The reader can note that domains $\widehat{\Omega}_i$ overlap (see for instance figure 3).
Before writing our 1D limit problem, we consider additional hypothesis on f^ε and g^ε, namely:

Hypothesis 1.2. There exists ε_0 such that, for $0 < \varepsilon < \varepsilon_0$, for $v^\varepsilon = f^\varepsilon$ or $v^\varepsilon = g^\varepsilon$,

- supp $v^\varepsilon \cap J^\varepsilon = \emptyset$,

- on each Ω_i^ε, $v^\varepsilon(s, \nu_i) = v^\varepsilon(s)$ (called in the following v_i^ε),

- v^ε converges (in an appropriate space) to a function v^0 (which is a 1D function thanks to the two previous points).
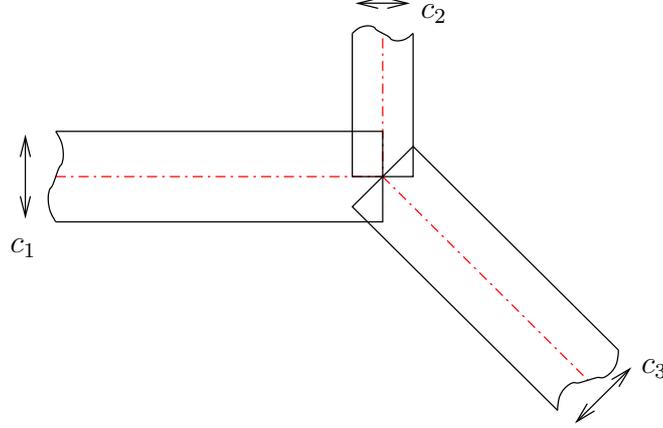
Figure 3: Configuration of the limit domain (red dotdashed) and the 2D rectanges $\widehat{\Omega}_i$ (black boxes)

Under the hypothesis, let us define the following problem: find $(u_i^0)_{1 \leqslant i \leqslant N}$ such that

$$
\begin{cases}
\dfrac{\partial^2 u_i^0}{\partial t^2}(t, s_i) - \dfrac{\partial^2 u_i^0}{\partial s_i^2}(t, s_i) & = \quad 0 \quad \text{for } (t, s_i) \in \mathbb{R}_+^* \times]0, +\infty[\\[2ex]
u_i^0(0, s_i) & = \quad f_i^0 \quad \text{for } s_i \in]0, +\infty[\\[2ex]
\dfrac{\partial u_i^0}{\partial t}(0, s_i) & = \quad g_i^0 \quad \text{for } s_i \in]0, +\infty[
\end{cases}
\tag{6}
$$

and

$$
u_j^0(t, 0) = u_k^0(t, 0) \quad \text{for } j \neq k \qquad \text{and} \qquad \sum_j c_j \frac{\partial u_j^0}{\partial s_j}(t, 0) = 0 \tag{7}
$$

The system (6) can be seen as the formal limit of the system (1) when ε tends to 0. The coupling conditions (7), known also in the literature as the Kirchhoff conditions, link the functions at the node of the graph.

Intuition says that, on any subdomain in the slots, error between the exact solution and the approximated solution is in $O(\varepsilon)$. More precisely, one has the following proposition:

Proposition 1.1. *Given $(\delta_i)_{1 \leqslant i \leqslant N}$, one can define:*

- *the sets $\Omega_{i,\delta}^\varepsilon$ given as (see for instance figure 4):*

$$
\Omega_{i,\delta}^\varepsilon = \{ \mathbf{x} \in \Omega_i^\varepsilon \text{ such that } \mathbf{x} \cdot \mathbf{t}_i \geqslant \delta_i \}
$$

- *the 2D function $\widetilde{u_i^0}$ defined as*

$$
\widetilde{u_i^0}(t, \mathbf{x}) = u_i^0(t, \mathbf{x} \cdot \mathbf{t}_i) \quad \text{for } t \in \mathbb{R}_+ \text{ and } \mathbf{x} \in \Omega_{i,\delta}^\varepsilon
$$
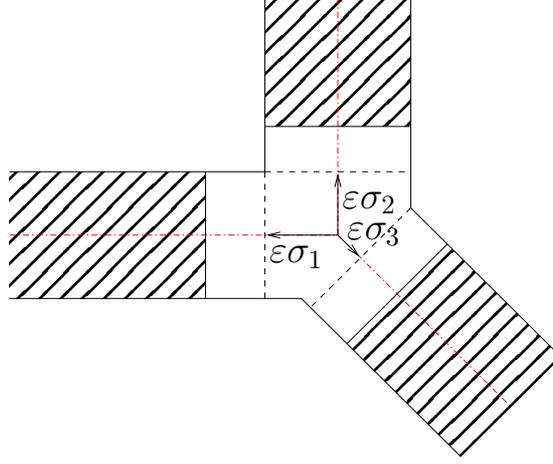
Figure 4: Configuration of the sets $\Omega_{i,\delta}^\varepsilon$

Then there exist some 1D function $(\widetilde{u_i^1})_{1 \leqslant i \leqslant N}$ in space such that

$$
\sum_{i=1}^N \frac{1}{\varepsilon} \int_{\Omega_{i,\delta}^\varepsilon} \left(\left| \frac{\partial (u^\varepsilon - \widetilde{u_i^0})}{\partial t} \right|^2 + \left| \nabla (u^\varepsilon - \widetilde{u_i^0}) \right|^2 \right)
$$
$$
= \left[\sum_{i=1}^N \int_{\delta_i}^{+\infty} \left(\left| \varepsilon \frac{\partial \widetilde{u_i^1}}{\partial t} \right|^2 + \left| \varepsilon \nabla \widetilde{u_i^1} \right|^2 \right) \right] + O(\varepsilon^4) \tag{8}
$$

Proof. The proof of this proposition is a direct consequence of corollary 3.8 taking $k = 1$. $\qquad\square$

Remark 1.3. We can find Cauchy data such that $\widetilde{u_i^1} = 0$ (for example, we choose Cauchy data such that the solution is a single wave contained in the first slot and going to infinity - this solution does not see the junction and does not have any interest for us). But one can see that when the junction is "excited", one has $\widetilde{u_i^1} \neq 0$ for any later time.

1.3 An improved 1D approximate model

To describe our improved problem, we need to introduce some additional notations for our problem.

1.3.1 Additional information about the geometry

Definition 1.4 (Definition of σ_i)**.** We introduce σ_i, for $1 \leqslant i \leqslant N$, as the smallest value for which (see figure 5 for an example)

$$
\{ \mathbf{x} \in \Omega_i^\varepsilon \text{ such that } \mathbf{x} \cdot \mathbf{t}_i \geqslant \varepsilon \sigma_i \} \cap \left(J^\varepsilon \cup \bigcup_{j \neq i} \Omega_j^\varepsilon \right) = \emptyset
$$

One can easily see that σ_i does not depend of ε.
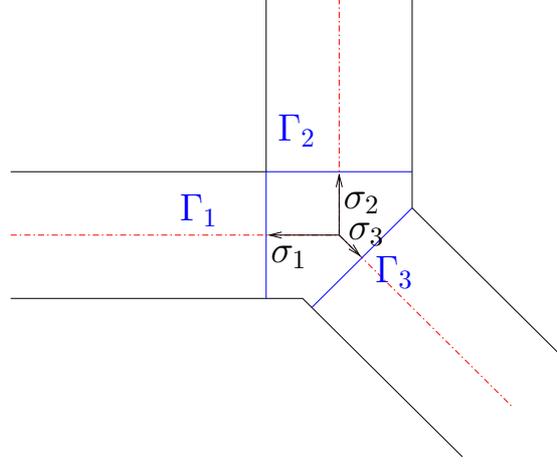
Figure 5: Configuration of an example of junction for three slots (in scaled coordinates).

From the definition of σ_i, we add additional sets: we define $\Omega_{i,\text{ext}}^{\varepsilon}$ as (see for instance figure 6)

$$\Omega_{i,\text{ext}}^{\varepsilon} = \{ \mathbf{x} \in \Omega_i^{\varepsilon} \quad / \quad \mathbf{x} \cdot \mathbf{t}_i > \varepsilon \sigma_i \}$$

and we define our new junction $\widetilde{J}^{\varepsilon}$ as

$$\widetilde{J}^{\varepsilon} = J^{\varepsilon} \cup \bigcup_{i=1}^{N} \{ \mathbf{x} \in \Omega^{\varepsilon} \quad / \quad \mathbf{x} \cdot \mathbf{t}_i < \varepsilon \sigma_i \}$$

Figure 6: Configuration of the new sets $\Omega_{i,\text{ext}}^{\varepsilon}$ and $\widetilde{J}^{\varepsilon}$

Note that we also have the decomposition

$$\overline{\Omega^{\varepsilon}} = \overline{\Omega_{1,\text{ext}}^{\varepsilon}} \cup \overline{\Omega_{2,\text{ext}}^{\varepsilon}} \cup \cdots \cup \overline{\Omega_{N,\text{ext}}^{\varepsilon}} \cup \overline{\widetilde{J}^{\varepsilon}}$$

with domains that do not intersect each others. We keep $\widetilde{J}^\varepsilon = \varepsilon \widehat{J}$, where $\widehat{J}$ is a fixed domain (i.e. not depending of ε) of $\mathbb{R}^2$. On domain $\widehat{J}$, we also define
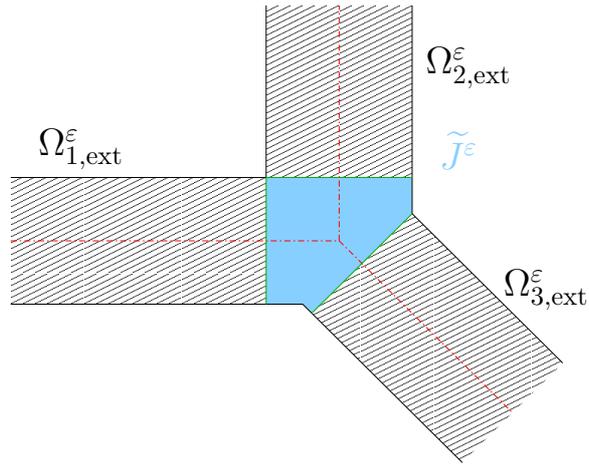
- Γ_i, for $1 \leqslant i \leqslant N$ (see again figure 5) as

$$\Gamma_i = \left\{ \mathbf{x} \in \partial \widehat{J} \quad / \quad \mathbf{x} \cdot \mathbf{t}_i = \sigma_i \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{n}_i| < \frac{c_i}{2} \right\},$$

- Γ_{neu} as the remaining boundary (if exists).

Remark 1.5. One can easily imagine situations where boundary Γ_N does not exist (for instance, a junction of three slots of same width with an angle of $2\pi/3$ between two slots: the junction is an equilateral triangle whose edges can be identified to some Γ_i).

1.3.2   Solution of auxiliary problems and DtN maps

To allow writing of Improved Kirchhoff conditions, one uses the solution of $N-1$ auxiliary problems. Let us introduce, once Σ_i has been identified to the segment $]0, c_i[$, T_i as the non-local DtN operator defined as

$$\left| \begin{aligned} T_i : \mathrm{H}^{\frac{1}{2}}(]0, c_i[) &\;\rightarrow\; \mathrm{H}^{-\frac{1}{2}}(]0, c_i[) \\[2mm] \varphi &\;\mapsto\; T_i\varphi \;=\; \sum_{p=1}^{\infty} \frac{p\pi}{c_i} \left(\int_0^{c_i} \varphi(\widehat{\nu}') w_{i,p}(\widehat{\nu}') d\widehat{\nu}' \right) w_{i,p} \end{aligned} \right. \qquad (9)$$

where $w_{i,p}$ is an orthonormal family of $\mathrm{L}^2(]0, c_i[)$ given by

$$w_{i,0}(\widehat{\nu}) = \sqrt{\frac{1}{c_i}}, \qquad w_{i,p}(\widehat{\nu}) = \sqrt{\frac{2}{c_i}} \cos\left(\frac{p\pi\widehat{\nu}}{c_i} \right) \qquad (10)$$

Proposition 1.2. *We have the following properties*

- $T_i \in \mathcal{L}\left(\mathrm{H}^{\frac{1}{2}}(]0, c_i[), \mathrm{H}^{-\frac{1}{2}}(]0, c_i[) \right)$

- *for any* $\varphi \in \mathrm{H}^{\frac{1}{2}}(]0, c_i[)$*, one has* $< T_i\varphi, \varphi >_{-\frac{1}{2}, \frac{1}{2}} \geqslant 0$

Proof. The proof is detailed in the appendix A.1. $\qquad\qquad\square$

Once we defined our DtN operators, we can define the following problem: find $W_i \in \mathrm{H}^1(\widehat{J})$, for $1 \leqslant i \leqslant N-1$, such that

$$\begin{cases} \Delta W_i &=\; 0 \quad \text{in } \widehat{J} \\[1mm] c_i \left(\nabla W_i \cdot \vec{n} + T_i W_i \right) &=\; 1 \quad \text{on } \Gamma_i \\[1mm] c_{i+1} \left(\nabla W_i \cdot \vec{n} + T_{i+1} W_i \right) &=\; -1 \quad \text{on } \Gamma_{i+1} \\[1mm] c_k \left(\nabla W_i \cdot \vec{n} + T_k W_i \right) &=\; 0 \quad \text{on } \Gamma_k, k \neq \{i, i+1\} \\[1mm] \nabla W_i \cdot \vec{n} &=\; 0 \quad \text{on } \Gamma_{\mathrm{neu}} \end{cases} \qquad (11)$$

with the additional condition

$$\int_{\widehat{J}} W_i = 0 \qquad (12)$$

The additional condition (12) has been added to claim the following proposition (whose proof is detailed in appendix A.2):

Proposition 1.3. *There exists a unique W_i in $\mathrm{H}^1(\widehat{J})$ satisfying (11) and (12).*

We can then compute the matrix $K \in \mathcal{M}_{N-1}(\mathbb{R})$ given by

$$K_{i,j} = \frac{1}{c_j} \int_{\Gamma_j} W_i - \frac{1}{c_{j+1}} \int_{\Gamma_{j+1}} W_i \tag{13}$$

Another useful property is given by the following proposition (whose proof is detailed in appendix A.3)

Proposition 1.4. *K is a symmetric definite positive matrix.*

1.3.3   Build the approximate problem

Thanks to sections 1.3.1 and 1.3.2, we can now give our improved Kirchhoff conditions. First, we introduce, for a family of functions $\left(v_i \in \mathrm{C}^1(\mathbb{R}_+^*, \mathrm{H}^1(S_i))\right)_{1 \leqslant i \leqslant N}$, with S_i is the segment defined in section 1, the following vectors:

$$V^\varepsilon(t) = (v_j(t, \varepsilon \sigma_j))_{1 \leqslant j \leqslant N} \in \mathbb{R}^N \tag{14-i}$$

$$\partial_S V^\varepsilon(t) = (\partial_{s_j} v_j(t, \varepsilon \sigma_j))_{1 \leqslant j \leqslant N} \in \mathbb{R}^N \tag{14-ii}$$

They represent the value of the i-th function and its normal derivate at the point on the boundary $\varepsilon \Gamma_i$ (the blue circled point on figure 7).
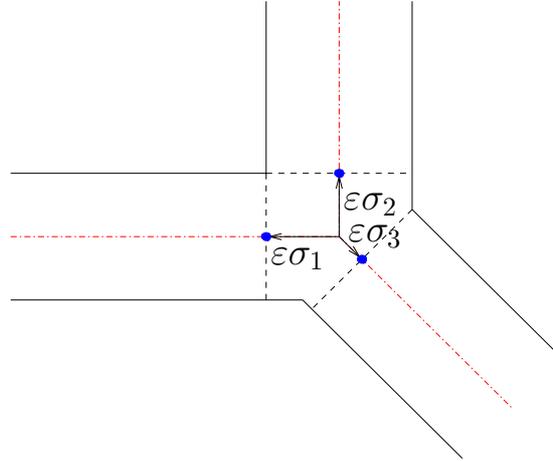
Figure 7: Points (blue circles) used for the definition of the vectors (14-i) and (14-ii).

One can see that this definition will remain valid even if we consider instead a family of functions $\left(v_i^\varepsilon \in \mathrm{C}^1(\mathbb{R}_+^*, \mathrm{H}^1(\varepsilon \sigma_i, +\infty))\right)_{1 \leqslant i \leqslant N}$ indexed by ε.

Let us now define the "jump matrix" $\mathcal{J}$, the "average matrix" $\mathcal{A}$ and the "weight matrix" $\mathcal{C}$ by the following formulas:

- $\mathcal{A}$ is the $N \times N$ matrix given by (denoting $|\widehat{J}|$ the area of $\widehat{J}$):

$$\mathcal{A} = \frac{|\widehat{J}|}{N^2} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \tag{15}$$

- $\mathcal{J}$ is the $N \times N$ matrix given by

$$\mathcal{J} = P^T K^{-1} P \tag{16}$$

with K is the previously defined matrix and P is defined by

$$P = \begin{pmatrix} 1 & -1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -1 & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & 1 & -1 & 0 \\ 0 & \dots & \dots & 0 & 1 & -1 \end{pmatrix} \tag{17}$$

- $\mathcal{C}$ is a diagonal matrix of $\mathcal{M}_N(\mathbb{R})$, given by

$$\mathcal{C} = \begin{pmatrix} c_1 & 0 & \dots & \dots & 0 \\ 0 & c_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & c_{N-1} & 0 \\ 0 & \dots & \dots & 0 & c_N \end{pmatrix}$$

The Improved Kirchhoff model that we propose is the following: find $(\tilde{u}_i^\varepsilon)_{1 \leqslant i \leqslant N}$ such that

$$\begin{cases} \dfrac{\partial^2 \tilde{u}_i^\varepsilon}{\partial t^2}(t, s_i) - \dfrac{\partial^2 \tilde{u}_i^\varepsilon}{\partial s_i^2}(t, s_i) &=& 0 & \text{for } (t, s_i) \in \mathbb{R}_+^* \times]\varepsilon \sigma_i, +\infty[\\[2mm] \tilde{u}_i^\varepsilon(0, s_i) &=& f_i^\varepsilon & \text{for } s_i \in]\varepsilon \sigma_i, +\infty[\\[2mm] \dfrac{\partial \tilde{u}_i^\varepsilon}{\partial t}(0, s_i) &=& g_i^\varepsilon & \text{for } s_i \in]\varepsilon \sigma_i, +\infty[\end{cases} \tag{18}$$

and

$$\mathcal{C} \partial_S \tilde{U}^\varepsilon(t) = \left(\dfrac{1}{\varepsilon} \mathcal{J} + \varepsilon \mathcal{A} \dfrac{\partial^2}{\partial t^2} \right) \tilde{U}^\varepsilon(t), \quad \text{for } t \in \mathbb{R}_+^* \tag{19}$$

Remark 1.6. It is clear that, when ε tends to 0, (18) gives (6). The not so clear point is that, when ε tends to 0, (19) gives (7). However, at least formally, $\tilde{U}^\varepsilon(t)$ tends to $\tilde{U}^0(t)$, and the formal limit of (19) multiplied by ε leads to the fact that $\tilde{U}^0(t)$ is in the kernel of $\mathcal{J}$, which is equal to

$$\text{Vect}\{(1, \dots, 1)^t\}$$

This gives the left part of (7). Then, going back to (19) and doing the scalar product with $(1, \dots, 1)^t$ leads to

$$\sum_j c_j \dfrac{\partial u_j^\varepsilon}{\partial s_j}(t, \varepsilon \sigma_j) = \varepsilon \dfrac{|\widehat{J}|}{N} \sum_j \dfrac{\partial^2 u_j}{\partial t^2}(t, \varepsilon \sigma_j)$$

The formal limit of this relation gives the right part of (7).

For the Improved Kirchhoff problem (18, 19), one may define the approximate energy

$$
\begin{aligned}
\widetilde{\mathcal{E}}^{\varepsilon}(t,v) \quad = \quad & \sum_{i=1}^{N} \frac{c_i}{2} \int_{\varepsilon\sigma_i}^{+\infty} \left(\left| \frac{\partial v}{\partial s_i}(t,s') \right|^2 + \left| \frac{\partial v}{\partial t}(t,s') \right|^2 \right) ds' \\
+ \quad & \frac{1}{2\varepsilon} \left(\mathcal{J}V^{\varepsilon}(t) \right) \cdot V^{\varepsilon}(t) + \frac{\varepsilon}{2} \left(\mathcal{A} \frac{\partial V^{\varepsilon}}{\partial t}(t) \right) \cdot \frac{\partial V^{\varepsilon}}{\partial t}(t)
\end{aligned}
\tag{20}
$$

Proposition 1.5. *We have the following energy conversation relation*

$$
\widetilde{\mathcal{E}}^{\varepsilon}(t, \tilde{u}^{\varepsilon}) = \widetilde{\mathcal{E}}^{\varepsilon}(0, \tilde{u}^{\varepsilon}) = \sum_{i=1}^{N} \frac{c_i}{2} \int_{\varepsilon\sigma_i}^{+\infty} \left(\left| \frac{\partial f^{\varepsilon}}{\partial s_i}(s') \right|^2 + |g^{\varepsilon}(s')|^2 \right) ds'
\tag{21}
$$

This proposition will be proved in section 4.2.
One idea, of course, is to compare the difference between the solution of the exact problem (1) and the solution of the 1D Improved problem (18, 19). We compare this in four steps:

- we estimate the difference between the solution of the exact problem and an expansion in powers of ε,

- we estimate the difference between the solution of the 1D Improved problem and another expansion in powers of ε,

- we remark in section 4.2 that the two expansions are the same one for at least the two first terms,

- we use some classical triangular inequality to conclude.

Finally, one gets the following proposition (which is an improvement of proposition 1.1):

Proposition 1.6. *Given* $(\delta_i)_{1 \leqslant i \leqslant N}$*, one can define:*

- *the set* $\Omega_{i,\delta}^{\varepsilon}$ *given as (see for instance figure 4):*

$$
\Omega_{i,\delta}^{\varepsilon} = \{ \mathbf{x} \in \Omega_i^{\varepsilon} \text{ such that } \mathbf{x} \cdot \mathbf{t}_i \geqslant \delta_i \}
$$

- *the 2D function* $\widetilde{u_i^{\varepsilon}}$ *defined as*

$$
\widetilde{u_i^{\varepsilon}}(t, \mathbf{x}) = \tilde{u}_i^{\varepsilon}(t, \mathbf{x} \cdot \mathbf{t}_i) \quad \text{for } t \in \mathbb{R}_+ \text{ and } \mathbf{x} \in \Omega_{i,\delta}^{\varepsilon}
$$

Then if $f_i^{\varepsilon} \in \mathrm{H}^5(S_i)$ *and* $g_i^{\varepsilon} \in \mathrm{H}^4(S_i)$*, for* $1 \leqslant i \leqslant N$ *(see back hypothesis 1.2), there exist some 1D functions* $(\mathfrak{u}_i)_{1 \leqslant i \leqslant N}$ *in space such that:*

$$
\begin{aligned}
& \sum_{i=1}^{N} \frac{1}{\varepsilon} \int_{\Omega_{i,\delta}^{\varepsilon}} \left(\left| \frac{\partial(u_i^{\varepsilon} - \widetilde{u_i^{\varepsilon}})}{\partial t}(t, \mathbf{x}) \right|^2 + \left| \nabla(u_i^{\varepsilon} - \widetilde{u_i^{\varepsilon}})(t, \mathbf{x}) \right|^2 \right) d\mathbf{x} \\
& = \left[\sum_{i=1}^{N} \int_{\delta_i}^{+\infty} \left(\left| \varepsilon^2 \frac{\partial \mathfrak{u}_i}{\partial t}(t, s_i) \right|^2 + \left| \varepsilon^2 \nabla \mathfrak{u}_i(t, s_i) \right|^2 \right) ds_i \right] + O(\varepsilon^6)
\end{aligned}
\tag{22}
$$

Proof. For this proof, we denote by $\mathfrak{E}_\delta$ the following quantity:

$$\mathfrak{E}_\delta^\varepsilon(v) = \sum_{i=1}^N \frac{1}{\varepsilon} \int_{\Omega_{i,\delta}^\varepsilon} \left(\left| \frac{\partial v}{\partial t}(t, \mathbf{x}) \right|^2 + |\nabla v(t, \mathbf{x})|^2 \right) d\mathbf{x}$$

Thanks to the hypothesis on the Cauchy data, we can bring the third order approximation for both u_i^ε and $\widetilde{u_i^\varepsilon}$. We denote then by $u_{i,2}^\varepsilon$ the approximation of u_i^ε to second order, and $\widetilde{u_{i,2}^\varepsilon}$ the approximation of $\widetilde{u_i^\varepsilon}$ to second order. Then one has that

$$\left| \mathfrak{E}_\delta^\varepsilon(u_i^\varepsilon - \widetilde{u_i^\varepsilon}) - \mathfrak{E}_\delta^\varepsilon(u_{i,2}^\varepsilon - \widetilde{u_{i,2}^\varepsilon}) \right| \leqslant \mathfrak{E}_\delta^\varepsilon(u_i^\varepsilon - u_{i,2}^\varepsilon) + \mathfrak{E}_\delta^\varepsilon(\widetilde{u_i^\varepsilon} * \widetilde{u_{i,2}^\varepsilon}) \qquad (23)$$

Now, for the left member size, since $u_i^0 = \widetilde{u_i^0}$ and $u_i^1 = \widetilde{u_i^1}$ (this point will be proved later in section 4.2), there exist a function $\mathfrak{u}_i$ such that

$$u_{i,2}^\varepsilon - \widetilde{u_{i,2}^\varepsilon} = \varepsilon^2 \mathfrak{u}_i$$

One can easily see that $\mathfrak{u}_i$ is 1D in space and does not depend on ε. To conclude, we use both corollaries 3.8 and 4.1 for $n = 2$ to treat the right member of (23), and we obtain the desired result. $\qquad \square$

2 The formal expansions

As we said in the introduction, as the problem is multi-scale (in space), it is not possible to write a uniform expansion for the solution everywhere in the domain Ω^ε. The method of matched asymptotics will lead us, we have to consider $N+1$ distinct zones, respectively N slot zones and a junction zone, in which different expansions will be obtained. However, contrarily to the naive intuition, this domain decomposition does not correspond to the partition (2) of Ω^ε: in the method of matched asymptotics, the different domains must overlap, the idea being that the different expansions must "coincide" in the overlapping zones.

2.1 An overlapping domain decomposition

In the following, we will denote by **C** the class of positive continuous functions of $\varepsilon > 0$ that tend to 0 when $\varepsilon \to 0$, less rapidly than $\varepsilon |\log(\varepsilon)|$ (a typical example is ε^β, with β strictly less than 1).

$$\mathbf{C} = \left\{ \varphi : \mathbb{R}_+^* \to \mathbb{R}_+^* \quad / \quad \lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0 \text{ and } \lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon)}{\varepsilon |\log(\varepsilon)|} = +\infty \right\} \qquad (24)$$
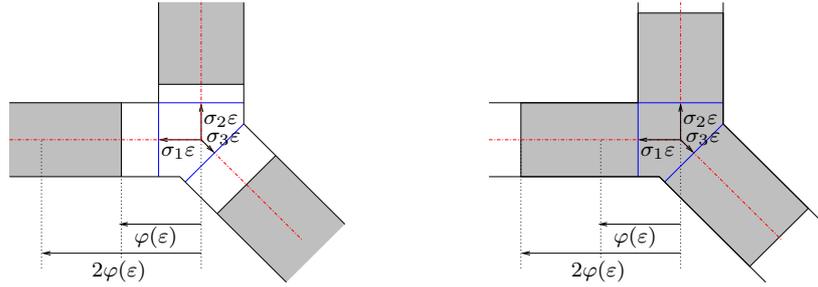

Figure 8: Slots zones $\boldsymbol{\Omega}_i^\varepsilon(\varepsilon)$ (left figure) and junction zone $\mathbf{J}^\varepsilon(\varepsilon)$ (right figure).

Given φ in **C**, we define, for $1 \leqslant i \leqslant N$, the i-th slot zone by (see figure 8):

$$\boldsymbol{\Omega}_i^\varepsilon(\varepsilon) = \{ \mathbf{x} \in \Omega_i^\varepsilon \quad / \quad \varphi(\varepsilon) \leqslant \mathbf{x} \cdot \mathbf{t}_i \} \quad (\subset \Omega_i^\varepsilon)$$

The junction zone is defined by (see again figure 8):

$$\mathbf{J}^\varepsilon(\varepsilon) = \varepsilon \widehat{J} \cup \bigcup_{i=1}^N \{ \mathbf{x} \in \Omega_i^\varepsilon \quad / \quad 0 \leqslant \mathbf{x} \cdot \mathbf{t}_i < 2\varphi(\varepsilon) \} \quad (\text{contains } J^\varepsilon)$$

in such a way that we have $\Omega^\varepsilon = \boldsymbol{\Omega}_1^\varepsilon(\varepsilon) \cup \boldsymbol{\Omega}_2^\varepsilon(\varepsilon) \cup \cdots \cup \boldsymbol{\Omega}_N^\varepsilon(\varepsilon) \cup \mathbf{J}^\varepsilon(\varepsilon)$ with N overlapping regions (see figure 9)

$$\begin{aligned} \mathcal{O}_i(\varepsilon) \quad &= \quad \boldsymbol{\Omega}_i^\varepsilon(\varepsilon) \cap \mathbf{J}^\varepsilon(\varepsilon) \\ &= \quad \left\{ (x,y) \in \Omega^\varepsilon \quad / \quad \varphi(\varepsilon) < \mathbf{x} \cdot \mathbf{t}_i < 2\varphi(\varepsilon) \text{ and } -\frac{c_i \varepsilon}{2} < \mathbf{x} \cdot \mathbf{n}_i < \frac{c_i \varepsilon}{2} \right\} \end{aligned}$$

The mapping

$$\mathbf{x} \mapsto (\mathbf{x} \cdot \mathbf{t}_i, \mathbf{x} \cdot \mathbf{n}_i / \varepsilon) \qquad (25)$$
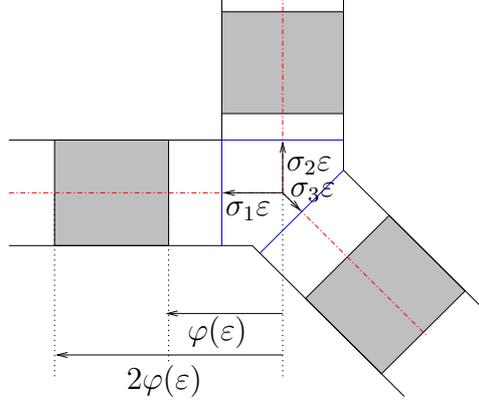
Figure 9: Configuration of the overlapping domains $\mathcal{O}_i(\varepsilon)$ (grey zones)

maps the domain $\boldsymbol{\Omega}_i^\varepsilon(\varepsilon)$ into the rectangle $\widehat{\boldsymbol{\Omega}}_i(\varepsilon)$ with

$$\widehat{\boldsymbol{\Omega}}_i(\varepsilon) =]\varphi(\varepsilon), +\infty[\times \left] -\frac{c_i}{2}, \frac{c_i}{2} \right[$$

Note that $\widehat{\boldsymbol{\Omega}}_i(\varepsilon)$ increases when ε decreases and converges to $\widehat{\Omega}_i$ when ε tends to 0. In the same way,

$$\mathbf{x} \mapsto \widehat{\mathbf{x}} = \mathbf{x}/\varepsilon \tag{26}$$

maps the domain $\mathbf{J}^\varepsilon(\varepsilon)$ onto $\widehat{\mathbf{J}}(\varepsilon)$, a domain which increases when ε decreases and converges to the unbounded domain:

$$\widehat{\mathbf{J}} = \widehat{J} \cup \bigcup_{i=1}^N \widehat{B}_i, \quad \widehat{B}_i = \{\widehat{\mathbf{x}} \ / \ \widehat{\mathbf{x}} \cdot \mathbf{t}_i \geqslant 0, \ |\widehat{\mathbf{x}} \cdot \mathbf{n}_i| < c_i/2\}$$

2.2   Local expansions and basic equations

We formulate our Ansatz for the asymptotics expansions which consists, in each zone after scaling ((25) or (26)), in looking for power series expansions with respect to ε[1]. In other words, we look for functions

$$u_i^k : \widehat{\Omega}_i \to \mathbb{C} \quad \text{and} \quad U^k : \widehat{\mathbf{J}} \to \mathbb{C}, \quad k \in \mathbb{N},$$

such that, at least formally

$$u^\varepsilon(t, s_i, \varepsilon\widehat{\nu}_i) \quad = \quad \sum_{k=0}^\infty \varepsilon^k u_i^k(t, s_i, \widehat{\nu}_i) + o(\varepsilon^\infty), \quad \text{in } \mathbb{R}_+^* \times \widehat{\boldsymbol{\Omega}}_i(\varepsilon) \tag{27}$$

$$u^\varepsilon(t, \varepsilon\widehat{\mathbf{x}}) \quad = \quad \sum_{k=0}^\infty \varepsilon^k U^k(t, \widehat{\mathbf{x}}) + o(\varepsilon^\infty), \quad \text{in } \mathbb{R}_+^* \times \widehat{\mathbf{J}}(\varepsilon) \tag{28}$$

It remains to obtain the equations that will determine the functions u_i^k and U^k. For the u_i^k's, we substitute formally the expansions (27) in the 2D acoustic

---

[1]This is a particular case. In [5], one can see that the authors look for power series expansions with respect to ε and $\log(\varepsilon)$. These expansions can be even more general.

equation written in $\mathbb{R}_+^* \times \widehat{\boldsymbol{\Omega}}_i(\varepsilon)$, using the scaled coordinates $(s, \widehat{\nu})$, and we identify the terms with the same power of ε. Straightforward manipulations lead to:

$$\frac{\partial^2 u_i^0}{\partial \widehat{\nu}_i^2} = 0, \quad \frac{\partial^2 u_i^1}{\partial \widehat{\nu}_i^2} = 0, \quad \frac{\partial^2 u_i^k}{\partial t^2} - \frac{\partial^2 u_i^k}{\partial s_i^2} - \frac{\partial^2 u_i^{k+2}}{\partial \widehat{\nu}_i^2} = 0, \quad k \geqslant 0 \qquad (29)$$

with the Neumann boundary condition along the "lateral" sides

$$\frac{\partial u_i^k}{\partial \widehat{\nu}_i}\left(t, s_i, -\frac{c_i}{2}\right) = \frac{\partial u_i^k}{\partial \widehat{\nu}_i}\left(t, s_i, \frac{c_i}{2}\right) = 0, \quad t \in \mathbb{R}_+^*, \quad s_i \in]0, +\infty[, \quad k \geqslant 0 \quad (30)$$

From (29) and (30), we deduce that

Proposition 2.1. *For* $1 \leqslant i \leqslant N$, *for* $k \in \mathbb{N}$, *one has*

$$u_i^k(t, s_i, \widehat{\nu}_i) = u_i^k(t, s_i), \quad k \geqslant 0 \qquad (31\text{-i})$$

$$\frac{\partial^2 u_i^k}{\partial t^2} - \frac{\partial^2 u_i^k}{\partial s_i^2} = 0, \quad k \geqslant 0 \qquad (31\text{-ii})$$

Proof. We prove (31-i) and (31-ii) for a given i by induction on k. For $k = 0, 1$, the first two equations of (29) combined with (30) show that u_i^0 and u_i^1 are independent of $\widehat{\nu}$. Then, integrating the third equation of (29) written for $k = 0$ (respectively $k = 1$) with respect to $\widehat{\nu}$ and using the boundary conditions (30) for $k = 2$ (respectively for $k = 3$), we see that u_i^0 (respectively u_i^1) satisfies (31-ii).
Assume that (31-i) and (31-ii) holds up to $k = p$. Then, the third equation of (29) written for $k = p - 1$ combined with (30) show that u_i^{p+1} is independent of $\widehat{\nu}$. Next, integrating the third equation of (29) written for $k = p+1$ with respect to $\widehat{\nu}$ and using the boundary conditions (30) leads to (31-ii) for $k = p + 1$. $\quad\square$

Remark 2.1. Now we may explain at least formally why the second line of hypothesis 1.2 is almost necessary. The proposition 2.1 tell us that each term of the asymptotic expansion (27) does not depend on $\widehat{\nu}_i$. Now assume that there exists a family of function $(f_i^k)_{k \in \mathbb{N}}$ defined on $\widehat{\Omega}_i$ such that

$$f^\varepsilon(s_i, \varepsilon \widehat{\nu}_i) = \sum_{k=0}^\infty \varepsilon^k f_i^k(s_i, \widehat{\nu}_i) + o(\varepsilon^\infty), \quad \text{in } \widehat{\boldsymbol{\Omega}}_i(\varepsilon)$$

We can see that $u_i^k(0, \cdot) = f_i^k$. Since u_i^k does not depend on $\widehat{\nu}$ and is continuous with respect to t, one can deduce that f_i^k should neither depends on $\widehat{\nu}$. In the same way, we get the same result for g_i^k.

Moreover, from Cauchy data in (1), we deduce

$$u_i^k(0, \cdot) = f_i^k, \quad 1 \leqslant i \leqslant N, \quad k \geqslant 0 \qquad (32\text{-i})$$

$$\frac{\partial u_i^k}{\partial t}(0, \cdot) = g_i^k, \quad 1 \leqslant i \leqslant N, \quad k \geqslant 0 \qquad (32\text{-ii})$$

To obtain the equations for the U^k's, we substitute formally the expansion (28) in the 2D acoustic equation written in $\widehat{\mathbf{J}}(\varepsilon)$, using the scaled coordinates $\widehat{\mathbf{x}}$, and

we identify the terms with the same power of ε. This permits us to see that the U^k's satisfy embedded Laplace's equations

$$\Delta U^0 = 0, \quad \Delta U^1 = 0, \quad \Delta U^{k+2} = \frac{\partial^2 U^k}{\partial t^2}, \quad k \in \mathbb{N}, \quad \text{in } \mathbb{R}_+^* \times \widehat{\mathbf{J}} \qquad (33)$$

with Neumann boundary conditions

$$\frac{\partial U^k}{\partial \vec{n}} = 0, \quad \text{on } \partial\widehat{\mathbf{J}}, \quad k \in \mathbb{N} \qquad (34)$$

Remark 2.2. In the sequel, we shall adopt the convention that all quantities super-indexed by k (such as U^k, u_i^k, ...) are 0 for negative values of k. This will be useful to simplify some formulas. For instance, with this convention the last equation of (33) is also valid for $k = 0, 1$.

2.3 Matching conditions

Equations (31) to (34) are not sufficient to characterize the functions (u_i^k, U^k)'s: we miss boundary conditions at $s = 0$ for the u_i^k's and additional conditions at infinity for the U^k's. These conditions, namely the matching conditions, will couple the u_i^k's and the U^k's. To derive them, one writes that the two expansions (27) and (28) must coincide in each overlapping zone $\mathcal{O}_i(\varepsilon)$. Introducing the canonical semi-strip $\mathcal{B}_i$ defined by

$$\mathcal{B}_i = \left\{ \widehat{\mathbf{x}} \in \widehat{B}_i \quad / \quad \widehat{\mathbf{x}} \cdot \mathbf{t}_i > \sigma_i \right\} \quad \text{(see the definition of } \sigma_i \text{ given by 1.4)}$$

and calling U_i^k the restriction of U^k on the i-th canonical semi-strip let us allow to express the following matching conditions:

$$\sum_{k=0}^{+\infty} \varepsilon^k U_i^k(t, s_i/\varepsilon, \nu_i/\varepsilon) + o(\varepsilon^\infty) = \sum_{k=0}^{+\infty} \varepsilon^k u_i^k(t, s_i, \nu_i/\varepsilon) + o(\varepsilon^\infty) \quad \text{in } \mathbb{R}_+^* \times \mathcal{O}_i(\varepsilon)$$
$$(35)$$

To express more precisely these matching conditions, it is useful to describe the form of the functions U_i^k's in the semi-strips $\mathcal{B}_i$: this is the object of the next section.

2.3.1 Modal expansion of solutions of embedded Laplace equations

Let consider, for a given i, $U_i^k : \mathcal{B}_i \to \mathbb{R}$ satisfying

$$\Delta U_i^0 = 0, \quad \Delta U_i^1 = 0, \quad \Delta U_i^{k+2} = \frac{\partial^2 U_i^k}{\partial t^2}, \quad \text{in } \mathcal{B}_i, \quad \frac{\partial U_i^k}{\partial \widehat{\nu}_i} = 0 \text{ for } |\widehat{\nu}_i| = c_i/2 \qquad (36)$$

Later, in section 2.3.2, the results of the present section will be applied to the restriction of the U^k's on $\mathcal{B}_i$, where U^k's are the coefficients of the expansion (28).

In $\mathcal{B}_i$, we shall use the local coordinates $(\widehat{s}_i, \widehat{\nu}_i) = (\widehat{\mathbf{x}} \cdot \mathbf{t}_i, \widehat{\mathbf{x}} \cdot \mathbf{n}_i)$ such that

$$\widehat{\mathbf{x}} \in \mathcal{B}_i \quad \Longleftrightarrow \quad (\widehat{s}_i, \widehat{\nu}_i) \in]\sigma_i, +\infty[\times] - c_i/2, c_i/2[$$

The behaviour of the fields U_i^k in the semi-strips $\mathcal{B}_i$ is easily described by using separation of variables in $(\widehat{s}_i, \widehat{\nu}_i)$ coordinates, which introduce naturally the orthonormal family $\widetilde{w}_{i,p}$ defined by

$$\widetilde{w}_{i,0}(\widehat{\nu}_i) = \sqrt{\frac{1}{c_i}}, \qquad \widetilde{w}_{i,p}(\widehat{\nu}_i) = \sqrt{\frac{2}{c_i}} \cos\left(\frac{p\pi\widehat{\nu}_i}{c_i} + \frac{p\pi}{2}\right) \tag{37}$$

Remark 2.3. One can easily see that

$$\widetilde{w}_{i,p}(\widehat{\nu}_i) = w_{i,p}(\widehat{\nu}_i + c_i/2)$$

where $w_{i,p}$ is the family defined by (10).

The "basis" $\widetilde{w}_{i,p}$ is adapted to the Neumann conditions at $\widehat{\nu}_i = \pm c_i/2$, and there exists 1D functions (in space) $U_{i,p}^k(t, \widehat{s}_i)$ such that

$$U_i^k(t, \widehat{s}_i, \widehat{\nu}_i) = \sum_{p \in \mathbb{N}} U_{i,p}^k(t, \widehat{s}_i) \widetilde{w}_{i,p}(\widehat{\nu}_i) \tag{38}$$

with (thanks to the fact that $\widetilde{w}_{i,p}$ is orthonormal):

$$U_{i,p}^k(t, \widehat{s}_i) = \int_{-c_i/2}^{c_i/2} U_i^k(t, \widehat{s}_i, \widehat{\nu}_i') \widetilde{w}_{i,p}(\widehat{\nu}_i') d\widehat{\nu}_i'$$

If we substitute formally the expression (38) into the equations (36), we obtain

$$\forall k \leqslant 1, \ \forall p \in \mathbb{N}, \quad \frac{\partial^2 U_{i,p}^k}{\partial \widehat{s}_i^2}(t, \widehat{s}_i) - \left(\frac{p\pi}{c_i}\right)^2 U_{i,p}^k(t, \widehat{s}_i) = 0 \tag{39-i}$$

$$\forall k \geqslant 2, \ \forall p \in \mathbb{N}, \quad \frac{\partial^2 U_{i,p}^k}{\partial \widehat{s}_i^2}(t, \widehat{s}_i) - \left(\frac{p\pi}{c_i}\right)^2 U_{i,p}^k(t, \widehat{s}_i) = \frac{\partial^2 U_{i,p}^{k-2}}{\partial t^2}(t, \widehat{s}_i) \tag{39-ii}$$

The resolution of (39) is a tedious but quite simple exercise on ordinary differential equations (the idea is to solve this system with respect to $\widehat{s}_i$ on a given time t fixed). In what follow, we reproduce some results of [5], that we present in a slightly different form, more adapted to the purpose of this report. After having remarked that the change of unknowns

$$U_{i,p}^k(t, \widehat{s}_i) = \exp(\pm \frac{p\pi}{c_i}\widehat{s}_i) V_{i,p,\pm}^k(t, \widehat{s}_i)$$

leads to the equations (with the convention of the remark 2.2)

$$\frac{\partial^2 V_{i,p,\pm}^k}{\partial \widehat{s}_i^2}(t, \widehat{s}_i) \pm 2\frac{p\pi}{c_i}\frac{\partial V_{i,p,\pm}^k}{\partial \widehat{s}_i}(t, \widehat{s}_i) = \frac{\partial^2 V_{i,p,\pm}^{k-2}}{\partial t^2}(t, \widehat{s}_i) \tag{40}$$

One can see, for $k = 0, 1$, that the equation (40) does not depend on time. We can write V^k as sum of products of functions depending on t and functions depending on s_i. The idea is to work with separation of variables and to search $V_{i,p,\pm}^k$, for $k \in \mathbb{N}$, under the form $V_{i,p,\pm}^k(t, \widehat{s}_i) = \phi_{i,p,\pm}^k(\widehat{s}_i)\psi_{i,p,\pm}^k(t)$, and one can see that the family ϕ has to satisfy the following equation (for this part, we don't care about dependance in time for functions $V_{i,p,\pm}^k$):

$$\frac{\partial^2 \phi_{i,p,\pm}^k}{\partial \widehat{s}_i^2}(\widehat{s}_i) \pm 2\frac{p\pi}{c_i}\frac{\partial \phi_{i,p,\pm}^k}{\partial \widehat{s}_i}(\widehat{s}_i) = C_{i,p,\pm}^k \phi_{i,p,\pm}^{k-2}(\widehat{s}_i) \tag{41}$$

Remark 2.4. One can see later (in the proof of lemma 2.3) that is it judicious to take $C^k_{i,p,\pm} = 1$, for each quartet $(i, p, k, \pm)$. In [3] we took $C^k_{i,p,\pm} = -1$, this coefficient being dictated by the coefficient of the right member of embedded Laplace equations.

We introduce, for each $p \in \mathbb{N}$, two sequences of polynomial solutions (in space) of (41)

$$(\mathfrak{n}^k_{i,p}, \mathfrak{d}^k_{i,p}), \quad k \in \mathbb{N},$$

which are defined inductively on k, for each $p \in \mathbb{N}$ and are identically 0 for odd values of k.

- The value $p = 0$ plays a particular role, since equation (41) degenerates (the equation with "$\pm = +$" is the same as the equation with "$\pm = -$"). For $k = 0, 1$ one has

$$\mathfrak{n}^0_{i,0}(\widehat{s}_i) = \widehat{s}_i, \ \mathfrak{n}^1_{i,0}(\widehat{s}_i) = 0, \qquad \mathfrak{d}^0_{i,0}(\widehat{s}_i) = 1, \ \mathfrak{d}^1_{i,0}(\widehat{s}_i) = 0 \qquad (42)$$

continuing for $k \geqslant 2$ with

$$\frac{\partial^2 \mathfrak{n}^k_{i,0}}{\partial \widehat{s}_i^2} = \mathfrak{n}^{k-2}_{i,0}, \qquad \mathfrak{n}^k_{i,0}(0) = \frac{\partial \mathfrak{n}^k_{i,0}}{\partial \widehat{s}_i}(0) = 0 \qquad (43\text{-i})$$

$$\frac{\partial^2 \mathfrak{d}^k_{i,0}}{\partial \widehat{s}_i^2} = \mathfrak{d}^{k-2}_{i,0}, \qquad \mathfrak{d}^k_{i,0}(0) = \frac{\partial \mathfrak{d}^k_{i,0}}{\partial \widehat{s}_i}(0) = 0 \qquad (43\text{-ii})$$

It is easy to see that, for even k, one recovers the monomials of the series expansion of $\exp(\widehat{s}_i)$:

$$\mathfrak{n}^{2m}_{i,0}(\widehat{s}_i) = \frac{\widehat{s}_i^{2m+1}}{(2m+1)!}, \qquad \mathfrak{d}^{2m}_{i,0}(\widehat{s}_i) = \frac{\widehat{s}_i^{2m}}{(2m)!}, \qquad (44)$$

- For $p \geqslant 1$, one starts from

$$\mathfrak{n}^0_{i,p}(\widehat{s}_i) = 1, \ \mathfrak{n}^1_{i,p}(\widehat{s}_i) = 0, \quad \mathfrak{d}^0_{i,p}(\widehat{s}_i) = 1, \ \mathfrak{d}^1_{i,p}(\widehat{s}_i) = 0, \quad \text{for } p \geqslant 1 \quad (45)$$

Then, $(\mathfrak{n}^k_{i,p}, \mathfrak{d}^k_{i,p})$ are defined as the polynomial solutions of

$$\frac{\partial^2 \mathfrak{n}^k_{i,p}}{\partial \widehat{s}_i^2} + \frac{2p\pi}{c_i} \frac{\partial \mathfrak{n}^k_{i,p}}{\partial \widehat{s}_i} = \mathfrak{n}^{k-2}_{i,p}, \qquad \mathfrak{n}^k_{i,p}(0) = 0 \qquad (46\text{-i})$$

$$\frac{\partial^2 \mathfrak{d}^k_{i,p}}{\partial \widehat{s}_i^2} - \frac{2p\pi}{c_i} \frac{\partial \mathfrak{d}^k_{i,p}}{\partial \widehat{s}_i} = \mathfrak{d}^{k-2}_{i,p}, \qquad \mathfrak{d}^k_{i,p}(0) = 0 \qquad (46\text{-ii})$$

Note the difference of sign between (46-i) and (46-ii) for the first derivative term.

The following proposition gives some properties about the functions $(\mathfrak{n}^k_{i,p}, \mathfrak{d}^k_{i,p})$:

Proposition 2.2. *Let $(\mathfrak{n}^k_{i,p}, \mathfrak{d}^k_{i,p})_{k \in \mathbb{N}, p \geqslant 1}$ be a polynomial family of functions satisfying (45) and (46), then*

- *this family is well-defined and is unique,*
- *$(\mathfrak{n}^{2m}_{i,p}, \mathfrak{d}^{2m}_{i,p})$ have degree m,*
- *$\mathfrak{n}^{2m+1}_{i,p} = \mathfrak{d}^{2m+1}_{i,p} = 0$,*

– *we have a relation between* $\mathfrak{n}_{i,p}^{2m}$ *and* $\mathfrak{n}_{i,p}^{2m}$ *given by:*

$$\mathfrak{n}_{i,p}^{2m}(\widehat{s}_i) = \mathfrak{d}_{i,p}^{2m}(-\widehat{s}_i)$$

– *we have a relation between* $\mathfrak{n}_{i,p}^{2m}$ *and* $\mathfrak{n}_{i,1}^{2m}$ *given by:*

$$\mathfrak{n}_{i,p}^{2m}(\widehat{s}_i) = p^{-k}\mathfrak{n}_{i,1}^{2m}(p\,\widehat{s}_i)$$

Proof. The proof of this proposition is very similar to the proof of the proposition 2.3 of [4]. □

Remark 2.5. The fact that the functions $\mathfrak{n}$ and $\mathfrak{d}$ are identically 0 for odd values of k is due to the fact that the equations satisfied by $U_{i,p}^{2m+1}$, for $m \in \mathbb{N}$, are identical to the equations satisfied by $U_{i,p}^{2m}$ (this is a choice of notation that permit us the writing of future lemmas, regardless whether k is odd or even).

Next, we construct two families of functions $\mathbf{n}_{i,p}^k$ and $\mathbf{d}_{i,p}^k$ from $\mathcal{B}_i$ into $\mathbb{R}$, for $(k,p) \in \mathbb{N}^2$, by

$$\mathbf{n}_{i,p}^k(\widehat{\mathbf{x}}) = \exp(p\pi\widehat{s}_i/c_i)\mathfrak{n}_{i,p}^k(\widehat{s}_i)\widetilde{w}_{i,p}(\widehat{\nu}_i) \qquad (47\text{-i})$$

$$\mathbf{d}_{i,p}^k(\widehat{\mathbf{x}}) = \exp(-p\pi\widehat{s}_i/c_i)\mathfrak{d}_{i,p}^k(\widehat{s}_i)\widetilde{w}_{i,p}(\widehat{\nu}_i) \qquad (47\text{-ii})$$

that constitute particular families of embedded Laplace's equations:

$$\forall p \in \mathbb{N}, \quad \forall k \in \mathbb{N}, \quad \Delta\mathbf{n}_{i,p}^{k+2} = \mathbf{n}_{i,p}^k, \quad \Delta\mathbf{d}_{i,p}^{k+2} = \mathbf{d}_{i,p}^k, \quad \text{in } \mathcal{B}_i \qquad (48)$$

with homogeneous Neumann boundary conditions at $\widehat{\nu}_i = \pm c_i/2$.

The families $(\mathbf{n}_{i,p}^k, \mathbf{d}_{i,p}^k)$ allow us to express the following lemma

Lemma 2.3 (Fundamental lemma for the expansion of U_i^k). *Let* $(U_i^k)_{k\in\mathbb{N}}$ *be a family of functions satisfying* (36), *then there exists two sequences* $(\eta_{i,p}^k, \delta_{i,p}^k)_{(k,p)\in\mathbb{N}^2}$ *of complex functions depending only on t such that:*

$$U_i^k(t,\widehat{\mathbf{x}}) = \sum_{p\in\mathbb{N}}\sum_{l=0}^{k}\left[\frac{\partial^l\eta_{i,p}^{k-l}}{\partial t^l}(t)\mathbf{n}_{i,p}^l(\widehat{\mathbf{x}}) + \frac{\partial^l\delta_{i,p}^{k-l}}{\partial t^l}(t)\mathbf{d}_{i,p}^l(\widehat{\mathbf{x}})\right] \quad \text{in } \mathbb{R}_+^* \times \mathcal{B}_i \quad (49)$$

Proof. We will prove the relation (49) by induction on k.

- **Initialization**: for $k = 0, 1$ and for a given $t \in \mathbb{R}_+^*$, one can see that $\Delta U_0^k(t,\cdot)$ vanishes. Hence, there exists two sequences of complex numbers, that we call $\eta_{i,p}^k(t)$ and $\delta_{i,p}^k(t)$, such that

$$U_i^k(t,\widehat{\mathbf{x}}) = \sum_{p\in\mathbb{N}}\left[\eta_{i,p}^k(t)\mathbf{n}_{i,p}^0(\widehat{\mathbf{x}}) + \delta_{i,p}^k(t)\mathbf{d}_{i,p}^0(\widehat{\mathbf{x}})\right] \quad \text{in } \mathbb{R}_+^* \times \mathcal{B}_i \qquad (50)$$

  For $k = 0$, the expressions (50) and (49) are the same ones. For $k = 1$, since $\mathbf{n}_{i,p}^1 = \mathbf{d}_{i,p}^1 = 0$ for all $p \in \mathbb{N}$, one can see also that the expressions (50) and (49) remain the same ones.

- **Heredity**: let us admit that the sequences $(\eta_{i,p}^k, \delta_{i,p}^k)$ have been constructed up to $l = k - 1$. Then let us introduce (remember that $\mathbf{n}_{i,p}^1 = \mathbf{d}_{i,p}^1 = 0$):

$$\widetilde{U}_i^k(t,\widehat{\mathbf{x}}) \quad = \quad \sum_{p\in\mathbb{N}}\sum_{l=1}^{k}\left[\frac{\partial^l \eta_{i,p}^{k-l}}{\partial t^l}(t)\mathbf{n}_{i,p}^l(\widehat{\mathbf{x}}) + \frac{\partial^l \delta_{i,p}^{k-l}}{\partial t^l}(t)\mathbf{d}_{i,p}^l(\widehat{\mathbf{x}})\right]$$

$$= \quad \sum_{p\in\mathbb{N}}\sum_{l=2}^{k}\left[\frac{\partial^l \eta_{i,p}^{k-l}}{\partial t^l}(t)\mathbf{n}_{i,p}^l(\widehat{\mathbf{x}}) + \frac{\partial^l \delta_{i,p}^{k-l}}{\partial t^l}(t)\mathbf{d}_{i,p}^l(\widehat{\mathbf{x}})\right]$$

We have (successively, we use (48), apply the change of index $l \to l-2$, and use (49) for $k-2$):

$$\Delta\widetilde{U}_i^k(t,\widehat{\mathbf{x}}) \quad = \quad \sum_{p\in\mathbb{N}}\sum_{l=2}^{k}\left[\frac{\partial^l \eta_{i,p}^{k-l}}{\partial t^l}(t)\Delta\mathbf{n}_{i,p}^l(\widehat{\mathbf{x}}) + \frac{\partial^l \delta_{i,p}^{k-l}}{\partial t^l}(t)\Delta\mathbf{d}_{i,p}^l(\widehat{\mathbf{x}})\right]$$

$$= \quad \sum_{p\in\mathbb{N}}\sum_{l=2}^{k}\left[\frac{\partial^l \eta_{i,p}^{k-l}}{\partial t^l}(t)\mathbf{n}_{i,p}^{l-2}(\widehat{\mathbf{x}}) + \frac{\partial^l \delta_{i,p}^{k-l}}{\partial t^l}(t)\mathbf{d}_{i,p}^{l-2}(\widehat{\mathbf{x}})\right]$$

$$= \quad \sum_{p\in\mathbb{N}}\sum_{l=0}^{k-2}\left[\frac{\partial^{l+2} \eta_{i,p}^{k-2-l}}{\partial t^{l+2}}(t)\mathbf{n}_{i,p}^{l}(\widehat{\mathbf{x}}) + \frac{\partial^{l+2} \delta_{i,p}^{k-2-l}}{\partial t^{l+2}}(t)\mathbf{d}_{i,p}^{l}(\widehat{\mathbf{x}})\right]$$

$$= \quad \frac{\partial^2 U_i^{k-2}}{\partial t^2}(t,\widehat{\mathbf{x}})$$

The function $U_i^k(t,\cdot) - \widetilde{U}_i^k(t,\cdot)$ being harmonic in $\mathcal{B}_i$, we know there exist two sequences complex numbers that we choose to denote respectively by $(\eta_{i,p}^k(t))_{p\in\mathbb{N}}$ and $(\delta_{i,p}^k(t))_{p\in\mathbb{N}}$, such that

$$U_i^k(t,\widehat{\mathbf{x}}) - \widetilde{U}_i^k(t,\widehat{\mathbf{x}}) = \sum_{p\in\mathbb{N}}\left[\eta_{i,p}^k(t)\mathbf{n}_{i,p}^0(\widehat{\mathbf{x}}) + \delta_{i,p}^k(t)\mathbf{d}_{i,p}^0(\widehat{\mathbf{x}})\right]$$

and the proof is complete. □

From the previous lemma, it is natural to introduce the (increasing) spaces

$$\mathcal{V}_i^k = \mathrm{span}\left\{(\mathbf{n}_{i,p}^l,\mathbf{d}_{i,p}^l),\ p\in\mathbb{N},\ l\leqslant k\right\}, \quad \mathcal{V}_i = \bigcup_{k\in\mathbb{N}}\mathcal{V}_i^k$$

and let us introduce, for the functions $U\in\mathcal{V}_i^0$, the two natural families of linear forms $\mathcal{N}_{i,p}$ and $\mathcal{D}_{i,p}$ defined such that:

$$\forall U\in\mathcal{V}_i^0, \quad U(\widehat{\mathbf{x}}) = \sum_{p=0}^{\infty}\mathcal{N}_{i,p}(U)\mathbf{n}_{i,p}^0(\mathbf{x}) + \mathcal{D}_{i,p}(U)\mathbf{d}_{i,p}^0(\mathbf{x})$$

Remark 2.6. One can see these linear forms as "traces" of the function on $\widehat{s}_i = 0$ (although $\widehat{s}_i = 0$ does not belong to $\mathcal{B}_i$).

Another way to write lemma 2.3 is

Corollary 2.4. *Let $(U_i^k)_{k\in\mathbb{N}}$ be a family of functions satisfying (36), then:*

- *$U_i^k(t,\cdot)\in\mathcal{V}_i^k$*

- *It can be written as*

$$U_i^k(t,\widehat{\mathbf{x}}) = \sum_{p\in\mathbb{N}}\sum_{l=0}^{k}\left[\frac{\partial^l \mathcal{N}_{i,p}(U_i^{k-l}(t,\cdot))}{\partial t^l}(t)\mathbf{n}_{i,p}^l(\widehat{\mathbf{x}}) + \frac{\partial^l \mathcal{D}_{i,p}(U_i^{k-l}(t,\cdot))}{\partial t^l}(t)\mathbf{d}_{i,p}^l(\widehat{\mathbf{x}})\right]$$

2.3.2 Derivation of the matching conditions

To express the matching conditions, we simply write that the two expansions (27) and (28) must coincide in the overlapping zones $\mathcal{O}_i(\varepsilon)$ (see back figure 9 page 16). In such zones, we have the following relations (we recall here (35) by denoting U_i^k the restriction of U^k to $\mathcal{B}_i$)

$$\sum_{k=0}^{+\infty} \varepsilon^k U_i^k(t, s_i/\varepsilon, \nu_i/\varepsilon) + o(\varepsilon^\infty) \;=\; u^\varepsilon(t, s_i, \nu_i) \;=\; \sum_{k=0}^{+\infty} \varepsilon^k u_i^k(t, s_i, \nu_i/\varepsilon) + o(\varepsilon^\infty)$$

for $t \in \mathbb{R}$, $\varphi(\varepsilon) < s_i < 2\varphi(\varepsilon)$ and $|\nu_i| < \varepsilon c_i/2$. We denote $\mathcal{R}$ the right equality of this relation and $\mathcal{L}$ the left one.

Treatment of $\mathcal{R}$ This is the simplest one. One has to consider that each function u_i^k does not depend on the third variable ν_i and the fact that ϕ belongs to **C** defined in (24), which implies that s_i tends to 0 as ε tends to 0. By using a Taylor development on $s_i = 0$ for each function u_i^k, one has

$$u^\varepsilon(t, s_i, \nu_i) = \sum_{k \in \mathbb{N}} \sum_{m \in \mathbb{N}} \left(\frac{1}{m!} \frac{\partial^m u_i^k}{\partial s_i^m}(t, 0) \right) \varepsilon^k s_i^m + o(\varepsilon^\infty) \tag{51}$$

Treatment of $\mathcal{L}$ Here one has more work to do. First, we use the corollary 2.4 of the fundamental lemma to express each U_i^k on the family $(\mathbf{n}_{i,p}^m, \mathbf{d}_{i,p}^m)_{(p,m) \in \mathbb{N}^2}$ defined in section 2.3.1, and we get

$$\begin{aligned} u^\varepsilon(t, s_i, \nu_i) \;=\;& \sum_{k \in \mathbb{N}} \varepsilon^k \sum_{p \in \mathbb{N}} \sum_{m=0}^{k} \left[\frac{\partial^m \mathcal{N}_{i,p}(U^{k-m}(t, \cdot))}{\partial t^m}(t) \mathbf{n}_{i,p}^m \left(\frac{s_i}{\varepsilon}, \frac{\nu_i}{\varepsilon} \right) \right. \\ &+ \left. \frac{\partial^m \mathcal{D}_{i,p}(U^{k-m}(t, \cdot))}{\partial t^m}(t) \mathbf{d}_{i,p}^m \left(\frac{s_i}{\varepsilon}, \frac{\nu_i}{\varepsilon} \right) \right] + o(\varepsilon^\infty) \end{aligned}$$

that can be rewritten as (summing for $m \leqslant k$ is like summing for $k \geqslant m$, for convergent series)

$$\begin{aligned} u^\varepsilon(t, s_i, \nu_i) \;=\;& \sum_{m \in \mathbb{N}} \sum_{k \geqslant m} \sum_{p \in \mathbb{N}} \varepsilon^k \left[\frac{\partial^m \mathcal{N}_{i,p}(U^{k-m}(t, \cdot))}{\partial t^m}(t) \mathbf{n}_{i,p}^m \left(\frac{s_i}{\varepsilon}, \frac{\nu_i}{\varepsilon} \right) \right. \\ &+ \left. \frac{\partial^m \mathcal{D}_{i,p}(U^{k-m}(t, \cdot))}{\partial t^m}(t) \mathbf{d}_{i,p}^m \left(\frac{s_i}{\varepsilon}, \frac{\nu_i}{\varepsilon} \right) \right] + o(\varepsilon^\infty) \end{aligned}$$

Since the function φ belongs to **C**, and since the functions $\mathbf{d}_{i,p}^m$ are exponentially decreasing at $\widehat{s}_i = +\infty$, the corresponding terms in the previous sum can be "put into" the $o(\varepsilon^\infty)$ part. For the rest of the sum, we distinguish the terms for $p = 0$, for which we use the formulas (44) and (47), from the terms corresponding

to $p \geqslant 1$ (which are exponentially increasing at infinity):

$$u^\varepsilon(t, s_i, \nu_i) = \sqrt{\frac{1}{c_i}} \sum_{m \in \mathbb{N}} \sum_{k \geqslant 2m} \varepsilon^k \frac{\partial^{2m} \mathcal{N}_{i,0}(U^{k-2m}(t, \cdot))}{\partial t^{2m}}(t) \frac{1}{(2m+1)!} \left(\frac{s_i}{\varepsilon}\right)^{2m+1}$$

$$+ \sqrt{\frac{1}{c_i}} \sum_{m \in \mathbb{N}} \sum_{k \geqslant 2m} \varepsilon^k \frac{\partial^{2m} \mathcal{D}_{i,0}(U^{k-2m}(t, \cdot))}{\partial t^{2m}}(t) \frac{1}{(2m)!} \left(\frac{s_i}{\varepsilon}\right)^{2m}$$

$$+ \sum_{p \geqslant 1} \sum_{m \in \mathbb{N}} \sum_{k \geqslant m} \frac{\partial^m \mathcal{N}_{i,p}(U^{k-m}(t, \cdot))}{\partial t^m}(t) \mathfrak{n}_{i,p}^m \left(\frac{s_i}{\varepsilon}\right) \exp\left(\frac{p\pi s_i}{\varepsilon c_i}\right) \widetilde{w}_{i,p}\left(\frac{\nu_i}{\varepsilon}\right)$$

Using the change of index $k \to k+2m+1$ in the first line (respectively $k \to k+2m$ in the second line) of the previous equation, one gets

$$u^\varepsilon(t, s_i, \nu_i) = \sqrt{\frac{1}{c_i}} \sum_{m \in \mathbb{N}} \sum_{k \geqslant -1} \frac{\partial^{2m} \mathcal{N}_{i,0}(U^{k+1}(t, \cdot))}{\partial t^{2m}}(t) \frac{1}{(2m+1)!} \varepsilon^k s_i^{2m+1}$$

$$+ \sqrt{\frac{1}{c_i}} \sum_{m \in \mathbb{N}} \sum_{k \in \mathbb{N}} \frac{\partial^{2m} \mathcal{D}_{i,0}(U^k(t, \cdot))}{\partial t^{2m}}(t) \frac{1}{(2m)!} \varepsilon^k s_i^m \qquad (52)$$

$$+ \sum_{p \geqslant 1} \sum_{m \in \mathbb{N}} \sum_{k \geqslant m} \mathcal{S}_{p,m,n}(\varepsilon, s_i, \nu_i)$$

with

$$\mathcal{S}_{p,m,n}(\varepsilon, s_i, \nu_i) = \frac{\partial^m \mathcal{N}_{i,p}(U^{k-m}(t, \cdot))}{\partial t^m}(t) \mathfrak{n}_{i,p}^m \left(\frac{s_i}{\varepsilon}\right) \exp\left(\frac{p\pi s_i}{\varepsilon c_i}\right) \widetilde{w}_{i,p}\left(\frac{\nu_i}{\varepsilon}\right)$$

Conclusion Finally, the formal identification of the expressions (51) and (52) in the overlapping zone $\mathcal{O}_i(\varepsilon)$, as functions of s_i and ε, will lead us to our matching conditions.

First, for $p \geqslant 1$, after multiplication of (51) and (52) by $\widetilde{w}_{i,p}\left(\frac{\nu_i}{\varepsilon}\right)$ and integration over ν_i, we get

$$\sum_{m \in \mathbb{N}} \mathfrak{n}_{i,p}^m \left(\frac{s_i}{\varepsilon}\right) \left[\sum_{k \geqslant m} \frac{\partial^m \mathcal{N}_{i,p}(U^{k-m}(t, \cdot))}{\partial t^m}(t)\right] = 0$$

The proposition 2.2 implies that the functions $\mathfrak{n}_{i,p}^m$ are linearly independent and one deduces from that $\mathcal{N}_{i,p}(U^{k-2m}(t, \cdot)) = 0, \forall m \geqslant 0, \forall k \geqslant 2m$, that is to say:

$$\mathcal{N}_{i,p}(U^k(t, \cdot)) = 0, \quad p \geqslant 1, \quad k \geqslant 0, \quad 1 \leqslant i \leqslant N \qquad (53)$$

which express the absence of exponentially growing terms. We can see that (53) leads to the following condition

$$U^k \text{ grows as most polynomially at infinity in } \widehat{\mathbf{J}} \qquad (54)$$

Next, it remains to identify power series expansions.The identification of the terms in $\varepsilon^k s_i^m$, distinguishing even and odd values of m, leads to

$$\frac{\partial^{2m} \mathcal{D}_{i,0}(U^k(t,\cdot))}{\partial t^{2m}}(t) = \sqrt{c_i}\frac{\partial^{2m} u_i^k}{\partial s_i^{2m}}(t,0)$$

$$\frac{\partial^{2m} \mathcal{N}_{i,0}(U^k(t,\cdot))}{\partial t^{2m}}(t) = \sqrt{c_i}\frac{\partial^{2m+1} u_i^{k-1}}{\partial s_i^{2m+1}}(t,0)$$

Using the fact that each u_i^k solves the $1D$ time wave equation, we have

$$\frac{\partial^{2m} u_i^k}{\partial s_i^{2m}}(t,0) = \frac{\partial^{2m} u_i^k}{\partial t^{2m}}(t,0)$$

and we get the "Dirichlet" and "Neumann" matching conditions, namely:

$$\mathcal{D}_{i,0}(U^k(t,\cdot)) = \sqrt{c_i}u_i^k(t,0), \qquad \forall k \in \mathbb{N}, \quad \forall t \in \mathbb{R}_+^* \qquad \text{(55-i)}$$

$$\mathcal{N}_{i,0}(U^k(t,\cdot)) = \sqrt{c_i}\frac{\partial u_i^{k-1}}{\partial s_i}(t,0), \qquad \forall k \in \mathbb{N}, \quad \forall t \in \mathbb{R}_+^* \qquad \text{(55-ii)}$$

2.4 Summary

Finally, one gets the coupled system of equations and matching conditions, that are (we recall here the equations (31), (32), (33) and (34), and the matching conditions (55)):

- the equations written on the slots zone:

$$u_i^k(t, s_i, \widehat{\nu}_i) = u_i^k(t, s_i), \quad k \geqslant 0$$

$$\frac{\partial^2 u_i^k}{\partial t^2} - \frac{\partial^2 u_i^k}{\partial s_i^2} = 0, \quad k \geqslant 0$$

$$u_i^k(0, \cdot) = f_i^k, \quad 1 \leqslant i \leqslant N, \quad k \geqslant 0$$

$$\frac{\partial u_i^k}{\partial t}(0, \cdot) = g_i^k, \quad 1 \leqslant i \leqslant N, \quad k \geqslant 0$$

- the equations written on the junction zone:

$$\Delta U^0 = 0, \quad \Delta U^1 = 0, \quad \Delta U^{k+2} = \frac{\partial^2 U^k}{\partial t^2}, \quad k \in \mathbb{N}, \quad \text{in } \mathbb{R}_+^* \times \widehat{\mathbf{J}}$$

$$\frac{\partial U^k}{\partial \vec{n}} = 0, \quad \text{on } \partial\widehat{\mathbf{J}}, \quad k \in \mathbb{N}$$

- the matching conditions between the slots zones and the junction zone:

$$\mathcal{D}_{i,0}(U^k(t,\cdot)) = \sqrt{c_i}u_i^k(t,0), \qquad \forall k \in \mathbb{N}, \quad \forall t \in \mathbb{R}_+^*$$

$$\mathcal{N}_{i,0}(U^k(t,\cdot)) = \sqrt{c_i}\frac{\partial u_i^{k-1}}{\partial s_i}(t,0), \qquad \forall k \in \mathbb{N}, \quad \forall t \in \mathbb{R}_+^*$$

3   Justification and error estimates

In this paragraph, our goal is to prove that the functions $(u_i^k)_{k\in\mathbb{N},1\leqslant i\leqslant N}$ and $(U^k)_{k\in\mathbb{N}}$ are uniquely defined, and that there exists an approximate function built from these functions which differs from the solution of the exact problem with some power of ε that is increasing with the order of the approximation we consider.

3.1   Existence and uniqueness of the formal expansion

In this section, our goal is to prove that the equations (31), (32), (33) and (34), together with the matching conditions (55), define unique families $(u_i^k)_{k\in\mathbb{N},1\leqslant i\leqslant N}$ and $(U^k)_{k\in\mathbb{N}}$. To reach our goal, we first formulate an equivalent problem where the unknowns $(U^k)_{k\in\mathbb{N}}$ are restricted to the junction $\widehat{J}$.

3.1.1   Restriction to a bounded domain of the problem for the U^k

Our goal in this section is to characterize the restrictions of the functions U^k's to the junction $\widehat{J}$ by giving exact Dirichlet to Neumann boundary conditions at the interfaces Γ_i (see again figure 5).

Let us start with the corollary 2.4 of the fundamental lemma, applied to the restriction of U^k to the semi-slot $\mathcal{B}_i$, and then apply (53), then we get, separating $l=0$ from $l\geqslant 1$:

$$
\begin{aligned}
U_i^k(t,\widehat{\mathbf{x}}) \quad &= \quad \mathcal{N}_{i,0}(U^k(t,\cdot))\mathbf{n}_{i,0}^0(\widehat{\mathbf{x}}) && + \sum_{p\in\mathbb{N}}\mathcal{D}_{i,p}(U^k(t,\cdot))\mathbf{d}_{i,p}^0(\widehat{\mathbf{x}}) \\
&\quad + \sum_{l=1}^{k}\frac{\partial^l\mathcal{N}_{i,0}(U^{k-l}(t,\cdot))}{\partial t^l}\mathbf{n}_{i,0}^l(\widehat{\mathbf{x}}) && + \sum_{p\in\mathbb{N}}\sum_{l=1}^{k}\frac{\partial^l\mathcal{D}_{i,p}(U^{k-l}(t,\cdot))}{\partial t^l}\mathbf{d}_{i,p}^l(\widehat{\mathbf{x}})
\end{aligned}
$$

Remember that, for our junction zone $\widehat{J}$, the boundary Γ_i is parametrized by

$$
\Gamma_i = \left\{\mathbf{x}\in\partial\widehat{J} \quad / \quad \mathbf{x}\cdot\mathbf{t}_i = \sigma_i \quad\text{and}\quad |\mathbf{x}\cdot\mathbf{n}_i| < \frac{c_i}{2}\right\},
$$

then, in the previous sum, we make the index change $l\to 2l$ (remember also that $\mathbf{n}_{i,p}^l$ and $\mathbf{d}_{i,p}^l$ vanish for odd values of l), and the use of analytically expressions of $\mathbf{n}_{i,p}^l$ and $\mathbf{d}_{i,p}^l$, leads to

$$
\begin{aligned}
U_i^k(t, \sigma_i, \widehat{\nu}_i) \;=\;& \mathcal{N}_{i,0}(U^k(t,\cdot))\frac{\sigma_i}{\sqrt{c_i}} + \mathcal{D}_{i,0}(U^k(t,\cdot))\frac{1}{\sqrt{c_i}} \\[2mm]
&+ \sum_{p \geqslant 1} \mathcal{D}_{i,p}(U^k(t,\cdot)) \exp\left(-\frac{p\pi\sigma_i}{c_i}\right) \widetilde{w}_{p,i}(\widehat{\nu}_i) \\[2mm]
&+ \frac{1}{\sqrt{c_i}} \sum_{l=1}^{k/2} \frac{\partial^{2l}\mathcal{N}_{i,0}(U^{k-2l}(t,\cdot))}{\partial t^{2l}} \frac{\sigma_i^{2l+1}}{(2l+1)!} \\[2mm]
&+ \frac{1}{\sqrt{c_i}} \sum_{l=1}^{k/2} \frac{\partial^{2l}\mathcal{D}_{i,0}(U^{k-2l}(t,\cdot))}{\partial t^{2l}} \frac{\sigma_i^{2l}}{(2l)!} \\[2mm]
&+ \sum_{l=1}^{k/2}\sum_{p \geqslant 1} \frac{\partial^{2l}\mathcal{D}_{i,p}(U^{k-2l}(t,\cdot))}{\partial t^{2l}} \exp\left(-\frac{p\pi\sigma_i}{c_i}\right) \mathfrak{d}_{i,p}^{2l}(\sigma_i)\widetilde{w}_{p,i}(\widehat{\nu}_i)
\end{aligned}
\tag{56}
$$

From (56), since $(\widetilde{w}_{p,i})_{p\in\mathbb{N}}$ is an orthonormal family of $\mathrm{L}^2(\Gamma_i)$, one can see that

$$
\begin{aligned}
\int_{\Gamma_i} U_i^k(t, \sigma_i, \widehat{\nu}_i)\widetilde{w}_{p,i}(\widehat{\nu}_i)d\widehat{\nu}_i =\;& \mathcal{D}_{i,p}(U^k(t,\cdot))e^{\left(-\frac{p\pi\sigma_i}{c_i}\right)} \\[2mm]
&+ \sum_{l=1}^{k/2} \frac{\partial^{2l}\mathcal{D}_{i,p}(U^{k-2l}(t,\cdot))}{\partial t^{2l}}e^{\left(-\frac{p\pi\sigma_i}{c_i}\right)}\mathfrak{d}_{i,p}^{2l}(\sigma_i)
\end{aligned}
\tag{57}
$$

Moreover, on Γ_i, the normal derivative of U^k is the same as the derivative among s_i. After some computations, one has

$$
\begin{aligned}
\frac{\partial U_i^k}{\partial s_i}(t, \sigma_i, \widehat{\nu}_i) \;=\;& \mathcal{N}_{i,0}(U^k(t,\cdot))\frac{1}{\sqrt{c_i}} \\[2mm]
&- \sum_{p \geqslant 1} \frac{p\pi}{c_i}\mathcal{D}_{i,p}(U^k(t,\cdot))e^{\left(-\frac{p\pi\sigma_i}{c_i}\right)}\widetilde{w}_{p,i}(\widehat{\nu}_i) \\[2mm]
&+ \frac{1}{\sqrt{c_i}} \sum_{l=1}^{k/2} \frac{\partial^{2l}\mathcal{N}_{i,0}(U^{k-2l}(t,\cdot))}{\partial t^{2l}} \frac{\sigma_i^{2l}}{(2l)!} \\[2mm]
&+ \frac{1}{\sqrt{c_i}} \sum_{l=1}^{k/2} \frac{\partial^{2l}\mathcal{D}_{i,0}(U^{k-2l}(t,\cdot))}{\partial t^{2l}} \frac{\sigma_i^{2l-1}}{(2l-1)!} \\[2mm]
&- \sum_{l=1}^{k/2}\sum_{p \geqslant 1} \frac{p\pi}{c_i}\frac{\partial^{2l}\mathcal{D}_{i,p}(U^{k-2l}(t,\cdot))}{\partial t^{2l}}e^{\left(-\frac{p\pi\sigma_i}{c_i}\right)}\mathfrak{d}_{i,p}^{2l}(\sigma_i)\widetilde{w}_{p,i}(\widehat{\nu}_i) \\[2mm]
&+ \sum_{l=1}^{k/2}\sum_{p \geqslant 1} \frac{\partial^{2l}\mathcal{D}_{i,p}(U^{k-2l}(t,\cdot))}{\partial t^{2l}}e^{\left(-\frac{p\pi\sigma_i}{c_i}\right)}\frac{\partial\mathfrak{d}_{i,p}^{2l}}{\partial s_i}(\sigma_i)\widetilde{w}_{p,i}(\widehat{\nu}_i)
\end{aligned}
\tag{58}
$$

The most important point is that, using (57), the sum of the second and fifth lines of (58) is equal to

$$
-\sum_{p \geqslant 1} \frac{p\pi}{c_i}\int_{\Gamma_i} U_i^k(t, \sigma_i, \widehat{\nu}_i)\widetilde{w}_{p,i}(\widehat{\nu}_i)d\widehat{\nu}_i
$$

We can now, by using (55-ii) and by using the definition of non-local DtN operator T_i given by (9), sum up the previous information by writing the following relation: on Γ_i, one has

$$
\begin{vmatrix}
\dfrac{\partial U^k}{\partial \vec{n}}(t,\cdot) + T_i U^k(t,\cdot) &=& \dfrac{\partial u_i^{k-1}}{\partial s_i}(t,0) \\[2ex]
&& + \displaystyle\sum_{l=1}^{k/2} \dfrac{\partial^{2l+1} u_i^{k-2l-1}}{\partial t^{2l}\partial s_i}(t,0)\dfrac{\sigma_i^{2l}}{(2l)!} \\[2ex]
&& + \dfrac{1}{\sqrt{c_i}}\displaystyle\sum_{l=1}^{k/2}\dfrac{\partial^{2l}\mathcal{D}_{i,0}(U^{k-2l}(t,\cdot))}{\partial t^{2l}}\dfrac{\sigma_i^{2l-1}}{(2l-1)!} \\[2ex]
&& + \displaystyle\sum_{l=1}^{k/2}\mathcal{S}_{i,k,l}(\sigma_i)
\end{vmatrix}
\tag{59}
$$

with

$$
\mathcal{S}_{i,k,l}(\sigma_i) = \sum_{p\geqslant 1}\frac{\partial^{2l}\mathcal{D}_{i,p}(U^{k-2l}(t,\cdot))}{\partial t^{2l}}\exp\left(-\frac{p\pi\sigma_i}{c_i}\right)\frac{\partial\mathfrak{d}_{i,p}^{2l}}{\partial s_i}(\sigma_i)\widetilde{w}_{p,i}
$$

For the right hand side of (59), one can see that we have two parts:

- one part linked to the knowledge of $U^{k'}$, for $k' < k$,

- one more interesting part linked to the derivative of $u_i^{k-1}(t,s_i)$ at $s_i = 0$.

Finally, we obtain a problem "equivalent" to ((31), (32), (33), (34), (55)) by replacing (55-ii) to the DtN condition (59). The precise statement is the following (denoting $\mathrm{C}_{1,2}(V) = \mathrm{C}^0(\mathbb{R}_+^*, \mathrm{H}^1(V)) \cap \mathrm{C}^1(\mathbb{R}_+^*, \mathrm{L}^2(V))$):

Theorem 3.1. *Let* $\left((u_j^k) \in C_{1,2}(]0,L_j[)\right)_{1\leqslant j\leqslant n}$ *and* $\left(U^k \in C_{1,2,loc}(\widehat{\mathbf{J}})\right)$ *be a solution of* $((31),(32),(33),(34))$ *with the matching conditions* (55)*, then*

$$
\left\{(u_1^k,\ldots,u_n^k,U^k) \in C_{1,2}(]0,L_1[) \times \cdots \times C_{1,2}(]0,L_N[) \times C_{1,2}(\widehat{J}), \quad k\geqslant 0\right\}
$$

is solution of $((31),(32))$*,* $((33),(34))_{\widehat{J}}$*,* (55-i) *and* (59)*, where* $((33),(34))_{\widehat{J}}$ *holds the restriction of* $((33),(34))$ *respectively to* $\widehat{J}$ *and* $\partial\widehat{\mathbf{J}}\cap\partial\widehat{J}$*.*
Reciprocally, if $\left((u_j^k) \in C_{1,2}(]0,L_j[)\right)_{1\leqslant j\leqslant n}$ *and* $\left(U^k \in C_{1,2,loc}(\widehat{\mathbf{J}})\right)$ *are solution of* $((31),(32))$*,* $((33),(34))_{\widehat{J}}$*,* (55-i) *and* (59)*, then by extending* U^k *to each* $\mathcal{B}_i$ *via*

$$
\begin{vmatrix}
U_i^k(t,\widehat{\mathbf{x}}) &=& \sqrt{c_i}\dfrac{\partial u_i^{k-1}}{\partial s_i}(t,0)\mathbf{n}_{i,0}^0(\widehat{\mathbf{x}}) & + \displaystyle\sum_{p\in\mathbb{N}}\mathcal{D}_{i,p}(U^k(t,\cdot))\mathbf{d}_{i,p}^0(\widehat{\mathbf{x}}) \\[2ex]
&& + \sqrt{c_i}\displaystyle\sum_{l=1}^{k}\dfrac{\partial^{l+1}u^{k-l-1}}{\partial t^l\partial s_i}(t,0)\mathbf{n}_{i,0}^l(\widehat{\mathbf{x}}) & + \displaystyle\sum_{p\in\mathbb{N}}\sum_{l=1}^{k}\dfrac{\partial^l\mathcal{D}_{i,p}(U^{k-l}(t,\cdot))}{\partial t^l}\mathbf{d}_{i,p}^l(\widehat{\mathbf{x}})
\end{vmatrix}
$$

Then $\left\{(u_1^k,\ldots,u_n^k,U^k) \in C_{1,2}(]0,L_1[) \times \cdots \times C_{1,2}(]0,L_N[) \times C_{1,2,loc}(\widehat{\mathbf{J}}), \quad k\geqslant 0\right\}$ *is a solution of* $((31),(32),(33),(34))$ *with the matching conditions* (55)

Proof. The direct statement has been proved. For the reciprocal, let us consider $\left\{ (u_1^k, \ldots, u_n^k, U^k) \in \mathrm{C}_{1,2}(]0, L_1[) \times \cdots \times \mathrm{C}_{1,2}(]0, L_N[) \times \mathrm{C}_{1,2,loc}(\widehat{\mathbf{J}}), \quad k \geqslant 0 \right\}$ solution of $((31), (32))$, $((33), (34))_{\widehat{j}}$, (55-i) and (59), and let us extend U^k to each $\mathcal{B}_i$. It is quite easy to prove that U^k satisfies embedded Laplace equations on $\mathcal{B}_i$ (the proof of this point is the same as the proof of the fundamental lemma 2.3). By looking the modal expansion of the corollary and the modal expansion written above, (55-i) is immediate. *A priori*, the most difficult point is to show that Dirichlet and Neumann traces are continuous on Γ_j; however these conditions are satisfied thanks to (59) (this condition has been built to satisfy such things). $\qquad\square$

3.1.2   Auxiliary lemmas

Existence and uniqueness is done by induction on k. According to theorem 3.1, it suffices to consider the problem $((31), (32))$, $((33), (34))_{\widehat{j}}$, (55-i) and (59). To clearly identify the recursion process, it is useful to reformulate this problem in a more decoupled way (we mean between u_i^k and U^k, at each step k), which is also useful from the computational point of view.

To achieve such a decoupling the idea is first to consider (59) as a boundary condition for U^k, next to formulate a $1D$ transmission problem for u_i^k. In this sense, we have to prove first the following two technical lemmas.

Lemma 3.2. *Given $\Phi \in \mathrm{L}^2(\widehat{J})$ and $(g_i \in \mathrm{H}^{-\frac{1}{2}}(\Gamma_i))_{1 \leqslant i \leqslant N}$, there exists $U \in \mathrm{H}^1(\widehat{J})$, which is unique up to an additive constant, such that*

$$
\begin{cases}
\Delta U = \Phi, & in \ \widehat{J}, \qquad \dfrac{\partial U}{\partial \vec{n}} = 0, \quad on \ \partial\widehat{J} \setminus \left(\bigcup_i \Gamma_i \right) \\[2ex]
\dfrac{\partial U}{\partial \vec{n}} + T_i U = g_i, & on \ \Gamma_i
\end{cases}
\tag{60}
$$

if and only if one satisfies the compatibility condition (in a weak sense)

$$
\sum_i \int_{\Gamma_i} g_i = \int_{\widehat{J}} \Phi
\tag{61}
$$

Moreover, W_i being defined by (11) and (12), any solution of (60) satisfies

$$
\frac{1}{c_i} \int_{\Gamma_i} U - \frac{1}{c_{i+1}} \int_{\Gamma_{i+1}} U = \int_{\widehat{J}} W_i \Phi + \sum_{j=1}^{N} \int_{\Gamma_j} W_i g_j
\tag{62}
$$

Proof. The existence-uniqueness proof is a classical exercise about Lax-Milgram lemma and Poincaré-Virtinger's inequality (the important point is that $T_i : \mathrm{H}^{\frac{1}{2}}(\Gamma_i) \to \mathrm{H}^{-\frac{1}{2}}(\Gamma_i)$ is a positive symmetric operator whose kernel contains the space of constant functions - see appendix A.1). The compatibility condition (61) is obtained by integrating the first equation of (60), using Green's formula and the symmetry of T_i (see again appendix A.1).

To obtain (62), we multiply the equation for U by W_i and integrate over $\widehat{J}$. Using Green's formula twice, and the fact that W_i is harmonic, we get

$$\sum_j \int_{\Gamma_j} \frac{\partial W_i}{\partial \vec{n}} U - \frac{\partial U}{\partial \vec{n}} W_i = \int_{\widehat{J}} W_i \Phi$$

Using the boundary conditions on Γ_j for the Neumann traces of U and W_i together with the symmetry of T_i, we obtain (62). $\qquad\square$

Lemma 3.3. *Given* $(\delta_i)_{1 \leqslant i \leqslant N-1} \in \mathrm{H}^1_{loc}(\mathbb{R}^*_+)$ *and* $\alpha \in \mathrm{L}^2_{loc}(\mathbb{R}^*_+)$ *such that* $\delta_i(0) = 0$ *for any* i, *and given* $(f_i)_{1 \leqslant i \leqslant N-1} \in \mathrm{H}^1(]0, +\infty[)$ *and* $(g_i)_{1 \leqslant i \leqslant N-1} \in \mathrm{L}^2(]0, +\infty[)$ *, there exists a unique family* $(u_i)_{1 \leqslant i \leqslant N} \in \mathrm{C}^0(\mathbb{R}^*_+, \mathrm{H}^1(]0, +\infty[) \cap \mathrm{C}^1(\mathbb{R}^*_+, \mathrm{L}^2(]0, +\infty[))$ *such that*

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial t^2} - \dfrac{\partial^2 u_i}{\partial s_i^2} = 0, & in \ \mathbb{R}^*_+ \times]0, +\infty[\\[2mm] u_i(0, \cdot) = f_i, & \dfrac{\partial u_i}{\partial t}(0, \cdot) = g_i, \\[2mm] \displaystyle\sum_{i=1}^{N} c_i \dfrac{\partial u_i}{\partial s_i}(\cdot, 0) = \alpha(\cdot) \\[2mm] u_{i+1}(\cdot, 0) - u_i(\cdot, 0) = \delta_i(\cdot), & \forall 1 \leqslant i \leqslant N-1 \end{cases} \tag{63}$$

Moreover, the norm of the solution is bounded by the norm of the Cauchy data, the norms of the Kirchhoff data and $\sqrt{t}$.

Proof. The idea is to say that solution of problem (63) can be decomposed into three functions:

- the solution (u_i^0) of the following problem:

$$\begin{cases} \dfrac{\partial^2 u_i^0}{\partial t^2} - \dfrac{\partial^2 u_i^0}{\partial s_i^2} = 0, & in \ \mathbb{R}^*_+ \times]0, \infty[\\[2mm] u_i^0(0, \cdot) = f_i, & \dfrac{\partial u_i^0}{\partial t}(0, \cdot) = g_i, \\[2mm] \displaystyle\sum_{i=1}^{N} c_i \dfrac{\partial u_i^0}{\partial s_i}(\cdot, 0) = 0 \\[2mm] u_{i+1}^0(\cdot, 0) - u_i^0(\cdot, 0) = 0, & \forall 1 \leqslant i \leqslant N-1 \end{cases}$$

  This problem admits a unique solution $(u_i^0)_{1 \leqslant i \leqslant N} \in \mathrm{C}^0(\mathbb{R}^*_+, \mathrm{H}^1(]0, +\infty[) \cap \mathrm{C}^1(\mathbb{R}^*_+, \mathrm{L}^2(]0, +\infty[))$ (and one has *a priori* estimates about the solution).

- the solution (u_i^1) of the following problem:

$$\begin{cases} \dfrac{\partial^2 u_i^1}{\partial t^2} - \dfrac{\partial^2 u_i^1}{\partial s_i^2} = 0, & in \ \mathbb{R}^*_+ \times]0, +\infty[\\[2mm] u_i^1(0, \cdot) = 0, & \dfrac{\partial u_i^1}{\partial t}(0, \cdot) = 0, \\[2mm] u_i^1(\cdot, 0) = \displaystyle\sum_{j<i} \delta_j(\cdot), & \forall 1 \leqslant i \leqslant N \end{cases}$$

  The solution of this system is simply written as (extending each $\delta_j(\tau)$ to 0 when $\tau < 0$):

$$u_i^1(t, s_i) = \sum_{j<i} \delta_j(t - s_i)$$

and it satisfies

$$u_{i+1}^1(\cdot, 0) - u_i^1(\cdot, 0) = \delta_i(\cdot)$$

Hypothesis on each δ_j ensure that $(u_i^1)_{1 \leqslant i \leqslant N} \in C^0(\mathbb{R}_+^*, H^1(]0, +\infty[) \cap C^1(\mathbb{R}_+^*, L^2(]0, +\infty[))$. However, we have

$$\sum_{i=1}^N c_i \frac{\partial u_i^1}{\partial s_i}(\cdot, 0) = -\sum_{i=1}^N \sum_{j<i} c_i \delta_j'(\cdot)$$

- the solution (u_i^2) of the following problem:

$$\begin{cases} \dfrac{\partial^2 u_i^2}{\partial t^2} - \dfrac{\partial^2 u_i^2}{\partial s_i^2} = 0, & \text{in } \mathbb{R}_+^* \times]0, +\infty[\\ u_i^2(0, \cdot) = 0, & \dfrac{\partial u_i^2}{\partial t}(0, \cdot) = 0, \\ \displaystyle\sum_{i=1}^N c_i \dfrac{\partial u_i^2}{\partial s_i}(\cdot, 0) = \alpha(\cdot) + \sum_{i=1}^N \sum_{j<i} c_i \delta_j'(\cdot) \\ u_{i+1}^2(\cdot, 0) - u_i^2(\cdot, 0) = 0, & \forall 1 \leqslant i \leqslant N-1 \end{cases}$$

Solution of this equation can be written as follow:

$$u_i^2(t, s_i) = -\frac{1}{\sum c_j} \left(\int_0^{t-s_i} \alpha(\tau) d\tau + \sum_{i=1}^N \sum_{j<i} c_i \delta_j(t-s_i) \right)$$

Under the hypothesis about α and each δ_j, one has $(u_i^2)_{1 \leqslant i \leqslant N} \in C^0(\mathbb{R}_+^*, H^1(]0, +\infty[) \cap C^1(\mathbb{R}_+^*, L^2(]0, +\infty[))$.

Finally, one can check that $u_i = u_i^0 + u_i^1 + u_i^2$ satisfies problem (63), and uniqueness is obtained thanks to uniqueness of the problem with classical Kirchoff conditions (if there exists two solutions of (63), the difference satisfies the wave equation with classical Kirchhoff conditions and null initial conditions).
Now, we compute for a given time $t \in \mathbb{R}_+^*$ the differents norms of the solution of problem (63). One can easily check that

$$\|u_i(t, \cdot)\|_{L^2(]0, +\infty[)} \leqslant \|u_i^0(t, \cdot)\|_{L^2(]0, +\infty[)} + \|u_i^1(t, \cdot)\|_{L^2(]0, +\infty[)} + \|u_i^2(t, \cdot)\|_{L^2(]0, +\infty[)}$$

$$\left\|\frac{\partial u_i}{\partial t}(t, \cdot)\right\|_{L^2(]0, +\infty[)} \leqslant \left\|\frac{\partial u_i^0}{\partial t}(t, \cdot)\right\|_{L^2(]0, +\infty[)} + \left\|\frac{\partial u_i^1}{\partial t}(t, \cdot)\right\|_{L^2(]0, +\infty[)} + \left\|\frac{\partial u_i^2}{\partial t}(t, \cdot)\right\|_{L^2(]0, +\infty[)}$$

$$\left\|\frac{\partial u_i}{\partial s_i}(t, \cdot)\right\|_{L^2(]0, +\infty[)} \leqslant \left\|\frac{\partial u_i^0}{\partial s_i}(t, \cdot)\right\|_{L^2(]0, +\infty[)} + \left\|\frac{\partial u_i^1}{\partial s_i}(t, \cdot)\right\|_{L^2(]0, +\infty[)} + \left\|\frac{\partial u_i^2}{\partial s_i}(t, \cdot)\right\|_{L^2(]0, +\infty[)}$$

By using some classical estimates, we can say that the norm $\|u_i^0(t, \cdot)\|_{L^2(]0, +\infty[)}$ is bounded by square root of order 1 polynom (in t) depending on the Cauchy datas, and the norm of its derivatives is bounded by a constant depending only on Cauchy data. For the functions u_i^1 and u_i^2, by using explicit computation, one has:

- for terms u_i^1:

$$\left\| u_i^1(t, \cdot) \right\|_{L^2(]0,+\infty[)} \leqslant \sum_{j<i} c_i \left\| \delta_j \right\|_{L^2(]0,t[)}$$

$$\left\| \frac{\partial u_i^1}{\partial t}(t, \cdot) \right\|_{L^2(]0,+\infty[)} \leqslant \sum_{j<i} c_i \left\| \delta_j' \right\|_{L^2(]0,t[)}$$

$$\left\| \frac{\partial u_i^1}{\partial s_i}(t, \cdot) \right\|_{L^2(]0,+\infty[)} \leqslant \sum_{j<i} c_i \left\| \delta_j' \right\|_{L^2(]0,t[)}$$

- for terms u_i^2:

$$\left\| u_i^2(t, \cdot) \right\|_{L^2(]0,+\infty[)} \leqslant \frac{1}{\sum c_j} \left(\sqrt{t} \left\| \alpha \right\|_{L^2(]0,t[)} + \sum_{i=1}^{N} \sum_{j<i} c_i \left\| \delta_j \right\|_{L^2(]0,t[)} \right)$$

$$\left\| \frac{\partial u_i^2}{\partial t}(t, \cdot) \right\|_{L^2(]0,+\infty[)} \leqslant \frac{1}{\sum c_j} \left(\left\| \alpha \right\|_{L^2(]0,t[)} + \sum_{i=1}^{N} \sum_{j<i} c_i \left\| \delta_j' \right\|_{L^2(]0,t[)} \right)$$

$$\left\| \frac{\partial u_i^2}{\partial s_i}(t, \cdot) \right\|_{L^2(]0,+\infty[)} \leqslant \frac{1}{\sum c_j} \left(\left\| \alpha \right\|_{L^2(]0,t[)} + \sum_{i=1}^{N} \sum_{j<i} c_i \left\| \delta_j' \right\|_{L^2(]0,t[)} \right)$$

and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.1. In the case where $\alpha \in L^2(]0,+\infty[)$ and $\delta_i \in H^1(][0,+\infty)$, we can see that the L^2 norm of derivatives of solution is uniformly bounded over time.

3.1.3 Existence and uniqueness

According to $((33),(34))_{\hat{\jmath}}$ and (59), we can apply, for a given $t \in \mathbb{R}_+^*$, lemma 3.2 with $U = U^k(t, \cdot)$, $\Phi = \frac{\partial^2 U^{k-2}}{\partial t^2}(t, \cdot)$ and $g_i = g_i^{k-1}(t, \cdot)$ with

$$
\left|
\begin{aligned}
g_i^{k-1}(t, \cdot) \;=\; & \frac{\partial u_i^{k-1}}{\partial s_i}(t, 0) \\[2mm]
& + \sum_{l=1}^{k/2} \frac{\partial^{2l+1} u_i^{k-2l-1}}{\partial t^{2l} \partial s_i}(t, 0) \frac{\sigma_i^{2l}}{(2l)!} \\[2mm]
& + \sum_{l=1}^{k/2} \frac{\partial^{2l} u_i^{k-2l}}{\partial t^{2l}}(t, 0) \frac{\sigma_i^{2l-1}}{(2l-1)!} \\[2mm]
& + \sum_{l=1}^{k/2} \sum_{p \geqslant 1} \frac{\partial^{2l} \mathcal{D}_{i,p}(U^{k-2l}(t, \cdot))}{\partial t^{2l}} e^{\left(-\frac{p\pi\sigma_i}{c_i}\right)} \frac{\partial \mathfrak{d}_{i,p}^{2l}}{\partial s_i}(\sigma_i) \widetilde{w}_{p,i}
\end{aligned}
\right.
\tag{64}
$$

where the index $k-1$ in g_i^{k-1} is "justified" by the fact that g_i^{k-1} is known explicitly when the u_i^{k-1} and the U^m for $m \leqslant k-1$ are known. Writing (61)

gives, once we made the change of index $k = k + 1$,

$$
\left|
\begin{aligned}
\sum_{i=1}^{N} c_i \frac{\partial u_i^k}{\partial s_i}(t,0) &= \int_{\widehat{J}} \frac{\partial^2 U^{k-1}}{\partial t^2} \\
&- \sum_{i=1}^{N} c_i \sum_{l=1}^{(k+1)/2} \frac{\partial^{2l+1} u_i^{k-2l}}{\partial t^{2l} \partial s_i}(t,0) \frac{\sigma_i^{2l}}{(2l)!} \\
&- \sum_{i=1}^{N} c_i \sum_{l=1}^{(k+1)/2} \frac{\partial^{2l} u_i^{k+1-2l}}{\partial t^{2l}}(t,0) \frac{\sigma_i^{2l-1}}{(2l-1)!}
\end{aligned}
\right.
\tag{65}
$$

Remark 3.2. One could see that if we take $(\sigma_i')_{1 \leqslant i \leqslant N}$ such that $\sigma_i' \geqslant \sigma_i$, and if we define $\widehat{J}'$ as

$$
\widehat{J}' = \widehat{J} \cup \bigcup_{i=1}^{N} \left\{ \mathbf{x} \in \mathbb{R}^2 / \ /\sigma_i \leqslant \mathbf{x} \cdot \mathbf{t}_i < \sigma_i' \text{ and } |x \cdot \mathbf{n}_i| < \frac{c_i}{2} \right\},
$$

one should have

$$
\left|
\begin{aligned}
\sum_{i=1}^{N} c_i \frac{\partial u_i^k}{\partial s_i}(t,0) &= \int_{\widehat{J}} \frac{\partial^2 U^{k-1}}{\partial t^2} \\
&- \sum_{i=1}^{N} c_i \sum_{l=1}^{(k+1)/2} \frac{\partial^{2l+1} u_i^{k-2l}}{\partial t^{2l} \partial s_i}(t,0) \frac{\sigma_i^{2l}}{(2l)!} \\
&- \sum_{i=1}^{N} c_i \sum_{l=1}^{(k+1)/2} \frac{\partial^{2l} u_i^{k+1-2l}}{\partial t^{2l}}(t,0) \frac{\sigma_i^{2l-1}}{(2l-1)!}
\end{aligned}
\right.
$$

This point can be proved by using the modal expansion of U^{k-1} on each rectangle $(\widehat{s}_i, \widehat{\nu}_i) \in]\sigma_i, \sigma_i'[\times] - c_i/2, c_i/2[$.

Given $1 \leqslant i \leqslant N$, integrating U^k on Γ_i by using modal expansion (56) and "Dirichlet" condition (55-i) - "Neumann" condition (55-ii) gives

$$
\left|
\begin{aligned}
\frac{1}{c_i} \int_{\Gamma_i} U^k(t,\cdot) &= u_i^k(t,0) + \frac{\partial u_i^{k-1}}{\partial s_i}(t,0)\sigma_i \\
&+ \sum_{l=1}^{k/2} \frac{\partial^{2l+1} u_i^{k-2l-1}}{\partial t^{2l} \partial s_i}(t,0) \frac{\sigma_i^{2l+1}}{(2l+1)!} \\
&+ \sum_{l=1}^{k/2} \frac{\partial^{2l} u_i^{k-2l}}{\partial t^{2l}}(t,0) \frac{\sigma_i^{2l}}{(2l)!}
\end{aligned}
\right.
\tag{66}
$$

Then using (62) with $U = U^k$, $\Phi = \dfrac{\partial^2 U^{k-2}}{\partial t^2}$ and g_i^{k-1} defined in (64), one has, for $1 \leqslant i \leqslant N-1$

$$
\left|
\begin{aligned}
u_i^k(t,0) - u_{i+1}^k(t,0) &= \int_{\widehat{J}} \frac{\partial^2 U^{k-2}}{\partial t^2} W_i + \sum_{j=1}^{n} \int_{\Gamma_j} W_i g_j^{k-1} \\
&\quad + \frac{\partial u_{i+1}^{k-1}}{\partial s_{i+1}}(t,0)\sigma_{i+1} - \frac{\partial u_i^{k-1}}{\partial s_i}(t,0)\sigma_i \\
&\quad + \sum_{l=1}^{k/2} \frac{\partial^{2l+1} u_{i+1}^{k-2l-1}}{\partial t^{2l}\partial s_{i+1}}(t,0) - \sum_{l=1}^{k/2} \frac{\partial^{2l+1} u_i^{k-2l-1}}{\partial t^{2l}\partial s_i}(t,0) \\
&\quad + \sum_{l=1}^{k/2} \frac{\partial^{2l} u_{i+1}^{k-2l}}{\partial t^{2l}}(t,0)\frac{\sigma_{i+1}^{2l}}{(2l)!} - \sum_{l=1}^{k/2} \frac{\partial^{2l} u_i^{k-2l}}{\partial t^{2l}}(t,0)\frac{\sigma_i^{2l}}{(2l)!}
\end{aligned}
\right. \tag{67}
$$

For each k, we have succeeded to decouple the calculation of u_i^k since jump conditions (67) and average condition (65) written for $k = k+1$ are sufficient, when associated to equations (63), to determine u_i^k uniquely (lemma 3.3).

As the solution of problem (60) with $\Phi = \dfrac{\partial^2 U^{k-2}}{\partial t^2}$ and $g_i = g_i^{k-1}$, U^k is defined up to an additive constant. To fix this constant we can use again (55-i) (in a symmetric way with respect to i):

$$
\left|
\begin{aligned}
\sum_{i=1}^{N} \frac{1}{c_i} \int_{\Gamma_i} U^k(t,\cdot) \;=\;& \sum_{i=1}^{N} u_i^k(t,0) + \sum_{i=1}^{N} \frac{\partial u_i^{k-1}}{\partial s_i}(t,0)\sigma_i \\
&+ \sum_{i=1}^{N}\sum_{l=1}^{k/2} \frac{\partial^{2l+1} u_i^{k-2l-1}}{\partial t^{2l}\partial s_i}(t,0)\frac{\sigma_i^{2l+1}}{(2l+1)!} \\
&+ \sum_{i=1}^{N}\sum_{l=1}^{k/2} \frac{\partial^{2l} u_i^{k-2l}}{\partial t^{2l}}(t,0)\frac{\sigma_i^{2l}}{(2l)!}
\end{aligned}
\right. \tag{68}
$$

Finally, we obtain a problem equivalent to $((31),(32))$, $((33),(34))_{\widehat{J}}$, (55-i) and (59) by replacing (55-i) by (65), (67) and (68). More precisely

Theorem 3.4. *The following two propositions are equivalent (for the clarity of notation, we omit to mention again the functional setting):*

(i) $\left\{(u_1^k,\ldots,u_N^k,U^k),\; k \geqslant 0\right\}$ *is solution of* $((31),(32))$, $((33),(34))_{\widehat{J}}$, *(55-i) and* (59)

(ii) $\left\{(u_1^k,\ldots,u_N^k,U^k),\; k \geqslant 0\right\}$ *is solution of* $((31),(32),(65),(67))$ *and* $((33),(34))_{\widehat{J}}$, *(59), (68) with* g_i^{k-1} *defined as (64)*

Proof. We just proved the implication $(i) \Rightarrow (ii)$. We will prove the implication $(ii) \Rightarrow (i)$. Let $\left\{(u_1^k,\ldots,u_N^k,U^k),\; k \geqslant 0\right\}$ be a solution of $((31),(32),(65),(67))$ and $((33),(34))_{\widehat{J}}$, (59), (68) with g_i^{k-1} defined as (64). The only point we have to prove is (55-i) is satisfied. In fact, by taking the modal extension of U^k

on Γ_i, we can rely $\mathcal{D}_{i,0}(U^k(t,\cdot))$ and the integration of U^k over Γ_i. However, (68) gives some information about the sum of the integrals of U^k over the Γ_i's, and (62) gives some information about the difference of the integrals of U^k over two consecutive Γ_i's. By taking some linear combination, we can retrieve the integral of U^k over a given Γ_i, and by identification, (55-i) is satisfied. $\qquad\square$

Next, we show that $((31), (32), (65), (67))$ and $((33),(34))_{\widehat{J}}$, (59), (68), with g_i^{k-1} defined as (64), admits a unique solution $\left\{(u_1^k,\ldots,u_N^k,U^k),\ k\geqslant 0\right\}$, by induction on $k\in\mathbb{N}$.

The case $k=0$. With the convention of 2.2, we see from $((31), (32), (65), (67))$ that u_i^0 is, as expected, the solution of the $1D$ wave equation with classical Kirchhoff conditions at the node (see (6), (7)).

Moreover, we see from $((33),(34))_{\widehat{j}}$, (59) that U^0 solves (60) with $\Phi=0$ and $g_i=0$, which implies that $U^0(t,\cdot)$ is constant. Next, (68) gives

$$U^0(t,\cdot) = \frac{1}{N}\sum_{i=1}^N \frac{1}{c_i}\int_{\Gamma_i} U^0(t,\cdot) = \frac{1}{N}\sum_{i=1}^N u_i^0(t,0) \quad \text{on } \widehat{J}$$

The general case $k\geqslant 1$. Assume that $\left\{(u_1^l,\ldots,u_N^l,U^l),\ l\leqslant k-1\right\}$ are known, then, according to theorem 3.4,

- We compute g_i^{k-1} thanks to (64),

- We determine $(u_i^k)_{1\leqslant i\leqslant N}$ as the unique solution of the $1D$ transmission problem (63) with the transmission conditions (67, 68) (cf. lemma 3.3),

- We determine U^k as the solution, cf. lemma 3.2, of the boundary value problem $((33,34)_{\widehat{j}}, 59, 68)$. One must of course check the compatibility relation (61) (written for $g_i=g_i^{k-1}$ and $\Phi=\frac{\partial^2 U^{k-2}}{\partial t^2}$), which is a consequence of (65).

Finally, regrouping the above results with theorem 3.1 and 3.4, we have proved the following theorem

Theorem 3.5. *There exists a unique family*

$$\left\{(u_1^k,\ldots,u_n^k,U^k)\in C_{1,2}(]0,L_1[)\times\cdots\times C_{1,2}(]0,L_N[)\times C_{1,2,loc}(\widehat{\mathbf{J}}),\quad k\geqslant 0\right\}$$

satisfying (31, 32, 33, 34), the matching conditions (55) and the growth condition (54).

Remark 3.3. In fact, by construction, one can check that, for a given $k\leqslant 1$, the datas α and δ_i of problem (63) depends on $\frac{\partial^2 u^{k-1}}{\partial t^2}$ and $\frac{\partial u^{k-1}}{\partial s}$. By recurrence, one can check that if we want to buid u^k, the real regularity on u^0 must be

$$u_i^0\in C^0(\mathbb{R}_+^*, H^{k+1}(]0,+\infty[)\cap C^1(\mathbb{R}_+^*, H^k(]0,+\infty[))$$

and since the problem satisfied by u^0 is a classical problem, one can check that we should have

$$\begin{aligned} f_i^\varepsilon &\in H^{k+1}(]0,+\infty[) \\ g_i^\varepsilon &\in H^k(]0,+\infty[) \end{aligned}$$

3.2   Error estimates

In this section, we will give some error estimates between the solution of exact problem (1) and a function built from our matched asymptotic expansions (27, 28). We recall here the natural energy (4) associated to the exact problem:

$$\mathcal{E}^\varepsilon(t, u^\varepsilon) = \frac{1}{2\varepsilon} \int_{\Omega^\varepsilon} \left(|\nabla u^\varepsilon(t, \mathbf{x})|^2 + \left| \frac{\partial u^\varepsilon}{\partial t}(t, \mathbf{x}) \right|^2 \right) d\mathbf{x}$$

whose derivative is given by (denoting $\square := \frac{\partial^2}{\partial t^2} - \Delta$):

$$\frac{\partial \mathcal{E}^\varepsilon(t, u^\varepsilon)}{\partial t} = \frac{1}{\varepsilon} \int_{\Omega^\varepsilon} \square u^\varepsilon(t, \mathbf{x}) \frac{\partial u^\varepsilon}{\partial t}(t, \mathbf{x}) d\mathbf{x} + \frac{1}{\varepsilon} \int_{\partial \Omega^\varepsilon} \frac{\partial u^\varepsilon}{\partial \vec{n}}(t, \sigma) \frac{\partial u^\varepsilon}{\partial t}(t, \sigma) d\sigma = 0$$

Then we have $\mathcal{E}^\varepsilon(t, u^\varepsilon) = \mathcal{E}^\varepsilon(0, u^\varepsilon)$. Under the hypothesis 1.2, one can see that, for ε small enough,

$$\mathcal{E}^\varepsilon(0, u^\varepsilon) = \sum_{i=1}^{N} \frac{c_i}{2} \int_0^{+\infty} \left| \frac{\partial f^\varepsilon}{\partial s_i}(t, s_i) \right|^2 + |g^\varepsilon(t, s_i)|^2 \, ds_i$$

This last relation explain the $\frac{1}{\varepsilon}$ multiplicative coefficient in the definition of the energy.

To be able to give error estimates, let us introduce a C^∞ cut-off function χ defined on $\mathbb{R}$ satisfying the three following points:

- $\chi : \mathbb{R} \to [0, 1]$

- $\chi(]-\infty, 1]) = 0$

- $\chi([2, +\infty[) = 1$

From this $1D$ cut-off function, we can define N bi-dimensional cut-off functions $(\chi_i^\varepsilon)_{1 \leqslant \beta \leqslant N}$ defined as

- $\chi_i^\varepsilon(\mathbf{x}) = \chi(\varphi(\varepsilon)^{-1}\mathbf{x} \cdot \mathbf{t}_i)$ when $\mathbf{x} \cdot \mathbf{t}_i > 0$

- $\chi_i^\varepsilon(\mathbf{x}) = 0$ elsewhere

Since $\varphi(\varepsilon)\varepsilon^{-1}$ is continuous and tends to ∞ when ε tends to zero, one can assume that, for $\varepsilon < \varepsilon_0$, $\varepsilon^{-1}\varphi(\varepsilon) > \max \sigma_i$. This ensures that $\chi_i^\varepsilon \chi_j^\varepsilon = 0$, for $\varepsilon < \varepsilon_0$ and $i \neq j$.

Once we introduced our cut-off functions, we can define our approximated solution by

$$
\begin{vmatrix}
\tilde{u}_n^\varepsilon(t, \mathbf{x}) &=& \displaystyle\sum_{i=1}^{N} \chi_i^\varepsilon(\mathbf{x}) \sum_{k=0}^{n} \varepsilon^k u_i^k(t, \mathbf{x} \cdot \mathbf{t}_i) \\
\\
&+& \left(1 - \displaystyle\sum_{i=0}^{N} \chi_i^\varepsilon(\mathbf{x}) \right) \sum_{k=o}^{n} \varepsilon^k U^k(t, \varepsilon^{-1}\mathbf{x})
\end{vmatrix}
\tag{69}
$$

One can easily see that, for ε small enough, and by using some Taylor expansion, we get the following inequality for the intial time estimate:

$$\mathcal{E}^\varepsilon(0, u^\varepsilon - \tilde{u}_n^\varepsilon) \quad = \quad \varepsilon^{n+1} \left(\|f'\|_{L^2(]0, +\infty[)}^2 + \|g\|_{L^2(]0, +\infty[)}^2 \right) + O(\varepsilon^{n+2}) \tag{70}$$

We are first looking for the normal derivative of the approximated function on boundary $\partial\Omega^\varepsilon$: because of the properties of the cut-off funtions, we have

$$\frac{\partial \tilde{u}_n^\varepsilon}{\partial \vec{n}}(t, \mathbf{x}) = 0 \quad \forall \mathbf{x} \in \partial\Omega^\varepsilon$$

then it gives

$$\frac{\partial(\tilde{u}_n^\varepsilon - u^\varepsilon)}{\partial \vec{n}}(t, \mathbf{x}) = 0 \quad \forall \mathbf{x} \in \partial\Omega^\varepsilon$$

On can see, thanks to the Green-Riemann formula, that the derivate of the energy $\mathcal{E}^\varepsilon(\cdot, u^\varepsilon - \tilde{u}_n^\varepsilon)$ with respect to time gives

$$\frac{\partial \mathcal{E}^\varepsilon(\cdot, u^\varepsilon - \tilde{u}_n^\varepsilon)}{\partial t}(t) = \frac{1}{\varepsilon} \int_{\Omega^\varepsilon} \Box(u^\varepsilon - \tilde{u}_n^\varepsilon)(t, \mathbf{x}) \frac{\partial(u^\varepsilon - \tilde{u}_n^\varepsilon)}{\partial t}(t, \mathbf{x}) d\mathbf{x} \qquad (71)$$

Next point is to compute the term $\Box \tilde{u}_n^\varepsilon(t, \mathbf{x})$, one has

$$
\begin{aligned}
\Box(\tilde{u}_n^\varepsilon - u^\varepsilon)(t, \mathbf{x}) \quad &= \quad \Box \tilde{u}_n^\varepsilon(t, \mathbf{x}) \\
&= \quad \sum_{i=1}^N \chi_i^\varepsilon(\mathbf{x}) \sum_{k=0}^n \varepsilon^k \Box u_i^k(t, \mathbf{x}) \\
&+ \quad \left(1 - \sum_{i=1}^N \chi_i^\varepsilon(\mathbf{x})\right) \sum_{k=0}^n \varepsilon^k \Box\left(U^k(\cdot, \varepsilon^{-1}\cdot)\right)(t, \varepsilon^{-1}\mathbf{x}) \\
&+ \quad 2\sum_{i=1}^N \varphi(\varepsilon)^{-1}\chi'\left(\varphi(\varepsilon)^{-1}\mathbf{x}\cdot\mathbf{t}_i\right) \sum_{k=0}^n \left(\varepsilon^k \frac{\partial u_i^k}{\partial s_i}(t, \mathbf{x})\right. \qquad (72) \\
&- \quad \left.\varepsilon^{k-1}\frac{\partial U^k}{\partial \widehat{s}_i}(t, \varepsilon^{-1}\mathbf{x})\right) \\
&+ \quad \sum_{i=1}^N \varphi(\varepsilon)^{-2}\chi''\left(\varphi(\varepsilon)^{-1}\mathbf{x}\cdot\mathbf{t}_i\right) \sum_{k=0}^n \left(\varepsilon^k u_i^k(t, \mathbf{x})\right. \\
&- \quad \left.\varepsilon^k U^k(t, \varepsilon^{-1}\mathbf{x})\right)
\end{aligned}
$$

On the relation (72), we treat separately the four lines:

- The first line, that is the simpliest one. For each function u_i^k, proposition 2.1 ensures that $\Box u_i^k = 0$, the the first line vanishes.

- The second line. Each function U^k does not satisfy $\Box U^k(\cdot, \varepsilon^{-1}\cdot) = 0$. However, thanks to (33), one has

$$\Box U^k(\cdot, \varepsilon^{-1}\cdot) = \frac{\partial^2 U^k}{\partial t^2} - \frac{1}{\varepsilon^2}\frac{\partial^2 U^{k-2}}{\partial t^2}$$

then the second line becomes

$$\left(1 - \sum_{i=1}^N \chi_i^\varepsilon(\mathbf{x})\right)\varepsilon^k \frac{\partial^2 U^k}{\partial t^2}(t, \varepsilon^{-1}\mathbf{x}) + \varepsilon^{k-1}\frac{\partial^2 U^{k-1}}{\partial t^2}(t, \varepsilon^{-1}\mathbf{x}) \qquad (73)$$

- The two groups of last lines (lines 3-4 and lines 5-6), that we can deal with the same manner. The main point is to use the relations (53, 55) associated

to corollary 2.4, and the fact that $\chi'(\varphi(\varepsilon)^{-1}\cdot)$ and $\chi''(\varphi(\varepsilon)^{-1}\cdot)$ do vanish when $\mathbf{x} \cdot \mathbf{t}_i \notin]\varphi(\varepsilon), 2\varphi(\varepsilon)[$. We do the computations with the fourth line (treatment of this line is simplier than treatment of the third line). For U_k, one starts with the modal expansion of corollary 2.4, and let us call Π^k the associated function given by the sum of terms for $p \geqslant 1$. Since $\chi''(\varphi(\varepsilon)^{-1}s_i)$ do vanish for $s_n \leqslant \varphi(\varepsilon)$, Π^k is a finite sum of exponentially decreasing functions and one can see that, given $q \in \mathbb{N}$, that we can bound uniformly Π^k for $0 \leqslant k \leqslant n$ by a constant (depending of the choice of ϕ, q and n) times ε^q. For the function u_i^k, since $\chi''(\varphi(\varepsilon)^{-1}s_i)$ do vanish for $s_i \geqslant 2\varphi(\varepsilon)$, which tends to 0, we simply use a Taylor expansion on $s_n = 0$ with Lagrange remainder at order $n - k$ (the reason is that the l-th derivative coefficient of the expansion appears, thanks to the relations (55), in the modal expansion of the function U^{k+l}): one gets

$$
\left| \begin{aligned}
u_i^k(t, s_i) &= \sum_{l=0}^{n-k} \frac{s_i^l}{l!} \frac{\partial^l u_i^k}{\partial s_i^l}(t, 0) \\
&+ \int_0^{s_i} \frac{1}{(n-k)!} \frac{\partial^{n+1-k} u_i^k}{\partial s_i^{n+1-k}}(t, \sigma)(s_i - \sigma)^{n-k} d\sigma
\end{aligned} \right.
\tag{74}
$$

and by using, as expected, the relations (55), the fifth and sixth lines become

$$
\sum_{i=1}^n \varphi(\varepsilon)^{-2} \chi'' \left(\varphi(\varepsilon)^{-1}\mathbf{x} \cdot \mathbf{t}_i \right) \sum_{k=0}^n \varepsilon^k \mathcal{I}_{i,n,k} + o(\varepsilon^q)
\tag{75}
$$

with

$$
\mathcal{I}_{i,n,k} = \int_0^{\mathbf{x}\cdot\mathbf{t}_i} \frac{1}{(n-k)!} \frac{\partial^{n+1-k} u_i^k}{\partial s_i^{n+1-k}}(t, \sigma)(\mathbf{x} \cdot \mathbf{t}_i - \sigma)^{n-k} d\sigma
$$

and, for $\mathbf{x} \cdot \mathbf{t}_i \leqslant 2\varphi(\varepsilon)$:

$$
|\mathcal{I}_{i,n,k}| \leqslant C_{i,n,k} \varphi(\varepsilon)^{n-k+1}
$$

with $C_{i,n,k}$ which does not depend on ε. In a same way, the third and fourth lines become

$$
\sum_{i=1}^n \varphi(\varepsilon)^{-1} \chi' \left(\varphi(\varepsilon)^{-1}\mathbf{x} \cdot \mathbf{t}_i \right) \left[\varepsilon^k \frac{\partial u_i^k}{\partial s_i}(t, \mathbf{x} \cdot \mathbf{t}_i) \right.
$$
$$
\left. + \sum_{k=0}^{n-1} \varepsilon^k \int_0^{\mathbf{x}\cdot\mathbf{t}_i} \frac{1}{(n-k-1)!} \frac{\partial^{n-k+1} u_i^k}{\partial s_i^{n-k+1}}(t, \sigma)(\mathbf{x} \cdot \mathbf{t}_i - \sigma)^{n-k} d\sigma \right]
\tag{76}
$$

with the same type of increase.

Remark 3.4. One can easily see that the constant use for bound Π^k blows up when $\phi(\varepsilon)$ tends to ε (in sense of functions) . This is due to the fact that the exponentials $\exp(-p\pi s_i/\varepsilon)$ are not small when s_i tends to ε.

Next, we multiply $\Box \tilde{u}_n^\varepsilon(t, \mathbf{x})$ by the derivate of $\tilde{u}_n^\varepsilon$ with respect to time, and we integrate over the space domain Ω^ε, after dividing by ε to get the derivative of the energy (see back (71)). The main point is to use the fact that, for any function $\Psi \in \mathrm{L}^2(\Omega^\varepsilon)$ supported for $\mathbf{x} \cdot \mathbf{t}_i \leqslant 2\varphi(\varepsilon)$ (whose proof is done by

using the Cauchy-Schwartz inequality and doing some calculs similar of those of lemma 3.7 of [4]):

$$\int_{\Omega^\varepsilon} \Psi(\mathbf{x}) \frac{\partial \tilde{u}_n^\varepsilon}{\partial t}(t, \mathbf{x}) \leqslant \|\Psi\|_{\mathrm{L}^2(\Omega^\varepsilon)} \left\| \frac{\partial \tilde{u}_n^\varepsilon}{\partial t}(t, \cdot) \right\|_{\mathrm{L}^2(\Omega^\varepsilon) \cap |\mathbf{x} \cdot \mathbf{t}_i \leqslant 2\varphi(\varepsilon)|} \tag{77}$$

$$\leqslant C \|\Psi\|_{\mathrm{L}^2(\Omega^\varepsilon)} \sqrt{\mathcal{E}^\varepsilon(t, \tilde{u}_n^\varepsilon)}$$

We apply (77) with Ψ given by various terms of (73), (75) and (76), and after some tedious computations, one gets that

Lemma 3.6. *There exist a function M depending on time, n and Cauchy data (f, g) such that, for ε small enough,*

$$\left| \frac{\partial \mathcal{E}^\varepsilon(\cdot, \tilde{u}_n^\varepsilon - u^\varepsilon)}{\partial t}(t) \right| \leqslant \phi(\varepsilon)^{n-\frac{1}{2}} M(t) \sqrt{\mathcal{E}^\varepsilon(t, \tilde{u}_n^\varepsilon - u^\varepsilon)} \tag{78}$$

By integrating the relation (78), and by using error estimate about initial state (70), one has

$$\mathcal{E}^\varepsilon(t, \tilde{u}_n^\varepsilon - u^\varepsilon) \leqslant \mathcal{E}^\varepsilon(0, \tilde{u}_n^\varepsilon - u^\varepsilon) + \phi(\varepsilon)^{n-\frac{1}{2}} \int_0^t M(\tau) \sqrt{\mathcal{E}^\varepsilon(\tau, \tilde{u}_n^\varepsilon)} d\tau \tag{79}$$

To conclude, one has to use the following variant of the Gronwall lemma:

Lemma 3.7 (admitted - see [1] for a complete proof in a more general case)**.** *Let $C > 0$ given, and $\phi(t), m(t)$ be two continuous positive functions defined on $[0, T]$, satisfying*

$$\forall t \in]0, T[, \phi(t) \leqslant C + \int_0^t m(\tau) \sqrt{\phi(\tau)} d\tau$$

then we have

$$\forall t \in]0, T[, \phi(t) \leqslant \left[\sqrt{C} + \frac{1}{2} \int_0^t m(\tau) d\tau \right]^2$$

Using this lemma for and (79) leads to the following formula

$$\mathcal{E}^\varepsilon(t, \tilde{u}_n^\varepsilon - u^\varepsilon) \leqslant \left[\sqrt{\mathcal{E}^\varepsilon(0, \tilde{u}_n^\varepsilon - u^\varepsilon)} + \frac{1}{2} \phi(\varepsilon)^{n-\frac{1}{2}} \int_0^t M(\tau) d\tau \right]^2 \tag{80}$$

Taking (80) with looking at the fact that $\mathcal{E}^\varepsilon(0, \tilde{u}_n^\varepsilon - u^\varepsilon) \leqslant C\varepsilon^{n+1} \leqslant \phi(\varepsilon)^{n-\frac{1}{2}}$ leads to

$$\mathcal{E}^\varepsilon(t, \tilde{u}_n^\varepsilon - u^\varepsilon) \leqslant C(f, g, t) \phi(\varepsilon)^{2n-1} \tag{81}$$

Remark 3.5. This result is underoptimal. In fact, we can see that the term responsible for the behaviour $\phi(\varepsilon)^{2n-1}$ is "located" in the junction zone. The idea is to get some error estimates on the slots zones, far from the junction zone. This is the object of the following corollary

Corollary 3.8. *Given $n \in \mathbb{N}$. Let us assume that the terms of the asymptotic expansion may be built up to term $n + 2$ (this implies that $f_i^\varepsilon \in \mathrm{H}^{n+3}(]0, +\infty[)$ and $g_i^\varepsilon \in \mathrm{H}^{n+2}(]0, +\infty[)$). Given $(\delta_i > 0)_{1 \leqslant i \leqslant N}$, one can build up*

- *the sets* $\Omega_{i,\delta}^{\varepsilon}$ *given as (see back figure 4 page 8):*

$$\Omega_{i,\delta}^{\varepsilon} = \{\mathbf{x} \in \Omega_i^{\varepsilon} \ \text{such that} \ \mathbf{x} \cdot \mathbf{t}_i \geqslant \delta_i\}$$

- *the following 2D function*

$$u_{i,n}^{\varepsilon}(t,\mathbf{x}) = \sum_{n=0}^{N} \varepsilon^n u_i^n(t, \mathbf{x} \cdot \mathbf{t}_i) \quad \text{for} \quad t \in \mathbb{R}_+^* \ \text{and} \ \mathbf{x} \in \Omega_{i,\delta}^{\varepsilon}$$

Then, one has the following error estimate

$$\sum_{i=1}^{N} \frac{1}{\varepsilon} \int_{\Omega_{i,\delta}^{\varepsilon}} \left(\left| \frac{\partial (u^{\varepsilon} - u_{i,n}^{\varepsilon})}{\partial t} \right|^2 + \left| \nabla(u^{\varepsilon} - u_{i,n}^{\varepsilon}) \right|^2 \right) \leqslant \tilde{C}(f,g,t) \varepsilon^{2n+2}$$

Proof. By using inequality $(a-b)^2 \leqslant 2a^2 + 2b^2$, one has

$$
\begin{aligned}
&\sum_{i=1}^{N} \frac{1}{\varepsilon} \int_{\Omega_{i,\delta}^{\varepsilon}} \left(\left| \frac{\partial (u^{\varepsilon} - u_{i,n}^{\varepsilon})}{\partial t} \right|^2 + \left| \nabla(u^{\varepsilon} - u_{i,n}^{\varepsilon}) \right|^2 \right) \\
&\leqslant \sum_{i=1}^{N} \frac{2}{\varepsilon} \int_{\Omega_{i,\delta}^{\varepsilon}} \left(\left| \frac{\partial (u^{\varepsilon} - u_{i,n+2}^{\varepsilon})}{\partial t} \right|^2 + \left| \nabla(u^{\varepsilon} - u_{i,n+2}^{\varepsilon}) \right|^2 \right) \qquad (82) \\
&+ \sum_{i=1}^{N} \frac{2}{\varepsilon} \int_{\Omega_{i,\delta}^{\varepsilon}} \left(\left| \frac{\partial (u_{i,n+2}^{\varepsilon} - u_{i,n}^{\varepsilon})}{\partial t} \right|^2 + \left| \nabla(u_{i,n+2}^{\varepsilon} - u_{i,n}^{\varepsilon}) \right|^2 \right)
\end{aligned}
$$

To treat the first term of the right member of (82), we see that the sums of the differents integrals is bounded by the integral on the whole domain Ω^{ε}, and this integral is no other than the energy (the fact that δ_i is fixed let us allow to choose ε small enough such that $2\phi(\varepsilon) \leqslant \delta_i \forall i$, i.e. the function $\tilde{u}_n^{\varepsilon}$ coincides with $u_{i,n+2}^{\varepsilon}$ on each domain $\Omega_{i,\delta}^{\varepsilon}$). We use then (81) with $n+2$ instead of n. The second term is easier to treat: one can easily see that $u_{i,n+2}^{\varepsilon} - u_{i,n}^{\varepsilon} = \varepsilon^{n+1} u_i^{n+1} + \varepsilon^{n+2} u_i^{n+2}$ with u_i^{n+1} and u_i^{n+2} known: one can bound the energy of $\varepsilon^{n+1} u_i^{n+1} + \varepsilon^{n+2} u_i^{n+2}$ by $C'(f,g,t)\varepsilon^{2n+2}$, for ε small enough. Finally, one has

$$
\begin{aligned}
\sum_{i=1}^{N} \frac{1}{\varepsilon} \int_{\Omega_{i,\delta}^{\varepsilon}} \left(\left| \frac{\partial (u^{\varepsilon} - u_{i,n}^{\varepsilon})}{\partial t} \right|^2 + \left| \nabla(u^{\varepsilon} - u_{i,n}^{\varepsilon}) \right|^2 \right) &\leqslant 2C(f,g,t)\phi(\varepsilon)^{2n+3} \\
&+ 2C'(f,g,t)\varepsilon^{2n+2}
\end{aligned} \qquad (83)
$$

Now, we choose $\phi : \varepsilon \mapsto \varepsilon^{\frac{2n+2}{2n+3}}$. One gets that $\phi(\varepsilon)^{2n+3} = \varepsilon^{2n+2}$ and the proof is complete. $\qquad\square$

Remark 3.6. In this proof, one can check why we need to get two orders further (one order would be not enough to get optimal estimate).

4 Construction of an improved 1D model

4.1 Construction of Improved Kirchhoff condition

In this section, once we proved by the theorem 3.5 that the functions u_i^k and U^k are uniquely defined, we explain how we can get the improved 1D problem (18, 19, 70).

In the following, we shall denote by $\tilde{u}_i^\varepsilon$ the function equal to $u_i^0 + \varepsilon u_i^1$ on the ith slot. We can see easily that

$$
\begin{cases}
\dfrac{\partial^2 \tilde{u}_i^\varepsilon}{\partial t^2} - \dfrac{\partial^2 \tilde{u}_i^\varepsilon}{\partial s_i^2} & = \quad 0 \quad \text{in } \mathbb{R}_+^* \times]\varepsilon\sigma_i, +\infty[\\[2ex]
\tilde{u}_i^\varepsilon & = \quad f^\varepsilon + O(\varepsilon^2) \quad \text{on } \{0\} \times]\varepsilon\sigma_i, +\infty[\\[2ex]
\dfrac{\partial \tilde{u}_i^\varepsilon}{\partial t} & = \quad g^\varepsilon + O(\varepsilon^2) \quad \text{on } \{0\} \times]\varepsilon\sigma_i, +\infty[
\end{cases}
\tag{84}
$$

One can see that, neglecting the $O(\varepsilon^2)$ term and for ε small enough, that $\tilde{u}_i^\varepsilon$ satisfies the same Cauchy data as u^ε. Next point to conclude is to link the values of $(\tilde{u}_i^\varepsilon)_{1 \leqslant i \leqslant N}$ near $s_i = 0$.

Remark 4.1. Ideally, the link of the values of $(\tilde{u}_i^\varepsilon)_{1 \leqslant i \leqslant N}$ would be at $s_i = 0$. However, one can see that associated problem would be ill-posed in terms of energy (for example, with a negative term of the form $1/\varepsilon$).

4.1.1 Average condition

Writing (65), for $k = 0$ and $k = 1$, gives respectively:

$$
\sum_{i=1}^{N} c_i \frac{\partial u_i^0}{\partial s_i}(t, 0) \quad = \quad 0
\tag{85-i}
$$

$$
\sum_{i=1}^{N} c_i \frac{\partial u_i^1}{\partial s_i}(t, 0) \quad = \quad \int_{\widehat{\mathcal{J}}} \frac{\partial^2 U^0}{\partial t^2}(t, \mathbf{x}) d\mathbf{x} - \sum_{i=1}^{N} \sigma_i c_i \frac{\partial^2 u_i^0}{\partial t^2}(t, 0)
\tag{85-ii}
$$

In the proof of theorem 3.4, we show that U^0 is constant and equal to $N^{-1} \sum u_i^0$. Rewritting (85-ii) leads to

$$
\sum_{i=1}^{N} c_i \frac{\partial u_i^1}{\partial s_i}(t, 0) + \sum_{i=1}^{N} \sigma_i c_i \frac{\partial^2 u_i^0}{\partial s_i^2}(t, 0) \quad = \quad \frac{|\widehat{\mathcal{J}}|}{N} \sum_{i=1}^{N} \frac{\partial^2 u_i^0}{\partial t^2}
\tag{86}
$$

Finally, one gets

$$
\begin{aligned}
\sum_{i=1}^{N} c_i \frac{\partial \tilde{u}_i^\varepsilon}{\partial s_i}(t, \varepsilon \sigma_i) &= \sum_{i=1}^{N} c_i \frac{\partial u_i^0}{\partial s_i}(t, 0) + \varepsilon \sum_{i=1}^{N} c_i \sigma_i \frac{\partial^2 u_i^0}{\partial s_i^2}(t, 0) + \varepsilon \sum_{i=1}^{N} c_i \frac{\partial u_i^1}{\partial s_i}(t, 0) + O(\varepsilon^2) \\
&= \varepsilon \int_{\hat{\jmath}} \frac{\partial^2 U^0}{\partial t^2}(t, \mathbf{x}) d\mathbf{x} + O(\varepsilon^2) \\
&= \varepsilon \frac{|\hat{J}|}{N} \sum_{i=1}^{N} \frac{\partial^2 u_i^0}{\partial t^2}(t, 0) + O(\varepsilon^2) \\
&= \varepsilon \frac{|\hat{J}|}{N} \sum_{i=1}^{N} \frac{\partial^2 \tilde{u}_i^\varepsilon}{\partial t^2}(t, \varepsilon \sigma_i) + O(\varepsilon^2)
\end{aligned}
$$

Let us now use notations (14-i) and (14-ii) of section 1.3.3, and let us introduce the vector $\mathbb{1}$ of $\mathbb{R}^N$ equal to $(1, \ldots, 1)^T$. Let us also introduce the canonical scalar product in $\mathbb{R}^N$

$$
\forall A, B \in (\mathbb{R}^N)^2, (A, B) = \sum_{i=1}^{N} A_i B_i
$$

The last line of the previous equation can be rewritten, neglecting the $O(\varepsilon^2)$ term

$$
\frac{\left(\mathcal{C} \partial_S \tilde{U}^\varepsilon(t), \mathbb{1} \right)}{(\mathbb{1}, \mathbb{1})} \mathbb{1} = \varepsilon \mathcal{A} \frac{\partial^2}{\partial t^2} \tilde{U}^\varepsilon(t) \tag{87}
$$

where $\mathcal{C}$ and $\mathcal{A}$ are the matrices introduced in the section 1.3.3.

4.1.2 Jump conditions

We now take the problem (60) satisfied by $U^1(t, \cdot)$ with $\Phi = 0$ and $g_i = \frac{\partial u_i^0}{\partial s_i}$, and difference of (66) for $k = 0, 1$ and $i = j, j+1$ gives

$$
u_j^0(t, 0) - u_{j-1}^0(t, 0) = 0 \tag{88-i}
$$

$$
\begin{aligned}
u_j^1(t, 0) + \sigma_j \frac{\partial u_j^0}{\partial s_j}(t, 0) & \\
-u_{j+1}^0(t, 0) - \sigma_{j+1} \frac{\partial u_{j+1}^0}{\partial s_{j+1}} &= \frac{1}{c_j} \int_{\Gamma_{j+1}} U^1(t, \cdot) - \frac{1}{c_{j+1}} \int_{\Gamma_{j+1}} U^1(t, \cdot) \tag{88-ii}
\end{aligned}
$$

The idea now is to express right member side of (88-ii) with respect to u^0. One can see first that this member does not depend on the additive constant added to U^1. Then, (85-i) gives that, since $\left(c_1 \frac{\partial u_1^0}{\partial s_1}(t, 0), \ldots, c_N \frac{\partial u_N^0}{\partial s_N}(t, 0) \right) \cdot \mathbb{1} = 0$,

$$
\left(c_1 \frac{\partial u_1^0}{\partial s_1}(t, 0), \ldots, c_N \frac{\partial u_N^0}{\partial s_N}(t, 0) \right)^T \in \operatorname{Im} P^T \tag{89}
$$

So there exists a vector $\Phi^0(t) = (\Phi_1^0(t), \ldots, \Phi_{N-1}^0(t)) \in \mathbb{R}^{n-1}$ such that

$$
\left(c_1 \frac{\partial u_1^0}{\partial s_1}(t, 0), \ldots, c_N \frac{\partial u_N^0}{\partial s_N}(t, 0) \right)^T = P^T \Phi^0(t)
$$

By the definition of P^T, one can see, modulo an additive constant, that

$$U^1(t, \cdot) = \sum_{i=1}^{N-1} \Phi_i^0(t) W_i$$

with W_i defined in (11), then one has

$$\frac{1}{c_j} \int_{\Gamma_{j+1}} U^1(t, \cdot) - \frac{1}{c_{j+1}} \int_{\Gamma_{j+1}} U^1(t, \cdot) = \sum_{i=1}^{N-1} K_{i,j} \Phi_i^0(t) \qquad (90)$$

with $K_{i,j}$ defined in (13). Combining (88-i) + ε(88-ii) with (90) leads to (using proposition 1.4 that ensures K is invertible):

$$P\tilde{U}^\varepsilon + O(\varepsilon^2) = \varepsilon K \Phi^0 \implies \varepsilon \Phi^0 = K^{-1} P \tilde{U}^\varepsilon + O(\varepsilon^2) \qquad (91)$$

Looking for (89) make us with to say that $\left(\mathcal{C} \partial_S \tilde{U}^\varepsilon + O(\varepsilon^2) \right) \cdot \mathbb{1}$, and this point is not certain. One better idea is to say that

$$\left(\mathcal{C} \partial_S \tilde{U}^\varepsilon(t) - \frac{\left(\mathcal{C} \partial_S \tilde{U}^\varepsilon(t), \mathbb{1} \right)}{(\mathbb{1}, \mathbb{1})} \mathbb{1} \right) \cdot \mathbb{1} = 0 \qquad (92)$$

Then we have, since $\mathcal{C} \partial_S \tilde{U}^\varepsilon - \frac{(\mathcal{C} \partial_S \tilde{U}^\varepsilon(t), \mathbb{1})}{(\mathbb{1}, \mathbb{1})} \mathbb{1} = \left(c_1 \frac{\partial u_1^0}{\partial s_1}(t, 0), \ldots, c_N \frac{\partial u_N^0}{\partial s_N}(t, 0) \right)^T + O(\varepsilon)$ and P^T is continuous injective:

$$\mathcal{C} \partial_S \tilde{U}^\varepsilon - \frac{\left(\mathcal{C} \partial_S \tilde{U}^\varepsilon(t), \mathbb{1} \right)}{(\mathbb{1}, \mathbb{1})} \mathbb{1} = P^T \left(\Phi^0 + O(\varepsilon) \right)$$

combining with (91) and neglecting the $O(\varepsilon^2)$ term leads to

$$\varepsilon \left\{ \mathcal{C} \partial_S \tilde{U}^\varepsilon - \frac{\left(\mathcal{C} \partial_S \tilde{U}^\varepsilon(t), \mathbb{1} \right)}{(\mathbb{1}, \mathbb{1})} \mathbb{1} \right\} = P^T K^{-1} P \tilde{U}^\varepsilon \qquad (93)$$

Finally, dividing (93) by ε and introducing $\mathcal{J} = P^T K^{-1} P$ leads to

$$\mathcal{C} \partial_S \tilde{U}^\varepsilon - \frac{\left(\mathcal{C} \partial_S \tilde{U}^\varepsilon(t), \mathbb{1} \right)}{(\mathbb{1}, \mathbb{1})} \mathbb{1} = \frac{1}{\varepsilon} \mathcal{J} \tilde{U}^\varepsilon \qquad (94)$$

4.1.3 Conclusion

Finally, adding (94) and (87) leads to the Improved Kirchhoff conditions (19):

$$\mathcal{C} \partial_S \tilde{U}^\varepsilon = \frac{1}{\varepsilon} \mathcal{J} \tilde{U}^\varepsilon + \varepsilon \mathcal{A} \frac{\partial^2}{\partial t^2} \tilde{U}^\varepsilon(t) \qquad (95)$$

4.2 Analysis of Improved Kirchhoff conditions

Here, we have to show that problem (84) with Improved Kirchhoff conditions (19) gives a well-posed problem whose solution differs from the solution of the exact problem (1) by a smaller error than the error between the limit solution and the exact solution. In this way, we show that the the solution of this improved model admits an asymptotic expansion, whose two first terms (and not only the first one) are equal to the two first terms of the matched asymptotic expansion of the slots terms for the solution of the exact problem.

4.2.1   Asymptotic expansions

One starts from the following ansatz:

Ansatz 4.2. For $1 \leqslant i \leqslant N$, there exists a family of functions $(\tilde{u}_i^k)_{k \in \mathbb{N}}$ defined on $\mathbb{R}_+ \times]0, +\infty[$ such that

$$\tilde{u}_i^\varepsilon(t, s_i) = \sum_{k \in \mathbb{N}} \varepsilon^k \tilde{u}_i^k(t, s_i) \quad \text{on} \quad \mathbb{R}_+ \times]\varepsilon \sigma_i, +\infty[\tag{96}$$

The idea is to prove that the error between the solution of the Improved model and its truncated approximation is small, and the two first terms $\tilde{u}_i^0$, $\tilde{u}_i^1$ are the same than the two first terms u_i^0, u_i^1 of the development of exact solution.

Equations   This is the simpliest part. Using ansatz (96) in system (84), separating in powers of ε leads to the fact that $\tilde{u}_i^k$ satisfy the equations (31-ii), (32-i) and (32-ii) on each domain $]\varepsilon \sigma_i, +\infty[$. Tending ε to 0 ensure us that $\tilde{u}_i^k$ satisfy the equations (31-ii), (32-i) and (32-ii) on the same domain as the functions u_i^k.

Matching conditions   One start from the Improved Kirchhoff conditions (19), and do the scalar product with $\mathbb{1}$. Then one gets that

$$\sum_{i=1}^N c_i \frac{\partial \tilde{u}_i^\varepsilon}{\partial s_i}(t, \varepsilon \sigma_i) = \varepsilon \frac{|\widehat{J}|}{N} \sum_{i=1}^N \frac{\partial^2 \tilde{u}_i^\varepsilon}{\partial t^2}(t, \varepsilon \sigma_i) \tag{97}$$

Using ansatz (96) and taking Taylor expansion of each function $\tilde{u}_i^n$ with respect to s_i at the point $s_i = 0$ leads to

$$\sum_{i=1}^N c_i \sum_{n \in \mathbb{N}} \varepsilon^n \left(\sum_{k \in \mathbb{N}} \frac{\varepsilon^k \sigma_i^k}{k!} \frac{\partial^{k+1} \tilde{u}_i^n}{\partial s_i^{k+1}}(t, 0) \right) = \varepsilon \frac{|\widehat{J}|}{N} \sum_{i=1}^N \sum_{n \in \mathbb{N}} \varepsilon^n \left(\sum_{k \in \mathbb{N}} \frac{\varepsilon^k \sigma_i^k}{k!} \frac{\partial^{k+2} \tilde{u}_i^n}{\partial s_i^k t^2}(t, 0) \right)$$

One can see that we have infinite polynomial sum with respect to ε. Taking the terms in ε^0 and ε^1 leads to this two relations

$$\sum_{i=1}^N c_i \frac{\partial \tilde{u}_i^0}{\partial s_i}(t, 0) = 0$$

$$\sum_{i=1}^N c_i \frac{\partial \tilde{u}_i^1}{\partial s_i}(t, 0) + \sum_{i=1}^N \sigma_i c_i \frac{\partial^2 \tilde{u}_i^0}{\partial s_i^2}(t, 0) = \frac{|\widehat{J}|}{N} \sum_{i=1}^N \frac{\partial^2 \tilde{u}_i^0}{\partial t^2}(t, 0)$$

which is nothing but (85-i) and (86).

Dealing with Dirichlet jumps is more technical, however, using some remarks about the Neumann jump will help us. Using again the remark that $\mathbb{1} \in \mathrm{Ker}(\mathcal{J})$, 97 (under its matricial form) multiplied by $\mathbb{1}$ gives

$$\left(\mathbb{1}^T \mathcal{C} \partial_S \tilde{U}^\varepsilon(t) \right) \mathbb{1} = \varepsilon \left(\mathbb{1}^T \mathcal{A} \frac{\partial^2 \tilde{U}^\varepsilon}{\partial t^2}(t) \right) \mathbb{1}$$

One better idea is to say that

$$\frac{\left(\mathcal{C} \partial_S \tilde{U}^\varepsilon(t), \mathbb{1} \right)}{(\mathbb{1}, \mathbb{1})} \mathbb{1} = \varepsilon \frac{\left(\mathcal{A} \frac{\partial^2 \tilde{U}^\varepsilon}{\partial t^2}(t), \mathbb{1} \right)}{(\mathbb{1}, \mathbb{1})} \mathbb{1} \tag{98}$$

and to say the fact that scalar product of $\mathbb{1}$ and any vector which we removed its average part is equal to zero. Then, is exists two functions Φ^ε and Ψ^ε depending on time (and ε) such that

$$\mathcal{C}\partial_S \tilde{U}^\varepsilon(t) - \frac{\left(\mathcal{C}\partial_S \tilde{U}^\varepsilon(t), \mathbb{1}\right)}{(\mathbb{1}, \mathbb{1})}\mathbb{1} \quad = \quad P^T \Phi^\varepsilon(t) \tag{99-i}$$

$$\mathcal{A}\frac{\partial^2 \tilde{U}^\varepsilon}{\partial t^2}(t) - \frac{\left(\mathcal{A}\frac{\partial^2 \tilde{U}^\varepsilon}{\partial t^2}(t), \mathbb{1}\right)}{(\mathbb{1}, \mathbb{1})}\mathbb{1} \quad = \quad P^T \Psi^\varepsilon(t) \tag{99-ii}$$

One can even see that $\Phi^0(t)$ is nothing but the one defined in section 4.1.2 by replacing each u_i^0 by $\tilde{u}_i^0$. Now, taking $\varepsilon(19) - (98)$, using (99-i) and (99-ii), leads to

$$\varepsilon P^T \Phi^\varepsilon(t) = \mathcal{J}\tilde{U}^\varepsilon(t) + \varepsilon^2 P^T \Psi^\varepsilon(t) \tag{100}$$

Now, using the fact that P^T is injective and the definition of $\mathcal{J}$, and using the fact that K is invertible leads to:

$$\varepsilon K \Phi^\varepsilon(t) = P\tilde{U}^\varepsilon(t) + \varepsilon^2 K \Psi^\varepsilon(t) \tag{101}$$

Finally, using ansatz (96) and developments of Φ^ε and Ψ^ε with respect to ε, using Taylor expansion of each function $\tilde{u}_i^n$ with respect to s_i at the point $s_i = 0$ and looking at the terms in ε^0 and ε^1 leads to (88), writing $\tilde{u}_i^0$ and $\tilde{u}_i^1$ instead of u_i^0 and u_i^1 and using (90) for the right member side of (88-ii).

Conclusion One can see first that $(\tilde{u}_i^0)_{1\leqslant i\leqslant N}$ and $(u_i^0)_{1\leqslant i\leqslant N}$ share the same equation, the same Cauchy data and the same jump conditions. Lemma 3.3 ensures that the two families are the same one (this point is normal, since the limit problem does only depend on the topology of the graph). Then, one can see that $(\tilde{u}_i^1)_{1\leqslant i\leqslant N}$ and $(u_i^1)_{1\leqslant i\leqslant N}$ share the same equation, the same Cauchy data and the same jump conditions. Lemma 3.3 ensures again that the two families are the same one (this point is the expected bonus point).

Matching conditions under the general form Let us keep ansatz (96), and for a given k in $\mathbb{N}$, we denote by $\partial_{s^l}\tilde{U}^k(t)$ the vector in $\mathbb{R}^N$ whose i-th coordinate is equal to $\frac{\partial^l \tilde{u}_i^k}{\partial s_i^l}(t, 0)$. Injecting this ansatz in (95) and taking a Taylor development of each vector $\tilde{U}^{k,\varepsilon}$ with respect to each s_i gives, denoting $\mathcal{S}$ the diagonal matrix whose i-th coefficient is equal to σ_i:

$$\begin{aligned}
\frac{1}{\varepsilon}\mathcal{J}\sum_{n\in\mathbb{N}}\varepsilon^n\sum_{k\in\mathbb{N}}\frac{\varepsilon^k}{k!}\mathcal{S}^k\partial_{s^k}\tilde{U}^n(t) & \\
+\varepsilon\mathcal{A}\sum_{n\in\mathbb{N}}\varepsilon^n\sum_{k\in\mathbb{N}}\frac{\varepsilon^k}{k!}\mathcal{S}^k\partial_{t^2 s^k}\tilde{U}^n(t) \quad &= \quad \mathcal{C}\sum_{n\in\mathbb{N}}\varepsilon^n\sum_{k\in\mathbb{N}}\frac{\varepsilon^k}{k!}\mathcal{S}^k\partial_{s^{k+1}}\tilde{U}^n(t)
\end{aligned} \tag{102}$$

Identifying coefficients of (102) with the same power of ε gives:

- power -1 of ε gives

$$\mathcal{J}\tilde{U}^0(t) = 0$$

- power 0 of ε gives

$$\mathcal{J}\left(\mathcal{S}\partial_s \tilde{U}^0(t) + \tilde{U}^1(t)\right) = \mathcal{C}\partial_s \tilde{U}^0(t)$$

- power $n+1$ of ε, with $n \in \mathbb{N}$, gives

$$\mathcal{J}\left(\sum_{k=0}^{n+2} \frac{1}{(n+2-k)!}\mathcal{S}^{n+2-k}\partial_{s^{n+2-k}}\tilde{U}^k(t)\right)$$
$$+\mathcal{A}\left(\sum_{k=0}^{n} \frac{1}{(n-k)!}\mathcal{S}^{n-k}\partial_{s^{n-k}}\frac{\partial^2\tilde{U}^k}{\partial t^2}(t)\right) = \mathcal{C}\left(\sum_{k=0}^{n+1} \frac{1}{(n+1-k)!}\mathcal{S}^{n+1-k}\partial_{s^{n+2-k}}\tilde{U}^k(t)\right)$$

4.2.2 Error estimates

As for section 3.2, we will give some error estimates between the solution of Improved problem (18,19) and its development. Let us recall the 1D energy (20):

$$\widetilde{\mathcal{E}}^\varepsilon(t,v) = \sum_{i=1}^{N} \frac{c_i}{2}\int_{\varepsilon\sigma_i}^{+\infty}\left(\left|\frac{\partial v}{\partial s_i}(t,s')\right|^2 + \left|\frac{\partial v}{\partial t}(t,s')\right|^2\right)ds'$$
$$+ \frac{1}{2\varepsilon}\left(\mathcal{J}V^\varepsilon(t)\right)\cdot V^\varepsilon(t) + \frac{\varepsilon}{2}\left(\mathcal{A}\frac{\partial V^\varepsilon}{\partial t}(t)\right)\cdot\frac{\partial V^\varepsilon}{\partial t}(t)$$

How do we get this energy? Let us consider the first line of (18). Let us multiply by a sequence of functions $(v_i)_{1\leqslant i\leqslant N}$, and integer over $s_i \in]\varepsilon\sigma_i, +\infty[$:

$$0 = \sum_{i=1}^{N} c_i\int_{\varepsilon\sigma_i}^{+\infty}\left(\frac{\partial^2\tilde{u}^\varepsilon}{\partial t^2}(t,s')v(s') - \frac{\partial^2\tilde{u}^\varepsilon}{\partial s_i^2}(t,s')v(s')\right)ds'$$

On the second part of each integral, we use the Green formula, and one gets

$$0 = \sum_{i=1}^{N} c_i\int_{\varepsilon\sigma_i}^{+\infty}\left(\frac{\partial^2\tilde{u}^\varepsilon}{\partial t^2}(t,s')v(s') + \frac{\partial\tilde{u}^\varepsilon}{\partial s_i}(t,s')\frac{\partial v}{\partial s_i}(s')\right)ds'$$
$$+ \left(\mathcal{C}\partial_S\tilde{U}^\varepsilon(t)\right)\cdot V^\varepsilon$$

Finally, using (19) leads to

$$0 = \sum_{i=1}^{N} c_i\int_{\varepsilon\sigma_i}^{+\infty}\left(\frac{\partial^2\tilde{u}^\varepsilon}{\partial t^2}(t,s')v(s') + \frac{\partial\tilde{u}^\varepsilon}{\partial s_i}(t,s')\frac{\partial v}{\partial s_i}(s')\right)ds'$$
$$+ \left(\left(\frac{1}{\varepsilon}\mathcal{J} + \varepsilon\mathcal{A}\frac{\partial^2}{\partial t^2}\right)\tilde{U}^\varepsilon(t)\right)\cdot V^\varepsilon \tag{103}$$

Finally, taking $v = \frac{\partial\tilde{u}^\varepsilon}{\partial t}$ leads to the derivative of the 1D energy (20).
Moreover, one can see that we have, since matrices $\mathcal{J}$ and $\mathcal{A}$ are symmetric:

$$\frac{\partial\mathcal{E}^\varepsilon(t,v)}{\partial t} = \sum_{i=1}^{N} c_i\int_{\varepsilon\sigma_i}^{+\infty}\left(\frac{\partial^2 v}{\partial t\partial s_i}(t,s')\frac{\partial v}{\partial s_i}(t,s') + \frac{\partial^2 v}{\partial t^2}(t,s')\frac{\partial v}{\partial t}(t,s')\right)ds'$$
$$+ \left(\frac{1}{\varepsilon}\mathcal{J}V^\varepsilon(t) + \varepsilon\mathcal{A}\frac{\partial^2 V^\varepsilon}{\partial t^2}(t)\right)\frac{\partial V^\varepsilon}{\partial t}(t)$$

Using the Green-Riemann formula, one gets that

$$
\begin{aligned}
\frac{\partial \mathcal{E}^{\varepsilon}(t,v)}{\partial t} &= \sum_{i=1}^{N} c_i \int_{\varepsilon \sigma_i}^{+\infty} \left(-\frac{\partial v}{\partial t}(t,s') \frac{\partial^2 v}{\partial s_i^2}(t,s') + \frac{\partial^2 v}{\partial t^2}(t,s') \frac{\partial v}{\partial t}(t,s') \right) ds' \\
&\quad + \left(\frac{1}{\varepsilon} \mathcal{J} V^{\varepsilon}(t) + \varepsilon \mathcal{A} \frac{\partial^2 V^{\varepsilon}}{\partial t^2}(t) - \mathcal{C} \partial_S V^{\varepsilon}(t) \right) \frac{\partial V^{\varepsilon}}{\partial t}(t)
\end{aligned}
$$

(104)

Now suppose that the terms of ansatz (96) exist up to $k = n+1$ (this implies that $f \in \mathrm{H}^{n+2}$ and $g \in \mathrm{H}^{n+1}$), and let us build $\tilde{u}^{\varepsilon,n}$ the function defined by

$$
\tilde{u}_i^{\varepsilon,n}(t,s_i) = \sum_{k=0}^{n} \varepsilon^k \tilde{u}_i^k(t,s_i) \quad \text{on} \quad \mathbb{R}_+ \times]\varepsilon \sigma_i, +\infty[
$$

Since each term of the development satisfy time-domain wave equation on each $\mathbb{R}_+ \times]\varepsilon \sigma_i, +\infty[$, the truncated expansion satisfies also this equation. However, for the node condition, one can see that, when developing (102) with powers of ε, the term in ε^n uses $\tilde{U}^{n+1}$. More precisely, there exists a vector $\Phi^{\varepsilon,n}(t)$ bounded with respect of ε, for ε small enough, such that

$$
\frac{1}{\varepsilon} \mathcal{J} \tilde{U}^{\varepsilon,n}(t) + \varepsilon \mathcal{A} \frac{\partial^2 \tilde{U}^{\varepsilon,n}}{\partial t^2}(t) - \mathcal{C} \partial_S \tilde{U}^{\varepsilon,n} = -\varepsilon^n \mathcal{J} \tilde{U}^{n+1}(t) - \varepsilon^{n+1} \Phi^{\varepsilon,n}(t) \quad (105)
$$

Using (105) in (104) gives

$$
\begin{aligned}
\frac{\partial \mathcal{E}^{\varepsilon}(t, \tilde{u}^{\varepsilon} - \tilde{u}^{\varepsilon,n})}{\partial t}(t) &= \varepsilon^n \left(\mathcal{J} \tilde{U}^{n+1}(t) \right) \cdot \frac{\partial(\tilde{U}^{\varepsilon} - \tilde{U}^{\varepsilon,n})}{\partial t}(t) \\
&\quad + \varepsilon^{n+1} \Phi^{\varepsilon,n}(t) \cdot \frac{\partial(\tilde{U}^{\varepsilon} - \tilde{U}^{\varepsilon,n})}{\partial t}(t)
\end{aligned}
$$

(106)

Finally, using that some properties on matrices $\mathcal{J}$ and $\mathcal{A}$ let us allow to express the following result: there exist two vectors $\Phi_{\mathcal{J}}^{\varepsilon,n}$ and $\Phi_{\mathcal{A}}^{\varepsilon,n}$ such that (proof of this result is detailed in appendix B):

$$
\Phi^{\varepsilon,n}(t) = \mathcal{J} \Phi_{\mathcal{J}}^{\varepsilon,n}(t) + \mathcal{A} \Phi_{\mathcal{A}}^{\varepsilon,n}(t)
$$

Thanks to this decomposition, one has

$$
\begin{aligned}
\frac{\partial \mathcal{E}^{\varepsilon}(t, \tilde{u}^{\varepsilon} - \tilde{u}^{\varepsilon,n})}{\partial t}(t) &= \varepsilon^n \left(\tilde{U}^{n+1}(t) + \varepsilon \Phi_{\mathcal{J}}^{\varepsilon,n}(t) \right)^T \mathcal{J} \frac{\partial(\tilde{U}^{\varepsilon} - \tilde{U}^{\varepsilon,n})}{\partial t}(t) \\
&\quad + \varepsilon^{n+1} (\Phi_{\mathcal{A}}^{\varepsilon,n}(t))^T \mathcal{A} \frac{\partial(\tilde{U}^{\varepsilon} - \tilde{U}^{\varepsilon,n})}{\partial t}(t)
\end{aligned}
$$

(107)

Finally, integrating (107) between 0 and T leads to (thanks to the fact that $\mathcal{E}^{\varepsilon}(0, \tilde{u}^{\varepsilon} - \tilde{u}^{\varepsilon,n}) = 0$):

$$
\begin{aligned}
\mathcal{E}^{\varepsilon}(T, \tilde{u}^{\varepsilon} - \tilde{u}^{\varepsilon,n}) &= \int_0^T \varepsilon^n \left(\tilde{U}^{n+1}(t) + \varepsilon \Phi_{\mathcal{J}}^{\varepsilon,n}(t) \right)^T \mathcal{J} \frac{\partial(\tilde{U}^{\varepsilon} - \tilde{U}^{\varepsilon,n})}{\partial t}(t) dt \\
&\quad + \int_0^T \varepsilon^{n+1} (\Phi_{\mathcal{A}}^{\varepsilon,n}(t))^T \mathcal{A} \frac{\partial(\tilde{U}^{\varepsilon} - \tilde{U}^{\varepsilon,n})}{\partial t}(t)
\end{aligned}
$$

(108)

Using a Green-Riemann formula on the first part of right member in (108) leads to

$$\mathcal{E}^\varepsilon(T, \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon,n}) = \varepsilon^n \left(\tilde{U}^{n+1}(t) + \varepsilon \Phi_{\mathcal{J}}^{\varepsilon,n}(t) \right)^T \mathcal{J} \left(\tilde{U}^\varepsilon(t) - \tilde{U}^{\varepsilon,n}(t) \right)$$

$$- \int_0^T \varepsilon^n \left(\frac{\partial(\tilde{U}^{n+1} + \varepsilon \Phi_{\mathcal{J}}^{\varepsilon,n})}{\partial t}(t) \right)^T \mathcal{J} \left(\tilde{U}^\varepsilon(t) - \tilde{U}^{\varepsilon,n}(t) \right) dt$$

$$+ \int_0^T \varepsilon^{n+1} \left(\Phi_{\mathcal{A}}^{\varepsilon,n}(t) \right)^T \mathcal{A} \frac{\partial(\tilde{U}^\varepsilon - \tilde{U}^{\varepsilon,n})}{\partial t}(t) dt$$

$$(109)$$

By using some Cauchy-Schwartz inequality (allowed even if $\mathcal{J}$ and $\mathcal{A}$ are not invertible), one gets that, for any vector $V \in \mathbb{R}^N$

$$V^T \mathcal{J} \left(\tilde{U}^\varepsilon(t) - \tilde{U}^{\varepsilon,n}(t) \right) \leqslant \varepsilon^{1/2} \left(V^T \mathcal{J} V \right)^{1/2} \sqrt{\mathcal{E}^\varepsilon(t, \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon,n})(t)}$$

$$V^T \mathcal{A} \frac{\partial(\tilde{U}^\varepsilon - \tilde{U}^{\varepsilon,n})}{\partial t}(t) \leqslant \varepsilon^{-1/2} \left(V^T \mathcal{A} V \right)^{1/2} \sqrt{\mathcal{E}^\varepsilon(t, \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon,n})(t)}$$

$$(110)$$

Then, there exists two functions $C_1^\varepsilon(t)$ and $C_2^\varepsilon(t)$ bounded with respect to ε near 0 such that

$$\mathcal{E}^\varepsilon(T, \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon,n}) \leqslant \varepsilon^{n+\frac{1}{2}} C_1^\varepsilon(T) \sqrt{\mathcal{E}^\varepsilon(T, \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon,n})}$$

$$+ \varepsilon^{n+\frac{1}{2}} \int_0^T C_2^\varepsilon(t) \sqrt{\mathcal{E}^\varepsilon(t, \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon,n})} dt$$

$$(111)$$

with

$$C_1^\varepsilon(t) = \sqrt{\left(\tilde{U}^{n+1}(t) + \varepsilon \Phi_{\mathcal{J}}^{\varepsilon,n}(t) \right)^T \mathcal{J} \left(\tilde{U}^{n+1}(t) + \varepsilon \Phi_{\mathcal{J}}^{\varepsilon,n}(t) \right)}$$

$$C_2^\varepsilon(t) = \sqrt{\left(\frac{\partial(\tilde{U}^{n+1} + \varepsilon \Phi_{\mathcal{J}}^{\varepsilon,n})}{\partial t}(t) \right)^T \mathcal{J} \left(\frac{\partial(\tilde{U}^{n+1} + \varepsilon \Phi_{\mathcal{J}}^{\varepsilon,n})}{\partial t}(t) \right)} \qquad (112)$$

$$+ \sqrt{\left(\Phi_{\mathcal{A}}^{\varepsilon,n}(t) \right)^T \mathcal{A} \left(\Phi_{\mathcal{A}}^{\varepsilon,n}(t) \right)}$$

using that $2ab \leqslant a^2 + b^2$ let us allow to write that

$$\mathcal{E}^\varepsilon(T, \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon,n}) \leqslant \frac{1}{2} \varepsilon^{2n+1} (C_1^\varepsilon(T))^2 + \frac{1}{2} \mathcal{E}^\varepsilon(T, \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon,n})$$

$$+ \varepsilon^{n+\frac{1}{2}} \int_0^T C_2^\varepsilon(t) \sqrt{\mathcal{E}^\varepsilon(t, \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon,n})} dt$$

which gives

$$\mathcal{E}^\varepsilon(t, \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon,n})(T) \leqslant \varepsilon^{2n+1} (C_1^\varepsilon(T))^2$$

$$+ 2 \varepsilon^{n+\frac{1}{2}} \int_0^T C_2^\varepsilon(t) \sqrt{\mathcal{E}^\varepsilon(t, \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon,n})} dt$$

$$(113)$$

using back lemma 3.7 gives that that, for any $t \in]0, T[$:

$$\mathcal{E}^\varepsilon(t, \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon,n})(t) \leqslant \varepsilon^{2n+1} \left(\sup_{\tau \in]0,t[} C_1^\varepsilon(\tau) + \int_0^t C_2^\varepsilon(\tau) d\tau \right)$$

Finally, using a triangular inequality (as for corollary 3.8) let us allows to express the following error estimate:

Corollary 4.1. *Given $n \in \mathbb{N}$. Let us assume that the terms of the asymptotic expansion may be built up to term $n + 2$ (this implies that $f_i^\varepsilon \in \mathrm{H}^{n+3}(]0, +\infty[)$ and $g_i^\varepsilon \in \mathrm{H}^{n+2}(]0, +\infty[))$. Given $(\delta_i > 0)_{1 \leqslant i \leqslant N}$, one defines the function $\tilde{u}^{\varepsilon, n}$ by*

$$\tilde{u}_i^{\varepsilon, n}(t, s_i) = \sum_{k=0}^{n} \varepsilon^k \tilde{u}_i^k(t, s_i) \quad on \quad \mathbb{R}_+ \times]\varepsilon\sigma_i, +\infty[$$

Then, there exists a constant $C(f, g, t)$ depending on time and Cauchy data such that, for ε small enough,

$$\sum_{i=1}^{N} \int_{\delta_i}^{\infty} \left| \frac{\partial \tilde{u}^\varepsilon - \tilde{u}^{\varepsilon, n}}{\partial t}(t, s_i) \right|^2 + |\nabla(\tilde{u}^\varepsilon - \tilde{u}^{\varepsilon, n})(t, s_i)|^2 \, ds_i \leqslant C(f, g, t)\varepsilon^{2n+2}$$

Remark 4.3. One thing that can be remark is that we need the same regularity for both theorems 3.8 and 4.1.

Conclusion

We proved in this report that we are able to give conditions at the "node" of our 1D graph that are more precise that standard Kirchhoff conditions (with respect to the exact solution). One can see that results obtained are a generalization of those obtained in [3], directly in time domain case.

One could say that no numerical results are presented here. Indeed, we get the same results as those observed in [4] for the time domain case, and there exists some works where we detail the numerical resolution of these type of problems on a more general geometry (with more than one junction, see [9]).

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A Some properties about the auxiliary problems

A.1 Some properties about the DtN map

Here, we prove some properties about the DtN maps T_i we used. We first recall the definition of T_i

$$T_i : \varphi \mapsto T_i\varphi = \sum_{p=1}^{\infty} \frac{p\pi}{c_i} \left(\int_0^{c_i} \varphi(\widehat{\nu}')w_{i,p}(\widehat{\nu}')d\widehat{\nu}' \right) w_{i,p}$$

Proposition A.1 (recall of the proposition 1.2). *T_i is a symmetric positive-average continuous operator from $\mathrm{H}^{\frac{1}{2}}(]0, c_i[)$ to $\mathrm{H}^{-\frac{1}{2}}(]0, c_i[)$ vanishing for constant functions.*

Proof. We have several points to prove. To prove the continuity from $\mathrm{H}^{\frac{1}{2}}(]0, c_i[)$ to $\mathrm{H}^{-\frac{1}{2}}(]0, c_i[)$, let us take $\varphi \in \mathrm{H}^{\frac{1}{2}}(]0, c_i[)$, and let us compute $< T_i\varphi, \psi >$ for any test function $\psi \in \mathrm{H}^{\frac{1}{2}}(]0, c_i[)$. One has

$$< T_i\varphi, \psi > \quad = \quad \sum_{p\in\mathbb{N}^*} \frac{p\pi}{c_i} \left(\int_0^{c_i} \varphi(\widehat{\nu}')w_{i,p}(\widehat{\nu}')d\widehat{\nu}' \right) \left(\int_0^{c_i} \psi(\widehat{\nu}')w_{i,p}(\widehat{\nu}')d\widehat{\nu}' \right)$$

By using some trivial inequalities, one has

$$| < T_i\varphi, \psi > | \quad \leqslant \quad \sum_{p\in\mathbb{N}^*} \frac{\sqrt{1+p^2}\pi}{c_i} \left| \int_0^{c_i} \varphi(\widehat{\nu}')w_{i,p}(\widehat{\nu}')d\widehat{\nu}' \right| \left| \int_0^{c_i} \psi(\widehat{\nu}')w_{i,p}(\widehat{\nu}')d\widehat{\nu}' \right|$$

Then we use the Cauchy-Schwartz inequality, to obtain

$$| < T_i\varphi, \psi > | \quad \leqslant \quad \frac{\pi}{c_i} \left(\sum_{p\in\mathbb{N}} \sqrt{1+p^2} \left| \int_0^{c_i} \varphi(\widehat{\nu}')w_{i,p}(\widehat{\nu}')d\widehat{\nu}' \right|^2 \right)^{1/2}$$

$$\left(\sum_{p\in\mathbb{N}} \sqrt{1+p^2} \left| \int_0^{c_i} \psi(\widehat{\nu}')w_{i,p}(\widehat{\nu}')d\widehat{\nu}' \right|^2 \right)^{1/2}$$

We can recognize product of parts of the $\mathrm{H}^{\frac{1}{2}}(]0, c_i[)$ of functions φ and ψ (see [4] for more details). This closes proof of the continuity.
If we look back the first relation of this proof, which is

$$< T_i\varphi, \psi > \quad = \quad \sum_{p\in\mathbb{N}^*} \frac{p\pi}{c_i} \left(\int_0^{c_i} \varphi(\widehat{\nu}')w_{i,p}(\widehat{\nu}')d\widehat{\nu}' \right) \left(\int_0^{c_i} \psi(\widehat{\nu}')w_{i,p}(\widehat{\nu}')d\widehat{\nu}' \right)$$

we can see that the right member is symmetric (one can invert φ and ψ), so we have

$$< T_i\varphi, \psi > = < T_i\psi, \varphi >$$

Moreover, taking $\psi = \varphi$ shows that the right member of the first relation is positive, which gives $< T_i\varphi, \varphi > \geqslant 0$. Finally, taking $T_i = w_{i,0}$ leads to

$$T_i w_{i,0} = \sum_{p\in\mathbb{N}^*} \frac{p\pi}{c_i} \left(\int_0^{c_i} w_{i,0}(\widehat{\nu}')w_{i,p}(\widehat{\nu}')d\widehat{\nu}' \right) w_{i,p} = 0 \qquad \square$$

A.2   Existence and uniqueness of functions W_i

Here, we prove some properties about the functions W_i we claimed in section 1.3.2.

Proposition A.2 (recall of the proposition 1.3). *There exists a unique W_i in* $\mathrm{H}^1(\widehat{J})$ *satisfying* (11) *and* (12)

Proof. The idea is to use the Lax-Milgram theorem on the space

$$\mathcal{V} = \left\{ V \in \mathrm{H}^1(\widehat{J}) \quad / \quad \int_{\widehat{J}} V = 0 \right\}$$

On this space, the H^1 semi-norm is equivalent to the H^1 norm, thanks to the Poincare's inequality. We multiply the first line of (11) by a test function V, and we use the Green-Riemann formula:

$$\int_{\widehat{J}} \nabla W_i \nabla V - \int_{\partial \widehat{J}} V(\nabla W_i \cdot \vec{n}) = 0$$

We use then second, third and fourth lines, and the equation written above can be rewritten as

$$\int_{\widehat{J}} \nabla W_i \nabla V + \sum_j \int_{\Gamma_j} V T_j W_i = \frac{1}{c_i} \int_{\Gamma_i} W - \frac{1}{c_{i+1}} \int_{\Gamma_{i+1}} W$$

Finally, by using definition of T_i, we can rewrite this equation as

$$\int_{\widehat{J}} \nabla W_i \nabla V + \sum_j \sum_{p \in \mathbb{N}^*} \frac{p\pi}{c_j} \left(\int_{\Gamma_j} W_i \widetilde{w}_{j,p} \right) \left(\int_{\Gamma_j} V \widetilde{w}_{j,p} \right) \\ = \frac{1}{c_i} \int_{\Gamma_i} W - \frac{1}{c_{i+1}} \int_{\Gamma_{i+1}} W \tag{114}$$

Let us call $a(W_i, V)$ the left part of (114) and $\ell_i(V)$ the right one. It is an evidence that ℓ_i is a continuous linear form on $\mathcal{V}$. Thanks to the Poincare's inequality, a is a bilinear coercive form. For the continuity of this form, just use the proposition A.1 and the fact that the trace operator is continuous from $\mathrm{H}^1(\widehat{J})$ to $\mathrm{H}^{\frac{1}{2}}(\Gamma_i)$. Finally, using the Lax-Milgram theorem ensure the existence and the uniqueness of W_i. □

A.3   Properties about the matrix K

We recall here the definition of matrix $K \in \mathcal{M}_{N-1}(\mathbb{R})$, given by

$$K_{i,j} = \frac{1}{c_j} \int_{\Gamma_j} W_i - \frac{1}{c_{j+1}} \int_{\Gamma_{j+1}} W_i$$

Here, we prove another property we claimed in section 1.3.2.

Proposition A.3 (recall of the proposition 1.4). *K is a symmetric definite positive matrix*

Proof. The central point of the proof os this proposition is to use the proof of proposition 1.3 given in the previous part of this appendix, in particular equation (114) taking $V = W_j$. One can see that we have, thanks to definition of $K_{i,j}$:

$$\int_{\widehat{J}} \nabla W_i \nabla W_j + \sum_{k=1}^{N} \sum_{p \in \mathbb{N}^*} \frac{p\pi}{c_k} \left(\int_{\Gamma_k} W_i \widetilde{w}_{k,p} \right) \left(\int_{\Gamma_k} W_j \widetilde{w}_{k,p} \right) = K_{j,i} \qquad (115)$$

Equation (115) shows immediately that the matrix K is symmetric. Now let us show that K is definite positive. Given $U \in \mathbb{N}^{N-1}$, let us define $\Phi = \sum_{i=1}^{N-1} U_i W_i$. One can see easily that

$$U^t K U = \int_{\widehat{J}} |\nabla \Phi|^2 + \sum_{i=1}^{N} \int_{\Gamma_i} \Phi T_i \Phi \geqslant \int_{\widehat{J}} |\nabla \Phi|^2 \geqslant 0 \qquad (116)$$

We get immediately that K is positive. Then, if $U^T K U = 0$, one can see that $\Phi = 0$, and by looking at Neumann traces on each Γ_k, we deduce that $U = 0$. $\square$

B Decomposition on $\mathcal{J}$ and $\mathcal{A}$

Proposition B.1. *Given $V \in \mathbb{R}^N$, there exist two unique vectors $V_{\mathcal{J}}$ and $V_{\mathcal{A}}$ such that*

$$V = \mathcal{J} V_{\mathcal{J}} + \mathcal{A} V_{\mathcal{A}} \quad and \quad \mathcal{J} V_{\mathcal{A}} = 0 \quad and \quad \mathcal{A} V_{\mathcal{J}} = 0$$

Proof. We recall some properties about matrices $\mathcal{A}$ and $\mathcal{J}$: these two matrices are symetric positives matrices, are given by (15) and (16). We first decompose V on span($\mathbb{1}$) and its orthogonal:

$$V = \frac{(V \cdot \mathbb{1})}{(\mathbb{1} \cdot \mathbb{1})} \mathbb{1} + \left(V - \frac{(V \cdot \mathbb{1})}{(\mathbb{1} \cdot \mathbb{1})} \mathbb{1} \right) \qquad (117)$$

One can see that for any vector $W \in \mathbb{R}^N$, $\mathbb{1}^T \mathcal{J} W = 0$. Moreover, one has $\mathcal{A} = |\widehat{J}|(\mathbb{1} \cdot \mathbb{1})^{-2} \mathbb{1} \mathbb{1}^T$. Then by taking

$$V_{\mathcal{A}} = |\widehat{J}|^{-1} (V \cdot \mathbb{1}) \mathbb{1}$$

one has $\mathcal{J} V_{\mathcal{A}} = 0$ and

$$\mathcal{A} V_{\mathcal{A}} = |\widehat{J}|(\mathbb{1} \cdot \mathbb{1})^{-2} \mathbb{1} \mathbb{1}^T |\widehat{J}|^{-1} (V \cdot \mathbb{1}) \mathbb{1} = \frac{(V \cdot \mathbb{1})}{(\mathbb{1} \cdot \mathbb{1})} \mathbb{1}$$

We have partially written V under the form (by using the last relation in (117))

$$V = \mathcal{A} V_{\mathcal{A}} + \left(V - \frac{(V \cdot \mathbb{1})}{(\mathbb{1} \cdot \mathbb{1})} \mathbb{1} \right) \qquad (118)$$

Now, one has

$$\left(V - \frac{(V \cdot \mathbb{1})}{(\mathbb{1} \cdot \mathbb{1})} \mathbb{1} \right) \cdot \mathbb{1} = 0$$

So there exists a vector $\Phi \in \mathbb{R}^{N-1}$ (that we could even compute by the hand) such that

$$\left(V - \frac{(V \cdot \mathbb{1})}{(\mathbb{1} \cdot \mathbb{1})} \mathbb{1} \right) = P^T \Phi$$

Since the matrix K is invertible, we denote by Ψ the unique vector in $\mathbb{R}^{N-1}$ such that $\Phi = K^{-1}\Psi$. Finally, since P is surjective and its kernel is equal to span($\mathbb{1}$), there exists a unique vector $V_{\mathcal{J}} \in \mathbb{R}^N$ such that $\Psi = P V_{\mathcal{J}}$ and $\mathbb{1} \cdot V_{\mathcal{J}} = 0$. Finally, one gets that

$$\left(V - \frac{(V \cdot \mathbb{1})}{(\mathbb{1} \cdot \mathbb{1})} \mathbb{1} \right) = P^T K^{-1} P V_{\mathcal{J}} \quad \text{and} \quad \mathcal{A} V_{\mathcal{J}} = 0$$

This last result, associated to (118) and (16), gives the result of existence. For the uniqueness, let $(V_{\mathcal{J}}, V_{\mathcal{A}})$ and $(V'_{\mathcal{J}}, V'_{\mathcal{A}})$ be two decompositions such that

$$V = \mathcal{J} V_{\mathcal{J}} + \mathcal{A} V_{\mathcal{A}} \quad \text{and} \quad \mathcal{J} V_{\mathcal{A}} = 0 \quad \text{and} \quad \mathcal{A} V_{\mathcal{J}} = 0$$
$$V = \mathcal{J} V'_{\mathcal{J}} + \mathcal{A} V'_{\mathcal{A}} \quad \text{and} \quad \mathcal{J} V'_{\mathcal{A}} = 0 \quad \text{and} \quad \mathcal{A} V'_{\mathcal{J}} = 0$$

Multiplying each of these two lines by $V'_{\mathcal{J}} - V_{\mathcal{J}}$ and taking the difference leads to, thanks to the last properties on this vectors:

$$(V'_{\mathcal{J}} - V_{\mathcal{J}})^T \mathcal{J} (V'_{\mathcal{J}} - V_{\mathcal{J}}) = 0$$

Denoting $\Phi = P(V'_{\mathcal{J}} - V_{\mathcal{J}})$ leads to

$$\Phi^T K^{-1} \Phi = 0$$

and thanks to the fact that K is a symmetric definite positive matrix, K^{-1} is a symmetric definite positive matrix, then $\Phi = 0$; and we deduce that $V'_{\mathcal{J}} - V_{\mathcal{J}} = 0$. In a same way, since $\mathcal{J} V_{\mathcal{A}} = 0$, there exists $\alpha \in \mathbb{R}$ such that $V_{\mathcal{A}} = \alpha \mathbb{1}$; and there exists $\alpha' \in \mathbb{R}$ such that $V'_{\mathcal{A}} = \alpha' \mathbb{1}$. Multiplying the two lines by $\mathbb{1}$ and taking the difference leads to

$$\frac{|\mathcal{J}|}{N}(\alpha - \alpha') = 0$$

This gives the last desired relation. $\square$

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