

III – Pseudo-inverse

System to solve

$$m \times 1 \quad n \times 1$$

$$\underline{\dot{X}} = \underset{m \times n}{J(q)} \cdot \underline{\dot{q}} \quad \Rightarrow \quad \underline{\dot{q}} = J^{(-1)}(q) \cdot \underline{\dot{X}}$$

3 possible cases :

- J is square : $n = m$ and $\det(J) \neq 0$
- J is square : $n = m$ and $\det(J) = 0$
- J isn't square : $n > m$

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System to solve

- J is square : $n = m$ and $\det(J) \neq 0$


$$\mathbf{J}^{(-1)}(\mathbf{q}) = \mathbf{J}^{-1}(\mathbf{q})$$

- J is square: $n = m$ and $\det(J) = 0$

1. $\det(J) = 0$ only for some values of q (singularities)
2. $\det(J) = 0$ for all values of q



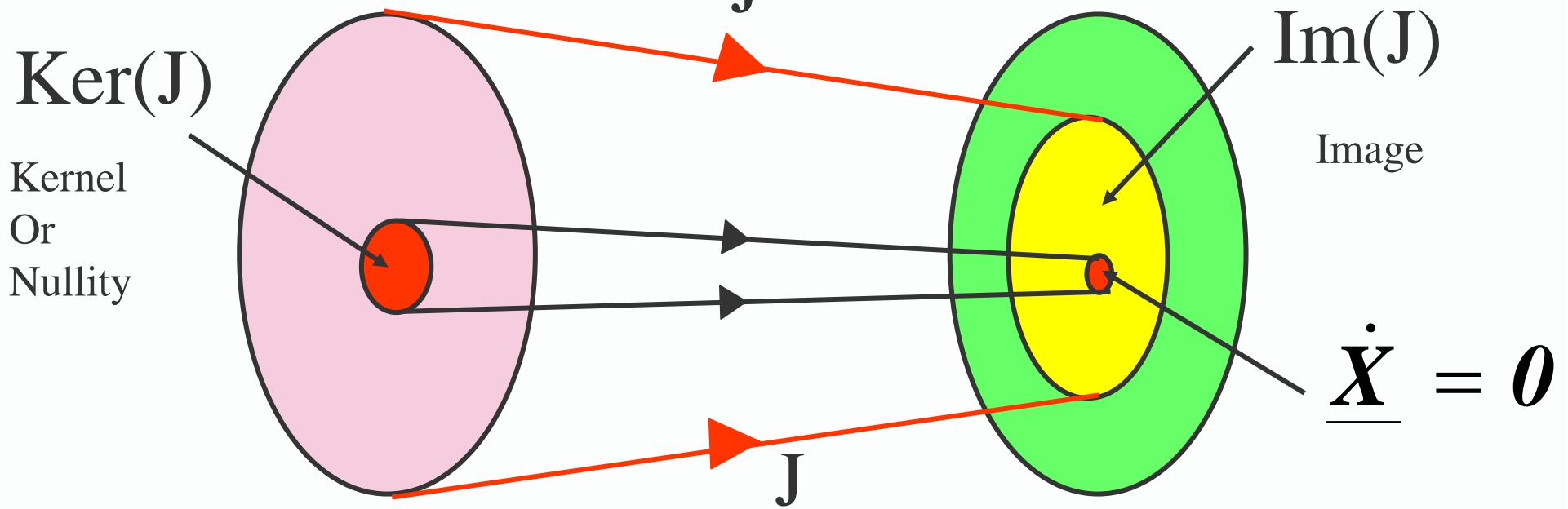
We degenerate the system by eliminating one or more variables in \mathbf{X} (Method of main variables)

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System to solve

$$\underline{\dot{q}} \in \mathcal{R}^n$$

$$\underline{\dot{X}} \in \mathcal{R}^m$$



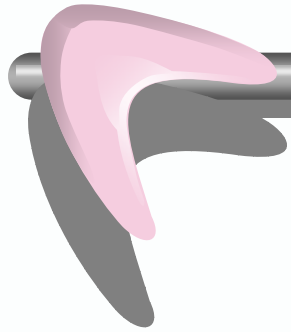
$$\text{Dim}(\text{Ker}(\mathbf{J})) + \text{Dim}(\text{Im}(\mathbf{J})) = n$$

$$\text{Dim}(\text{Ker}(\mathbf{J})) = n - r$$

$$\text{Dim}(\text{Im}(\mathbf{J})) = r \quad (r = \text{rank of the matrix } \mathbf{J})$$

*If \mathbf{J} is full rank
then $\text{Dim}(\text{Ker}(\mathbf{J})) = n - m$
Number of redundant dof*

III – Pseudo-inverse



System to solve

- J isn't square : $n > m$

$$\underline{\dot{q}} \in \mathcal{R}^n$$

$$\underline{\dot{X}} \in \mathcal{R}^m$$

When there is several solutions,
we minimize one criteria i.e.

$$C = \sum_{i=1}^n \Delta q_i^2$$

General form of a quadratic criteria (*y variable, a coefficient >0*)

$$C = a_1^2 \cdot y_1^2 + a_2^2 \cdot y_2^2 + \dots + a_t^2 \cdot y_t^2$$

In vectorial form we get

$$C = \underline{\Delta q}^T \cdot \mathbf{M} \cdot \underline{\Delta q}$$

M is a symmetric matrix
Defined and positive

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- \mathbf{J} isn't square : $n > m$

$$\underline{\dot{\mathbf{q}}} \in \mathfrak{R}^n$$

$$\underline{\dot{\mathbf{X}}} \in \mathfrak{R}^m$$

Finding $\mathbf{J}^{(-1)}$ is equivalent to find $\Delta \underline{\mathbf{q}}$ which minimize \mathbf{C}

Knowing that $\Delta \underline{\mathbf{q}}$ must satisfy $\Delta \underline{\mathbf{X}} = \mathbf{J}(\underline{\mathbf{q}}) \cdot \Delta \underline{\mathbf{q}}$

We build the lagrangian of the system

$$\mathbf{L} = \frac{1}{2} \Delta \underline{\mathbf{q}}^T \cdot \mathbf{M} \cdot \Delta \underline{\mathbf{q}} + \underline{\Lambda}^T \cdot \left(\Delta \underline{\mathbf{X}} - \mathbf{J}(\underline{\mathbf{q}}) \cdot \Delta \underline{\mathbf{q}} \right)$$

$\underline{\Lambda}^T$ = multiplier lagrange vector

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- J isn't square : $n > m$

$$\underline{\dot{q}} \in \mathfrak{R}^n$$
$$\underline{\dot{X}} \in \mathfrak{R}^m$$

Minimizing \mathbf{C} is minimizing the lagrangian (look optimization techniques)

Minimum (Lie75) is obtained when choosing

$$\Delta \underline{q} = \mathbf{M}^{-1} \cdot \mathbf{J}^T(\underline{q}) \cdot \left(\mathbf{J}(\underline{q}) \cdot \mathbf{M}^{-1} \cdot \mathbf{J}^T(\underline{q}) \right)^{-1} \cdot \Delta \underline{X}$$

For instance : $\mathbf{M} = \mathbf{I}_n$ $\mathbf{C} = \frac{1}{2} \Delta \underline{q}^T \cdot \Delta \underline{q} = \frac{1}{2} \sum_{i=1}^n \Delta q_i^2$

$$\Delta \underline{q} = \underbrace{\mathbf{J}^T(\underline{q}) \cdot \left(\mathbf{J}(\underline{q}) \cdot \mathbf{J}^T(\underline{q}) \right)^{-1}}_{\text{Pseudo-inverse}} \cdot \Delta \underline{X} = \mathbf{J}^+ \cdot \Delta \underline{X}$$

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System to solve

$$\underline{\dot{q}} \in \mathcal{R}^n$$

- J isn't square : $n > m$

$$\underline{\dot{X}} \in \mathcal{R}^m$$

The solutions for $\underline{\Delta q} = \mathbf{J}^{(-I)}(\mathbf{q}) \cdot \underline{\Delta X}$

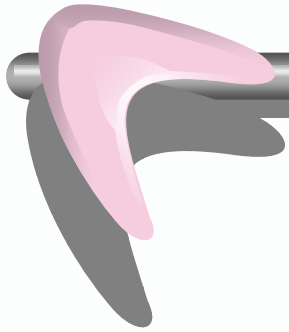

$$\underline{\Delta q} = \mathbf{J}^+ \cdot \underline{\Delta X} + \left(\mathbf{I}_n - \mathbf{J}^+ \cdot \mathbf{J} \right) \cdot \mathbf{Z}$$

\mathbf{J}^+ is unique, but \mathbf{Z} is not unique

$\left(\mathbf{I}_n - \mathbf{J}^+ \cdot \mathbf{J} \right)$ Orthogonal projector on the kernel of J

Choice of \mathbf{Z} : gradient vector of a scalar function depending of \mathbf{q} (can be cost function or potential function)

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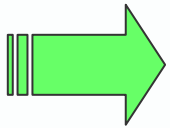
System to solve

$$\underline{\dot{q}} \in \mathcal{R}^n$$

- **J isn't square : $n > m$**

$$\underline{\dot{X}} \in \mathcal{R}^m$$

Choix de **Z** :



- **Avoiding obstacles**
- **Joint limit avoidance**
- **Singularities avoidance**
- **Force and torque distribution in joint space**

For instance, one cost function between current position and mean value is

$$\Phi(\underline{q}) = \frac{1}{n} \left[\frac{|\mathbf{q}_1 - \bar{\mathbf{q}}_1|^2}{|\mathbf{q}_{1 \max} - \mathbf{q}_{1 \min}|^2} + \frac{|\mathbf{q}_2 - \bar{\mathbf{q}}_2|^2}{|\mathbf{q}_{2 \max} - \mathbf{q}_{2 \min}|^2} + \dots \right]$$

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System to solve

- J isn't square : $n > m$

$$\underline{\dot{q}} \in \mathcal{R}^n$$
$$\underline{\dot{X}} \in \mathcal{R}^m$$

$$\Phi(\underline{q}) \in [0 : 1]$$

$$\underline{Z} = \alpha \cdot \text{grad}(\Phi(\underline{q}))^T = \alpha \cdot \left[\frac{\delta\Phi}{\delta q_1} \quad \frac{\delta\Phi}{\delta q_2} \quad \dots \quad \frac{\delta\Phi}{\delta q_n} \right]^T$$

The global solutions minimizing the two criterias

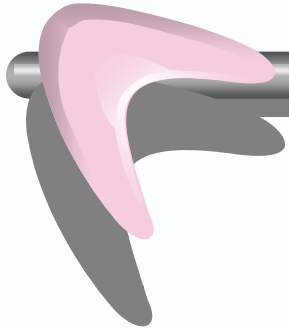
$$\underline{\Delta q}^* = \underline{\Delta q}^{(1)} + \underline{\Delta q}^{(2)}$$

Minimizing the distance between current position and mean value

Minimizing the quadratic criteria

$$\underline{C} = \frac{1}{2} \sum_{i=1}^n \Delta q_i^2$$

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Final remarks on pseudo-inverse of $A_{m \times n}$

Left pseudo inverse

$$m < n \quad \mathbf{A}^+ = \mathbf{A}^T \cdot \left(\mathbf{A} \cdot \mathbf{A}^T \right)^{-1}$$

$n \times m \quad n \times m \quad m \times n \quad n \times m$

Right pseudo inverse

$$m > n \quad \mathbf{A}^+ = \left(\mathbf{A}^T \cdot \mathbf{A} \right)^{-1} \cdot \mathbf{A}^T$$

$n \times m \quad n \times m \quad m \times n \quad n \times m$

III – Pseudo-inverse

Final remarks on pseudo-inverse of $A_{m \times n}$

Definition of the pseudo inverse [Penrose55]

$$\begin{aligned} \mathbf{A} \cdot \mathbf{A}^+ \cdot \mathbf{A} &= \mathbf{A} \\ \mathbf{A}^+ \cdot \mathbf{A} \cdot \mathbf{A}^+ &= \mathbf{A}^+ \\ \mathbf{A}^+ \text{ verify} \quad \mathbf{A} \cdot \mathbf{A}^+ &= \left(\mathbf{A} \cdot \mathbf{A}^+ \right)^T \\ \mathbf{A}^+ \cdot \mathbf{A} &= \left(\mathbf{A}^+ \cdot \mathbf{A} \right)^T \end{aligned}$$