

Linear Algebra & Geometry

why is linear algebra useful in computer vision?

References:

-Any book on linear algebra!

-[HZ] - chapters 2, 4

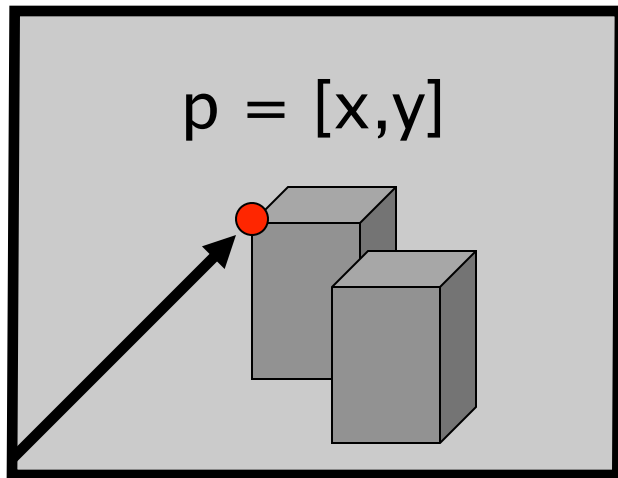
Why is linear algebra useful in computer vision?

- Representation
 - 3D points in the scene
 - 2D points in the image
- Coordinates will be used to
 - Perform geometrical transformations
 - Associate 3D with 2D points
- Images are matrices of numbers
 - Find properties of these numbers

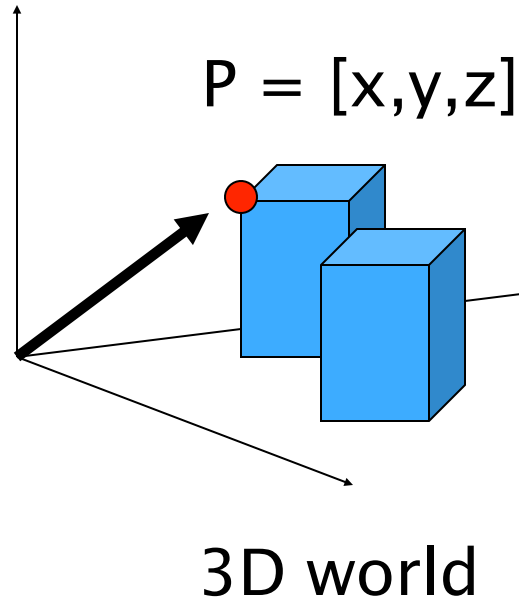
Agenda

1. Basics definitions and properties
2. Geometrical transformations
3. SVD and its applications

Vectors (i.e., 2D or 3D vectors)

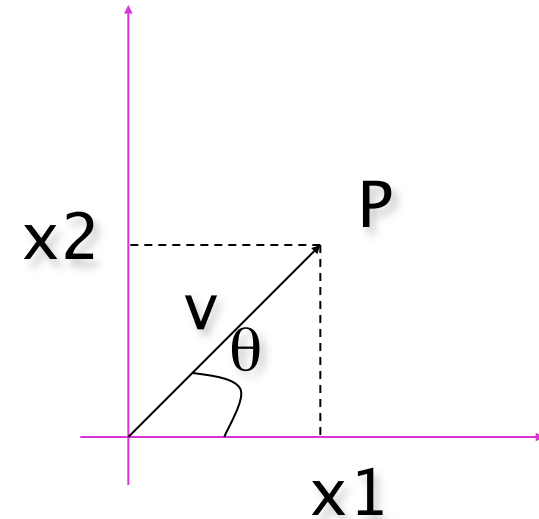


Image



Vectors (i.e., 2D vectors)

$$\mathbf{v} = (x_1, x_2)$$



Magnitude: $\|\mathbf{v}\| = \sqrt{x_1^2 + x_2^2}$

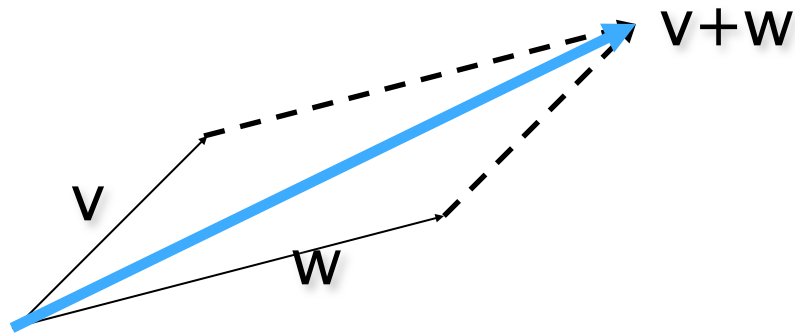
If $\|\mathbf{v}\| = 1$, \mathbf{v} is a UNIT vector

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{x_1}{\|\mathbf{v}\|}, \frac{x_2}{\|\mathbf{v}\|} \right) \text{ is a unit vector}$$

Orientation: $\theta = \tan^{-1} \left(\frac{x_2}{x_1} \right)$

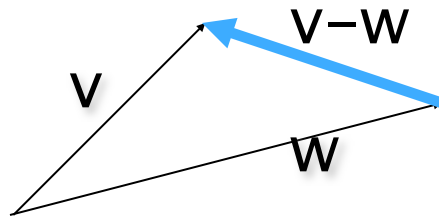
Vector Addition

$$\mathbf{v} + \mathbf{w} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$



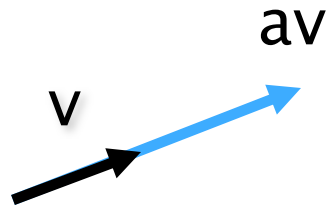
Vector Subtraction

$$\mathbf{v} - \mathbf{w} = (x_1, x_2) - (y_1, y_2) = (x_1 - y_1, x_2 - y_2)$$

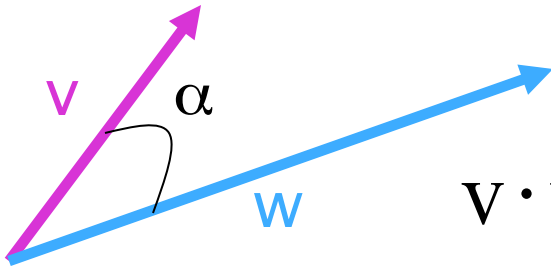


Scalar Product

$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$



Inner (dot) Product



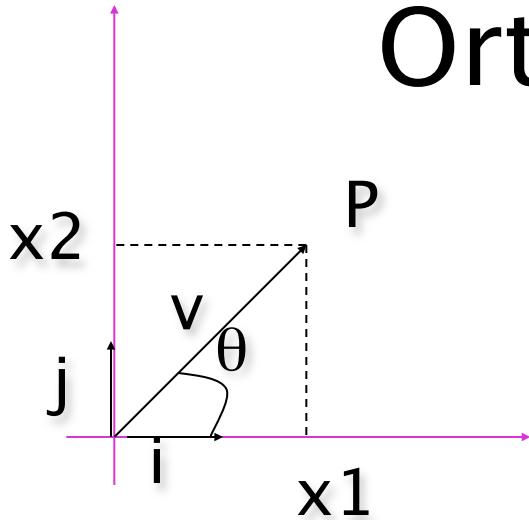
$$v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = x_1 y_1 + x_2 y_2$$

The inner product is a **SCALAR!**

$$v \cdot w = (x_1, x_2) \cdot (y_1, y_2) = \|v\| \cdot \|w\| \cos \alpha$$

$$\text{if } v \perp w, \quad v \cdot w = ? = 0$$

Orthonormal Basis



$$\begin{aligned} \mathbf{i} &= (1,0) & \|\mathbf{i}\| &= 1 & \mathbf{i} \cdot \mathbf{j} &= 0 \\ \mathbf{j} &= (0,1) & \|\mathbf{j}\| &= 1 & & \end{aligned}$$

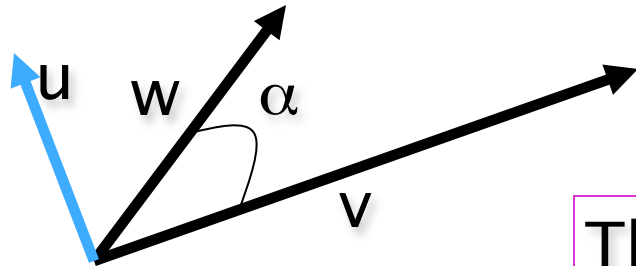
$$\mathbf{v} = (x_1, x_2)$$

$$\mathbf{v} = x_1 \mathbf{i} + x_2 \mathbf{j}$$

$$\mathbf{v} \cdot \mathbf{i} = ? = (x_1 \mathbf{i} + x_2 \mathbf{j}) \cdot \mathbf{i} = x_1 \cdot 1 + x_2 \cdot 0 = x_1$$

$$\mathbf{v} \cdot \mathbf{j} = (x_1 \mathbf{i} + x_2 \mathbf{j}) \cdot \mathbf{j} = x_1 \cdot 0 + x_2 \cdot 1 = x_2$$

Vector (cross) Product



$$u = v \times w$$

The cross product is a **VECTOR!**

$$\text{Magnitude: } \|u\| = \|v \times w\| = \|v\| \|w\| \sin \alpha$$

Orientation:

$$u \perp v \Rightarrow u \cdot v = (v \times w) \cdot v = 0$$
$$u \perp w \Rightarrow u \cdot w = (v \times w) \cdot w = 0$$

$$\text{if } v \parallel w ? \quad \rightarrow u = 0$$

Vector Product Computation

$$\mathbf{i} = (1,0,0) \quad \|\mathbf{i}\| = 1 \quad \mathbf{i} = \mathbf{j} \times \mathbf{k}$$

$$\mathbf{j} = (0,1,0) \quad \|\mathbf{j}\| = 1 \quad \mathbf{j} = \mathbf{k} \times \mathbf{i}$$

$$\mathbf{k} = (0,0,1) \quad \|\mathbf{k}\| = 1 \quad \mathbf{k} = \mathbf{i} \times \mathbf{j}$$

$$\begin{aligned} \mathbf{u} = \mathbf{v} \times \mathbf{w} &= (x_1, x_2, x_3) \times (y_1, y_2, y_3) \\ &= (x_2 y_3 - x_3 y_2) \mathbf{i} + (x_3 y_1 - x_1 y_3) \mathbf{j} + (x_1 y_2 - x_2 y_1) \mathbf{k} \end{aligned}$$

Matrices

$$A_{n \times m} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$



Pixel's intensity value

$$\text{Sum: } C_{n \times m} = A_{n \times m} + B_{n \times m} \quad c_{ij} = a_{ij} + b_{ij}$$

A and B must have the same dimensions!

$$\text{Example: } \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}$$

Matrices

$$A_{n \times m} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \mathbf{a}_i$$
$$B_{m \times p} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mp} \end{bmatrix} \mathbf{b}_j$$

Product:

$$C_{n \times p} = A_{n \times m} B_{m \times p}$$

$$c_{ij} = \mathbf{a}_i \cdot \mathbf{b}_j = \sum_{k=1}^m a_{ik} b_{kj}$$

A and B must have compatible dimensions!

$$A_{n \times n} B_{n \times n} \neq B_{n \times n} A_{n \times n}$$

Matrices

Transpose:

$$C_{m \times n} = A^T_{n \times m}$$

$$(A + B)^T = A^T + B^T$$

$$c_{ij} = a_{ji}$$

$$(AB)^T = B^T A^T$$

If $A^T = A$ A is symmetric

Examples:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \\ 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 \\ 1 & 5 \end{bmatrix} \text{ Symmetric? No!}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 7 \end{bmatrix} \text{ Symmetric? Yes!}$$

Matrices

Determinant:

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

A must be square

Example: $\det \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = 2 - 15 = -13$

Matrices

Inverse:

A must be square

$$A_{n \times n} A^{-1}_{n \times n} = A^{-1}_{n \times n} A_{n \times n} = I$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

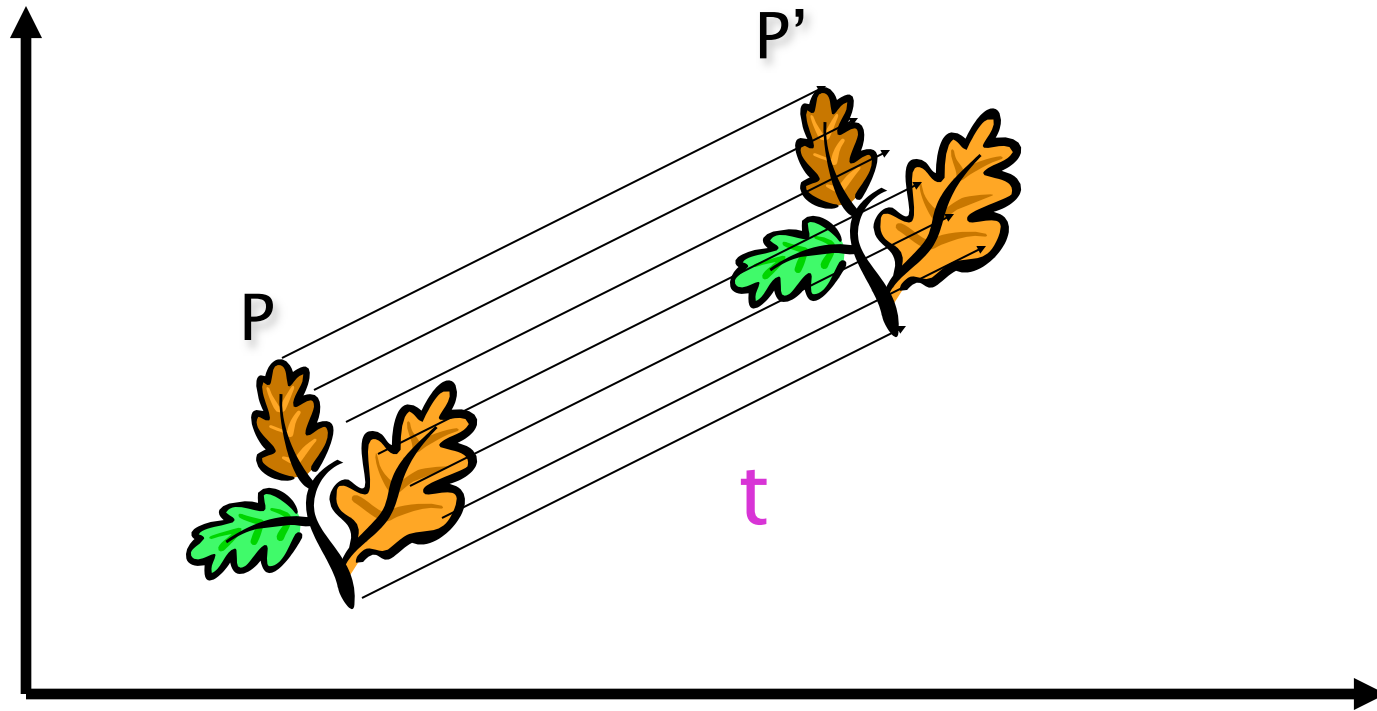
Example:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = ? = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$$

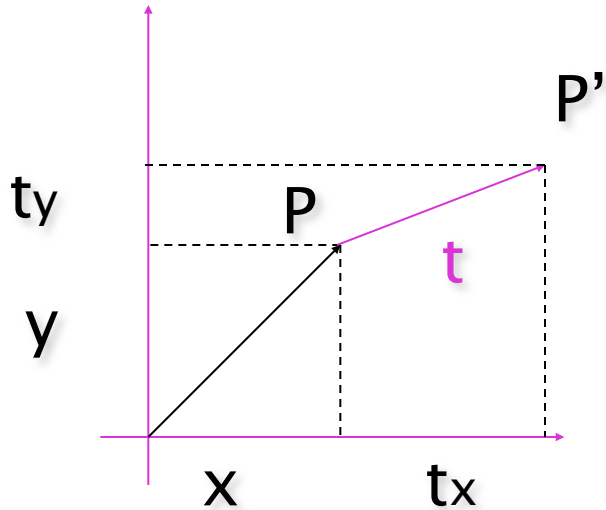
$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D Geometrical Transformations

2D Translation



2D Translation Equation

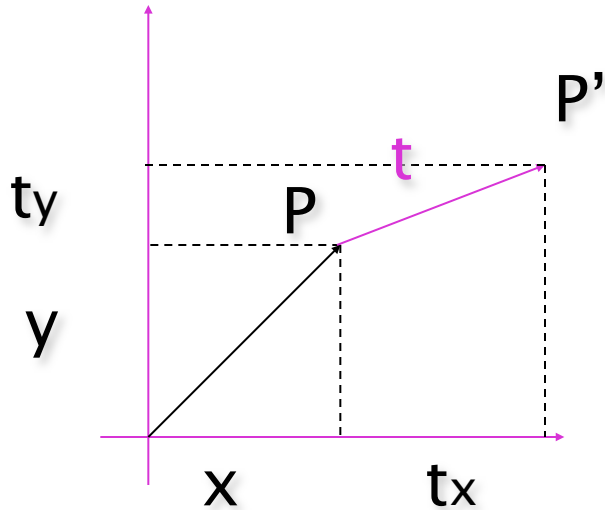


$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' = \mathbf{P} + \mathbf{t} = (x + t_x, y + t_y)$$

2D Translation using Matrices



$$\mathbf{P} = (x, y)$$

$$\mathbf{t} = (t_x, t_y)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- Multiply the coordinates by a non-zero scalar and add an extra coordinate equal to that scalar. For example,

$$(x, y) \rightarrow (x \cdot z, y \cdot z, z) \quad z \neq 0$$

$$(x, y, z) \rightarrow (x \cdot w, y \cdot w, z \cdot w, w) \quad w \neq 0$$

Back to Cartesian Coordinates:

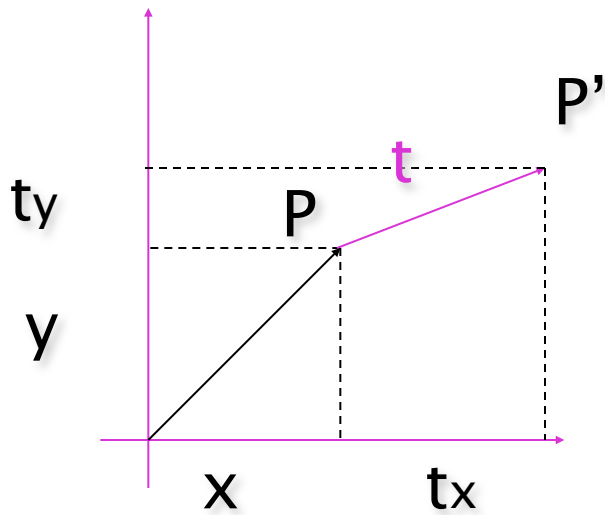
- Divide by the last coordinate and eliminate it. For example,

$$(x, y, z) \quad z \neq 0 \rightarrow (x/z, y/z)$$

$$(x, y, z, w) \quad w \neq 0 \rightarrow (x/w, y/w, z/w)$$

- NOTE: in our example the scalar was 1

2D Translation using Homogeneous Coordinates



$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

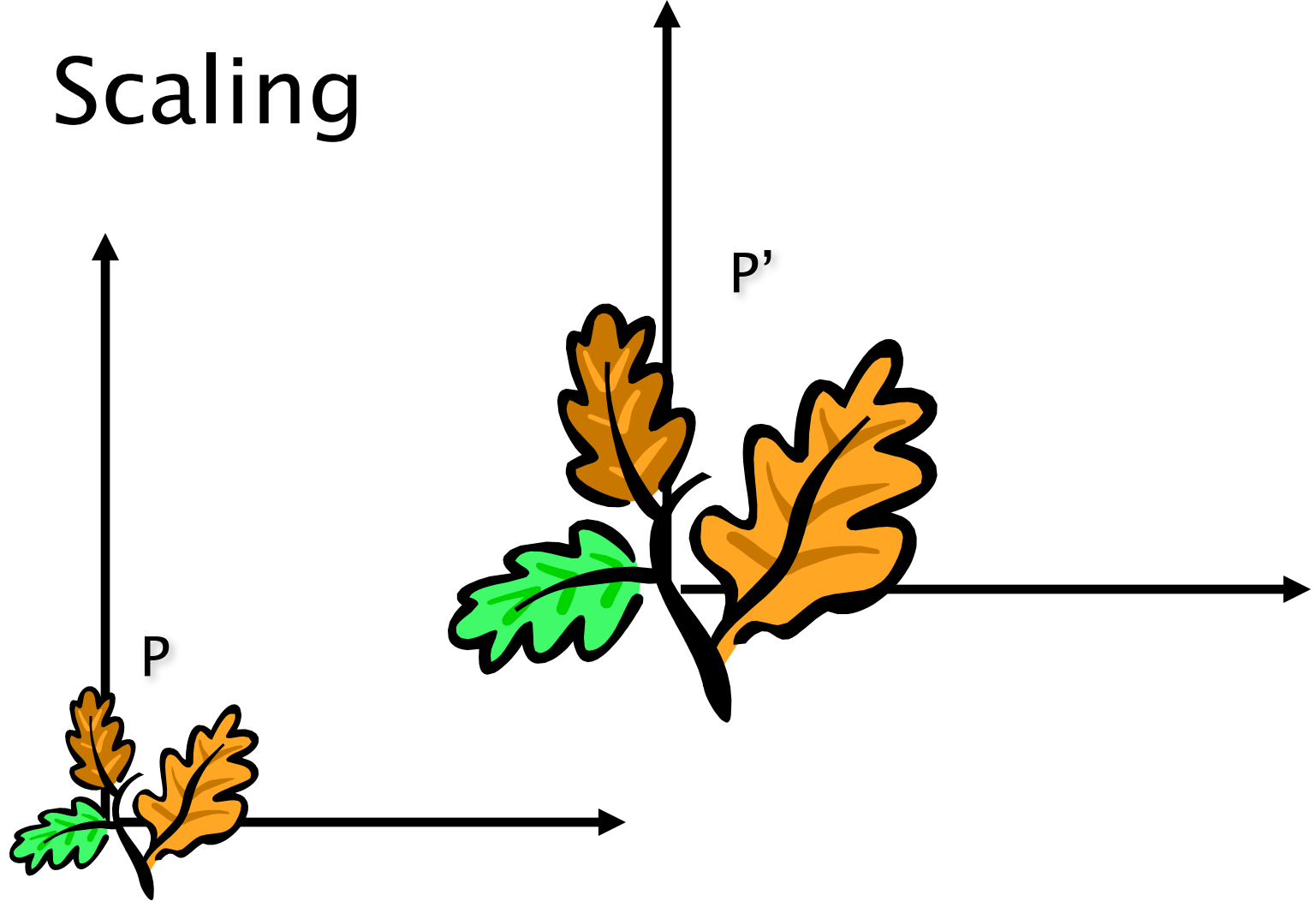
$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

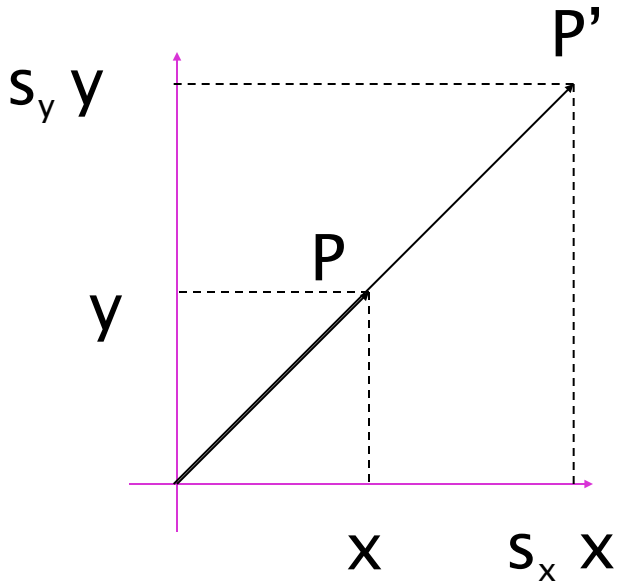
\swarrow \mathbf{t}
 \swarrow \mathbf{P}

$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P}$$

Scaling



Scaling Equation



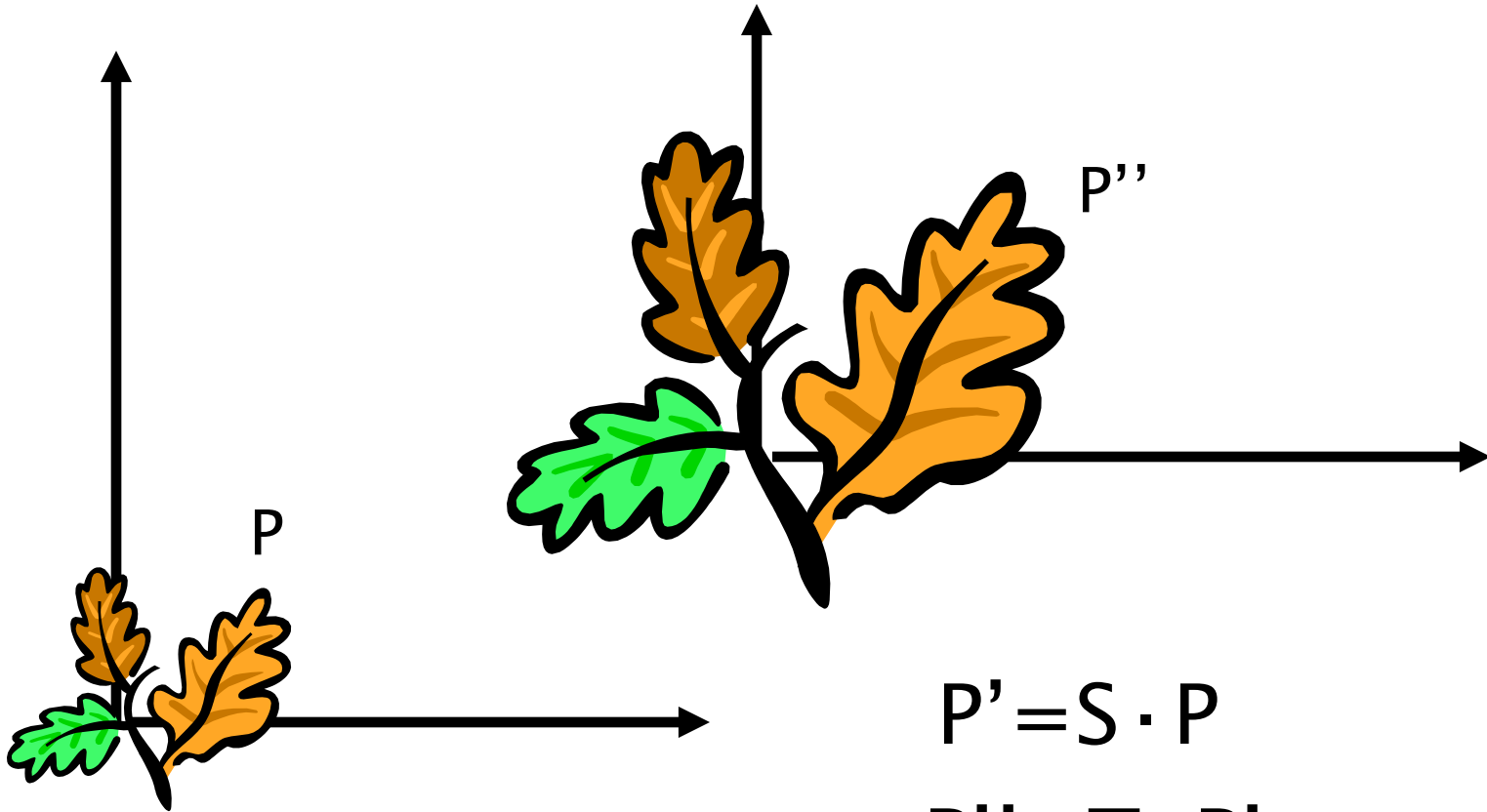
$$\mathbf{P} = (x, y) \rightarrow \mathbf{P}' = (s_x x, s_y y)$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P}' = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P}' \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}' & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$

Scaling & Translating



$$P' = S \cdot P$$

$$P'' = T \cdot P'$$

$$P'' = T \cdot P' = T \cdot (S \cdot P) = (T \cdot S) \cdot P = A \cdot P$$

Scaling & Translating

$$\begin{aligned} \mathbf{P}'' = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} &= \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \\ &= \underbrace{\begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix}}_A \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} S & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix} \end{aligned}$$

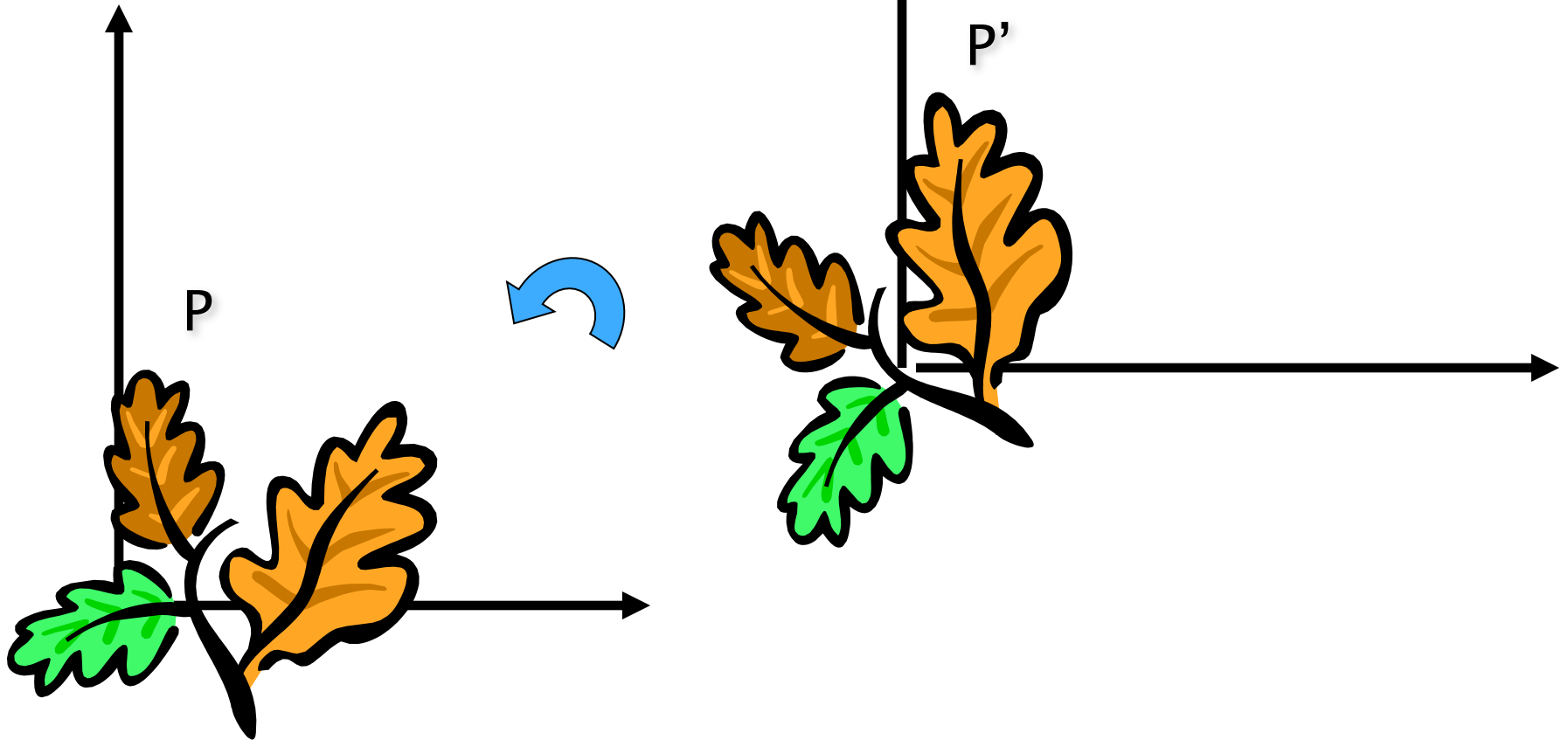
Translating & Scaling = Scaling & Translating ?

$$\mathbf{P}''' = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

$$\mathbf{P}''' = \mathbf{S} \cdot \mathbf{T} \cdot \mathbf{P} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

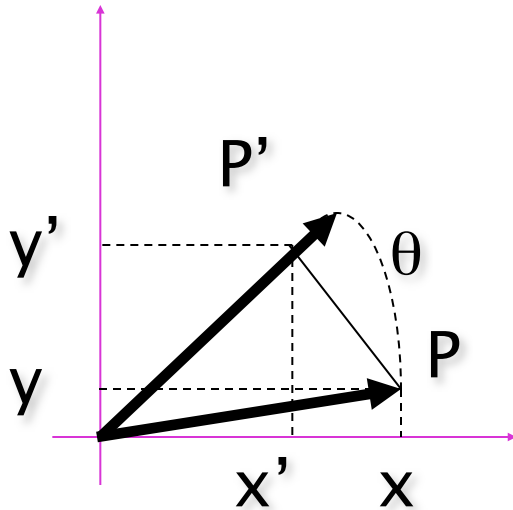
$$= \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}$$

Rotation



Rotation Equations

Counter-clockwise rotation by an angle θ



$$x' = \cos \theta x - \sin \theta y$$

$$y' = \cos \theta y + \sin \theta x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P}' = \mathbf{R} \mathbf{P}$$

Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2  4 elements

Note: R belongs to the category of normal matrices and satisfies many interesting properties:

$$\mathbf{R} \cdot \mathbf{R}^T = \mathbf{R}^T \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

Rotation+ Scaling + Translation

$$P' = (T R S) P$$

$$P' = T \cdot R \cdot S \cdot P = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos \theta & -\sin \theta & t_x \\ \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} R' & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} R' S & t \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

If $s_x = s_y$, this is a similarity transformation!

Transformation in 2D

-Isometries

-Similarities

-Affinity

-Projective

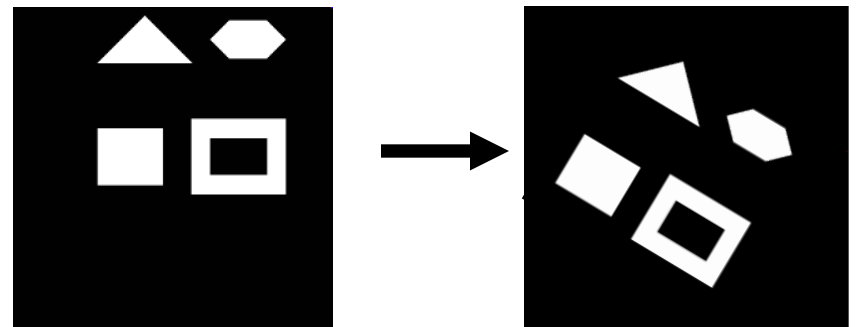
Transformation in 2D

Isometries:

[Euclidean]

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_e \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

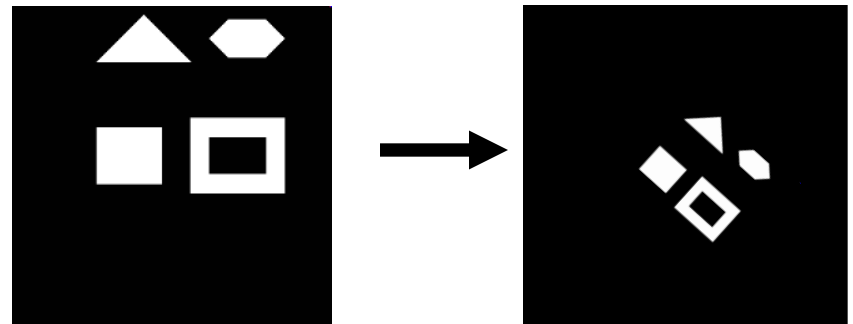
- Preserve distance (areas)
- 3 DOF
- Regulate motion of rigid object



Transformation in 2D

Similarities:
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s & R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_s \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

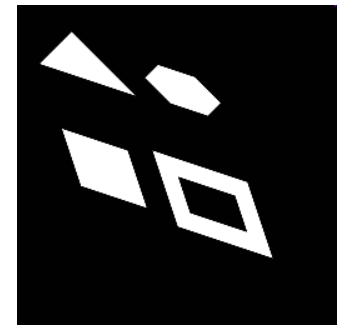
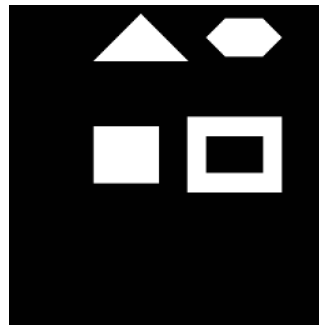
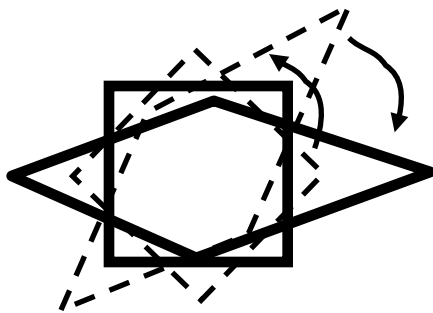
- Preserve
 - ratio of lengths
 - angles
- 4 DOF



Transformation in 2D

Affinities:
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_a \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = R(\theta) \cdot R(-\phi) \cdot D \cdot R(\phi) \quad D = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



Transformation in 2D

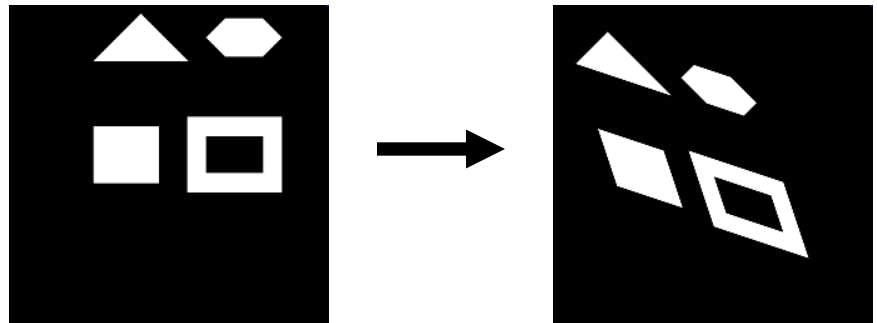
Affinities:
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_a \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = R(\theta) \cdot R(-\phi) \cdot D \cdot R(\phi) \quad D = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

-Preserve:

- Parallel lines
- Ratio of areas
- Ratio of lengths on collinear lines
- others...

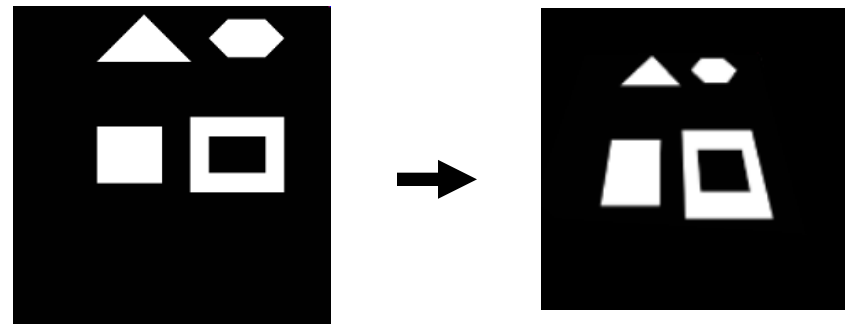
- 6 DOF



Transformation in 2D

Projective:
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} A & t \\ \boxed{v} & b \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = H_p \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- 8 DOF
- Preserve:
 - cross ratio of 4 collinear points
 - collinearity
 - and a few others...



Eigenvalues and Eigenvectors

- Eigen relation

$$\mathbf{A}\mathbf{u}=\lambda\mathbf{u}$$

- Matrix \mathbf{A} acts on vector \mathbf{u} and produces a scaled version of the vector.
- Eigen is a German word meaning “proper” or “specific”
- \mathbf{u} is the eigenvector while λ is the eigenvalue.

Eigenvalues and Eigenvectors

The eigenvalues of A are the roots of the *characteristic equation*

$$p(\lambda) = \det(\lambda I - A) = 0$$

$$\lambda_1, \dots, \lambda_N \quad S = [v_1 \quad \dots \quad v_N]$$

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdot & \\ & & & \lambda_N \end{bmatrix} \text{ diagonal form of matrix}$$

Eigenvectors of A are columns of S

Singular Value Decomposition

$$\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \mathbf{A}$$

- Where \mathbf{U} and \mathbf{V} are orthogonal matrices, and $\mathbf{\Sigma}$ is a diagonal matrix. For example:

$$\begin{matrix} U & & \Sigma & & V^T & & A \\ \begin{bmatrix} -.40 & .916 \\ .916 & .40 \end{bmatrix} & \times & \begin{bmatrix} 5.39 & 0 \\ 0 & 3.154 \end{bmatrix} & \times & \begin{bmatrix} -.05 & .999 \\ .999 & .05 \end{bmatrix} & = & \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} \end{matrix}$$

Singular Value decomposition

- **Singular values:** Non negative square roots of the eigenvalues of $\mathbf{A}^t\mathbf{A}$. Denoted $\sigma_i, i=1, \dots, n$
- **SVD:** If \mathbf{A} is a real m by n matrix then there exist orthogonal matrices \mathbf{U} ($\in \mathbb{R}^{m \times m}$) and \mathbf{V} ($\in \mathbb{R}^{n \times n}$) such that

$$\mathbf{A} = \mathbf{U} \mathbf{\Sigma} \mathbf{V}^{-1} \quad \mathbf{U}^{-1}\mathbf{A}\mathbf{V} = \mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \cdot & \\ & & & \sigma_N \end{bmatrix}$$

Properties of the SVD

- Suppose we know the singular values of \mathbf{A} and we know r are non zero

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq \sigma_{r+1} = \dots = \sigma_p = 0$$

- $\text{Rank}(\mathbf{A}) = r$.
- $\text{Null}(\mathbf{A}) = \text{span}\{\mathbf{v}_{r+1}, \dots, \mathbf{v}_n\}$
- $\text{Range}(\mathbf{A}) = \text{span}\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$
- $\|\mathbf{A}\|_F^2 = \sigma_1^2 + \sigma_2^2 + \dots + \sigma_p^2$ $\|\mathbf{A}\|_2 = \sigma_1$
- *Numerical rank*: If k singular values of A are larger than a given number ε . Then the ε rank of A is k .
- Distance of a matrix of rank n from being a matrix of rank $k = \sigma_{k+1}$

An Numerical Example

$$\begin{matrix} U & & \Sigma & & V^T & & A \\ \begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix} & \times & \begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix} & \times & \begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix} & = & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}
 \end{matrix}$$

- Look at how the multiplication works out, left to right:
- Column 1 of **U** gets scaled by the first value from Σ .

$$\begin{matrix} U\Sigma & & V^T & & A_{\text{partial}} \\ \begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix} & \times & \begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix} & & \begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}
 \end{matrix}$$

- The resulting vector gets scaled by row 1 of V^T to produce a contribution to the columns of **A**

An Numerical Example

$$\begin{array}{c}
 U\Sigma \\
 \begin{bmatrix} -3.67 & -0.71 & 0 \\ -8.8 & 0.30 & 0 \end{bmatrix} \times \begin{array}{c} V^T \\ \begin{bmatrix} -0.42 & -0.57 & -0.70 \\ 0.81 & 0.11 & -0.58 \\ 0.41 & -0.82 & 0.41 \end{bmatrix} \\
 A_{\text{partial}} \\
 \begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix} \\
 \\
 U\Sigma \\
 + \begin{bmatrix} -3.67 & -0.71 & 0 \\ -8.8 & 0.30 & 0 \end{bmatrix} \times \begin{array}{c} V^T \\ \begin{bmatrix} -0.42 & -0.57 & -0.70 \\ 0.81 & 0.11 & -0.58 \\ 0.41 & -0.82 & 0.41 \end{bmatrix} \\
 A_{\text{partial}} \\
 \begin{bmatrix} -0.6 & -0.1 & 0.4 \\ 0.2 & 0 & -0.2 \end{bmatrix} \\
 \\
 = \\
 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}
 \end{array}$$

- Each product of (column i of \mathbf{U}) \cdot (value i from Σ) \cdot (row i of \mathbf{V}^T) produces a

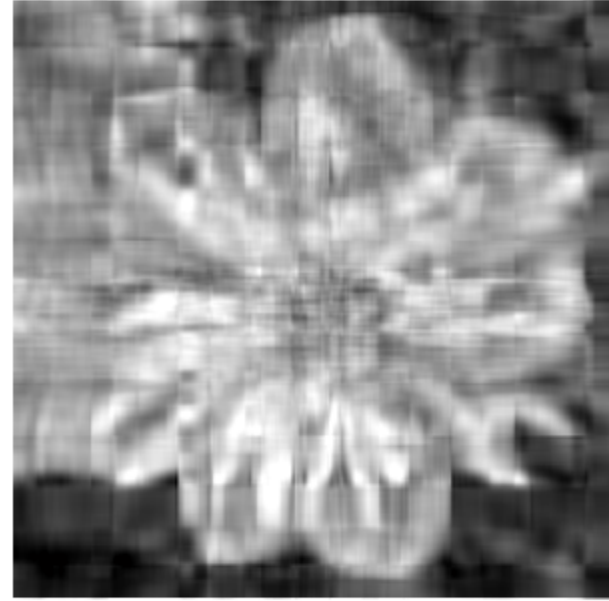
An Numerical Example

$$\begin{matrix} U & & \Sigma & & V^T & & A \\ \begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix} & \times & \begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix} & \times & \begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix} & = & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \end{matrix}$$

We can look at Σ to see that the first column has a large effect

while the second column has a much smaller effect in this example

SVD Applications



- For this image, using **only the first 10** of 300 singular values produces a recognizable reconstruction
- So, SVD can be used for image compression

Principal Component Analysis

$$\begin{matrix} U\Sigma \\ \begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix} \end{matrix} \times \begin{matrix} V^T \\ \begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix} \end{matrix} = \begin{matrix} A_{\text{partial}} \\ \begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix} \end{matrix}$$

- Remember, columns of **U** are the Principal Components of the data: the major patterns that can be added to produce the columns of the original matrix
- One use of this is to construct a matrix where each column is a separate data sample
- Run SVD on that matrix, and look at the first few columns of **U** to see patterns that are common among the columns
- This is called Principal Component Analysis (or PCA) of the data samples

Principal Component Analysis

$$\begin{matrix} U\Sigma \\ \begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix} \end{matrix} \times \begin{matrix} V^T \\ \begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix} \end{matrix} = \begin{matrix} A_{\text{partial}} \\ \begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix} \end{matrix}$$

- Often, raw data samples have a lot of redundancy and patterns
- PCA can allow you to represent data samples as weights on the principal components, rather than using the original raw form of the data
- By representing each sample as just those weights, you can represent just the “meat” of what’s different between samples.
- This minimal representation makes machine learning and other algorithms much more efficient

Why is it useful?

- Square matrix may be singular due to round-off errors.

Can compute a “regularized” solution

–
$$\mathbf{x} = \mathbf{A}^{-1} \mathbf{b} = (\mathbf{U} \Sigma \mathbf{V}^t)^{-1} \mathbf{b} = \sum_{i=1}^n \frac{\mathbf{u}_i^t \mathbf{b}}{\sigma_i} \mathbf{v}_i$$

- If σ_i is small (vanishes) the solution “blows up”
- Given a tolerance ε we can determine a solution that is “closest” to the solution of the original equation, but that does not “blow up”
$$\mathbf{x}_r = \sum_{i=1}^k \frac{\mathbf{u}_i^t \mathbf{b}}{\sigma_i} \mathbf{v}_i \quad \sigma_k > \varepsilon, \quad \sigma_{k+1} \leq \varepsilon$$

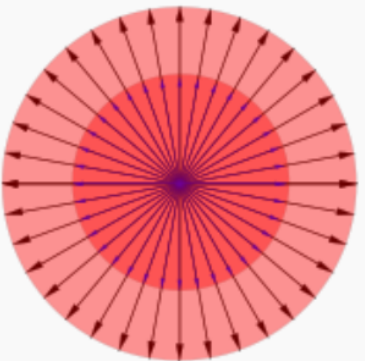
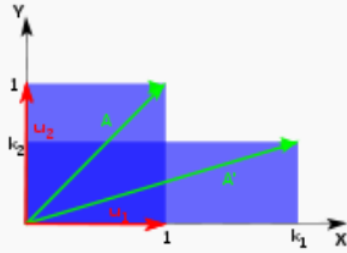
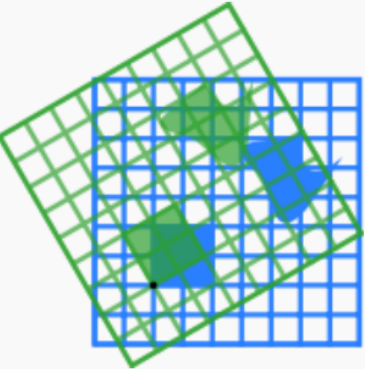
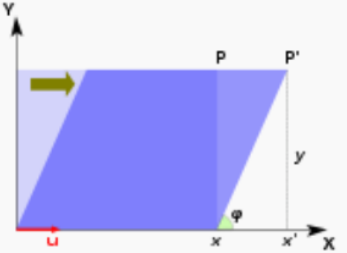
- Least squares solution is the \mathbf{x} that satisfies

$$\mathbf{A}^t \mathbf{A} \mathbf{x} = \mathbf{A}^t \mathbf{b}$$

- can be effectively solved using SVD

HW 0.1:

Compute eigenvalues and eigenvectors of the following transformations

	scaling	unequal scaling	counterclockwise rotation by φ	horizontal shear
illustration				
matrix	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$	$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$	$\begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$