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► **To cite this version:**

Frédéric Havet, Claudia Linhares Sales, Leonardo Sampaio. b-coloring of tight graphs. [Research Report] RR-7241, 2010, pp.11. inria-00468734v1

**HAL Id: inria-00468734**

**<https://inria.hal.science/inria-00468734v1>**

Submitted on 31 Mar 2010 (v1), last revised 20 Sep 2010 (v2)

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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

## *b-coloring of tight graphs*

Frédéric Havet — Cláudia Linhares Sales — Leonardo Sampaio

**N° 7241**

Mars 2010

Thème COM

A large blue rectangle occupies the lower half of the page. Overlaid on it is the text 'Rapport de recherche' in a white, serif font. The 'R' is significantly larger and more stylized than the other letters. A horizontal white brushstroke underline is positioned below the text.

**R**apport  
de recherche



## *b*-coloring of tight graphs \*

Frédéric Havet <sup>†</sup>, Cláudia Linhares Sales <sup>‡</sup>, Leonardo Sampaio <sup>§</sup>

Thème COM — Systèmes communicants  
Équipe-Projet Mascotte

Rapport de recherche n° 7241 — Mars 2010 — 11 pages

**Abstract:** A coloring  $c$  of a graph  $G = (V, E)$  is a *b-coloring* if in every color class there is a vertex colored  $i$  whose neighborhood intersects every other color classes. The *b-chromatic number* of  $G$ , denoted  $\chi_b(G)$ , is the greatest integer  $k$  such that  $G$  admits a *b-coloring* with  $k$  colors. A graph  $G$  is *tight* if it has exactly  $m(G)$  vertices of degree  $m(G) - 1$ , where  $m(G)$  is the largest integer  $m$  such that  $G$  has at least  $m$  vertices of degree at least  $m - 1$ . Determining the *b-chromatic number* of a tight graph  $G$  is NP-hard even for a connected bipartite graph [9]. In this paper we show that it is also NP-hard for a tight chordal graph. We also show that the *b-chromatic number* of a split graph can be computed in polynomial time. Then we define the *b-closure* and the *partial b-closure* of a tight graph, and use these concepts to give a characterization of tight graphs whose *b-chromatic number* is equal to  $m(G)$ . This characterization is used to develop polynomial time algorithms for deciding whether  $\chi_b(G) = m(G)$ , for tight graphs that are complement of bipartite graphs,  $P_4$ -sparse and block graphs. We generalize the concept of pivoted tree introduced by Irving and Manlove [6] and show its relation with the *b-chromatic number* of tight graphs. Finally, we give an alternative formulation of the Erdős-Faber-Lovász conjecture in terms of *b-colorings* of tight graphs.

**Key-words:** graph coloring, *b-coloring*, precoloring extension, tight graphs

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\* Research supported by the INRIA Equipe Associée EWIN.

## ***b*-coloration des graphes étriqués**

**Résumé :** Une  $k$ -coloration  $c$  d'un graphe  $G$  est une *b-coloration* si dans toute classe de couleur, il y a un sommet dont le voisinage intersecte toutes les autres classes de couleurs. The *nombre b-chromatique* d'un graphe est le plus grand entier  $k$  tel que  $G$  admette une *b-coloration* avec  $k$  couleurs. Un graphe est *étriqué* s'il a exactement  $m(G)$  sommet de degré  $m(G) - 1$ , avec  $m(G)$  le plus grand entier  $m$  tel que  $G$  ait au moins  $m$  sommets de degré au moins  $m - 1$ . Calculer le nombre *b-chromatique* d'un graphe étriqué est NP-dur même pour les graphes connexes bipartis [9]. Dans ce rapport, nous montrons que c'est également NP-difficile pour les graphes étriqués cordaux. Nous montrons également que le nombre *b-chromatique* d'un graphe split peut être calculé en temps polynomial. Ensuite nous définissons la *b-clôture* et la *b-clôture partielle* d'un graphe étriqué. Nous utilisons ces deux concepts pour concevoir des algorithmes en temps polynomial pour décider si  $\chi_b(G) = m(G)$  pour les graphes étriqués qui sont bipartis,  $P_4$ -sparse ou des block-graphes. Nous généralisons également le concept d'arbre pivoté de Irving and Manlove [6] et montrons sa relation avec le nombre *b-chromatique* des graphes étriqués. Enfin, nous donnons une formulation alternative de la conjecture d'Erdős-Faber-Lovász en termes de *b-coloration* des graphes étriqués.

**Mots-clés :** coloration de graphe, *b-coloration*, extension de précoloration, graphes étriqués

## 1 Introduction

A *k-coloring* of a graph  $G = (V, E)$  is a function  $c : V \rightarrow \{1, 2, \dots, k\}$ , such that  $c(u) \neq c(v)$  for all  $uv \in E(G)$ . The *color class*  $c_i$  is the subset of vertices of  $G$  that are assigned to color  $i$ . The *chromatic number* of  $G$ , denoted  $\chi(G)$ , is the least integer  $k$  such that  $G$  admits a  $k$ -coloring. Given a  $k$ -coloring  $c$ , a vertex  $v$  is a *b-vertex* of color  $i$ , if  $c(v) = i$  and  $v$  has at least one neighbor in every color class  $c_j$ ,  $j \neq i$ . A coloring of  $G$  is a *b-coloring* if every color class has a *b-vertex*. The *b-chromatic number* of a graph  $G$ , denoted  $\chi_b(G)$ , is the largest integer  $k$  such that  $G$  admits a *b-coloring* with  $k$  colors. These concepts were first defined in [6]. In that paper, Irving and Manlove proved that the problem of determining the *b-chromatic number* of a graph is NP-Hard. In fact, it was shown in [9] that deciding whether a graph admits a *b-coloring* with a given number of colors is an NP-complete problem, even for connected bipartite graphs. The following upper bound for the *b-chromatic number* of a graph, presented in [6], has been proved to be very useful. If  $G$  admits a *b-coloring* with  $m$  colors, then  $G$  must have at least  $m$  vertices with degree at least  $m - 1$  (since each color class has one *b-vertex*). The *m-degree* of a graph  $G$ , denoted by  $m(G)$ , is the largest integer  $m$  such that  $G$  has  $m$  vertices of degree at least  $m - 1$ . It is easy to see that  $\chi_b(G) \leq m(G)$  for every graph  $G$ . A vertex of  $G$  with degree at least  $m(G)$  is called a *dense* vertex. The preceding upper bound leads us to the definition of a class of graphs which are tight with respect to the number and degree of their dense vertices:

**Definition 1** (tight graph). A graph  $G$  is *tight* if it has exactly  $m(G)$  dense vertices, each of which has degree  $m(G) - 1$ .

In this paper, we mainly investigate the following decision problem:

TIGHT *b*-CHROMATIC PROBLEM

Instance: A tight graph  $G$ .

Question: Does  $\chi_b(G)$  equals  $m(G)$ ?

A direct consequence of the NP-completeness result shown in [9] is the following:

**Theorem 2.** *The TIGHT *b*-CHROMATIC PROBLEM is NP-complete for connected bipartite graphs.*

For any positive  $k$ ,  $P_k$  denotes a path with  $k$  vertices. A graph  $G$  is *P<sub>4</sub>-sparse* if every set of five vertices of  $G$  induces at most one  $P_4$ . Bonomo et al. [2] proved that the *b-chromatic number* of *P<sub>4</sub>-sparse* graphs can be determined in polynomial time. They asked if this result could be extended to *distance-hereditary graphs*, that are graphs in which every induced path is a shortest path. We answer in the negative to this question by showing the following stronger result (Theorem 3). *The TIGHT *b*-CHROMATIC PROBLEM is NP-complete for chordal distance-hereditary graphs.* We recall that a graph is *chordal* if it does not contain any induced cycle of size greater than 3.

The proof of our NP-completeness result is a reduction from 3-EDGE-COLORABILITY. We reduce an instance of this problem to a graph which is slightly more than a *split graph*, i.e. a graph whose vertex set may be partitioned into a clique and an independent set. Hence a natural question is to ask about the complexity of finding the *b-chromatic number* of a given split graph. We show in Theorem 4 that it can be solved in polynomial time.

In Section 3, we introduce the *b-closure*  $G^*$  of a graph  $G$ . We show that for a tight graph  $G$ ,  $\chi_b(G) = m(G)$  if and only if  $\chi(G^*) = m(G)$ . Hence if one can determine the chromatic number of the closure in polynomial time, one can also solve the TIGHT *b*-CHROMATIC PROBLEM in polynomial time. We show that it is the case for (tight) complement of bipartite graphs. Indeed, we prove that the closures of such graphs are also complements of bipartite graphs and the chromatic number of the complement of a bipartite graph can be determined in polynomial time. This was unknown since the characterization of complements of bipartite graphs with  $\chi_b(G) = k$  given by [7] does not lead to a polynomial algorithm for determining their *b-chromatic number*.

Moreover, we introduce the definition of pivoted tight graph and use this definition to give a sufficient condition for a tight graph to satisfy  $\chi_b(G) < m(G)$ .

The method of computing the  $b$ -closure of a graph and then the chromatic number of it does not yield polynomial-time algorithms to solve the TIGHT  $b$ -CHROMATIC PROBLEM for all classes of tight graphs. However, for some of them, we show in Section 4, that the TIGHT  $b$ -CHROMATIC PROBLEM may be solved in polynomial time using a slight modification of the closure, the *partial closure*. It is the case for block graphs and  $P_4$ -sparse graphs. It is already known that deciding if  $\chi_b(G) = m(G)$  is polynomial time solvable for  $P_4$ -sparse graphs [2]. However, our linear-time algorithm for tight  $P_4$ -sparse graphs is faster than the  $O(|V|^3)$  algorithm of [2]. It is also interesting to see how our general method can be used to solve these problems.

Finally, we give an alternative formulation of the Erdős-Faber-Lovász conjecture [1] in terms of  $b$ -colorings of tight graphs.

## 2 Chordal graphs

**Theorem 3.** *The TIGHT  $b$ -CHROMATIC PROBLEM is NP-complete for chordal distance-hereditary graphs.*

*Proof.* The problem belongs to NP since a  $b$ -coloring with  $m(G)$  colors is a certificate. To show that it is also NP-complete, we present a reduction from 3-EDGE-COLORABILITY of 3-regular graphs, which is known to be NP-complete [4]. Let  $G$  be a 3-regular graph with  $n$  vertices. Set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and  $E(G) = \{e_1, \dots, e_m\}$ . Let  $I$  be the vertex-edge incidence graph of  $G$ , that is the bipartite graph with vertex set  $V(I) = V(G) \cup E(G)$  in which an edge of  $G$  is adjacent to its two end-vertices. We construct from  $I$  a new graph  $H$  as follows. First, we add an edge between every pair of vertices in  $V(G)$  and then, we add three disjoint copies of  $K_{1,n+2}$ . One can easily see that  $d_H(v) = n - 1 + 3 = n + 2$ , for  $v \in V(G)$ , and that  $d_H(u) = 2$ , for  $u \in E(G)$ . Moreover, each copy of  $K_{1,n+2}$  has exactly one vertex with degree equal to  $n + 2$ . Consequently,  $m(H) = n + 3$  and  $H$  is tight. In  $H$ ,  $V(G)$  is a clique and  $E(G)$  is an independent set, so  $H[A \cup B]$  is a split graph, and so it is chordal. As the disjoint copies of  $K_{1,n+2}$  are themselves chordal graphs, we get that the entire graph  $H$  is chordal. One can easily check that  $H$  is also distance-hereditary. We now prove that  $G$  admits a 3-edge-coloring if and only if  $\chi_b(H) = m(H) = n + 3$ .

Let  $c$  be a 3-edge-coloring of  $E(G)$  that uses colors  $\{1, 2, 3\}$ . We shall construct a  $b$ -coloring  $c'$  of  $H$  with  $n + 2$  colors. Let  $c'(u) = c(u)$ , for  $u \in E(G)$ , and  $c'(v_i) = i + 3$ , for  $1 \leq i \leq n$ . Note that in this partial coloring, the vertices in  $V(G)$  are  $b$ -vertices of their respective colors. To obtain the remaining  $b$ -vertices, one just have to appropriately color the copies of  $K_{1,n+2}$ , which can be easily done. Then,  $c'$  is a  $b$ -coloring of  $H$  with  $m(H) = n + 3$  colors.

Now, let  $c'$  be a  $b$ -coloring of  $H$  that uses  $n + 3$  colors. Since  $V(G)$  is a clique, we may assume that  $c'(v_i) = i + 3$ , for  $1 \leq i \leq n$ . Since there are only  $n + 3$  vertices of degree  $n + 2$  in  $H$ , each vertex in  $V(G)$  is a  $b$ -vertex. But then, since every vertex in  $V(G)$  has degree exactly  $n + 2$  in  $H$ , all its neighbors must have distinct colors. As a consequence, since no vertex in  $V(G)$  is colored with one of the colors in  $\{1, 2, 3\}$ , for every vertex in  $V(G)$ , its 3 neighbours in  $E(G)$  are colored with distinct colors in  $\{1, 2, 3\}$ . This implies that  $G$  admits a 3-edge-coloring of  $G$ , and completes the proof.  $\square$

The three copies of  $K_{1,n+2}$  play an important role in this reduction, since one can show the following

**Theorem 4.** *If  $G$  is a split graph then  $\chi_b(G) = m(G)$ . Hence, the  $b$ -chromatic number of a split graph can be determined in polynomial time.*

*Proof.* Let  $G$  be a split graph and  $(K, S)$  a partition of  $V(G)$  with  $K$  a clique and  $S$  an independent set such that  $|K|$  is maximum. Every vertex in  $K$  has degree at least  $|K| - 1$  and every vertex  $s$  in  $S$  has degree at most  $|K| - 1$  otherwise  $(K \cup \{s\}, S \setminus \{s\})$  would contradict the maximality of  $|K|$ . Hence  $m(G) = |K|$ .

Coloring the vertices in  $K$  with  $|K|$  distinct colors and then extend it greedily to the vertices of  $S$  (This is possible since every vertex in  $S$  has degree smaller than  $|K|$ .) gives a  $b$ -colouring of  $G$  with  $m(G) = |K|$  colours.  $\square$

### 3 *b*-closure

**Definition 5** (*b*-closure). Let  $G$  be a tight graph. The *b*-closure of  $G$ , denoted by  $G^*$ , is the graph with vertex set  $V(G^*) = V(G)$  and edge set  $E(G^*) = E(G) \cup \{uv \mid u \text{ and } v \text{ are non-adjacent dense vertices}\} \cup \{uv \mid u \text{ and } v \text{ are vertices with a common dense neighbour}\}$ .

The next theorem proves the relation, for a tight graph  $G$ , between the parameters  $\chi_b(G)$  and  $\chi(G^*)$ :

**Lemma 6.** *Let  $G$  be a tight graph. Then  $\chi_b(G) = m(G)$  if and only if  $\chi(G^*) = m(G)$ .*

*Proof.* Set  $m = m(G)$ . Suppose that  $\chi_b(G) = m$ , and let  $c$  be a *b*-coloring of  $G$  with  $m$  colors. It is easy to see that the  $m$  dense vertices form a clique in  $G^*$  and so  $\chi(G^*) \geq m$ . Let us show that  $c$  is a proper coloring for  $G^*$ . Let  $uv \notin E$  be such that  $uv \in E(G^*)$ . If both  $u$  and  $v$  are dense, as there are exactly  $m$  dense vertices in  $G$ , they must have distinct colors in  $c$ . Now, suppose that  $u$  or  $v$  is not a dense vertex. By the definition of  $G^*$ ,  $u$  and  $v$  have a common dense neighbor, say  $d$ , in  $G$ . Since all dense vertices of  $G$  have degree  $m - 1$  and  $c$  is a *b*-coloring,  $u$  and  $v$  must have been assigned distinct colors in  $c$ . Hence,  $\chi(G^*) = m$ .

Conversely, let  $c'$  be a proper coloring of  $G^*$  with  $m$  colors. In this case, since  $E(G) \subseteq E(G^*)$ ,  $c'$  is also a proper coloring of  $G$ . It only remains to show that every color of  $c'$  has a *b*-vertex. As the dense vertices of  $G$  form a clique in  $G^*$ , they have distinct colors in  $c'$ . Moreover, for a dense vertex  $d$  of  $G$ , we have that  $N_{G^*}(d)$  is a clique. As a consequence,  $d$  is a *b*-vertex. Therefore,  $\chi_b(G) = m$ .  $\square$

Since  $\omega(G^*) > m$  implies that  $\chi(G^*) > m$ , it follows:

**Corollary 7.** *Let  $G$  be a tight graph. If  $\chi_b(G) = m(G)$ , then  $\omega(G^*) = \chi(G^*) = m(G)$ .*

#### 3.1 Complement of bipartite graphs

By Lemma 6, it is interesting to consider the *b*-closure of a tight graph  $G$  if the chromatic number of its closure can be determined in polynomial time. Indeed if so, one can decide in polynomial time if  $\chi_b(G) = m(G)$ . We now show that it is the case if  $G$  is the complement of bipartite graph.

**Lemma 8.** *The *b*-closure of the complement of a bipartite graph is a complement of a bipartite graph.*

*Proof.* Let  $G$  be a tight complement of a bipartite graph. Let  $V(G) = X \cup Y$  where  $X$  and  $Y$  are two disjoint cliques in  $G$ . As  $V(G^*) = V(G)$ , and since  $E(G) \subseteq E(G^*)$ , the sets  $X$  and  $Y$  are cliques in  $G^*$ . So they also form a partition of  $V(G^*)$  into two cliques.  $\square$

Computing the chromatic number of the complement  $G$  of a bipartite graph  $\overline{G}$  is equivalent to compute the maximum size of a matching in this bipartite graph. Hence it can be done in  $O(\sqrt{|V(G)|} \cdot |E(\overline{G})|)$  by the algorithm of Hopcroft and Karp [5] and in  $O(|V(G)|^{2.376})$  using an approach based on the fast matrix multiplication algorithm [11].

**Corollary 9.** *Let  $G$  be a tight complement of bipartite graph. It can be decided in  $O(\max\{\sqrt{|V(G)|} \cdot |E(\overline{G})|, |V(G)|^{2.376}\})$  if  $\chi_b(G) = m(G)$ .*

#### 3.2 Pivoted graphs

In the study of the *b*-chromatic number of trees, Irving and Manlove [6] introduced the notion of a *pivoted tree*, and showed that a tree  $T$  satisfies  $\chi_b(T) < m(T)$  if and only if it is pivoted. We generalize this notion and show how our generalization is related to the *b*-chromatic number of tight graphs.



**Definition 10** (Pivoted Graph). Let  $G$  be a tight graph. We say that  $G$  is *pivoted* if there is a set  $N$  of non-dense vertices, with  $|N| = k$ , and a set of dense vertices  $D$ , with  $|D| = m(G) - k + 1$ , satisfying:

1. For every pair  $u, v \in N$ ,  $u$  is adjacent to  $v$ , or there is a dense vertex  $w$  that is adjacent to both  $u$  and  $v$ .
2. For every pair  $u \in N, d \in D$ , either  $u$  is adjacent to  $d$  or  $u$  and  $d$  are both adjacent to a dense vertex  $w$  (not necessarily in  $D$ ).

**Theorem 11.** *Let  $G$  be a tight graph. Then  $G$  is a pivoted graph if and only if  $\omega(G^*) > m(G)$ .*

*Proof.* First, assume that  $G$  is a pivoted graph. Then Definitions 5 and 10 immediately imply that  $N \cup D$  is a clique of size  $m + 1$  in  $G^*$ .

Reciprocally, assume that  $\omega(G^*) > m$ . Let  $S \subseteq V(G^*)$  be a clique of size  $m + 1$  in  $G^*$ . Let  $N = \{v \in S \mid v \text{ is not dense in } G\}$  and  $D = \{v \in S \mid v \text{ is dense in } G\}$ . Let  $u, v \in S$ . If  $u, v \in D$ , there is nothing to show, since Definition 10 imposes no restrictions between dense vertices in  $G$ . If  $u \in N, v \in D \cup N$ , we have that either  $uv \in E(G)$ , or  $ud, vd \in E(G)$ , for a dense vertex  $d \in V(G)$ . So, it is easy to see that the sets  $N$  and  $D$  satisfy the requirements of Definition 10.  $\square$

Lemma 6 and Theorem 11 have the following corollary.

**Corollary 12.** *Let  $G$  be a tight graph. If  $G$  is a pivoted graph, then  $\chi_b(G) < m(G)$ .*

*Proof.* As  $G$  is pivoted, Theorem 11 implies that  $\omega(G^*) > m(G)$ , and therefore  $\chi(G^*) > m(G)$ . Then, by Lemma 6,  $\chi_b(G) < m(G)$ .  $\square$

There are graphs satisfying  $\chi(G^*) > m(G)$  but not  $\omega(G^*) > m(G)$ . Figure 1 shows a chordal non-pivoted graph  $G$  with exactly  $m(G) = 7$  dense vertices, each of degree 6, such that  $\chi_b(G) < m(G)$ .

In contrast to what happens with pivoted graphs, where a clique of size greater than  $m$  is formed in their  $b$ -closures, the graph of Figure 1 has clique number 7, but its  $b$ -closure produces an odd hole (by the five non-dense vertices in the bigger component) which causes  $\chi(G^*) > 7$ .

## 4 Partial $b$ -closure

**Definition 13** (partial  $b$ -closure). Let  $G$  be a tight graph. The *partial  $b$ -closure* of  $G$ , denoted  $G_p^*$ , is the graph with vertex set  $V(G^*) = V(G)$  and edge set  $E(G^*) = E(G) \cup \{uv \mid u \text{ and } v \text{ are vertices with a common dense neighbour}\}$ .

**Lemma 14.** *Let  $G_p^*$  be the partial  $b$ -closure of a graph  $G$ , and let  $D$  be the set of  $m(G)$  dense vertices of  $G$ . Then  $\chi_b(G) = m(G)$  if and only if  $G_p^*$  admits a  $m(G)$ -coloring where all the vertices in  $D$  have distinct colors.*

*Proof.* The proof is similar to the one of Lemma 6. In this case, since we do not add edges between all the pairs of dense vertices in  $G_p^*$ , we need the requirement that the  $m(G)$ -colouring of  $G_p^*$  is such that all dense vertices have distinct colours.  $\square$

By Lemma 14, one can decide in polynomial time if  $\chi_b(G) = m$  wherever it can be decided in polynomial time if the constrained coloring of its partial closure  $G_p^*$  exists. In particular, it is the case if the precoloring extension problem can be decided in polynomial time for  $G$ . We show that this is the case for block graphs and  $P_4$ -sparse graphs.

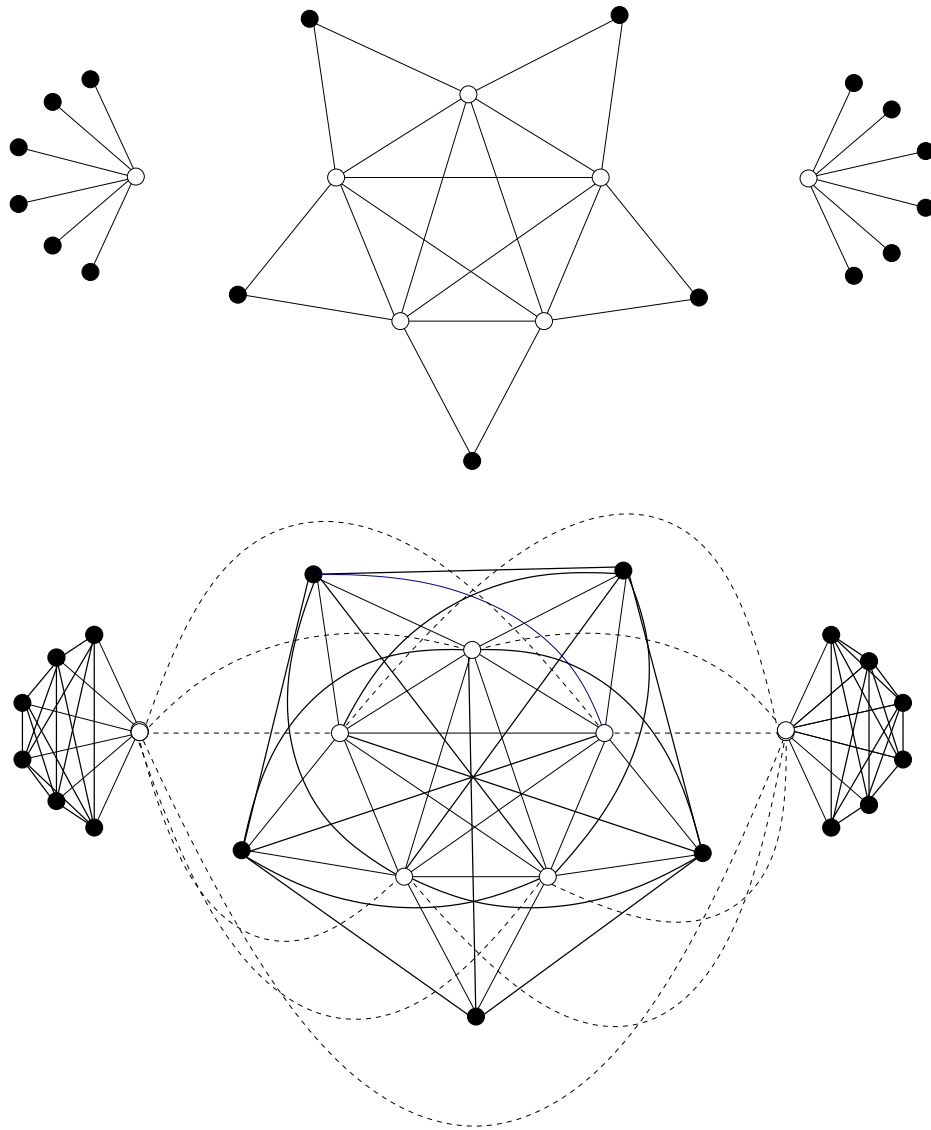


Figure 1: A non-pivoted chordal graph, satisfying  $\chi_b(G) < m(G)$ , and its  $b$ -closure  $G^*$ , satisfying  $\chi(G^*) > \omega(G^*) = m(G)$  (the new edges between the dense vertices are dashed).

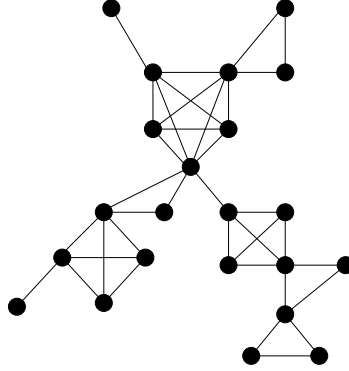


Figure 2: A block graph.

## 4.1 Block graphs

A graph  $G = (V, E)$  is a *block graph* if every of its blocks (maximal 2-connected subgraphs) is a complete graph. For an example, see Figure 2.

**Lemma 15.** *The partial  $b$ -closure of a block graph is chordal.*

*Proof.* By contradiction, assume that the partial  $b$ -closure  $G_p^*$  of a block graph  $G$  is not chordal. Then it has an induced cycle  $C = (v_1, v_2, \dots, v_k)$  of length  $k \geq 4$ . For every edge  $v_i v_{i+1}$  of  $C$  (indices must be taken modulo  $k$ ) either  $v_i v_{i+1} \in E(G)$  or there is a dense vertex  $w_i \in V(G)$  such that  $v_i w_i, w_i v_{i+1} \in E(G)$ . In the latter case, the vertex  $w_i$  is adjacent to no  $v_j$  for  $j \notin \{i, i+1\}$  in  $G$ , otherwise both  $v_j v_i$  and  $v_j v_{i+1}$  would be edges of  $G_p^*$  and  $C$  would not be induced. Furthermore, this implies that all the existing  $w_i$ 's are distinct. Let  $C'$  be the cycle obtained from  $C$  by replacing each edge  $v_i v_{i+1}$  by  $v_i w_i v_{i+1}$  whenever  $v_i v_{i+1} \notin E(G)$ . Observe that  $C'$  is a cycle of  $G$ .

But, since  $G$  is a block graph, the vertices of any cycle (in particular,  $C'$ ) form a clique in  $G$  and thus also in  $G_p^*$ . Hence the vertices of  $C$  form a clique in  $G_p^*$ , a contradiction.  $\square$

Marx [10] showed that the precoloring extension problem when all the  $C$  colours are used at most once is solvable in time  $O(C \cdot |V(G)|^3)$  for a chordal graph  $G$ . Hence,

**Corollary 16.** *the TIGHT  $b$ -CHROMATIC PROBLEM can be decided in time  $O(m(G)|V(G)|^3)$  for tight block graphs.*

**Remark 17.** A tree is a block graph, so using the partial closure method the TIGHT  $b$ -CHROMATIC PROBLEM for tight trees can be solved in time  $O(m(G)|V(G)|^3)$ . However, Irving and Manlove [6] gave a linear time algorithm to compute the  $b$ -chromatic number of any tree. Hence the TIGHT  $b$ -CHROMATIC PROBLEM can be solved in linear time for trees.

## 4.2 $P_4$ -sparse graphs

**Lemma 18.** *The partial  $b$ -closure of a  $P_4$ -sparse graph is  $P_4$ -sparse.*

*Proof.* Let  $G$  be a  $P_4$ -sparse graph. Suppose, by way of contradiction, that  $G_p^*$  is not  $P_4$ -sparse. Then there is at least one induced  $P_4$  in  $G_p^*$  that is not in  $G$ . Let  $P = (v_1, v_2, v_3, v_4)$  be such a  $P_4$  in  $G_p^*$ . We will show that there are 5 vertices that induces two  $P_4$ 's in  $G$ , thus getting a contradiction. By symmetry, it is enough to consider the following five cases.

Case 1 :  $v_1v_2 \in E(G)$ ,  $v_2v_3 \in E(G)$  and  $v_3v_4 \notin E(G)$ .

Then,  $v_3$  and  $v_4$  are both adjacent to a dense vertex  $w \in V(G)$  (by the definition of the partial *b*-closure). Note that  $v_1w \notin E(G)$  (resp.  $v_2w \notin E(G)$ ) otherwise  $v_1v_4 \in E(G_p^*)$  (resp.  $v_2v_4 \in E(G_p^*)$ ) and  $P$  would not be an induced  $P_4$  in  $G_p^*$ . Hence,  $\{v_1, v_2, v_3, w, v_4\}$  induces a  $P_5$  which contains two induced  $P_4$ .

Case 2 :  $v_1v_2 \in E(G)$ ,  $v_2v_3 \notin E(G)$  and  $v_3v_4 \in E(G)$ .

In this case,  $v_2$  and  $v_3$  are both adjacent to a dense vertex  $w \in V(G)$  (again, by the definition of the *b*-closure). Note that  $v_1w, v_4w \notin E(G)$ , for otherwise, this would imply that  $v_1v_3 \in E(G_p^*)$  ( $v_2v_4 \in E(G_p^*)$ ), by the definition of the partial *b*-closure. But then,  $\{v_1, v_2, w, v_3, v_4\}$  is an induced  $P_5$  in  $G$ .

Case 3 :  $v_1v_2 \notin E(G)$ ,  $v_2v_3 \in E(G)$  and  $v_3v_4 \notin E(G)$ .

As  $v_1v_2 \notin E(G)$ , the vertices  $v_1$  and  $v_2$  are both adjacent to a dense vertex  $w_1 \in V(G)$ . Moreover,  $w_1v_3 \notin E(G)$  (resp.  $w_1v_4 \notin E(G)$ ), since for otherwise  $v_1v_3 \in E(G_p^*)$  (resp.  $v_1v_4 \in E(G_p^*)$ ) and  $P$  would not be an induced  $P_4$  in  $G_p^*$ . By a similar argument,  $v_3$  and  $v_4$  are both adjacent to a dense vertex  $w_2 \in V(G)$ , which is not adjacent to  $v_1$  and  $w_2$ . Note that  $w_1$  and  $w_2$  are distinct since  $w_1v_4 \notin E(G)$ . If  $w_1w_2 \notin E(G)$ , then  $\{v_1, w_1, v_2, v_3, w_2\}$  is an induced  $P_5$  in  $G$ . If  $w_1w_2 \in E(G)$ , then  $\{v_1, w_1, v_2, w_2, v_4\}$  induces two  $P_4$ 's in  $G$ .

Case 4 :  $v_1v_2 \notin E(G)$ ,  $v_2v_3 \notin E(G)$  and  $v_3v_4 \in E(G)$ .

Using arguments similar to the ones in the previous cases, we obtain that there are distinct dense vertices  $w_1, w_2 \in V(G)$  satisfying  $v_1w_1, v_2w_1, v_2w_2, v_3w_2 \in E(G)$ , and  $v_1w_2, v_4w_2, v_3w_1, v_4w_1 \notin E(G)$ . If  $w_1w_2 \in E(G)$  then  $\{v_1, w_1, w_2, v_3, v_4\}$  induces a  $P_5$  in  $G$ . If  $w_1w_2 \notin E(G)$ , then the set  $\{v_1, w_1, v_2, w_2, v_3\}$  induces a  $P_5$  in  $G$ .

Case 5 :  $v_1v_2 \notin E(G)$ ,  $v_2v_3 \notin E(G)$  and  $v_3v_4 \notin E(G)$ .

Again, by similar arguments to the ones used in the previous cases, there are distinct dense vertices  $w_1, w_2, w_3 \in V(G)$  such that  $v_1w_1, v_2w_1, v_2w_2, v_3w_2, v_3w_3, v_4w_3 \in E(G)$ , and  $v_3w_1, v_4w_1, v_1w_2, v_4w_2, v_1w_3, v_2w_3 \notin E(G)$ . If  $w_1w_3 \in E(G)$ , the set  $\{v_1, w_1, w_3, v_3, v_4\}$  induces two  $P_4$ 's in  $G$ . Henceforth we may assume that  $w_1w_3 \notin E(G)$ . If  $w_1w_2, w_2w_3 \in E(G)$ , then the set  $\{v_1, w_1, w_2, w_3, v_4\}$  induces a  $P_5$  in  $G$ . Hence by symmetry, we may assume that  $w_2w_3 \in E(G)$ . If  $w_1w_2 \in E(G)$ , then the set  $\{v_1, w_1, v_2, w_2, v_3\}$  induces two  $P_4$ 's in  $G$ . If  $w_1w_2 \notin E(G)$  the set  $\{v_1, w_1, v_2, w_2, w_3\}$  induces two  $P_4$ 's in  $G$ .

□

Babel et al. [8] showed that the precoloring extension problem is linear-time solvable for  $(q, q - 4)$ -graphs, which are graphs where no set of at most  $q$  vertices induces more than  $q - 4$  different  $P_4$ 's. Hence,

**Corollary 19.** *The TIGHT *b*-CHROMATIC PROBLEM can be decided in linear time for tight  $P_4$ -sparse graphs.*

Consequently, for tight  $P_4$ -sparse graphs, this algorithm is faster than the  $O(|V|^3)$  algorithm given in [2], that solves the more general case where the input graph is not necessarily tight.

## 5 A relation with the Erdős-Faber-Lovász conjecture

As we saw in the previous sections, given a tight graph  $G$ , deciding if  $\chi_b(G) = m(G)$  is equivalent to deciding if  $\chi(G^*) = m(G)$ . It is easy to see that the graph  $G^*$  can be seen as the union of  $m(G) + 1$  cliques each of size  $m(G)$ , where these cliques may intersect at some vertices. On the other hand, the well know Erdős-Faber-Lovász conjecture [1] states that:

**Conjecture 20** (Erdős-Faber-Lovász). *Any graph  $G$  obtained by the union of  $K_m^1, K_m^2, \dots, K_m^m$ , where  $|K_m^i \cap K_m^j| \leq 1$ ,  $i \neq j$ , has chromatic number  $m$ .*

Now, consider the conjecture:

**Conjecture 21.** *Let  $G$  be a tight graph such that:*

1. *For every edge  $uv \in E(G)$ , one of its endpoints is dense, and the other is non-dense, and*
2. *If  $d'$  and  $d''$  are dense vertices in  $G$ , then  $|N(d') \cap N(d'')| \leq 1$ .*

*Then,  $\chi_b(G) = m(G)$ .*

We can prove that:

**Theorem 22.** *The Erdős-Faber-Lovász conjecture is true if and only if Conjecture 21 is true.*

*Proof.* Let  $G$  be a graph as in the Conjecture 21, and consider its  $b$ -closure  $G^*$ . It can be easily seen that  $G^*$  is the union of  $m(G) + 1$  cliques each of size  $m(G)$ . Notice that the restrictions imposed for the graph in Conjecture 21 imply that each pair of these cliques intersect in at most one vertex. Consider the graph  $H$  obtained from  $G^*$  by taking each maximal clique and adding a new vertex that it is adjacent to every other vertex in the clique. Then,  $H$  is the union of  $m(G) + 1$  cliques, each of size  $m(G) + 1$ . Then, if the Erdős-Faber-Lovász conjecture is true,  $\chi(H) = m(G) + 1$ . Since the vertices we added in the construction of  $H$  form a maximal independent set, we have that  $\chi(G^*) = \chi(H) - 1 = m(G)$ , and consequently, by Lemma 6,  $\chi_b(G) = m(G)$ .

Now, let  $G$  be a graph as described in the Erdős-Faber-Lovász conjecture. Then  $G$  is the union of  $K_m^1, K_m^2, \dots, K_m^m$ , where  $|K_m^i \cap K_m^j| \leq 1$ ,  $i \neq j$ . Since two cliques intersect in at most one vertex, and there are only  $m$  cliques, every clique, say  $K_i$ , has at least one vertex that is only contained in  $K_i$ . For each  $1 \leq i \leq m$ , let  $d_i$  be such a vertex. Consider the graph  $H$  obtained from  $G$  by taking, for each  $1 \leq i \leq m$ , the clique  $K_i$  and removing every edge on the clique such that  $d_i$  is not an endpoint of this edge. It can be easily seen that the resulting graph is a tight graph with the properties required by Conjecture 21. So, if this conjecture is true, we have that  $\chi_b(H) = m(H)$ . Observe that  $G$  is isomorphic to  $H^*$ . Then, by Lemma 6,  $\chi(G) = \chi_b(H) = m(G)$ .  $\square$

Once we proved this relationship between these problems, we can deduce results on one problem from results on the other. As an example, we can prove the following.

**Theorem 23.** *Let  $G$  be a tight graph, and let  $G[D \cup N(D)]$  be the graph induced by its dense vertices and their neighbours. If  $G[D \cup N(D)]$  is such that:*

1. *for every edge  $uv \in E(G[D \cup N(D)])$ , exactly one of its endpoints is dense, and*
2. *if  $d'$  and  $d''$  are dense vertices in  $G$ , then  $|N(d') \cap N(d'')| \leq 1$ , and*
3. *every non-dense vertex is either adjacent to only one dense vertex or to at least  $\sqrt{m(G)}$  dense vertices,*

*then,  $\chi_b(G) = m(G)$ .*

*Proof.* Since the vertices that are not in  $G[D \cup N(D)]$  have degree lower than  $m - 1$ , it can be easily seen that  $\chi_b(G[D \cup N(D)]) = \chi_b(G)$ . The result follows from the fact that  $G[D \cup N(D)]^*$ , the  $b$ -closure of  $G[D \cup N(D)]$ , is a graph as described in the Erdős-Faber-Lovász conjecture, with the additional property that every vertex in  $G[D \cup N(D)]^*$  is either in exactly one clique or in at least  $\sqrt{m(G)}$  cliques of size  $m$ . It was already proved that the conjecture of Erdős-Faber-Lovász is true in this case [12].  $\square$

We conclude by observing that the problem of determining if a tight graph  $G$  satisfies  $\chi_b(G) = m(G)$  is equivalent to the problem of determining if a graph composed of the union of  $m$  cliques of size  $m$  is  $m$ -colorable, where we have no restriction on the number of vertices that these cliques may share.

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ISSN 0249-6399