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Basic Knowledge

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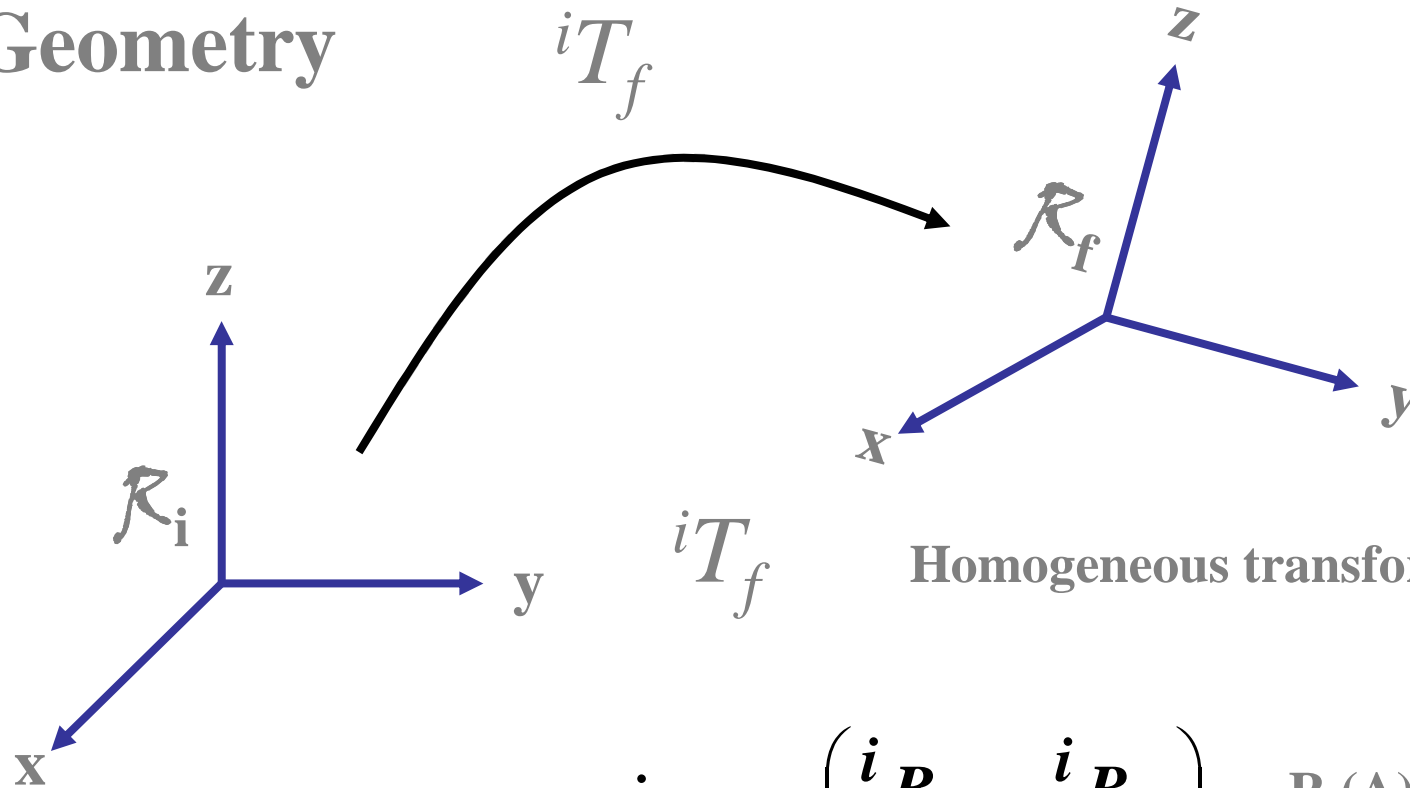
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Basic Knowledge *Homogeneous transformation*

Geometry



Homogeneous transformation matrix

$${}^i T_f = \begin{pmatrix} {}^i R_f & {}^i P_f \\ 000 & 1 \end{pmatrix}$$

R (A): Orientation
P : Position



Basic Knowledge *Homogeneous transformation*

Geometry

Consider a 3D point in space

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}_{R_f}$$

Then

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}_{R_i} \stackrel{=^i}{=} T_f \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}_{R_f} \quad \rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{R_i} \stackrel{=^i}{=} R_f \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{R_f} + {}^i P_f$$



Basic Knowledge Homogeneous transformation

Geometry

$${}^i T_f$$

Homogeneous transformation matrix

Different representations

R : Orientation

s, n, a Cosinus directors

RPY angles (Roll (z), Pitch(y), Yaw(x))

Briant angles (x,y,z)

Euler angles (z,x,z)

u. θ , u. $\sin(\theta)$, u. $\sin(\theta/2)$,

Quaternion $\lambda_1, \lambda_2, \lambda_3, \lambda_4$

P : Position

Cartesian coordinates

Cylindrical coordinates

Spherical coordinates



Basic Knowledge Homogeneous transformation

Geometry

$${}^i T_f$$

Homogeneous transformation matrix

Different representations (i.e)

R : Orientation

P : Position

s, n, a Director Cosinus

Cartesian coordinates

$$R = \begin{pmatrix} s_x & n_x & a_x \\ s_y & n_y & a_y \\ s_z & n_z & a_z \end{pmatrix}$$

$$P = \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}$$

No rotation
 $R = I_3$

No translation
 $P = (0, 0, 0)^T$



Basic Knowledge Homogeneous transformation

Geometry

$${}^i T_f$$

Homogeneous transformation matrix

Main properties of the rotation matrix

R : Orientation

s, n, a Director Cosinus

$$R = \begin{pmatrix} s_x & n_x & a_x \\ s_y & n_y & a_y \\ s_z & n_z & a_z \end{pmatrix}$$

$$\|\underline{s}\| = \|\underline{n}\| = \|\underline{a}\| = 1$$

$$\underline{s} \cdot \underline{n} = 0$$

$$\underline{s} \times \underline{n} = \underline{a}$$

$$\underline{s} \cdot \underline{a} = 0$$

$$\underline{n} \times \underline{a} = \underline{s}$$

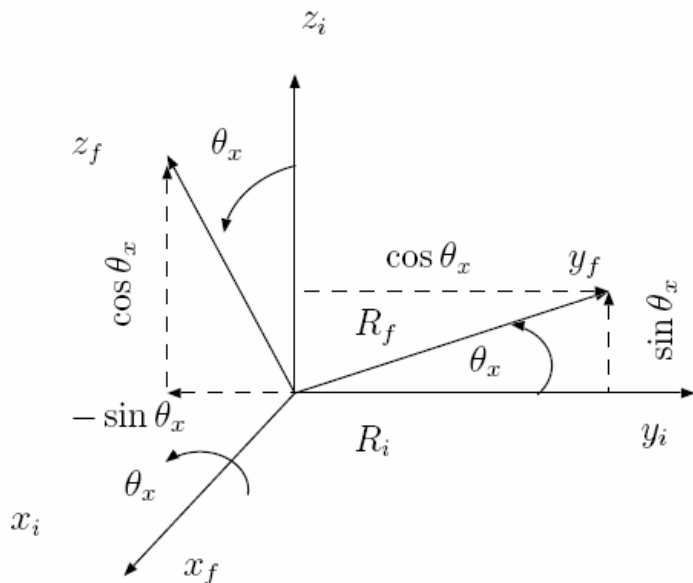
$$\underline{a} \cdot \underline{n} = 0$$

$$\underline{a} \times \underline{s} = \underline{n}$$

$$R^{-1} = R^T$$

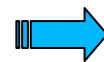
Basic Knowledge Homogeneous transformation: Rotation matrix

Geometry iR_f Rotation matrix $\text{Rot}(x, \theta_x)$



$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_x & -\sin \theta_x \\ 0 & \sin \theta_x & \cos \theta_x \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\theta_x & -s\theta_x \\ 0 & s\theta_x & c\theta_x \end{pmatrix}$$

$$\begin{aligned} \underline{i}_{R_f} &= 1 \cdot \underline{i}_{R_i} + 0 \cdot \underline{j}_{R_i} + 0 \cdot \underline{k}_{R_i} = \underline{i}_{S_f} \\ \underline{j}_{R_f} &= 0 \cdot \underline{i}_{R_i} + \cos \theta_x \cdot \underline{j}_{R_i} + \sin \theta_x \cdot \underline{k}_{R_i} = \underline{i}_{n_f} \\ \underline{k}_{R_f} &= 0 \cdot \underline{i}_{R_i} - \sin \theta_x \cdot \underline{j}_{R_i} + \cos \theta_x \cdot \underline{k}_{R_i} = \underline{i}_{a_f} \end{aligned}$$



$$\begin{aligned} \underline{i}_{R_i} &= \underline{f}_{S_i} \\ \underline{j}_{R_i} &= \underline{f}_{n_i} \\ \underline{k}_{R_i} &= \underline{f}_{a_i} \end{aligned}$$

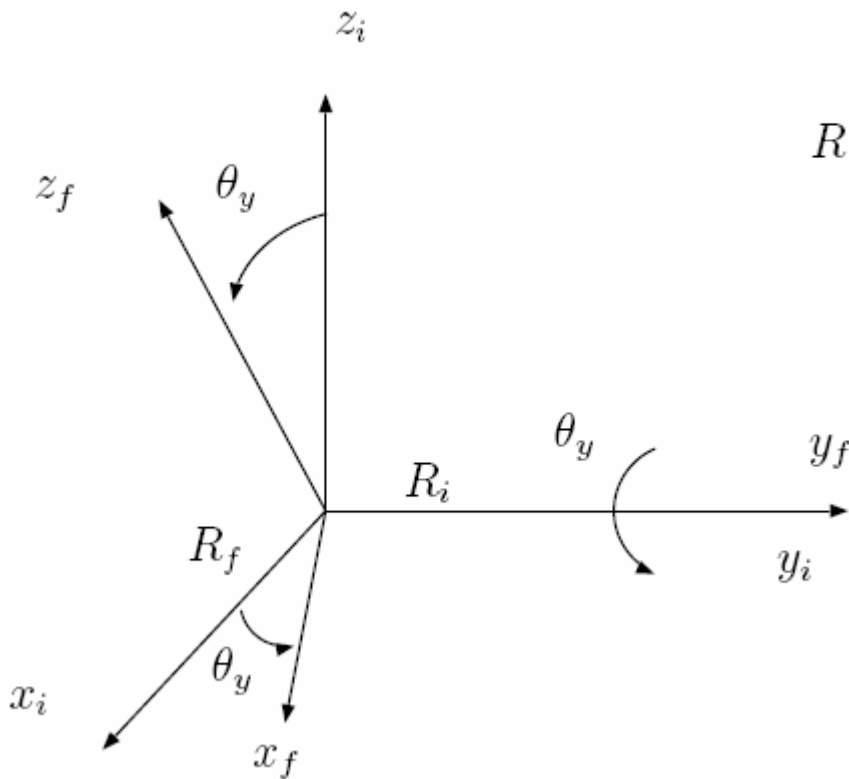
Matrix to change the
frame for one vector

Basic Knowledge Homogeneous transformation: Rotation matrix

Geometry

iR_f

Rotation matrix $\text{Rot}(y, \theta_y)$



$$R = \begin{pmatrix} \cos \theta_y & 0 & \sin \theta_y \\ 0 & 1 & 0 \\ -\sin \theta_y & 0 & \cos \theta_y \end{pmatrix} = \begin{pmatrix} c\theta_y & 0 & s\theta_y \\ 0 & 1 & 0 \\ -s\theta_y & 0 & c\theta_y \end{pmatrix}$$

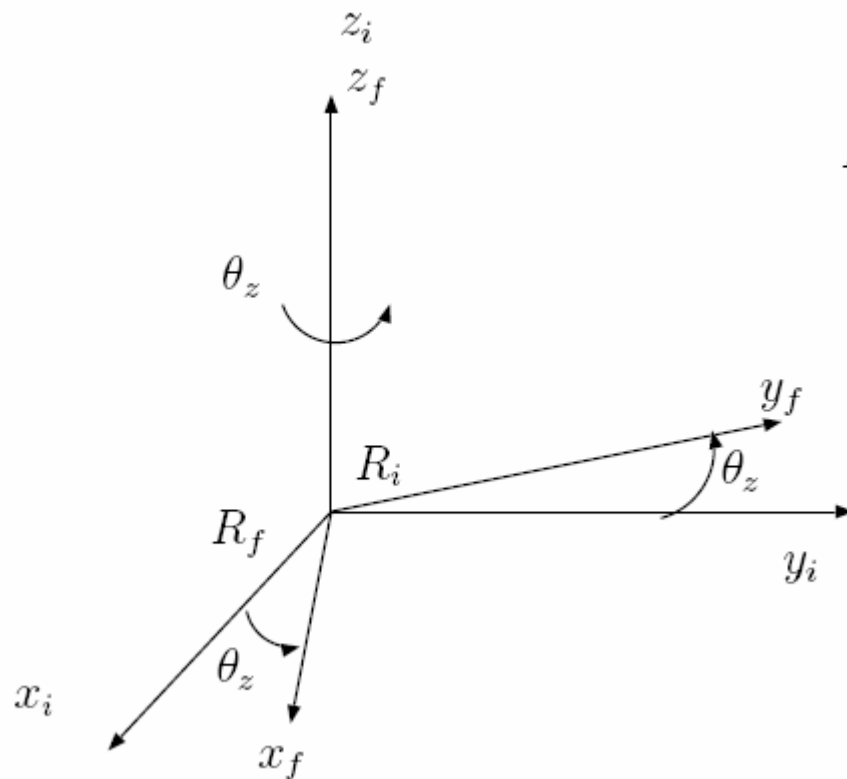


Basic Knowledge *Homogeneous transformation: Rotation matrix*

Geometry

iR_f

Rotation matrix $\text{Rot}(z, \theta_z)$

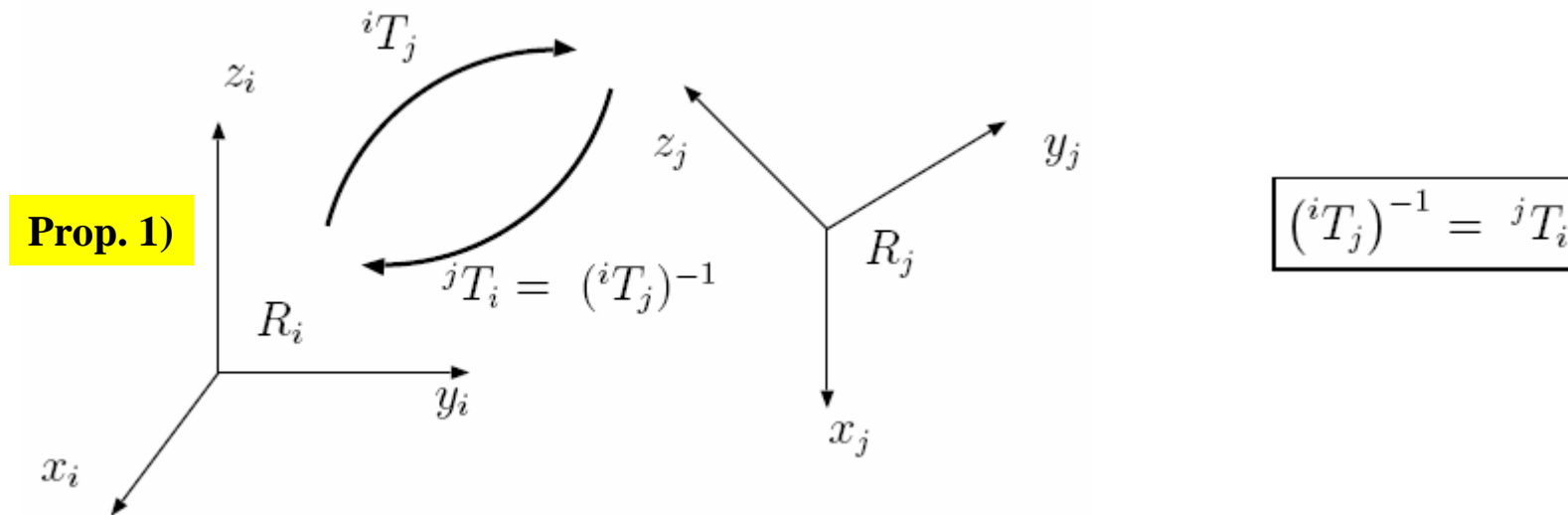


$$R = \begin{pmatrix} \cos \theta_z & -\sin \theta_z & 0 \\ \sin \theta_z & \cos \theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c\theta_z & -s\theta_z & 0 \\ s\theta_z & c\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Basic Knowledge *Homogeneous transformation properties*

Geometry ${}^i T_f$ Homogeneous transformation matrix



Prop. 2)

$$(T)^{-1} = (\text{Rot}(\underline{u}, \theta_u))^{-1} = \text{Rot}(\underline{u}, -\theta_u) = \text{Rot}(-\underline{u}, \theta_u)$$

$$\text{Rot}(x, -\theta_x) = (\text{Rot}(x, \theta_x))^T = (\text{Rot}(x, \theta_x))^{-1}$$

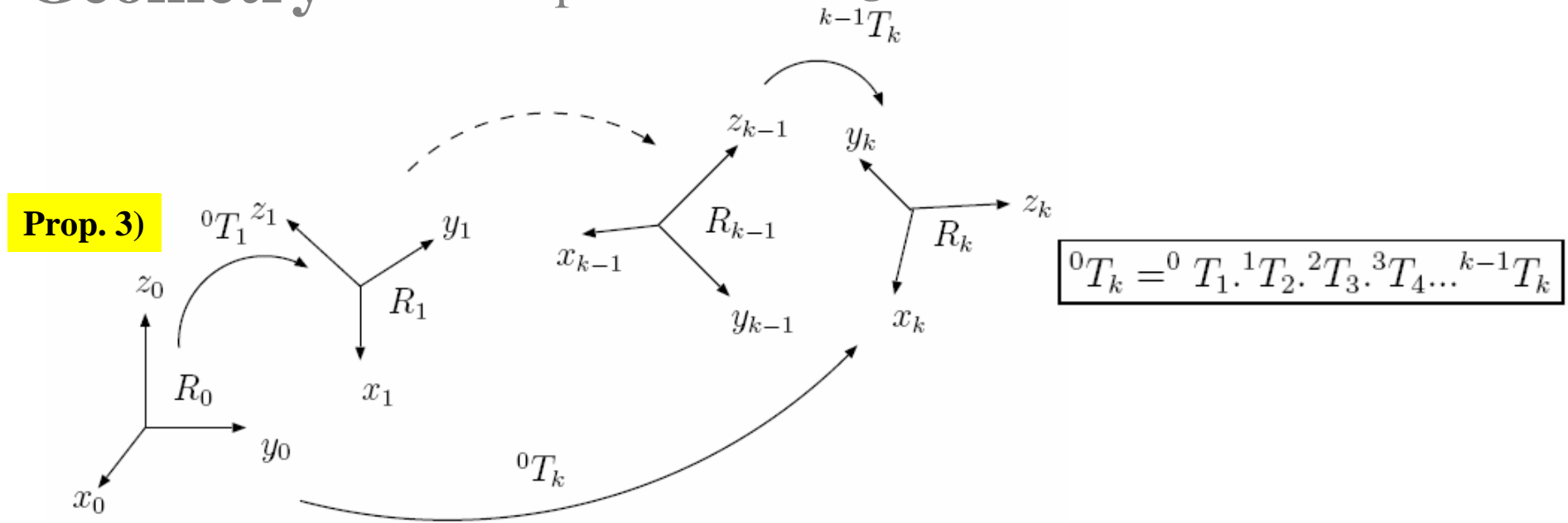
$$(\text{Trans}(u, d))^{-1} = \text{Trans}(-u, d) = \text{Trans}(u, -d)$$

Basic Knowledge Homogeneous transformation properties

Geometry

$${}^i T_f$$

Homogeneous transformation matrix



Prop. 3)

$${}^0 T_k = {}^0 T_1 \cdot {}^1 T_2 \cdot {}^2 T_3 \cdot {}^3 T_4 \dots {}^{k-1} T_k$$

Prop. 4)

$$T^{-1} = \begin{pmatrix} A^T & -\underline{s}^T \cdot P \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} A^T & -A^T \cdot P \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Basic Knowledge Homogeneous transformation properties

Geometry

 ${}^i T_f$

Homogeneous transformation matrix

$$T_1.T_2 = \begin{pmatrix} A_1 & P_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A_2 & P_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1.A_2 & A_1.P_2 + P_1 \\ 0 & 1 \end{pmatrix}$$

Prop. 5)

$$T_2.T_1 = \begin{pmatrix} A_2 & P_2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} A_1 & P_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_2.A_1 & A_2.P_1 + P_2 \\ 0 & 1 \end{pmatrix}$$

$$T_1.T_2 \neq T_2.T_1$$

$$\text{Rot}(\underline{u}, \theta_1). \text{Rot}(\underline{u}, \theta_2) = \text{Rot}(\underline{u}, \theta_1 + \theta_2)$$

Prop. 6)

$$\text{Trans}(\underline{u}, d). \text{Rot}(\underline{u}, \theta_1) = \text{Rot}(\underline{u}, \theta_1). \text{Trans}(\underline{u}, d)$$



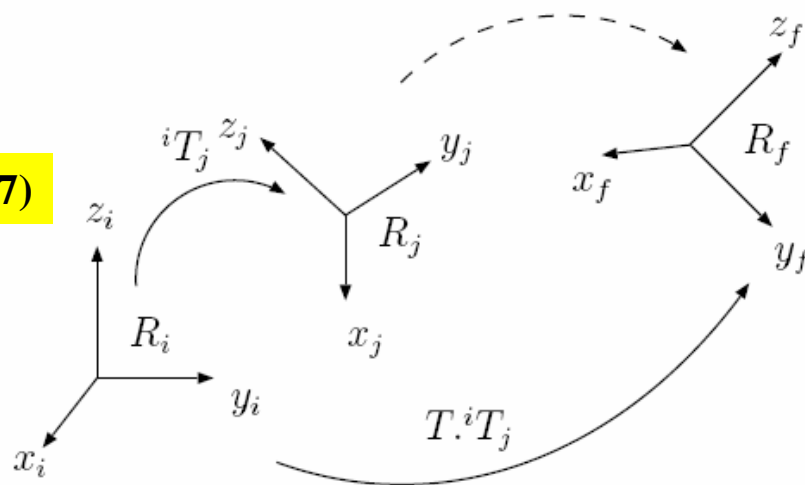
Basic Knowledge *Homogeneous transformation properties*

Geometry

$${}^i T_f$$

Homogeneous transformation matrix

Prop. 7)



$${}^i T_f = T \cdot {}^i T_j$$

T is defined in R_i

$${}^i T_f = {}^i T_j \cdot T'$$

T' is defined in R_j



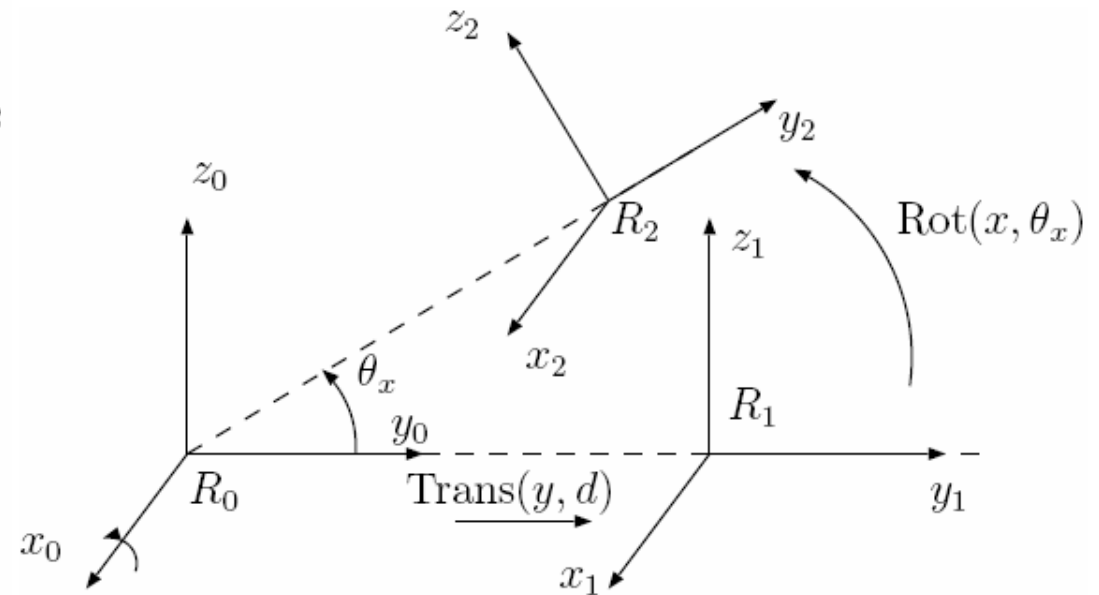
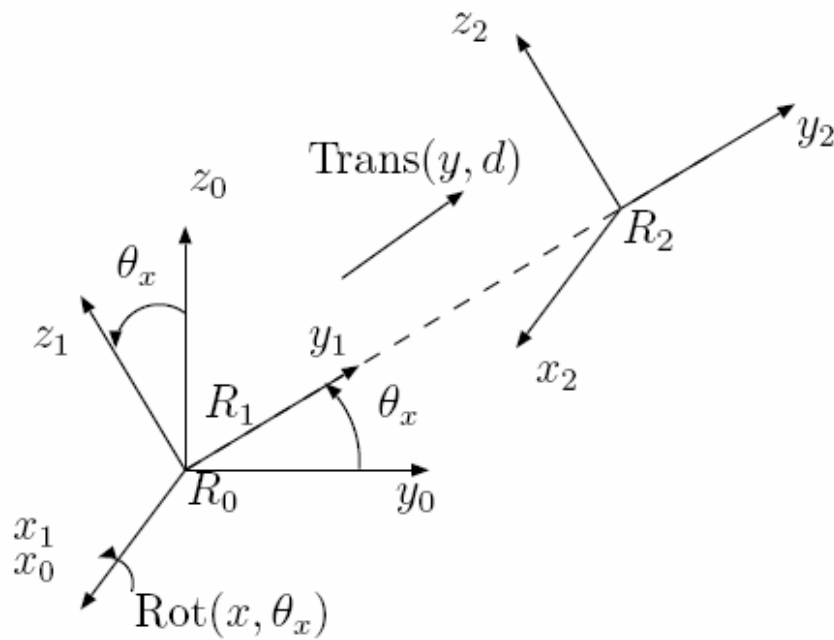
$$T \cdot {}^i T_j = {}^i T_j \cdot T'$$

Basic Knowledge *Homogeneous transformation properties*

Geometry

$${}^i T_f$$

Homogeneous transformation matrix





Basic Knowledge Homogeneous transformation properties

Geometry

 ${}^i T_f$

Homogeneous transformation matrix

$$\hat{u} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}$$

Prop. 8)

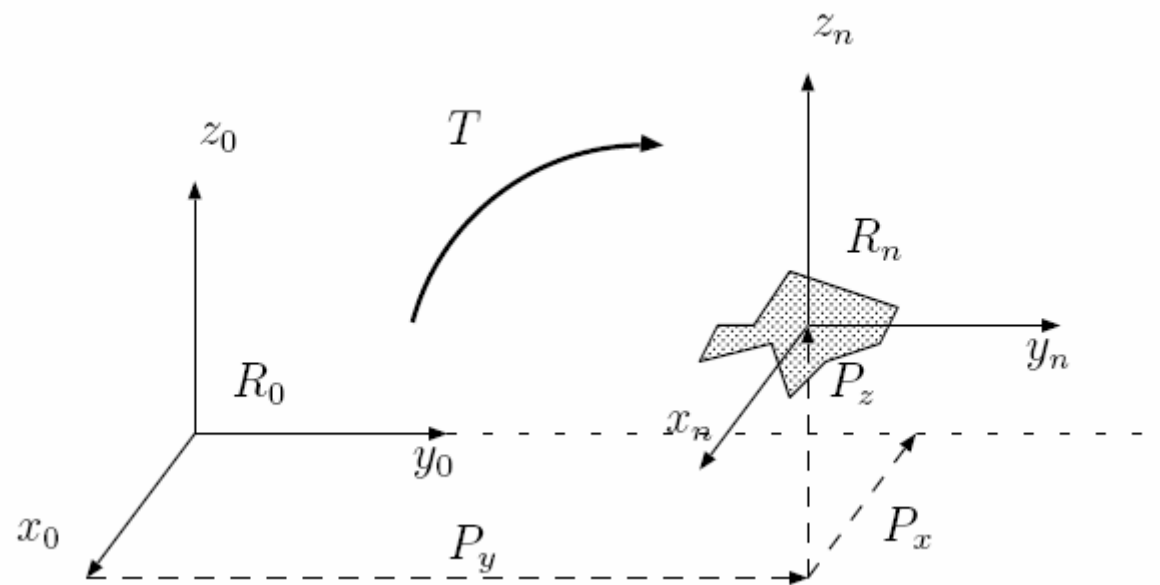
$$\underline{u} \wedge \underline{V} = \hat{u} \cdot \underline{V}$$

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \wedge \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} u_y \cdot V_z - u_z \cdot V_y \\ u_z \cdot V_x - u_x \cdot V_z \\ u_x \cdot V_y - u_y \cdot V_x \end{pmatrix} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix} \cdot \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$



Basic Knowledge *Rigid body pose parameterization: position*

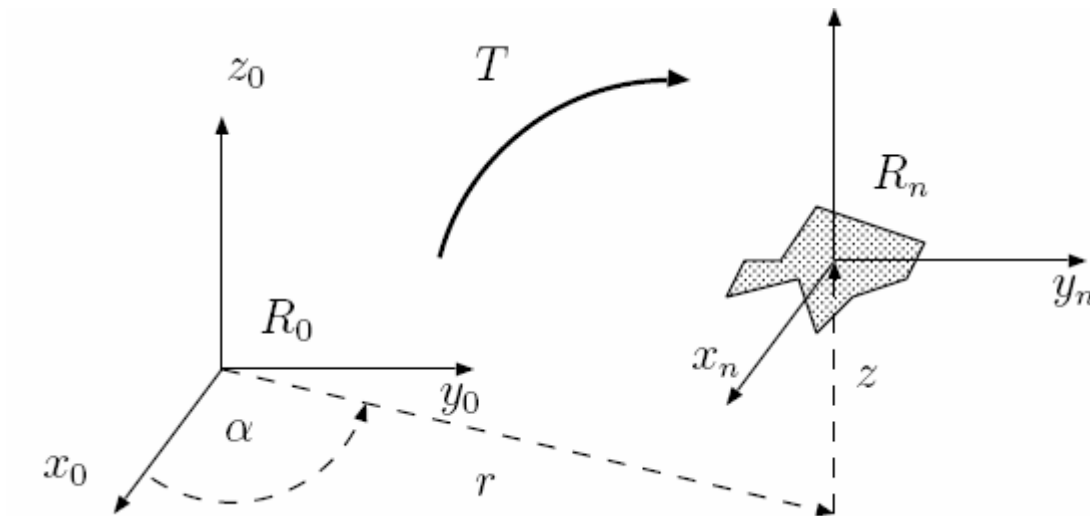
Cartesian coordinates



$$P_{\text{car}} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix}$$

Basic Knowledge Rigid body pose parameterization: position

Cylindrical coordinates



$$P_{\text{cyl}} = \begin{bmatrix} r \cdot \cos \alpha \\ r \sin \alpha \\ z \end{bmatrix}$$

$$r = \sqrt{P_x^2 + P_y^2}$$

$$\alpha = \text{atan2}(P_y, P_x)$$

$$z = P_z$$

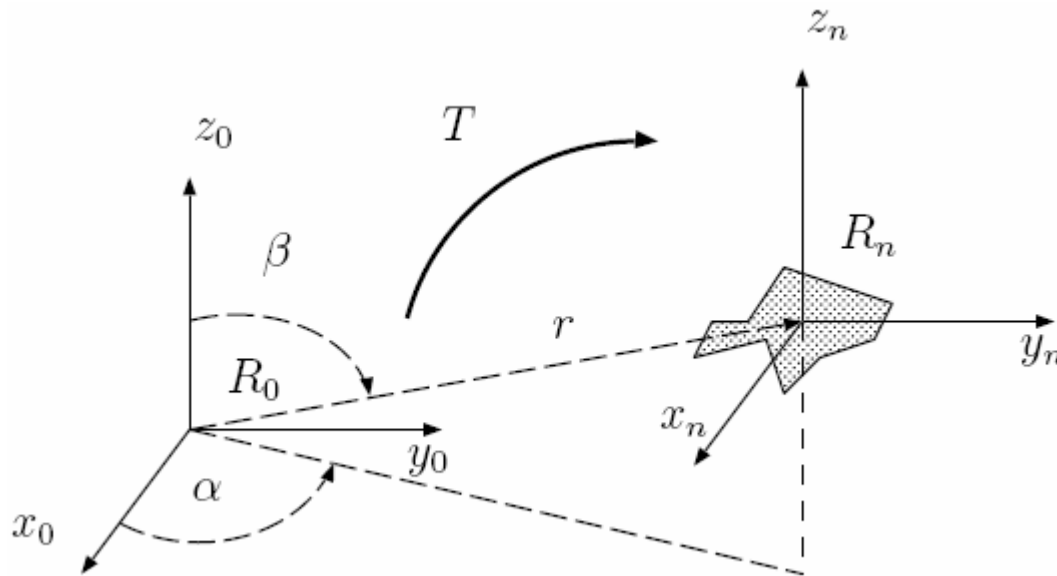
$$\alpha \in [-180; 180]$$

Singularity : $P_x = P_y = 0$



Basic Knowledge *Rigid body pose parameterization: position*

Spherical coordinates



$$P_{\text{sph}} = \begin{bmatrix} r \cdot c\alpha \cdot s\beta \\ r \cdot s\alpha \cdot s\beta \\ r \cdot c\beta \end{bmatrix}$$

$$r = \sqrt{P_x^2 + P_y^2 + P_z^2}$$

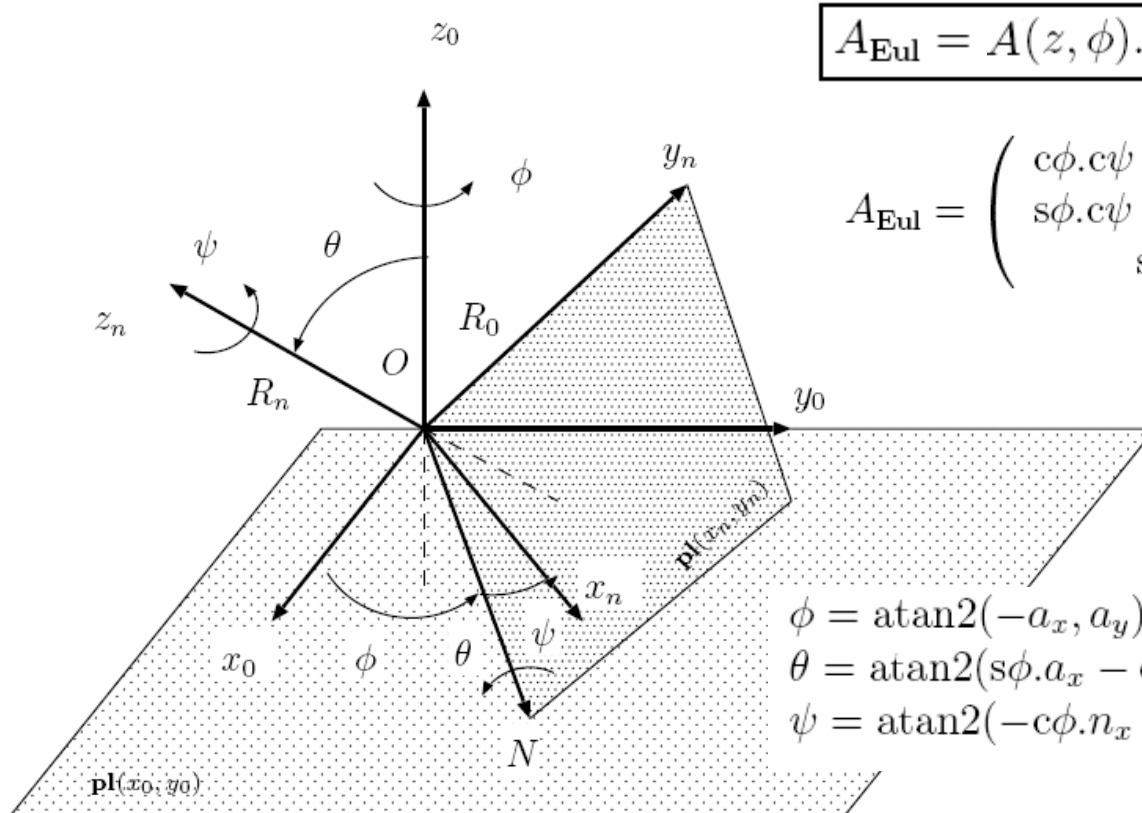
$$\alpha = \text{atan2}(P_y, P_x) \text{ if } \beta \neq 0 \text{ or } \alpha = 0 \text{ if } \beta = 0$$

$$\beta = \text{atan2}\left(\frac{P_y}{s\alpha}, P_z\right) \text{ if } \alpha \neq 0 \text{ or } \beta = \text{atan2}(P_x, P_z) \text{ if } \alpha = 0$$



Basic Knowledge *Rigid body pose parameterization: orientation*

Euler angles (z, x, z)



$$A_{\text{Eul}} = A(z, \phi).A(x, \theta).A(z, \psi)$$

$$A_{\text{Eul}} = \begin{pmatrix} c\phi.c\psi - s\phi.c\theta.s\psi & -c\phi.s\psi - s\phi.c\theta.c\psi & s\phi.s\theta \\ s\phi.c\psi + c\phi.c\theta.s\psi & -s\phi.s\psi + c\phi.c\theta.c\psi & -c\phi.s\theta \\ s\theta.s\psi & s\theta.c\psi & c\theta \end{pmatrix}$$

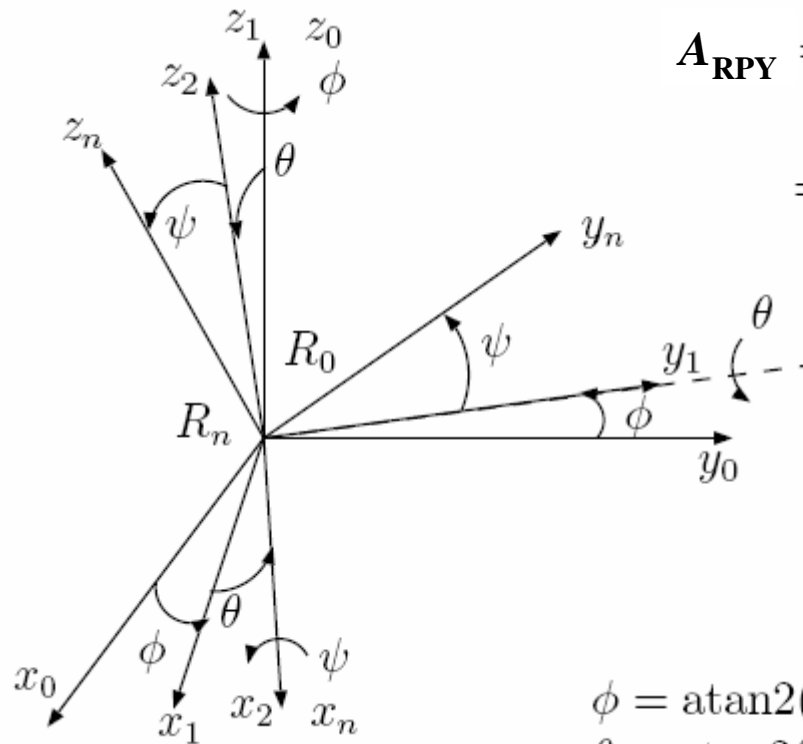
$$\begin{aligned} \phi &= \text{atan2}(-a_x, a_y) & \text{or} & & (\phi = \text{atan2}(a_x, -a_y) + 180^0) \\ \theta &= \text{atan2}(s\phi.a_x - c\phi.a_y, a_z) \\ \psi &= \text{atan2}(-c\phi.n_x - s\phi.n_y, c\phi.s_x + s\phi.s_y) \end{aligned}$$

Remarks: Euler angles can be defined also by (z,y,z) $\rightarrow A_{\text{Eul}} = A(z, \phi).A(y, \theta).A(z, \psi)$



Basic Knowledge *Rigid body pose parameterization: orientation*

RPY angles (z,y,x)



$$A_{\text{RPY}} = A(z, \phi) \cdot A(y, \theta) \cdot A(x, \psi)$$

$$= \begin{pmatrix} c\phi \cdot c\theta & c\phi \cdot s\theta \cdot s\psi - s\phi \cdot c\psi & c\phi \cdot s\theta \cdot c\psi + s\phi \cdot s\psi \\ s\phi \cdot c\theta & s\phi \cdot s\theta \cdot s\psi + c\phi \cdot c\psi & s\phi \cdot s\theta \cdot c\psi - c\phi \cdot s\psi \\ -s\theta & c\theta \cdot s\psi & c\theta \cdot c\psi \end{pmatrix}$$

$$\phi = \text{atan2}(s_y, s_x) \quad \text{or} \quad (\phi = \text{atan2}(-s_y, -s_x) + 180^\circ)$$

$$\theta = \text{atan2}(-s_z, c\phi \cdot s_x + s\phi \cdot s_y)$$

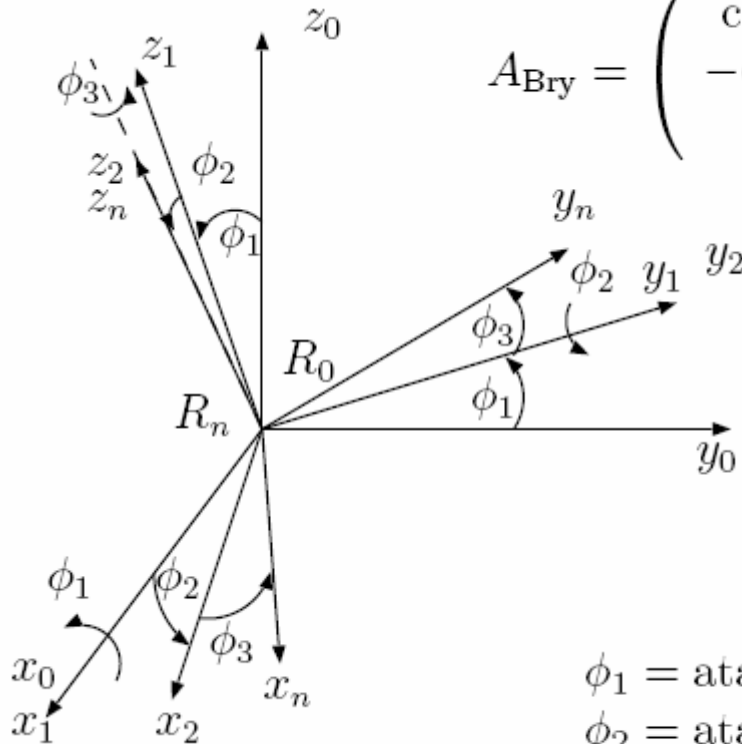
$$\psi = \text{atan2}(s\phi \cdot a_x - c\phi \cdot a_y, -s\phi \cdot n_x + c\phi \cdot n_y)$$



Basic Knowledge *Rigid body pose parameterization: orientation*

Bryant angles (x,y,z) $A_{Bry} = A(x, \phi_1).A(y, \phi_2).A(z, \phi_3)$

$$A_{Bry} = \begin{pmatrix} c\phi_2 \cdot c\phi_3 & c\phi_1 \cdot s\phi_3 + s\phi_1 \cdot s\phi_2 \cdot c\phi_3 & s\phi_1 \cdot s\phi_3 - c\phi_1 \cdot s\phi_2 \cdot c\phi_3 \\ -c\phi_2 \cdot s\phi_3 & c\phi_1 \cdot c\phi_3 - s\phi_1 \cdot s\phi_2 \cdot s\phi_3 & s\phi_1 \cdot c\phi_3 + c\phi_1 \cdot s\phi_2 \cdot s\phi_3 \\ s\phi_2 & -s\phi_1 \cdot c\phi_2 & c\phi_1 \cdot c\phi_2 \end{pmatrix}$$

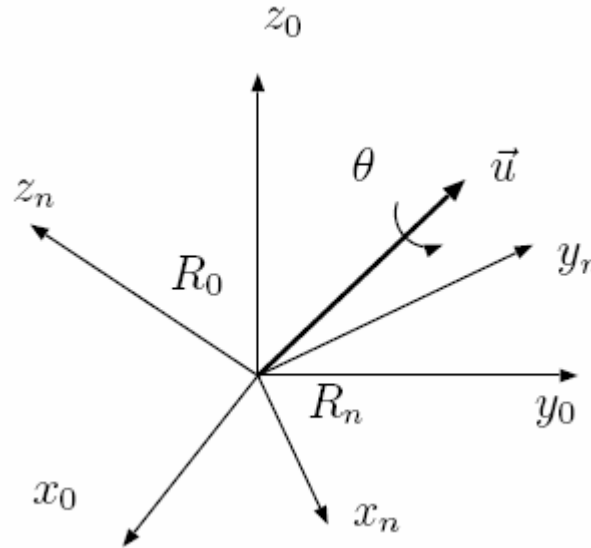


$$\begin{aligned} \phi_1 &= \text{atan2}(-n_z, a_z) \quad \text{or} \quad (\phi_1 = \text{atan2}(n_z, -a_z) + 180^0) \\ \phi_2 &= \text{atan2}(s_z, c\phi_1 \cdot a_z - s\phi_1 \cdot n_z) \\ \phi_3 &= \text{atan2}(c\phi_1 \cdot n_x + s\phi_1 \cdot a_x, c\phi_1 \cdot n_y + s\phi_1 \cdot a_y) \end{aligned}$$



Basic Knowledge Rigid body pose parameterization: orientation

Orientation (\underline{u}, θ)



$$A(\underline{u}, \theta) = \begin{pmatrix} u_x^2 \cdot (1 - c\theta) + c\theta & u_x u_y \cdot (1 - c\theta) - u_z \cdot s\theta & u_x u_z \cdot (1 - c\theta) + u_y \cdot s\theta \\ u_x u_y \cdot (1 - c\theta) + u_z \cdot s\theta & u_y^2 \cdot (1 - c\theta) + c\theta & u_y u_z \cdot (1 - c\theta) - u_x \cdot s\theta \\ u_x u_z \cdot (1 - c\theta) - u_y \cdot s\theta & u_y u_z \cdot (1 - c\theta) + u_x \cdot s\theta & u_z^2 \cdot (1 - c\theta) + c\theta \end{pmatrix}$$

$$A(\underline{u}, \theta) = \underline{u} \cdot \underline{u}^T \cdot (1 - c\theta) + I_3 \cdot c\theta + \hat{u} \cdot s\theta \quad \text{Rodrigues formula}$$

$$A(\underline{u}, \theta) = I_3 + \hat{u} \cdot s\theta + \hat{u}^2 \cdot (1 - c\theta)$$

Basic Knowledge *Rigid body pose parameterization: orientation*

Orientation (\underline{u}, θ)

$$A(\underline{u}, \theta) = \underline{u} \cdot \underline{u}^T \cdot (1 - c\theta) + I_3 \cdot c\theta + \hat{u} \cdot s\theta \quad \text{Rodrigues formula}$$

$$A(\underline{u}, \theta) = I_3 + \hat{u} \cdot s\theta + \hat{u}^2 \cdot (1 - c\theta)$$

$$\hat{u} = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix} \quad \underline{u} \cdot \underline{u}^T = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \cdot (u_x \ u_y \ u_z) = \begin{pmatrix} u_x^2 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 \end{pmatrix}$$

$$\hat{u}^2 = \hat{u} \cdot \hat{u} = \underline{u} \cdot \underline{u}^T - I_3 = \begin{pmatrix} u_x^2 - 1 & u_x u_y & u_x u_z \\ u_x u_y & u_y^2 - 1 & u_y u_z \\ u_x u_z & u_y u_z & u_z^2 - 1 \end{pmatrix}$$

Mac Laurin

$$A(\underline{u}, \theta) = I_3 + \hat{u} \cdot s\theta + \hat{u}^2 \cdot (1 - c\theta) \quad \Rightarrow \quad A(\underline{u}, \theta) = I_3 + \hat{u} \cdot \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots \right) + \hat{u}^2 \cdot \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \dots \right)$$

$$\hat{u}^3 = -\hat{u}, \quad \hat{u}^4 = -\hat{u}^2, \quad \hat{u}^5 = \hat{u}, \quad \hat{u}^6 = \hat{u}^2$$

$$\Rightarrow \quad A(\underline{u}, \theta) = \exp(\hat{u}, \theta) = e^{\hat{u} \cdot \theta}$$



Basic Knowledge *Rigid body pose parameterization: orientation*

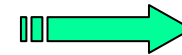
Orientation (\underline{u}, θ)

$$\left. \begin{aligned} A(\underline{u}, \theta) &= I_3 + \hat{u}.s\theta + \hat{u}^2.(1 - c\theta) \\ A(\underline{u}, \theta)^T &= I_3 - \hat{u}.s\theta + \hat{u}^2.(1 - c\theta) \\ A(\underline{u}, \theta) - A(\underline{u}, \theta)^T &= 2.\hat{u}.s\theta \end{aligned} \right\}$$



$$[u.s\theta]^\times = [u.s\theta]^\wedge = \frac{A(\underline{u}, \theta) - A(\underline{u}, \theta)^T}{2}$$

$$\left. \begin{aligned} \text{Trace}(A(\underline{u}, \theta)) &= \text{Trace}(I_3 + \hat{u}.s\theta + \hat{u}^2.(1 - c\theta)) \\ \text{Trace}(A(\underline{u}, \theta)) &= 3 + (1 - c\theta).\text{Trace}(\hat{u}^2) \\ \text{Trace}(A(\underline{u}, \theta)) &= 3 + (1 - c\theta).(-2) \\ \text{Trace}(A(\underline{u}, \theta)) &= 1 + 2.c\theta \end{aligned} \right\}$$



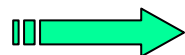
$$\begin{aligned} \text{Tr}(A(\underline{u}, \theta)) &= 1 + 2.c\theta \\ \cos \theta &= \frac{\text{Tr}(A(\underline{u}, \theta)) - 1}{2} \end{aligned}$$

$$\begin{pmatrix} s_x & n_x & a_x \\ s_y & n_y & a_y \\ s_z & n_z & a_z \end{pmatrix} - \begin{pmatrix} s_x & s_y & s_z \\ n_x & n_y & n_z \\ a_x & a_y & a_z \end{pmatrix} = \begin{pmatrix} 0 & n_x - s_y & a_x - s_z \\ s_y - n_x & 0 & a_y - n_z \\ s_z - a_x & n_z - a_y & 0 \end{pmatrix}$$

$$\mathbf{s}_y - \mathbf{n}_x = 2 \mathbf{u}_z s\theta$$

$$\mathbf{a}_x - \mathbf{s}_z = 2 \mathbf{u}_y s\theta$$

$$\mathbf{n}_z - \mathbf{a}_y = 2 \mathbf{u}_x s\theta$$



$$\sin \theta = \pm \frac{1}{2} \sqrt{(s_y - n_x)^2 + (a_x - s_z)^2 + (n_z - a_y)^2}$$

$$\underline{u} = \frac{1}{2.\sin \theta} \begin{pmatrix} n_z - a_y \\ a_x - s_z \\ s_y - n_x \end{pmatrix}$$

Basic Knowledge Rigid body pose parameterization: orientation

Quaternion $(\underline{\lambda}_1, \underline{\lambda}_2, \underline{\lambda}_3, \underline{\lambda}_4)$ $\theta \in [-180; 180]$

$$\lambda_1 = \cos\left(\frac{\theta}{2}\right)$$

$$\lambda_2 = u_x \cdot \sin\left(\frac{\theta}{2}\right)$$

$$\lambda_3 = u_y \cdot \sin\left(\frac{\theta}{2}\right)$$

$$\lambda_4 = u_z \cdot \sin\left(\frac{\theta}{2}\right)$$

$$\lambda_1 = \frac{1}{2} \cdot \sqrt{s_x + n_y + a_z + 1}$$

$$\lambda_2 = \frac{1}{2} \cdot \text{sign}(n_x - a_y) \cdot \sqrt{s_x - n_y - a_z + 1}$$

$$\lambda_3 = \frac{1}{2} \cdot \text{sign}(a_x - s_z) \cdot \sqrt{-s_x + n_y - a_z + 1}$$

$$\lambda_4 = \frac{1}{2} \cdot \text{sign}(s_y - n_x) \cdot \sqrt{-s_x - n_y + a_z + 1}$$

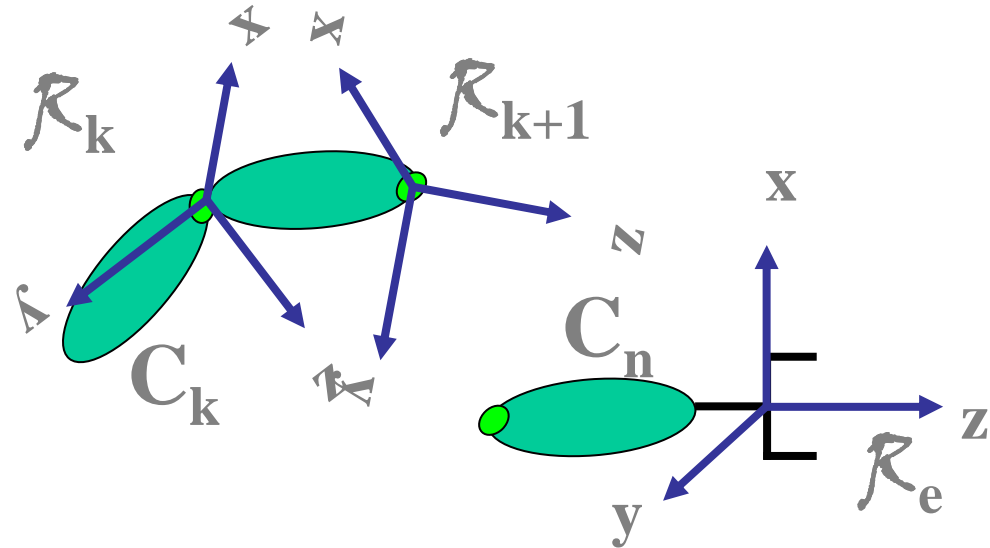
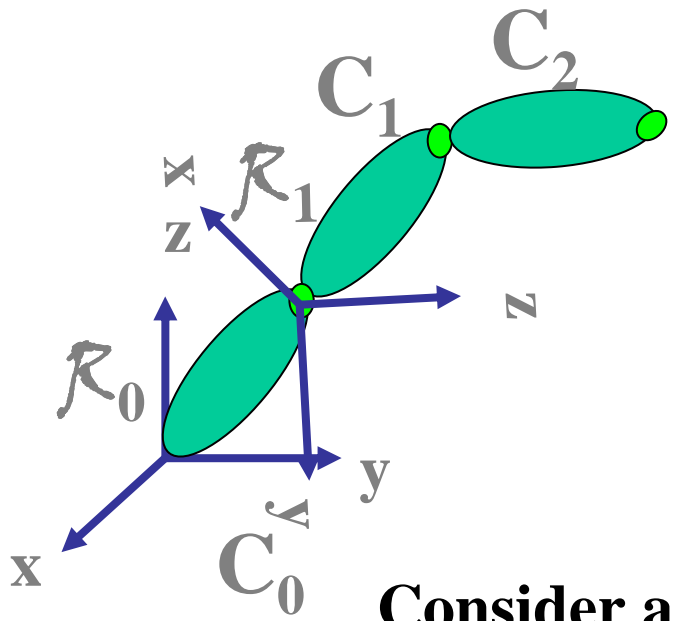
$$A_{Quat} = \begin{pmatrix} 2 \cdot (\lambda_1^2 + \lambda_2^2) - 1 & 2 \cdot (\lambda_2 \cdot \lambda_3 - \lambda_1 \cdot \lambda_4) & 2 \cdot (\lambda_2 \cdot \lambda_4 + \lambda_1 \cdot \lambda_3) \\ 2 \cdot (\lambda_2 \cdot \lambda_3 + \lambda_1 \cdot \lambda_4) & 2 \cdot (\lambda_1^2 + \lambda_3^2) - 1 & 2 \cdot (\lambda_3 \cdot \lambda_4 - \lambda_1 \cdot \lambda_2) \\ 2 \cdot (\lambda_2 \cdot \lambda_4 - \lambda_1 \cdot \lambda_3) & 2 \cdot (\lambda_3 \cdot \lambda_4 + \lambda_1 \cdot \lambda_2) & 2 \cdot (\lambda_1^2 + \lambda_4^2) - 1 \end{pmatrix}$$

$$2\lambda_1^2 - 1 = \cos \theta$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1$$

Basic Knowledge *Multi-Rigid bodies*

Case of serial manipulator robot



Consider a robot with $n+1$ rigid bodies C_k
 We associate $n+1$ frames
 C_0 is the base of the robot (fixed)



Basic Knowledge Multi-Rigid bodies

Case of serial manipulator robot

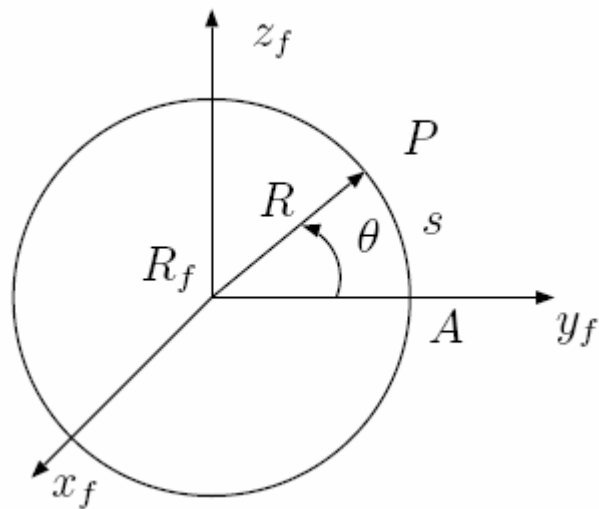
The problem to solve is to obtain the position and orientation of the end effector frame R_e in the fixed frame R_0

$${}^k T_{k+1} = \begin{pmatrix} {}^k R_{k+1} & {}^k P_{k+1} \\ 000 & 1 \end{pmatrix} \quad \text{Elementary frame transform}$$

$${}^0 T_e = {}^0 T_1 \cdot {}^1 T_2 \cdot {}^2 T_3 \cdots {}^{n-1} T_n \cdot {}^n T_e$$

Basic Knowledge *Rigid body kinematics*

Circular motion



$$s = R.\theta$$

$$v = \frac{ds}{dt} = R.\frac{d\theta}{dt} = R.\omega$$

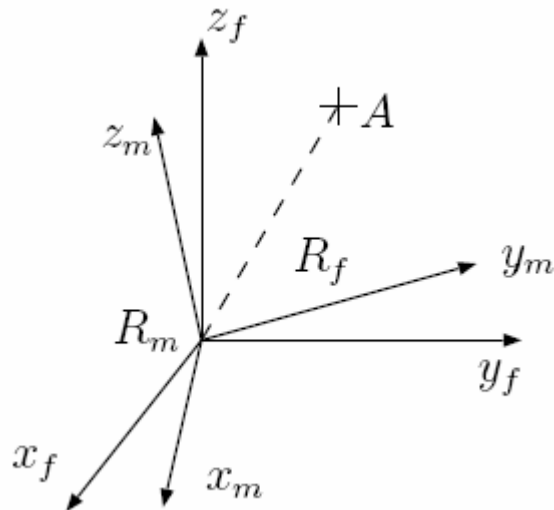
v: tangential velocity

ω : angular velocity



Basic Knowledge *Rigid body kinematics*

Rotating frame



R_f : fixed frame (origin O fixed)

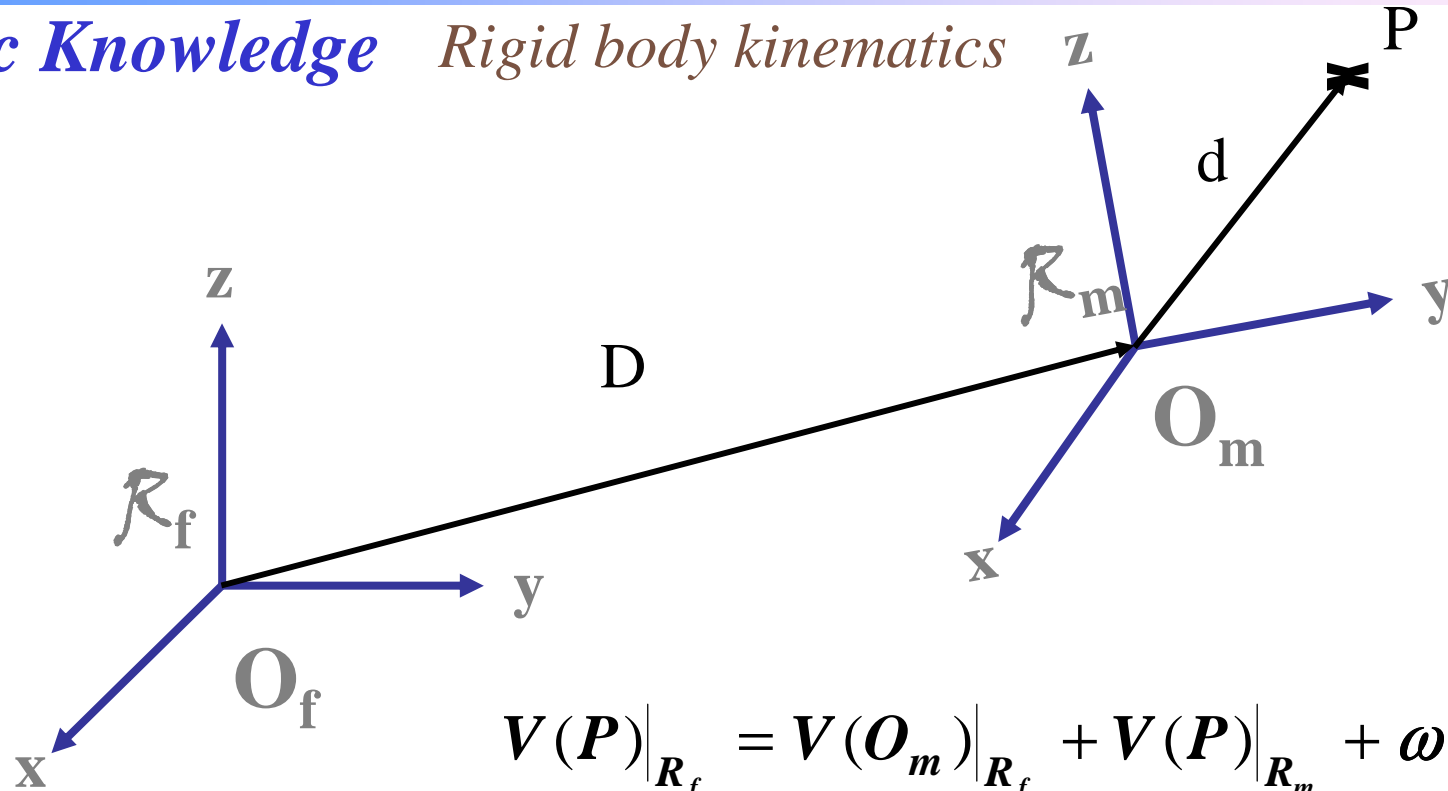
R_m : mobile frame just in rotation w.r.t R_f

$$\left(\frac{dA}{dt}\right)_{R_f} = \left(\frac{dA}{dt}\right)_{R_m} + \omega \wedge \underline{A}$$

$$\boxed{V(A)_{R_f} = V(A)_{R_m} + \omega \wedge \underline{A}}$$

ω : angular velocity

Basic Knowledge *Rigid body kinematics*



$$V(P)|_{R_f} = V(O_m)|_{R_f} + V(P)|_{R_m} + \omega \times \overline{O_m P}$$

\mathcal{R}_f : fixed frame
 \mathcal{R}_m : Mobile frame
 P : one point in \mathcal{R}_m

$$\frac{d}{dt}(d)|_{R_f} = \dot{D} + \frac{d}{dt}(d)|_{R_m} + \omega \times d$$

ω : angular velocity of \mathcal{R}_m relative to \mathcal{R}_f

Basic Knowledge *Rigid body kinematics*

Kinematic

Using this relation we can established
The kinematic evolution of a multi-rigidbody robot
See next slide



$$V(P)|_{R_f} = V(O_m)|_{R_f} + V(P)|_{R_m} + \omega \times \overline{O_m P}$$

Remarks :

If $D=0$ then

$$V(P)|_{R_f} = V(P)|_{R_m} + \omega \times \overline{O_m P}$$

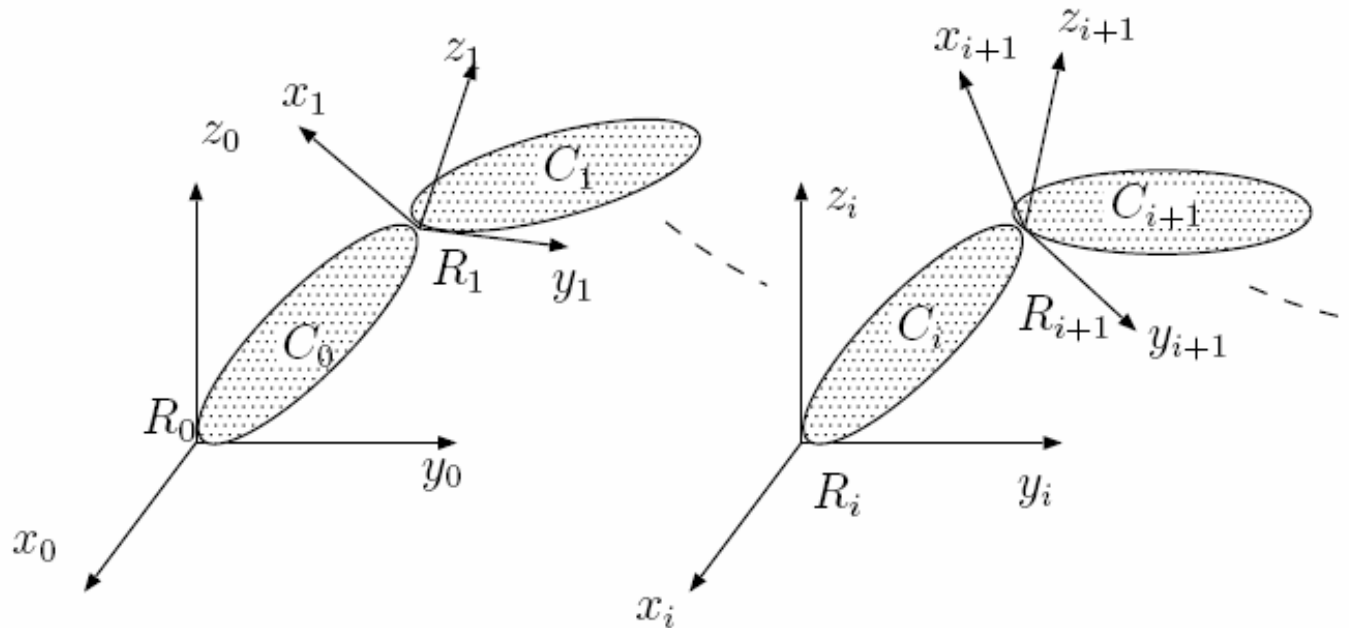
If $D=0$ and $V(P)|_{R_m} = 0$ then $V(P)|_{R_f} = \omega \times \overline{O_m P}$

$$= -\overline{O_m P} \times \omega$$

$$= \tilde{\omega} \cdot d$$

$$\tilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} = AS[\omega] = [\omega]^\wedge = [\hat{\omega}]$$

Basic Knowledge *Multi-Rigid bodies kinematics*

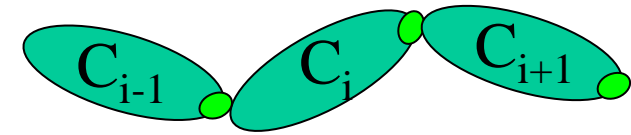


$$V(O_{i+1})_{(R_i)}^{(R_0)} = V(O_i)_{(R_i)}^{(R_0)} + V(O_{i+1})_{(R_i)}^{(R_i)} + \Omega_i^{(R_0)} \wedge O_i O_{i+1}$$

$\underbrace{\hspace{10em}}$ Velocity of O_{i+1} w.r.t R_0 expressed in R_i
 $\underbrace{\hspace{5em}}$ Velocity of O_i w.r.t R_0 expressed in R_i
 $\underbrace{\hspace{5em}}$ Velocity of O_{i+1} w.r.t R_i expressed in R_i
 $\underbrace{\hspace{5em}}$ Angular velocity of R_i w.r.t R_0 expressed in R_i

Basic Knowledge *Multi-Rigid bodies kinematics*

$$V(O_{i+1})_{(R_i)}^{(R_0)} = V(O_i)_{(R_i)}^{(R_0)} + V(O_{i+1})_{(R_i)}^{(R_i)} + \Omega_i^{(R_0)} \wedge O_i O_{i+1}$$

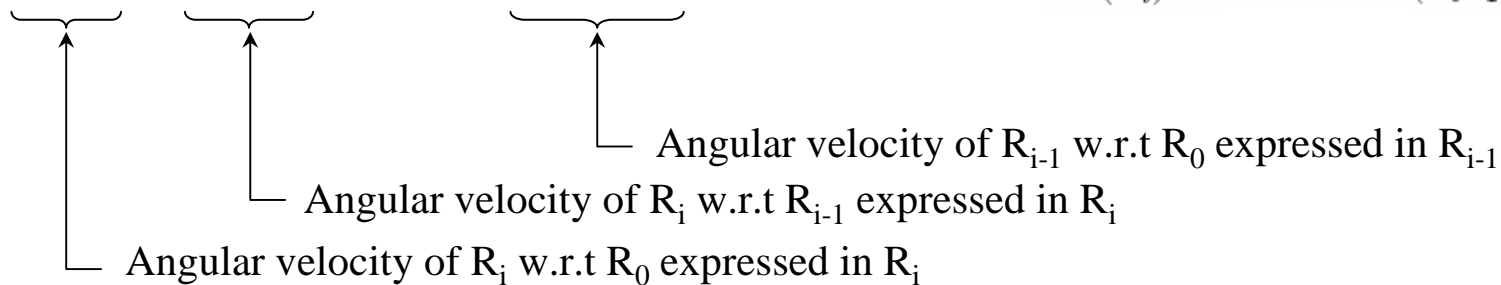


$$V(O_{i+1})_{(R_i)}^{(R_0)} = {}^i A_{i+1} \cdot V(O_{i+1})_{(R_{i+1})}^{(R_0)}$$

$$V(O_i)_{(R_i)}^{(R_0)} = {}^i A_{i-1} \cdot V(O_i)_{(R_{i-1})}^{(R_0)}$$

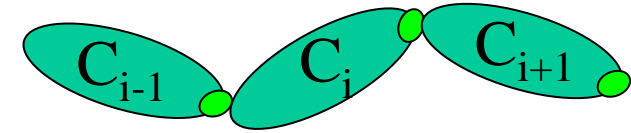
$$\Omega_i^{(R_0)} = \Omega_i^{(R_i)} + {}^i A_{i-1} \cdot \Omega_{i-1}^{(R_{i-1})}$$

$$\begin{aligned} \Omega_i^{(R_0)} &= \Omega_i^{(R_{i-1})} + \Omega_{i-1}^{(R_0)} \\ \Omega_i^{(R_i)} &= {}^i A_{i-1} \cdot \Omega_i^{(R_{i-1})} \end{aligned}$$



Basic Knowledge *Multi-Rigid bodies kinematics*

$$V(O_{i+1})_{(R_i)}^{(R_0)} = V(O_i)_{(R_i)}^{(R_0)} + V(O_{i+1})_{(R_i)}^{(R_i)} + \Omega_i^{(R_0)} \wedge O_i O_{i+1}$$



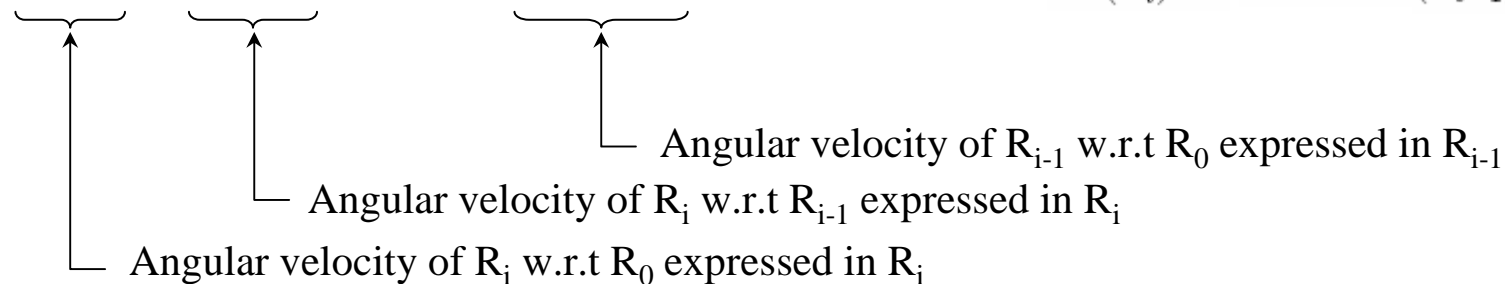
$$V(O_{i+1})_{(R_i)}^{(R_0)} = {}^i A_{i+1} \cdot V(O_{i+1})_{(R_{i+1})}^{(R_0)}$$

$$V(O_i)_{(R_i)}^{(R_0)} = {}^i A_{i-1} \cdot V(O_i)_{(R_{i-1})}^{(R_0)}$$

$$\Omega_i^{(R_0)} = \Omega_i^{(R_i)} + {}^i A_{i-1} \cdot \Omega_{i-1}^{(R_0)}$$

$$\Omega_i^{(R_0)} = \Omega_i^{(R_{i-1})} + \Omega_{i-1}^{(R_0)}$$

$$\Omega_i^{(R_0)} = {}^i A_{i-1} \cdot \Omega_i^{(R_{i-1})}$$





Basic Knowledge *Multi-Rigid bodies kinematics*

Considering two frames R_a and R_b rigidly linked (case for R_n and R_E)

$$\begin{aligned}
 V(O_a) \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} &= V(O_b) \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} + \cancel{V(O_a) \Big|_{\mathcal{R}_b}^{\mathcal{R}_b}} + \Omega_b \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} \wedge O_b O_a \Big|_{\mathcal{R}_b} \\
 &= V(O_b) \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} - AS[O_b O_a] \cdot \Omega_b \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} \\
 &= V(O_b) \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} - AS[{}^b P_a] \cdot \Omega_b \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} \\
 &= V(O_b) \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} - AS[-{}^b R_a \cdot {}^a P_b] \cdot \Omega_b \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} \\
 &= V(O_b) \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} + {}^b R_a \cdot AS[{}^a P_b] \cdot \Omega_b \Big|_{\mathcal{R}_b}^{\mathcal{R}_0}
 \end{aligned}
 \quad \Bigg| \quad
 \begin{aligned}
 \Omega_a \Big|_{\mathcal{R}_a}^{\mathcal{R}_0} &= \cancel{\Omega_a \Big|_{\mathcal{R}_b}^{\mathcal{R}_b}} + {}^a R_b \cdot \Omega_b \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} \\
 &= {}^a R_b \cdot \Omega_b \Big|_{\mathcal{R}_b}^{\mathcal{R}_0}
 \end{aligned}$$

$$V(O_a) \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} = {}^b R_a \cdot V(O_a) \Big|_{\mathcal{R}_a}^{\mathcal{R}_0} \Rightarrow V(O_a) \Big|_{\mathcal{R}_a}^{\mathcal{R}_0} = {}^a R_b \cdot V(O_a) \Big|_{\mathcal{R}_b}^{\mathcal{R}_0}$$

$$\begin{pmatrix} V_a \Big|_{\mathcal{R}_a}^{\mathcal{R}_0} \\ \Omega_a \Big|_{\mathcal{R}_a}^{\mathcal{R}_0} \end{pmatrix}_{\mathcal{R}_a}^{\mathcal{R}_0} = \begin{pmatrix} {}^a R_b & AS[{}^a P_b] \cdot {}^a R_b \\ O_{3 \times 3} & {}^a R_b \end{pmatrix} \cdot \begin{pmatrix} V_b \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} \\ \Omega_b \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} \end{pmatrix}_{\mathcal{R}_b}^{\mathcal{R}_0}$$

twist

$$T_a \Big|_{\mathcal{R}_a}^{\mathcal{R}_0} = \begin{pmatrix} {}^a R_b & AS[{}^a P_b] \cdot {}^a R_b \\ O_{3 \times 3} & {}^a R_b \end{pmatrix} \cdot T_b \Big|_{\mathcal{R}_b}^{\mathcal{R}_0} = {}^a \mathcal{W}_b \cdot T_b \Big|_{\mathcal{R}_b}^{\mathcal{R}_0}$$

$$\begin{aligned}
 T_a &= {}^a \mathcal{W}_b \cdot T_b \\
 {}^a \mathcal{W}_b &= \begin{pmatrix} {}^a R_b & AS[{}^a P_b] \cdot {}^a R_b \\ O_{3 \times 3} & {}^a R_b \end{pmatrix}
 \end{aligned}$$

Basic Knowledge Multi-Rigid bodies kinematics

Considering two frames R_i and R_j and a twist $V_i=(v_i,\omega_i)^T$ expressed in O_i
We wish to compute the corresponding twist $V_j=(v_j,\omega_j)^T$ expressed in O_j

$$\begin{aligned} \omega_j &= \omega_i \\ v_j &= v_i + \omega_i \times L_{i,j} \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} v_j \\ \omega_j \end{bmatrix} = \begin{bmatrix} I_3 & -\hat{L}_{i,j} \\ 0_3 & I_3 \end{bmatrix} \begin{bmatrix} v_i \\ \omega_i \end{bmatrix} \quad \xrightarrow{\text{Projection}} \quad \begin{bmatrix} {}^i v_j \\ {}^i \omega_j \end{bmatrix} = \begin{bmatrix} I_3 & -{}^i \hat{P}_j \\ 0_3 & I_3 \end{bmatrix} \begin{bmatrix} {}^i v_i \\ {}^i \omega_i \end{bmatrix}$$

Since ${}^j v_j = {}^j R_i {}^i v_j$ and ${}^j \omega_j = {}^j R_i {}^i \omega_j$

$${}^j V_j = {}^j S_i {}^i V_i \quad \text{with} \quad {}^j S_i = \begin{bmatrix} {}^j R_i & -{}^j R_i {}^i \hat{P}_j \\ 0_3 & {}^j R_i \end{bmatrix}$$

$${}^i S_j^{-1} = \begin{bmatrix} {}^i R_j & {}^i \hat{P}_j {}^i R_j \\ 0_3 & {}^i R_j \end{bmatrix} = {}^j S_i \quad {}^j S_i = \begin{bmatrix} {}^i R_j & {}^i \hat{P}_j {}^i R_j \\ 0_3 & {}^i R_j \end{bmatrix} = \begin{bmatrix} {}^i R_j & -{}^i R_j {}^j \hat{P}_i \\ 0_3 & {}^i R_j \end{bmatrix}$$

Basic Knowledge Differential translation and rotation of frames

Consider a differential translation vector $d\mathbf{P}_i$ expressing the translation of the origin of frame R_i , and of a differential rotation vector δ_i , equal to $\mathbf{u}_i \cdot d\theta$, representing the rotation of an angle $d\theta$ about an axis, with unit vector \mathbf{u}_i , passing through the origin O_i .

$${}^i\mathbf{T}_j + d{}^i\mathbf{T}_j = \text{Trans}({}^i dx_i, {}^i dy_i, {}^i dz_i) \text{Rot}({}^i \mathbf{u}_i, d\theta) {}^i\mathbf{T}_j$$

$${}^i\mathbf{T}_j + d{}^i\mathbf{T}_j = {}^i\mathbf{T}_j \text{Trans}({}^j dx_j, {}^j dy_j, {}^j dz_j) \text{Rot}({}^j \mathbf{u}_j, d\theta)$$

$$d{}^i\mathbf{T}_j = {}^i\mathbf{T}_j {}^j\Delta \quad d{}^i\mathbf{T}_j = {}^i\mathbf{T}_j [\text{Trans}({}^j dx_j, {}^j dy_j, {}^j dz_j) \text{Rot}({}^j \mathbf{u}_j, d\theta) - \mathbf{I}_4]$$

$$d{}^i\mathbf{T}_j = {}^i\Delta {}^i\mathbf{T}_j \quad d{}^i\mathbf{T}_j = [\text{Trans}({}^i dx_i, {}^i dy_i, {}^i dz_i) \text{Rot}({}^i \mathbf{u}_i, d\theta) - \mathbf{I}_4] {}^i\mathbf{T}_j$$

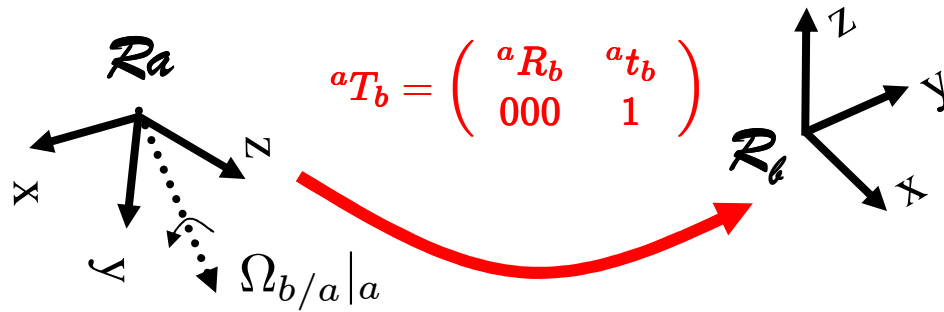
The differential transformation matrix Δ is defined as [Paul 81]

$$\Delta = [\text{Trans}(dx, dy, dz) \text{Rot}(\mathbf{u}, d\theta) - \mathbf{I}_4]$$

$${}^j\Delta = \begin{bmatrix} {}^j\delta_j & {}^j d\mathbf{P}_j \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} {}^j\hat{\mathbf{u}}_j d\theta & {}^j d\mathbf{P}_j \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} {}^j d\mathbf{P}_j \\ {}^j\delta_j \end{bmatrix} = {}^j\mathbf{S}_i \begin{bmatrix} {}^i d\mathbf{P}_i \\ {}^i\delta_i \end{bmatrix}$$



Basic Knowledge *Angular velocity properties*



$$R = {}^a R_b$$

$$R \cdot R^T = I_3$$

$$\dot{R} \cdot R^T + R \cdot \dot{R}^T = 0 \implies \dot{R} \cdot R^T = -R \cdot \dot{R}^T = -\left(\dot{R} \cdot R^T\right)^T$$

$$\dot{R} \cdot R^T = S(t) \implies \dot{R} = S(t) \cdot R$$

Consider $p_b(t)$ constant $p_a(t) = R(t) \cdot p_b(t)$

$$\dot{p}_a(t) = \dot{R}(t) \cdot p_b(t) \implies \dot{p}_a(t) = S(t) \cdot R \cdot p_b(t)$$

$$\dot{p}_a(t) = \Omega \times p_a(t) \implies \dot{p}_a(t) = [\Omega]_{\times} \cdot R \cdot p_b(t)$$

$$\dot{p}_a(t) = \Omega \times (R \cdot p_b(t)) \implies S(t) = [\Omega]_{\times}$$

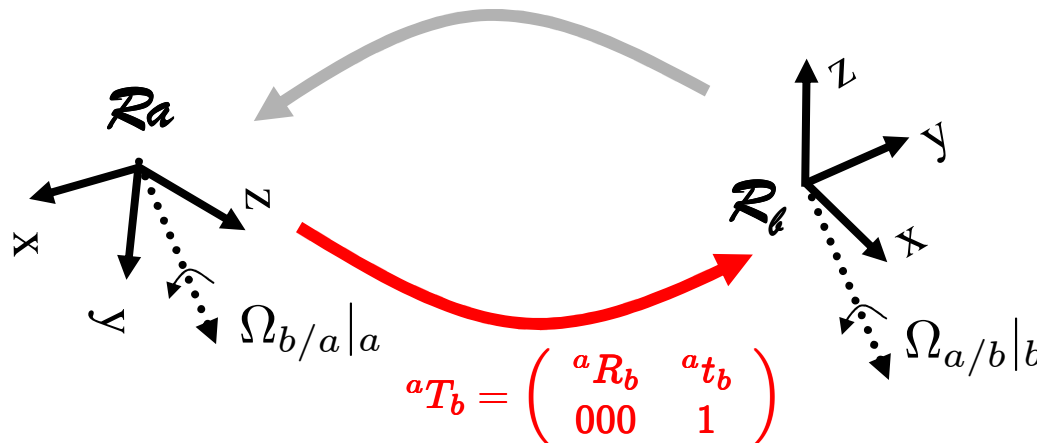
$$[\Omega_{b/a|a}]_{\times} = {}^a \dot{R}_b \cdot {}^a R_b^T$$



Basic Knowledge *Angular velocity properties*

$${}^b T_a = \begin{pmatrix} {}^b R_a & {}^b t_a \\ 000 & 1 \end{pmatrix}$$

${}^b R_a$ represents the orientation
 ${}^b t_a$ represents the position



$${}^a T_b = \begin{pmatrix} {}^a R_b & {}^a t_b \\ 000 & 1 \end{pmatrix}$$

$$\Omega_{b/a|a} \quad \Omega_{b/a|a} = -\Omega_{a/b|a} \quad \Omega_{b/a|a} = {}^a R_b \cdot \Omega_{b/a|b}$$

$$\Omega_{a/b|b} \quad \Omega_{a/b|b} = -\Omega_{b/a|b} \quad \Omega_{a/b|b} = {}^b R_a \cdot \Omega_{a/b|a}$$

$$\left[\Omega_{b/a|a} \right]_{\times} = {}^a \dot{R}_b \cdot {}^a R_b^T$$

$$\left[\Omega_{a/b|b} \right]_{\times} = {}^b \dot{R}_a \cdot {}^b R_a^T$$

Basic Knowledge Representation of forces (wrench)

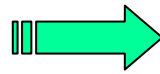
A collection of forces and moments acting on a body can be reduced to a *wrench* \mathbf{F}_i at point O_i , which is composed of a force \mathbf{f}_i at O_i and a moment \mathbf{m}_i about O_i :

$$\mathbf{F}_i = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{m}_i \end{bmatrix}$$

Note that the vector field of the moments constitutes a screw where the vector of the screw is \mathbf{f}_i . Thus, the wrench forms a screw.

Consider a given wrench ${}^i\mathbf{F}_i$, expressed in frame R_i . For calculating the equivalent wrench ${}^j\mathbf{F}_j$, we use the transformation matrix between screws such that:

$$\begin{bmatrix} {}^j\mathbf{m}_j \\ {}^j\mathbf{f}_j \end{bmatrix} = {}^j\mathbf{S}_i \begin{bmatrix} {}^i\mathbf{m}_i \\ {}^i\mathbf{f}_i \end{bmatrix}$$



$$\begin{aligned} {}^j\mathbf{f}_j &= {}^j\mathbf{R}_i {}^i\mathbf{f}_i \\ {}^j\mathbf{m}_j &= {}^j\mathbf{R}_i ({}^i\mathbf{f}_i \times {}^i\mathbf{P}_j + {}^i\mathbf{m}_i) \end{aligned}$$

$$\begin{bmatrix} {}^j\mathbf{f}_j \\ {}^j\mathbf{m}_j \end{bmatrix} = {}^i\mathbf{S}_j^T \begin{bmatrix} {}^i\mathbf{f}_i \\ {}^i\mathbf{m}_i \end{bmatrix}$$