

*Asymptotic analysis
for the solution to the Helmholtz problem
in the exterior of a finite thin straight wire*

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N° 6277

Juillet 2007

Thème NUM



*Rapport
de recherche*

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Thème NUM — Systèmes numériques
Projets POems

Rapport de recherche n° 6277 — Juillet 2007 — 66 pages

Abstract: In this document we are interested in the solution of the Helmholtz equation with Dirichlet boundary condition in the exterior of a thin elongated body. We suppose that the geometry is well described in ellipsoidal coordinates. We propose an asymptotic analysis of this problem, using matched expansions. This leads to the construction of an approximate field with more explicit expression. The approximate field is composed of the first terms of the asymptotic expansion of the exact solution. Our study also leads to a validation of an acoustic version of the Pocklington's equation.

Key-words: Helmholtz, asymptotic, Pocklington, wire, thin, matched expansions, elongated

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Résumé : Dans ce document nous nous intéressons à la solution de l'équation de Helmholtz avec condition de Dirichlet à l'extérieur d'un corps fin élancé figurant un fil. Nous supposons que la géométrie peut-être convenablement décrite en coordonnées ellipsoïdales. Nous proposons une analyse asymptotique de ce problème, en utilisant des développements raccordés. Ceci mène à la construction d'un champ approché dont l'expression est plus explicite et qui est composé des premiers termes du développement du champ exact. Notre étude mène également à la validation d'une version acoustique de l'équation de Pocklington.

Mots-clés : Helmholtz, asymptotique, Pocklington, fil, fin, développements raccordés, slender

1 Introduction

This document provides a study on the diffraction problem of an acoustic wave scattered by a thin wire. In order to be more precise, let us consider a domain containing, on the one hand, fixed obstacles denoted Ω_{ob} , and on the other hand, a thin elongated body Ω_ϵ^i with the shape of a wire (ϵ being the thickness of this wire). We wish to study the behavior of the solution to the Helmholtz problem with homogeneous Dirichlet boundary conditions in $\Omega_\epsilon = \mathbb{R}^3 \setminus \overline{\Omega_{ob} \cup \Omega_\epsilon^i}$, the source being given by a function f which support $\text{supp} f$ is disjoint from Ω_ϵ^i . This geometry is represented in the picture below.

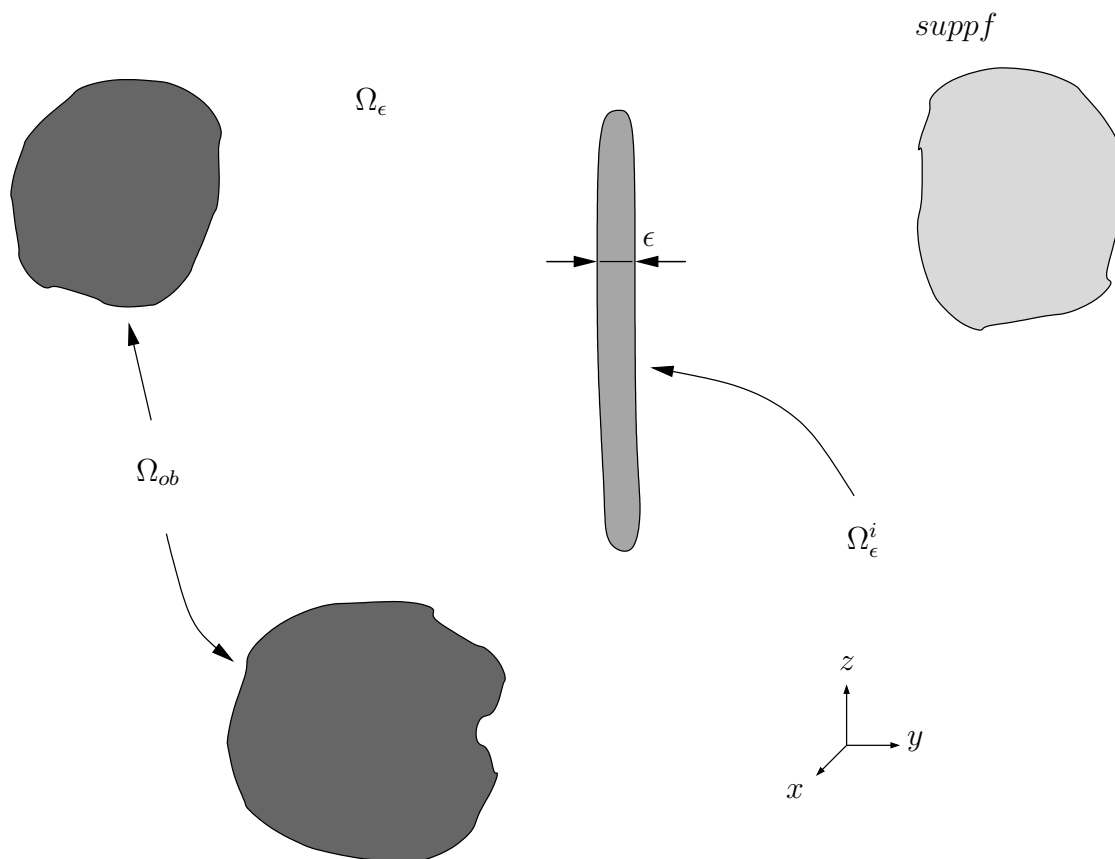


Figure 1: The geometry we consider

The problem of diffraction by a wire is an old problem that dates back to the end of the nineteenth century. In the physics literature, we have to cite the article of Pocklington [13]. It is a founding article for wire models. It derives a one-dimensional integral equation taking into account the coupling between the current on a wire and the electromagnetic field in the exterior of this wire. We have also to cite the article by R.Holland by L.Simpson [21]. It introduces a very simple numerical scheme able to take into account thin wires in a finite difference scheme for the computation of the propagation of an electromagnetic field. This method has been widely used so far, however it requires to determine empirically the value of a parameter called lineic inductance, and this raises many problems that led us to the present study. Finally, we can cite the book of Taflov [2] or D.S.Jones [3]. They are interested in the numerical resolution of Maxwell's equations in geometries containing wires using the Pocklington equation. It has to be noticed that the model of Pocklington and Holland have not yet received a full theoretical justification. To the best of our knowledge, the Holland model has not been proved to be consistent so far.

Concerning mathematical studies on this subject, a first work has been realized by Fedoryuk in [8] and [18]. Fedoryuk proposes an asymptotic model on the basis of an integral equation in order to solve the Laplace problem, in a geometry with symmetry of revolution. This work has been followed by two complementary articles [12] and [10], that consider geometries with symmetry of revolution and coercive problems with both Dirichlet and Neuman boundary condition. A similar work has been simultaneously provided by F.Rogier [9]. This time the geometry has not necessarily the symmetry of revolution, however error estimates are less precise. The thesis of A.Mazari [1] takes into account the work of Rogier and completes it with numerical considerations on the Pocklington's equation for Maxwell's equations in a simple geometrical setting (straight cylindrical antenna in free space).

As was suggested by these articles, we want to tackle this problem using asymptotic analysis. For a didactic introduction to asymptotic analysis, the book by J.Sanchez-Hubert and E.Sanchez-Palencia [4] seems a good starting point. Among the possible approach we decided to use matched asymptotics. But there exists also other possible approaches such as the multiscale expansion technique. Matched asymptotics and multiscale expansions are equivalent as was shown in [7]. The choice of matched asymptotics is a first difference between the present work, and the work contained in the papers cited above. A study of general problems, restricted though to cases where maximum principle can be applied, is available in [24]. The book by Il'in [14] contains a great number of studies that are the occasion of presenting the application of matched asymptotics in a wide variety of situations. Also the book of Mazy'a, Nazarov and Plamenevskii contains the asymptotic study of many situations in which singularities appear via geometry (which is not always the case in problems concerned by asymptotics). This book proposes a study comparable to the one presented in Fedoryuk's article [18]. Mazy'a, Nazarov and Plamenevskii use multiscale expansions. Finally we would like to cite the two PhD thesis of S.Tordeux [22] and G.Vial [23] that are very instructive and represent good introductions both to matched asymptotics and multiscale expansions.

We think that the present work contains innovative aspects in the way that we adopted to tackle the problem of thin wires. Concerning technical aspects, we have used matched asymptotics instead of multiscale expansions. In particular, we have adapted the work of Fedoryuk to a non-coercive situation (Helmholtz equation) which raised some technical difficulties. The ideas of section 5.3 are very similar to [18]. In error estimates, the proof of stability is classical in the literature of asymptotics. Concerning consistency, we have adapted technical results contained in [22] in the case of spheroidal coordinates (corollary 7.1 and lemma 7.5). Moreover section 8 come to an agreement with ideas contained in [1] (restreining the space of Lagrange multipliers to elements with symmetry of revolution), but the way of deriving these ideas is different. In particular, we did not encounter in literature any proof of stability on a mixed formulation like the result of 8.2. We believe that the interesting point of this work is, on the one hand, the result 7.1 that proves the validity of the matched expansion, and on the other hand, the result 8.2 that leads to a simpler formulation of the problem than the initial one: conceding an error in $O(\varepsilon)$ it is possible to consider that the fields are constant on each cross section of the wire. This provides a justification of the (scalar version of the) Pocklington equation. Moreover this work seems to provide a unifying theoretical setting for wire models.

1.1 Brief survey on wire models

Before strating with our main discution, we shall first give an overview of the existing models for thin wires. These models deal with Maxwell's equations, but they can be adapted for the treatment of other equations such as Laplace and Helmholtz equation. We will present these existing models for Helmholtz equation, but keep in mind that similar models can be proposed for Laplace and Maxwell's equation. Each of these models differs only by its equation corresponding to the boundary condition on the wire. Until the end of this sub-section we consider a straight thin wire with a surface Γ^ε described in cylindrical coordinates by $r = \Phi^\varepsilon(\theta, z)$ for $z \in [-1, 1]$

Pocklington's model

The founder of wire models is H.C Pocklington who proposed in 1897 an equation coupling the field diffracted by an antenna with the current at the surface of this antenna. To simplify, suppose that $\Omega_{ob} = \emptyset$ (antenna in free space). Denote J the median line in this wire, with a geometrical structure as in figure 1, and suppose that this line is on the z -axis. Let $\gamma^\varepsilon(\theta, z)$ be the surface measure on Γ^ε in cylindrical coordinates. The length of the boundary of a section corresponding to the coordinate z is then

$$\tilde{\gamma}^\varepsilon(z) = \int_0^{2\pi} \gamma^\varepsilon(\theta, z) d\theta$$

We consider an incident wave $u_{\text{inc}} \in H_{loc}^1(\Omega^\varepsilon)$ satisfying the homogeneous Helmholtz equation in the neighborhood of the wire. For exemple we can construct such a field by solving the equation $\Delta u_{\text{inc}} + k^2 u_{\text{inc}} = f$ with $\text{supp} f \cap \Omega_\varepsilon^i = \emptyset$. We are interested in the corresponding diffracted wave, that is to say the function $u_d^\varepsilon \in H_{loc}^1(\Omega_\varepsilon)$ satisfying:

$$\begin{cases} \Delta u_d^\varepsilon + k^2 u_d^\varepsilon = 0 & \text{in } \Omega_\varepsilon \\ u_d^\varepsilon = -u_{\text{inc}} & \text{on } \Gamma^\varepsilon \\ u_d^\varepsilon & \text{outgoing} \end{cases}$$

Moreover we denote $u_{\text{tot}}^\varepsilon = u_d^\varepsilon + u_{\text{inc}}$ the total field, and $G(\mathbf{x}; \mathbf{x}')$ the Green function of our acoustic problem and $I^\varepsilon = \tilde{\gamma}^\varepsilon \frac{\partial u_d^\varepsilon}{\partial n} |_{\Gamma^\varepsilon}$ that will represent the current on the wire. In order to write a relevant integral equation, Pocklington assumed that the current actually did not vary on a section of the wire, i.e $I^\varepsilon = I^\varepsilon(z)$ does not depend on z , and that the diffracted field in this region was also independant of z , $u_d^\varepsilon|_{\Gamma^\varepsilon} = u_d^\varepsilon|_{\Gamma^\varepsilon}(z)$. In the case of the Helmholtz problem, the integral equation of Pocklington is

$$\int_{z'=-1}^{+1} \left(\int_{\theta=0}^{2\pi} \int_{\theta'=0}^{2\pi} \frac{e^{ik|\mathbf{x}(\theta, z) - \mathbf{x}(\theta', z')|}}{4\pi|\mathbf{x}(\theta, z) - \mathbf{x}(\theta', z')|} \frac{\gamma^\varepsilon(\theta, z)}{\tilde{\gamma}^\varepsilon(z)} \frac{\gamma^\varepsilon(\theta', z')}{\tilde{\gamma}^\varepsilon(z')} d\theta d\theta' \right) I^\varepsilon(z') dz' = -u_{\text{inc}}|_{\Gamma^\varepsilon}(z).$$

We denoted $\mathbf{x}(\theta, z)$ the point of Γ^ε with cylindrical coordinates θ and z . This equation is a one dimensional version of the integral equation corresponding to a single layer potential used for the resolution of the Helmholtz problem. This equation is what we call the model of Pocklington.

Model of Pocklington with regularized kernel

The kernel of Pocklington's integral equation is singular (as a function of z'). Its singularity is logarithmic. Serveral authors have replaced it by a regularized kernel. Let us denote by $\tilde{\mathbf{x}}(z)$ the point of J corresponding to the coordinate z . These authors have prefered to consider the equation

$$\int_{z'=-1}^{+1} \left(\int_{\theta'=0}^{2\pi} \frac{e^{ik|\tilde{\mathbf{x}}(z) - \mathbf{x}(\theta', z')|}}{4\pi|\tilde{\mathbf{x}}(z) - \mathbf{x}(\theta', z')|} \gamma^\varepsilon(\theta', z') d\theta' \right) I^\varepsilon(z') dz' = -u_{\text{inc}}|_{\Gamma^\varepsilon}(z)$$

Since the kernel of this equation is very smooth, the associated integral operator is compact and this new integral equation is ill-posed. As a consequence this equation can lead to realistic results only if the smallest mesh step used for the discretization is more than three times larger than the radius ε of the wire. This equation is what we call the model of Pocklington with regularized kernel. For the design of the code Magellan which purpose is to study conducting wires and shells, Bendali & al. have proposed a variational formulation in which the homogeneous Dirichlet boundary condition on the wire is taken into account using a one dimensional space of Lagrange multipliers (containing elements that do not depend on the θ variable). This restricted space is

$$H_{\text{moy}}^{-1/2}(\Gamma^\varepsilon) = \{q \in H^{-1/2}(\Gamma^\varepsilon) \mid \frac{\partial q}{\partial \theta} = 0 \}$$

where derivatives have to be understood in the sense of distributions. For the total field (incident+diffracted) an ‘‘averaged’’ Dirichlet boundary condition is then written on the median line

$$\int_{-1}^{+1} u_{\text{tot}}^\varepsilon(\tilde{\mathbf{x}}(z)) q(z) dz = 0 \quad \forall q \in H_{\text{moy}}^{-1/2}(\Gamma^\varepsilon).$$

Writing an integral representation for $u_{\text{tot}}^\varepsilon$, this equation leads to a variational formulation of the regularized version of the Pocklington’s equation. We call this model the regularized fictitious domain model. Mazari used this idea but tried to impose a Dirichlet boundary condition using an integral equation with a singular kernel, that corresponds to the model of Pocklington. This equation writes

$$\int_0^{2\pi} \int_{-1}^{+1} u_{\text{tot}}^\varepsilon(\mathbf{x}(\theta, z)) q(z) dz d\theta = 0 \quad \forall q \in H_{\text{moy}}^{-1/2}(\Gamma^\varepsilon).$$

We call this model, the singular fictitious domain model.

Model of Holland

In order to solve numerically the problem of diffraction by an antenna, Holland & al. in [21] have proposed simple modifications of a classical FDTD scheme sufficient for simulating the presence of an antenna in the medium of propagation. As was proved by Collino and Millot in [5], the new resulting scheme can be inherited from a continuous variational formulation of the Maxwell’s equations that differs from the usual formulation by the presence of an averaging operator and a parameter. This model assumes that the section of the wire is circular, the variations of the current on a transverse section are negligible, and the field has an electrostatic behavior in the neighborhood of the wire. From these assumptions, Holland deduces that $u_{\text{tot}}^\varepsilon(r, \theta, z) \simeq I^\varepsilon(\theta, z) \ln(\frac{r}{\varepsilon})$. Then a function δ^ε has to be chosen in order to regularize the surface measure on Γ^ε . This function has to verify

$$\begin{aligned} \int_0^{+\infty} \delta^\varepsilon(r) 2\pi r dr &= 1, \\ \forall \varphi \text{ continuous, } \int \varphi(x, y) \delta^\varepsilon(\sqrt{x^2 + y^2}) dx dy &\xrightarrow{\varepsilon \rightarrow 0_+} \varphi(0), \\ \delta^\varepsilon(r) &= 0, \quad r_\varepsilon > r > \varepsilon \quad \text{and} \quad r^\varepsilon \xrightarrow{\varepsilon \rightarrow 0_+} 0. \end{aligned}$$

The choice of the function δ^ε lies on practical considerations. Instead of the usual homogeneous Dirichlet condition on the wire, one writes an equation averaged by δ^ε and makes use of the radial electrostatic behavior of the field. So the variational formulation that has to be discretized is

$$\begin{cases} \int_{\Omega_R} \nabla u_{\text{tot}}^\varepsilon \cdot \nabla \bar{v} - k^2 \int_{\Omega_R} u_{\text{tot}}^\varepsilon \bar{v} + \int_{\Gamma_R} \bar{v} T_R u_{\text{tot}}^\varepsilon + \int_{\Omega_R} \delta^\varepsilon(r) I^\varepsilon(z) \bar{v}(r, \theta, z) = - \int_{\Omega_R} f \bar{v} \quad \forall v \in H^1(\Omega_R), \\ \int_{\Omega_R} \delta^\varepsilon(r) q(z) u_{\text{tot}}^\varepsilon(r, \theta, z) = \int_{\Omega_R} \delta^\varepsilon(r) q(z) I^\varepsilon(z) \ln(\frac{r}{\varepsilon}) \quad \forall q \in H_{\text{moy}}^{-1/2}(\Gamma^\varepsilon). \end{cases}$$

where T_R is a usual DTN map acting on the boundary of Ω_R .

1.2 Method of matched asymptotic expansions

We are now going to briefly present the matched asymptotic expansions method and the way we wish to apply it to our problem. This will give an outline of our study. In this document we are interested in the expansion with respect to ε of the function u^ε satisfying

$$\begin{cases} \Delta u^\varepsilon + k^2 u^\varepsilon = f & \text{in } \Omega_\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \\ u^\varepsilon & \text{outgoing.} \end{cases}$$

In order to study the behavior of u^ε we can first ask ourselves whether u^ε admits a limit (as $\varepsilon \rightarrow 0$). As the wire shrinks and disappears, one might guess that this limit exists and is the function u_0 satisfying

$$\begin{cases} \Delta u_0 + k^2 u_0 = f & \text{in } \Omega, \\ u_0 & \text{outgoing.} \end{cases}$$

Then one can ask what is the behavior of $u^\varepsilon - u_0, \dots$ etc. We will then seek the first term of the expansion of u^ε . For this purpose, we will proceed in 5 steps

- **step 0**

We describe in detail the geometrical setting of our problem, rigorously formulate the diffraction problem using variational formulations, and introduce a suitable functional setting. We will also define two regions $\mathcal{Z}_n^\varepsilon$ and $\mathcal{Z}_f^\varepsilon$ called near and far field regions. We will postulate a form in each of these regions for the first term of the expansion of u^ε :

$$u^\varepsilon \simeq u_0 + u_1^\varepsilon \quad \text{in } \mathcal{Z}_f^\varepsilon \quad \text{with } u_1^\varepsilon = o(u_0) \quad (1)$$

We will call u_0 and u_1^ε “far field terms”. For the near field region, as we don’t really know the behavior of u^ε , We will simply write

$$u^\varepsilon \simeq U_1^\varepsilon \quad \text{in } \mathcal{Z}_n^\varepsilon \quad (2)$$

We will call U_1^ε “near field term”.

- **step 1**

This step is treated in section 3, where we seek in a formal and intuitive manner necessary conditions for $u_0 + u_1^\varepsilon$ to be a good approximation of u^ε in $\mathcal{Z}_{10}^\varepsilon$. We will characterize $u_0 + u_1^\varepsilon$ up to the choice of a function $a^\varepsilon \in L^2([-1; 1])$. The definition of a^ε is delayed to section 5.

- **step 2**

This step is treated in section 4 where we seek necessary conditions for U_1^ε to be an approximation of u^ε in $\mathcal{Z}_{pr}^\varepsilon$. We will characterize U_1^ε up to the choice of a function $b^\varepsilon \in L^2([-1; 1])$.

- **step 3**

To end the definition of a suitable approximate field we need to define a^ε and b^ε . This will be the subject of this step treated in section 5 where we prove that these functions have to satisfy an integral equation.

- **step 4**

We would have *formally* constructed $u_0 + u_1^\varepsilon$ and U_1^ε in the preceding step, so this step will deal with defining an approximate field \tilde{u}^ε using the far and near field (section 6), and then prove in section 7 that $\|\tilde{u}^\varepsilon - u^\varepsilon\|_{H^1(\Omega_R)}$ is small, which is the subject of theorem 7.1 that states

$$\|u^\varepsilon - \tilde{u}^\varepsilon\|_{H^1(\Omega_R)} \leq \kappa \varepsilon^{1/2} \ln 1/\varepsilon$$

for a constant $\kappa > 0$ independant of ε .

Once we have applied this method, we would have obtained an approximation of the exact solution. The question is what to do with this approximation. We will use it to prove that u^ε is also very close to another function u^ε that is solution to the same diffraction problem with simplified boundary conditions. This problem will be written using a mixed formulation that we call the simplified fictitious domain formulation. We will prove a result very similar to 7.1 with u^ε , which will yield a validation of the model of Pocklington.

2 Step 0: geometrical setting and initial problem

2.1 General remarks

Consider a space of reference $\mathcal{E} = \mathbb{R}^3$. So \mathcal{E} has a natural manifold structure. When we write “ \mathbf{x} ” we refer to a point of \mathcal{E} as a manifold, thus independantly of any coordinate system. We will often refer to objects related to \mathcal{E} and so independant of any coordinate system, such as functions for which we write $u(\mathbf{x})$.

We wish to emphasize the difference between the manifold structure of \mathcal{E} , and its representation in coordinate systems. We will refer to a point $\mathbf{x} \in \mathcal{E}$ with 3 coordinates chosen in a set different from \mathcal{E} . For exemple we will naturally refer to a point using its cartesian coordinates $(x; y; z) \in \mathbb{R}^3$. We will use four coordinate systems. The first is the cartesian system with coordinates denoted $(x; y; z)$. We will write $\mathbf{x}(x; y; z)$ in order to mean that the cartesian coordinates \mathbf{x} are $(x; y; z)$. The second system is the spherical coordinates restricted to the unit sphere, with coordinates denoted (ν, θ) (be careful we don't use the usual coordinates (θ, φ) in the present case). The third system is called in litterature prolate spheroidal coordinates. It belongs to the family of ellipsoidal coordinates, so we call it ellipsoidal in order not to make any confusion with spherical coordinates. The coordinates associated with this system are denoted $(\xi; \nu; \theta)$. In accordance with our notation for cartesian coordinates, we write $\mathbf{x}(\xi; \nu; \theta)$ to mean that \mathbf{x} has $(\xi; \nu; \theta)$ as ellipsoidal coordinates. We finally consider a last coordinate system that we call scaled coordinates system. It will be defined below. The corresponding coordinates are denoted (ζ, ν, θ) . Again we write $\mathbf{x}(\zeta; \nu; \theta)$ in order to mean that \mathbf{x} has $(\zeta; \nu; \theta)$ for scaled coordinates.

Sometimes we will write “ $u(\xi; \nu; \theta)$ ” instead of writing “ $u(\mathbf{x})$ where \mathbf{x} has $(\xi; \nu; \theta)$ for ellipsoidal coordinates” (of course the same remark is true for cartesian, spherical or scaled coordinates). Also when considering a function depending only on one variable, let say u depending only on ξ , then we simply write “ $u(\xi)$ ” instead of “ $u(\mathbf{x})$ where \mathbf{x} has $(\xi; \nu; \theta)$ for ellipsoidal coordinates.”

Now we recall the precise definition of our coordinate systems, and define the scaled coordinates.

2.2 Coordinate systems

In this sub-section, we shall remind the reader some definitions concerning usual cordinate systems, and also introduce some others. As we stated before, we will be interested in spherical coordinates, ellipsoidal coordinates, and a new coordinate system called scaled coordinates.

2.2.1 Spherical coordinates

In this document we will consider coordinates on the unit sphere $S^2 = \{\mathbf{x} \in \mathcal{E} \text{ t.q } |\mathbf{x}| = 1\}$ slightly different from usual spherical coordinates. Let us denote these coordinates (ν, θ) . One such pair will refer to a point on S^2 with cartesian coordinates given by the formula

$$\begin{cases} x = \sqrt{1 - \nu^2} \cos \theta & \nu \in [-1; +1], \\ y = \sqrt{1 - \nu^2} \sin \theta & \theta \in [0; 2\pi], \\ z = \nu. \end{cases}$$

We denote $O_+^{S^2}$ and $O_-^{S^2}$ the upper and lower poles of S^2 . We also denote $\phi_{\text{sp}}^{S^2} : S^2 \setminus \{O_+^{S^2}, O_-^{S^2}\} \rightarrow]-1, +1[\times (\mathbb{R}/2\pi\mathbb{Z})$ the chart that associates to any point $\mathbf{x} \in S^2$ its spherical coordinates (ν, θ) given by the above formula. $\phi_{\text{sp}}^{S^2}$ is a C^∞ -difféomorphism. Here is the expression of the Laplace operator on the unit sphere,

$$\Delta_{S^2} = \frac{\partial}{\partial \nu} (1 - \nu^2) \frac{\partial}{\partial \nu} + \frac{1}{1 - \nu^2} \frac{\partial^2}{\partial \theta^2}.$$

2.2.2 Ellipsoidal coordinates

Let us recall how these coordinates $(\xi; \nu; \theta)$ are defined using cartesian coordinates,

$$\begin{cases} x = \sqrt{(\xi^2 - 1)(1 - \nu^2)} \cos \theta & \xi \in [1; +\infty[\\ y = \sqrt{(\xi^2 - 1)(1 - \nu^2)} \sin \theta & \nu \in [-1; +1] \\ z = \xi \nu & \theta \in [0; 2\pi] \end{cases}$$

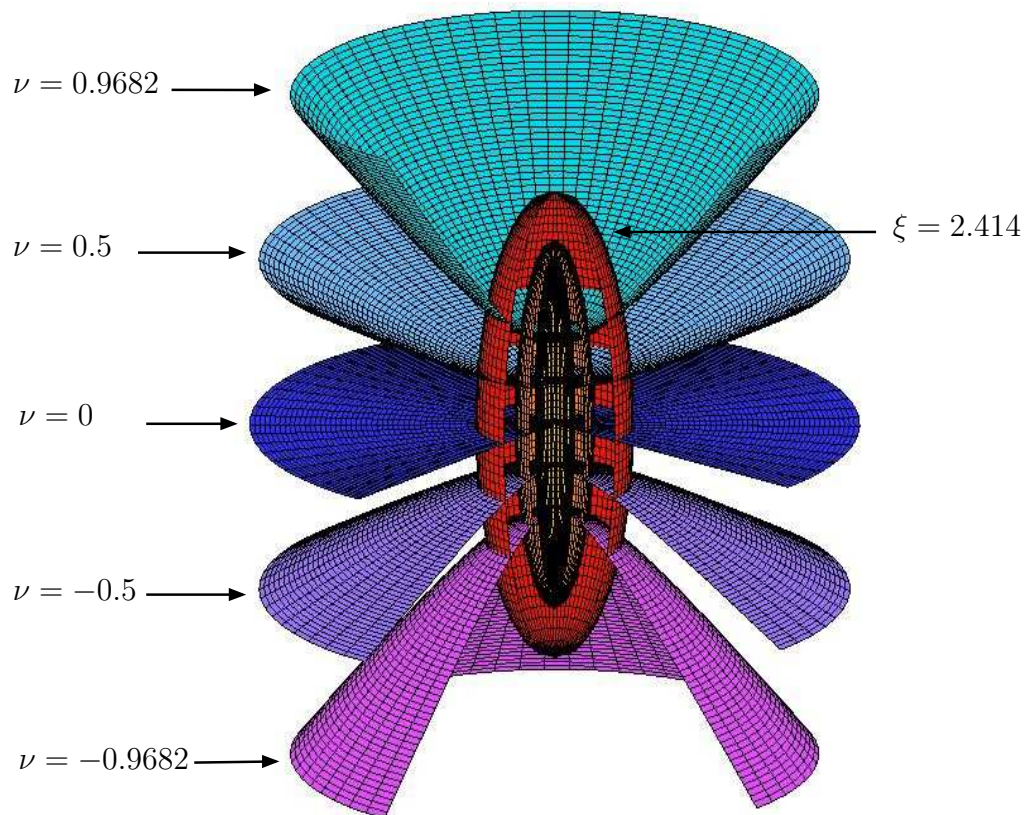
Let $U_{el} = \mathcal{E} \setminus \{\mathbf{x}(x, y, z) \mid x^2 + y^2 = 0\}$. We denote $\phi_{el} : U_{el} \rightarrow]1, +\infty[\times]-1, +1[\times (\mathbb{R}/2\pi\mathbb{Z})$ the function that associates to any point its ellipsoidal coordinates. ϕ_{el} is a C^∞ -difféomorphism. The iso- ξ and iso- ν surfaces corresponding to this coordinate system are represented on the figure below. Let us find the explicit equation of these surfaces. First we deal with a surface of equation $\xi = \xi_0$. It is possible to make ν and θ disappear in the formulas defining ellipsoidal coordinates. One then obtain

$$\frac{z^2}{\xi_0^2} + \frac{x^2 + y^2}{\xi_0^2 - 1} = 1,$$

which is the equation of an ellipsoid of revolution. Now we treat the case of the surface of equation $\nu = \nu_0$. The same argument as before yields the equation

$$\frac{z^2}{\nu_0^2} - \frac{x^2 + y^2}{1 - \nu_0^2} = 1,$$

which is the equation of a hyperboloid of revolution.



Concerning the metric tensor we compute

$$\begin{aligned} \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2 &= \frac{\xi^2 - \nu^2}{\xi^2 - 1}, \\ \left(\frac{\partial x}{\partial \nu}\right)^2 + \left(\frac{\partial y}{\partial \nu}\right)^2 + \left(\frac{\partial z}{\partial \nu}\right)^2 &= \frac{\xi^2 - \nu^2}{1 - \nu^2}, \\ \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 &= (\xi^2 - 1)(1 - \nu^2). \end{aligned}$$

Moreover, as the coordinate lines are orthogonal, as is easily verified, we directly deduce the expression of the metric tensor and the measure in ellipsoidal coordinates

$$d^2m = \frac{\xi^2 - \nu^2}{\xi^2 - 1} d^2\xi + \frac{\xi^2 - \nu^2}{1 - \nu^2} d^2\nu + (\xi^2 - 1)(1 - \nu^2) d^2\theta \quad \text{and} \quad dx dy dz = (\xi^2 - \nu^2) d\xi d\nu d\theta$$

We also deduce the expression of laplacian and gradient operators,

$$\begin{aligned} \nabla \psi &= \frac{\xi^2 - 1}{\xi^2 - \nu^2} \frac{\partial \psi}{\partial \xi} \mathbf{e}_\xi + \frac{1 - \nu^2}{\xi^2 - \nu^2} \frac{\partial \psi}{\partial \nu} \mathbf{e}_\nu + \frac{1}{(\xi^2 - 1)(1 - \nu^2)} \frac{\partial \psi}{\partial \theta} \mathbf{e}_\theta \\ \Delta \psi &= \frac{1}{\xi^2 - \nu^2} \frac{\partial}{\partial \xi} (\xi^2 - 1) \frac{\partial \psi}{\partial \xi} + \frac{1}{\xi^2 - \nu^2} \frac{\partial}{\partial \nu} (1 - \nu^2) \frac{\partial \psi}{\partial \nu} + \frac{1}{(\xi^2 - 1)(1 - \nu^2)} \frac{\partial^2 \psi}{\partial \theta^2}. \end{aligned}$$

where the vectors \mathbf{e}_ξ , \mathbf{e}_ν and \mathbf{e}_θ are defined using the cartesian vectors \mathbf{e}_x , \mathbf{e}_y and \mathbf{e}_z according to the relations

$$\mathbf{e}_\xi = \frac{\partial x}{\partial \xi} \mathbf{e}_x + \frac{\partial y}{\partial \xi} \mathbf{e}_y + \frac{\partial z}{\partial \xi} \mathbf{e}_z, \quad \mathbf{e}_\nu = \frac{\partial x}{\partial \nu} \mathbf{e}_x + \frac{\partial y}{\partial \nu} \mathbf{e}_y + \frac{\partial z}{\partial \nu} \mathbf{e}_z \quad \text{and} \quad \mathbf{e}_\theta = \frac{\partial x}{\partial \theta} \mathbf{e}_x + \frac{\partial y}{\partial \theta} \mathbf{e}_y + \frac{\partial z}{\partial \theta} \mathbf{e}_z.$$

2.2.3 Scaled coordinates

The scaled coordinates of a point will be denoted $(\zeta; \nu; \theta)$ (compared to the cartesian coordinates only the first coordinate change) and will be defined using ellipsoidal coordinates by the formula

$$\xi^2 = 1 + \varepsilon^2 \zeta^2$$

or by means of cartesian coordinates using the formulas

$$\begin{cases} x = \varepsilon \zeta \sqrt{1 - \nu^2} \cos \theta & \zeta \in [0; +\infty[\\ y = \varepsilon \zeta \sqrt{1 - \nu^2} \sin \theta & \nu \in] - 1; +1[\\ z = \sqrt{1 + \varepsilon^2 \zeta^2} \nu & \theta \in [0; 2\pi]. \end{cases}$$

The expression of the measure with these coordinates is $\frac{\varepsilon^2 \zeta}{\xi} (\xi^2 - \nu^2) d\zeta d\nu d\theta$.

2.3 Geometrical setting

2.3.1 Two remarks on the choice of the coordinate system

We have chosen the use of ellipsoidal coordinates in order to describe the geometrical elements of our problem because neither the spherical system nor the cylindrical one are well suited to the geometry of the tips. Our choice for the scaled coordinates comes from the fact that the laplacian has a very simple expression with these coordinates, as we shall see in section 4.

2.3.2 Description of the geometry

We will now describe the geometry we want to consider for our analysis. First of all we wish to consider an open set $\Omega \subset \mathcal{E}$ that we assume to be C^∞ . We will refer to the unit segment located between the points $(0; 0; 1)$ and $(0; 0; -1)$ by the letter J , and assume that $J \subset \Omega$. In ellipsoidal coordinates, the segment J is described by the equation $\xi = 1$. We will also consider an open set Ω_ε^i with a boundary Γ^ε described in ellipsoidal coordinates by the equation

$$(\Gamma^\varepsilon) : \quad \xi^2 = 1 + \varepsilon^2 \Phi^2(\nu; \theta) \quad , \quad \nu \in [-1; +1] \text{ and } \theta \in [0; 2\pi].$$

This equation written in scaled coordinates becomes very simple

$$(\Gamma^\varepsilon) : \quad \zeta = \Phi(\nu; \theta) \quad , \quad \nu \in [-1; +1] \text{ and } \theta \in [0; 2\pi].$$

This surface will represent the wire during our study. We make the following assumptions on Φ :

- **H 1** : Φ is C^∞ ,
- **H 2** : There exists a constant Φ_0 such that $\Phi_0 < \Phi(\nu; \theta) \quad \forall \nu \in [-1; +1] , \forall \theta \in [0; 2\pi]$,
- **H 3** : There exists a $\nu_0 \in]0; 1[$ such that $\Phi(\nu; \theta) = \Phi(\nu)$ for $|\nu| > \nu_0$.

Note that hypothesis **H1** excludes the presence of geometrical singularities such as corners or edges at the surface of Γ^ε . Moreover hypothesis **H3** assumes that the wire has symmetry of revolution in the neighborhood of the tips. In spite of these restrictions, a certain variety of shapes for the wire can be represented by this type of equation if we suppose that ε is fixed.

Let us denote Ω_ε^i the interior set of Γ^ε , and denote $\Omega_\varepsilon = \Omega \setminus \overline{\Omega_\varepsilon^i}$. We also denote B_R the open ball of center 0 and radius R , and $\Gamma_R = \partial B_R$. We define $\Omega_\varepsilon^R = \Omega_\varepsilon \cap B_R$ and $\Omega_R = \Omega \cap B_R$. When $\varepsilon \rightarrow 0$, $\Omega_\varepsilon^i \rightarrow J$ in the sense of Hausdorff: the obstacle shrinks around the segment J .

We denote $O_+^{\Gamma^\varepsilon}$ and $O_-^{\Gamma^\varepsilon}$ the upper and lower poles on Γ^ε . Since the equation of Γ^ε is given in ellipsoidal coordinates by $(\Gamma^\varepsilon) : \xi^2 = 1 + \varepsilon^2 \Phi^2(\nu; \theta)$, we see that (ν, θ) is a good coordinate system on $\Gamma^\varepsilon \setminus \{O_+^{\Gamma^\varepsilon}, O_-^{\Gamma^\varepsilon}\}$. Let us denote $\phi_{\text{el}}^{\Gamma^\varepsilon} : \Gamma^\varepsilon \setminus \{O_+^{\Gamma^\varepsilon}, O_-^{\Gamma^\varepsilon}\} \rightarrow]-1, +1[\times (\mathbb{R}/2\pi\mathbb{Z})$ the chart that maps any point $\mathbf{x} \in \Gamma^\varepsilon$ with ellipsoidal coordinates $(\sqrt{1 + \varepsilon^2 \Phi^2(\nu, \theta)}, \nu, \theta)$ to its coordinates (ν, θ) . $\phi_{\text{el}}^{\Gamma^\varepsilon}$ is a C^∞ -difféomorphism.

2.3.3 Several useful sets

During our analysis, we will need to refer to three particular sets, three zones in Ω_ε .

$$\mathcal{Z}_{\text{pr}}^\varepsilon = \{\mathbf{x}(\xi; \nu; \theta) \in \mathbb{R}^3 \mid \varepsilon^2 \Phi^2(\nu; \theta) < \xi^2 - 1 < 2\varepsilon\},$$

$$\mathcal{Z}_{\text{lo}}^\varepsilon = \{\mathbf{x}(\xi; \nu; \theta) \in \mathbb{R}^3 \mid \varepsilon < \xi^2 - 1\},$$

$$\mathcal{T}^\varepsilon = \mathcal{Z}_{\text{pr}}^\varepsilon \cap \mathcal{Z}_{\text{lo}}^\varepsilon = \{\mathbf{x}(\xi; \nu; \theta) \in \mathbb{R}^3 \mid \varepsilon < \xi^2 - 1 < 2\varepsilon\}.$$

We will call $\mathcal{Z}_{\text{pr}}^\varepsilon$ the near field zone, and $\mathcal{Z}_{\text{lo}}^\varepsilon$ the far field zone, and \mathcal{T}^ε the transition zone. These zones are delimited by ellipsoids. \mathcal{T}^ε is like an ellipsoidal shell shrinking around J . For purely practical reasons, we will also need to refer to an open neighborhood of J in Ω . We assume that there exists $\xi_0 > 1$ such that $\{\mathbf{x}(\xi; \nu; \theta) \in \mathbb{R}^3 \mid \xi < 3\xi_0\} \subset \Omega$. Then we define $\mathcal{O} = \{\mathbf{x}(\xi; \nu; \theta) \mid \xi < 2\xi_0\}$ and note that $J \subset \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega$.

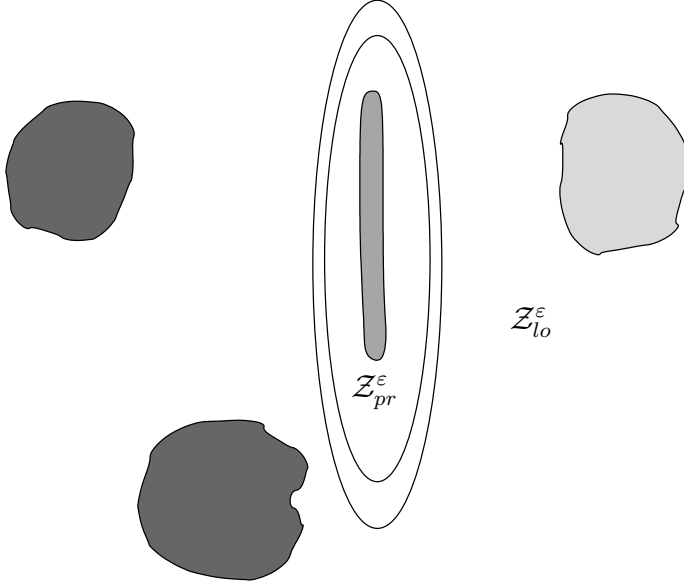


Figure 2: Far field and near field zones

2.4 Initial problem and functional setting

2.4.1 Standard space

We simply denote $\mathcal{H} = \{v \in H^1(\Omega_R) \text{ t.q. } v|_{\partial\Omega} = 0\}$. We define a sesquilinear form on \mathcal{H} by $(u; v)_{\mathcal{H}} = \int_{\Omega_R} \nabla u \cdot \nabla \bar{v} + \int_{\Omega_R} u \bar{v}$. With this sesquilinear form, \mathcal{H} is an Hilbert space. We will also refer to the operator $\mathcal{A} : \mathcal{H} \rightarrow \mathcal{H}$ defined by $(\mathcal{A}u; v)_{\mathcal{H}} = \int_{\Omega_R} \nabla u \cdot \nabla \bar{v} - k^2 \int_{\Omega_R} u \bar{v} + \int_{\Gamma_R} \bar{v} T_R u \quad \forall u, v \in \mathcal{H}$. T_R is a usual Dirichlet-to-Neumann map for which we give the explicit expression and the properties in the appendix A. We also denote γ^{ϵ} the surface measure on Γ^{ϵ} . So if $v \in C^{\infty}(\Omega_R)$, then

$$\int_{\Gamma^{\epsilon}} v(\mathbf{x}) d\sigma(\mathbf{x}) = \int_{\nu=-1}^{+1} \int_{\theta=0}^{2\pi} v \circ (\phi_{\text{el}}^{\Gamma^{\epsilon}})^{-1}(\nu; \theta) \gamma^{\epsilon}(\nu; \theta) d\nu d\theta.$$

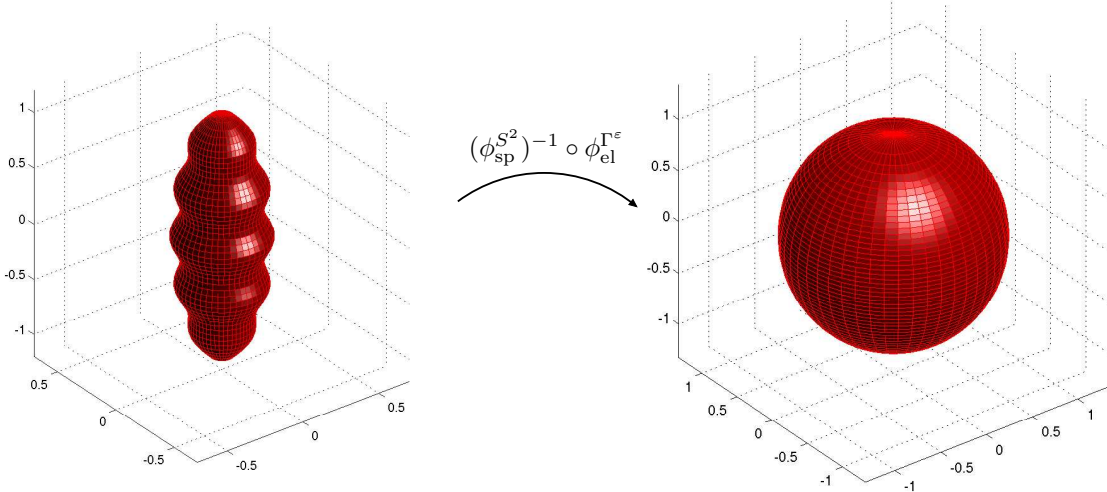
We denote $\mathcal{H}_0^{\epsilon} = \{v \in \mathcal{H} \text{ t.q. } v|_{\Gamma^{\epsilon}} = 0\}$. In §7.1 we use the operator $\mathcal{A}_0^{\epsilon} : \mathcal{H}_0^{\epsilon} \rightarrow \mathcal{H}_0^{\epsilon}$ defined by $(\mathcal{A}_0^{\epsilon}u; v)_{\mathcal{H}} = (\mathcal{A}u; v)_{\mathcal{H}} \quad \forall u, v \in \mathcal{H}_0^{\epsilon}$.

2.4.2 Functional space on the wire

We will need a functional setting for functions “living on Γ^{ϵ} ” that will in practice be function on J . First denote $\mathcal{L} = \frac{\partial}{\partial \nu}(1 - \nu^2) \frac{\partial}{\partial \nu}$ unbounded operator on $L^2(I)$. Define $E^n(I) = \{u \in L^2(I) \mid \langle (-\mathcal{L})^k u; u \rangle_{L^2(I)} < +\infty \quad k = 0 \dots n\}$ with the norm $\|u\|_{E^n(I)}^2 = \sum_{q=0}^n \langle (-\mathcal{L})^q u; u \rangle_{L^2(I)}$.

2.4.3 Trace space

Now we will study in detail the space $H^{1/2}(\Gamma^{\epsilon})$. We begin with a technical lemma of geometry that establishes topological equivalence between the wire Γ^{ϵ} and the sphere S^2 . This enables us to study $H^{1/2}(S^2)$ instead of $H^{1/2}(\Gamma^{\epsilon})$. In particular we will define an explicit norm on $H^{1/2}(\Gamma^{\epsilon})$ using this equivalence.

**Lemma 2.1**

The map $(\phi_{sp}^{S^2})^{-1} \circ \phi_{el}^{\Gamma^\epsilon} : \Gamma^\epsilon \setminus \{O_+^{\Gamma^\epsilon}, O_-^{\Gamma^\epsilon}\} \rightarrow S^2 \setminus \{O_+^{S^2}, O_-^{S^2}\}$ can be extended in a C^∞ -diffeomorphism of Γ^ϵ into S^2 .

Proof:

Defining $(\phi_{sp}^{S^2})^{-1} \circ \phi_{el}^{\Gamma^\epsilon}(O_+^{\Gamma^\epsilon}) = O_+^{S^2}$ and $(\phi_{sp}^{S^2})^{-1} \circ \phi_{el}^{\Gamma^\epsilon}(O_-^{\Gamma^\epsilon}) = O_-^{S^2}$, it is easy to see that $(\phi_{sp}^{S^2})^{-1} \circ \phi_{el}^{\Gamma^\epsilon}$ is a bijective map between Γ^ϵ and S^2 . If we can prove the regularity of $(\phi_{sp}^{S^2})^{-1} \circ \phi_{el}^{\Gamma^\epsilon}$ extended in this manner, then the inverse mapping theorem will yield the desired result. We construct a function $\mathcal{G} : \mathcal{E} \rightarrow \mathcal{E}$ such that $\mathcal{G}|_{\Gamma^\epsilon} = (\phi_{sp}^{S^2})^{-1} \circ \phi_{el}^{\Gamma^\epsilon}$ and show that \mathcal{G} is a C^∞ function on a neighborhood of Γ^ϵ . Define $\phi = \text{pr} \circ \phi_{el} \circ \phi_{car}^{-1}$, where pr is the projection on the first ellipsoidal coordinate ξ , and ϕ_{car} is the map associated with cartesian coordinates. Thus for a point $\mathbf{x} \in \mathcal{E}$ with cartesian coordinates (x, y, z) and ellipsoidal coordinates (ξ, ν, θ) , we have $\xi = \phi(x, y, z)$. We show that $\phi^2 - 1$ is a C^∞ function on $\phi_{car}(\mathcal{E} \setminus J)$, and $\phi^2(x, y, z) - 1 > 0$ for any $(x, y, z) \in \phi_{car}(\mathcal{E} \setminus J)$. Indeed, according to the formulas defining the ellipsoidal system, the following equality always holds

$$\frac{x^2 + y^2}{\xi^2 - 1} + \frac{z^2}{\xi^2} = 1. \quad (3)$$

So by means of a simple computation we obtain the explicit expression of ϕ ,

$$\phi^2(x, y, z) - 1 = \frac{-(1 - x^2 - y^2 - z^2) + \sqrt{(1 - x^2 - y^2 - z^2)^2 + 4(x^2 + y^2)}}{2}.$$

This yields the desired properties. We also obtain the same properties for ϕ and $\sqrt{\phi^2 - 1}$. We define \mathcal{G} in cartesian coordinates by

$$\phi_{car} \circ \mathcal{G} \circ \phi_{car}^{-1}(x, y, z) = \left(\frac{x}{\sqrt{\phi^2(x, y, z) - 1}}, \frac{y}{\sqrt{\phi^2(x, y, z) - 1}}, \frac{z}{\phi(x, y, z)} \right).$$

Using the equation (3), we see that $S^2 = \mathcal{G}(\mathcal{E} \setminus J)$ and that \mathcal{G} is a C^∞ on $\mathcal{E} \setminus J$. We also easily see that $\mathcal{G}|_{\Gamma^\epsilon} = (\phi_{sp}^{S^2})^{-1} \circ \phi_{el}^{\Gamma^\epsilon}$, hence the result.

■

The preceding result leads to a very simple characterization of $H^{1/2}(\Gamma^\varepsilon)$.

Corollary 2.1

$$u \in H^{1/2}(\Gamma^\varepsilon) \Leftrightarrow \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} (1+l^2)^{1/2} \left| \int_{\nu=-1}^{+1} \int_{\theta=0}^{2\pi} u(\nu, \theta) P_l^m(\nu) e^{im\theta} d\nu d\theta \right|^2 < +\infty.$$

Note that the expression “ $u(\nu, \theta)$ ” is not perfectly rigorous, and we should write “ $u \circ (\phi_{\text{el}}^{\Gamma^\varepsilon})^{-1}(\nu, \theta)$ ” instead. We can reformulate this corollary writing

$$u \in H^{1/2}(\Gamma^\varepsilon) \Leftrightarrow ((\phi_{\text{el}}^{\Gamma^\varepsilon})^{-1} \circ \phi_{\text{sp}}^{S^2})^* u \in H^{1/2}(S^2). \quad (4)$$

Here we have used the notation $*$ defined by $\phi^* u = u \circ \phi$. The preceding characterization depends on ε only through the chart $\phi_{\text{el}}^{\Gamma^\varepsilon}$. From this characterization, we obtain a natural norm

$$\|u\|_{H^{1/2}(\Gamma^\varepsilon)}^2 = \sum_{l=0}^{+\infty} \sum_{m=-l}^{+l} (1+l^2)^{1/2} \left| \int_{\nu=-1}^{+1} \int_{\theta=0}^{2\pi} u(\nu, \theta) P_l^m(\nu) e^{im\theta} d\nu d\theta \right|^2.$$

We also define the dual norm in the following manner

$$\|u\|_{H^{-1/2}(\Gamma^\varepsilon)} = \sup_{v \in H^{1/2}(\Gamma^\varepsilon)} \frac{\int_{\Gamma^\varepsilon} u \bar{v}}{\|v\|_{H^{1/2}(\Gamma^\varepsilon)}}.$$

We will now construct a lifting operator continuously mapping $H^{1/2}(\Gamma^\varepsilon)$ to $H^1(\Omega_R)$, with an operator norm bounded as $\varepsilon \rightarrow 0$.

Lemma 2.2

There exists $\kappa, \varepsilon_0 > 0$ such that $\forall \varepsilon \in]0, \varepsilon_0[$, $\forall u \in H^{1/2}(\Gamma^\varepsilon)$ there exists $\mathcal{R}(u) \in H^1(\Omega_R)$ satisfying $\mathcal{R}(u)|_{\Gamma^\varepsilon} = u$ and $\|\mathcal{R}(u)\|_{H^1(\Omega_R)} \leq \kappa \|u\|_{H^{1/2}(\Gamma^\varepsilon)}$.

Proof:

To prove this lemma we will transform our actual wire into an ellipsoid which has a simpler (separable) shape. Denote $\check{\Gamma}^\varepsilon$ the ellipsoid with equation $\xi^2 = 1 + \varepsilon^2$. We introduce a geometrical transformation mapping Γ^ε into the ellipsoid $\check{\Gamma}^\varepsilon$. Let $\mathcal{G} : \mathcal{E} \rightarrow \mathcal{E}$ be a bijective function (take care that this \mathcal{G} has nothing to do with the \mathcal{G} of lemma 2.1). \mathcal{G} is described in ellipsoidal coordinates by the formula

$$\mathcal{G}_{\text{el}}(\xi', \nu', \theta') = \phi_{\text{el}} \circ \mathcal{G} \circ \phi_{\text{el}}^{-1}(\xi', \nu', \theta') = (\xi, \nu, \theta) = \left(\sqrt{1 + \frac{\xi'^2 - 1}{\Phi^2(\nu', \theta')}} , \nu', \theta' \right)$$