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To cite this version:

Alin Bostan. Algebraicity and transcendence of power series: combinatorial and computational aspects. Doctoral. 3rd Algorithmic and Enumerative Combinatorics Summer School, Hagenberg, 2016 Austria. 2016. cel-01391907

HAL Id: cel-01391907

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Algebraicity and transcendence of power series: combinatorial and computational aspects

Alin Bostan

Inria informatics mathematics

Algorithmic and Enumerative Combinatorics
RISC, Hagenberg, August 1–5, 2016

Overview

1. Monday: Context and Examples
2. Tuesday: Properties and Criteria (1)
3. Wednesday: Properties and Criteria (2)
4. Thursday: Algorithmic Proofs of Algebraicity
5. Friday: Transcendence in Lattice Path Combinatorics

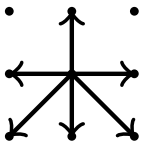
Part V: Transcendence in Lattice Path Combinatorics

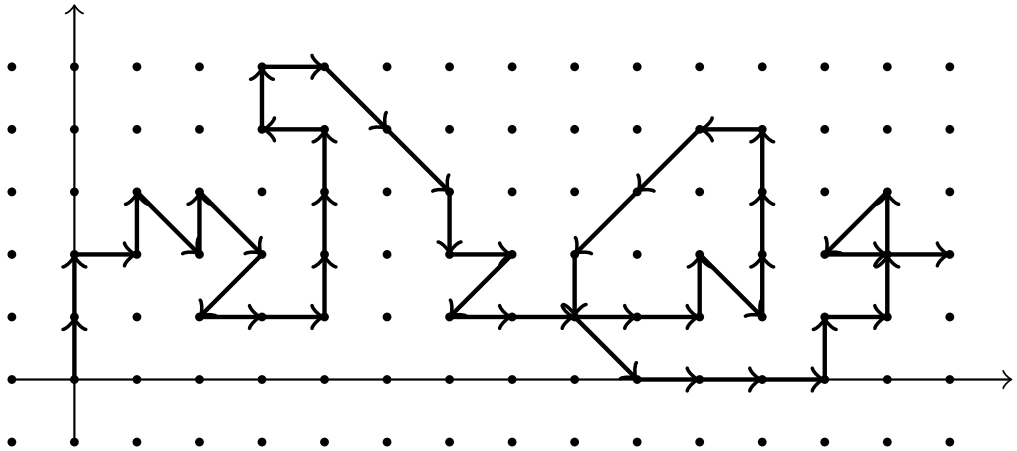
Lattice walks with small steps in the quarter plane

▷ We focus on nearest-neighbor walks in the quarter plane, i.e. walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a *fixed* subset $\mathfrak{S}$ of

$$\{\swarrow, \leftarrow, \nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}.$$

▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

$\mathfrak{S} =$ 



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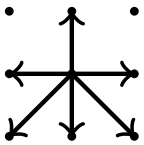
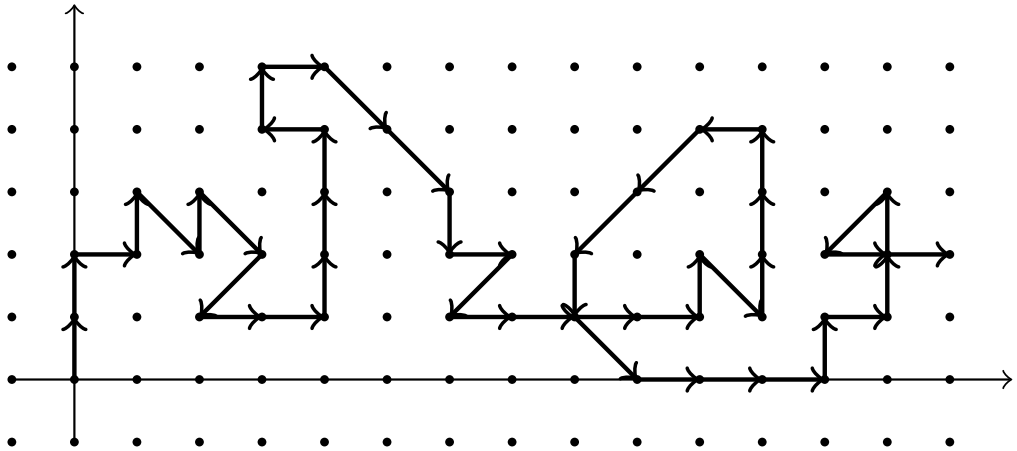
$\mathfrak{S} =$ 



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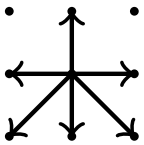
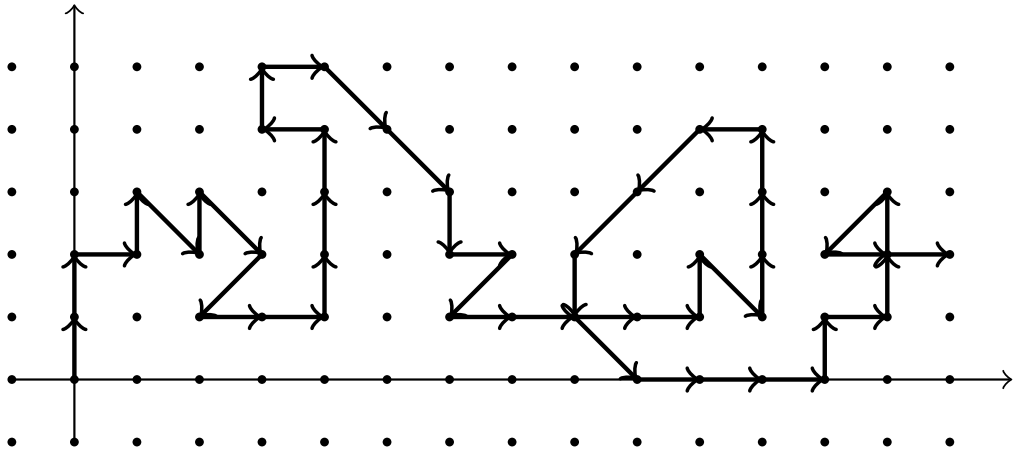
$\mathfrak{S} =$ 



▷ Counting sequence: $f_{n;i,j}$ = number of walks of length n ending at (i,j).

▷ Specializations:

- $f_{n;0,0}$ = number of walks of length n returning to origin ("excursions");
- $f_n = \sum_{i,j\geq 0} f_{n;i,j}$ = number of walks with prescribed length n.

Generating series and combinatorial problems

▷ Complete generating series:

$$F(t;x,y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n;i,j} x^i y^j \right) t^n \ \in \mathbb{Q}[x,y][[t]].$$

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Given $\mathfrak{S}$, what can be said about $F(t;x,y)$, resp. $f_{n;i,j}$, and their variants?

- Structure of F: algebraic? transcendental?
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Our goal: Use computer algebra to give computational answers.

Small-step models of interest

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1,0,1\}^2 \setminus \{(0,0)\}$, some are:

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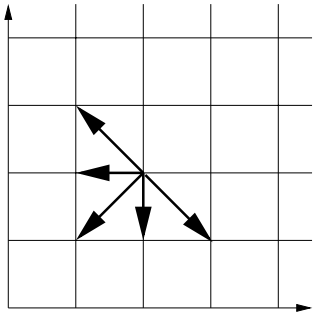

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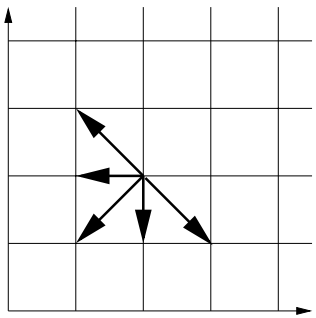

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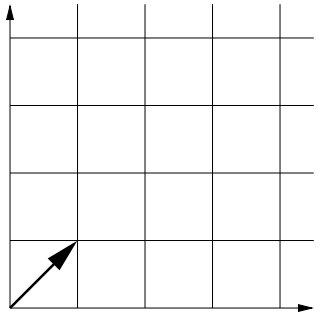

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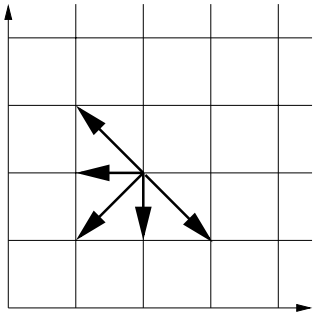

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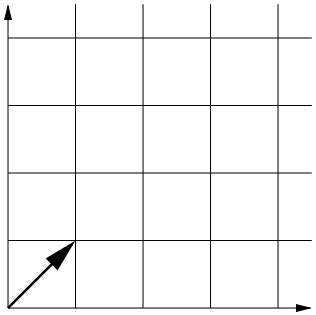

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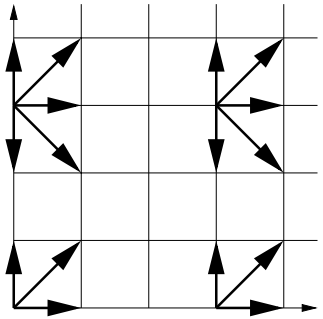

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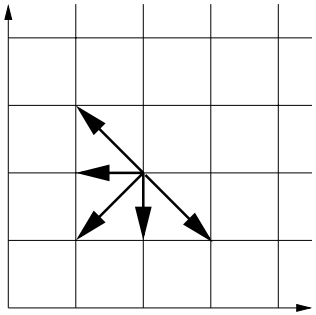

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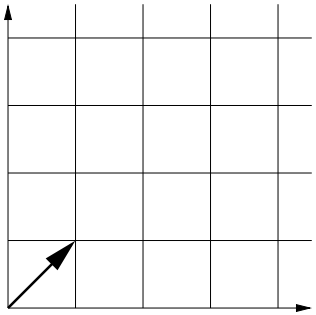

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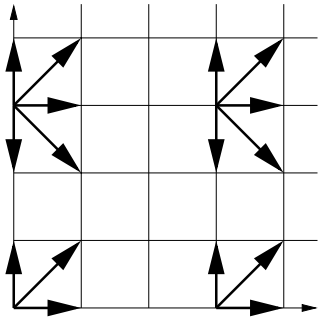

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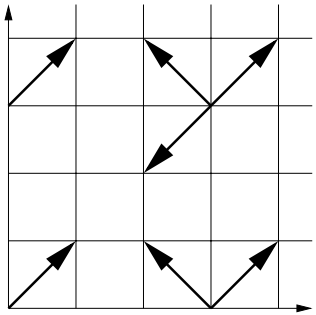

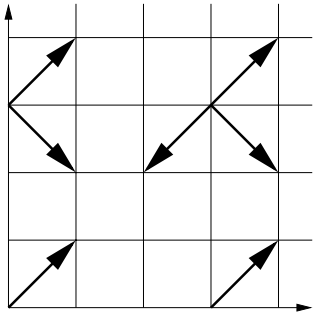

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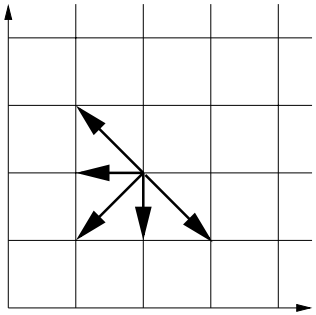

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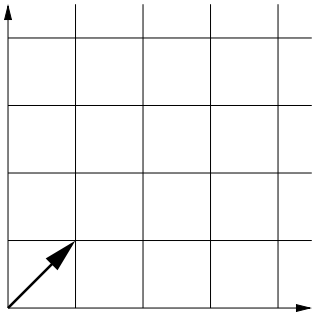

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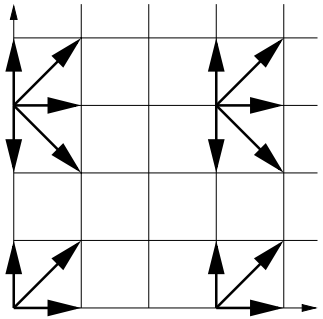

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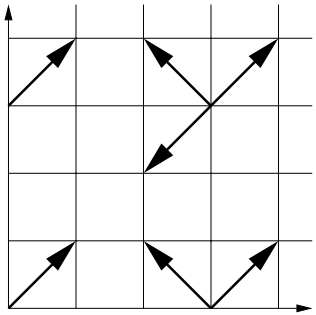

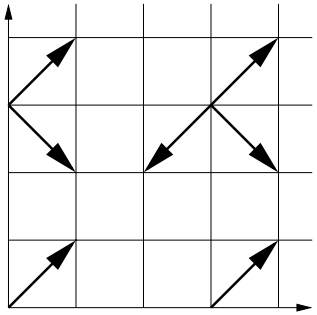

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One is left with 79 interesting distinct models.

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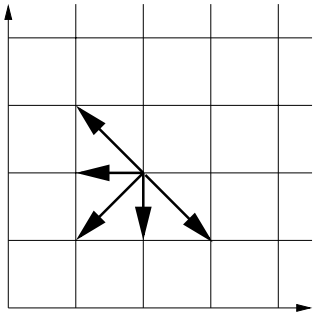

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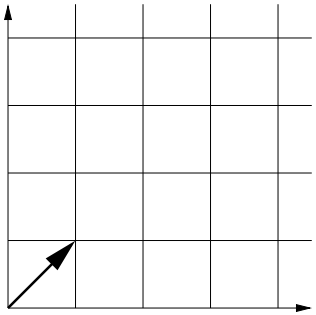

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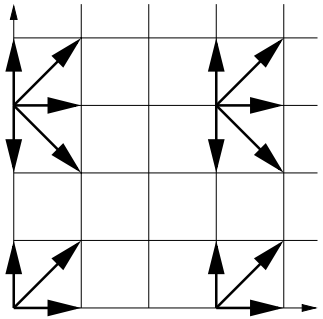

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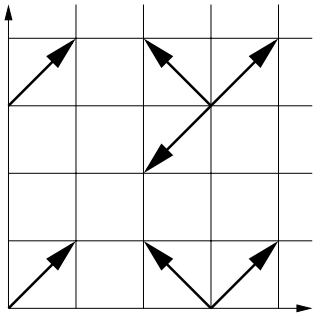

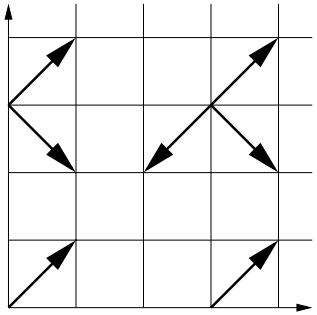

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Is any further classification possible?

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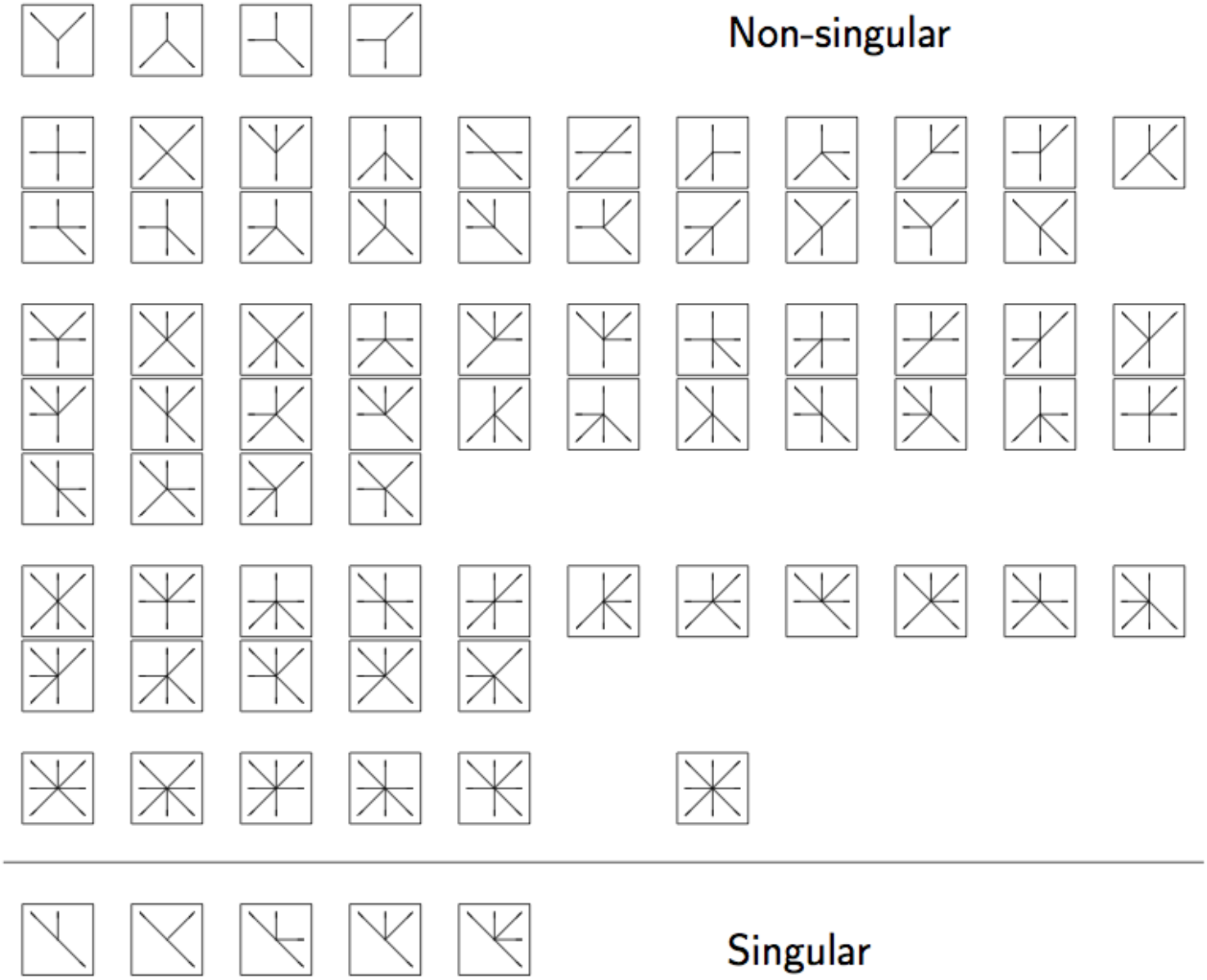

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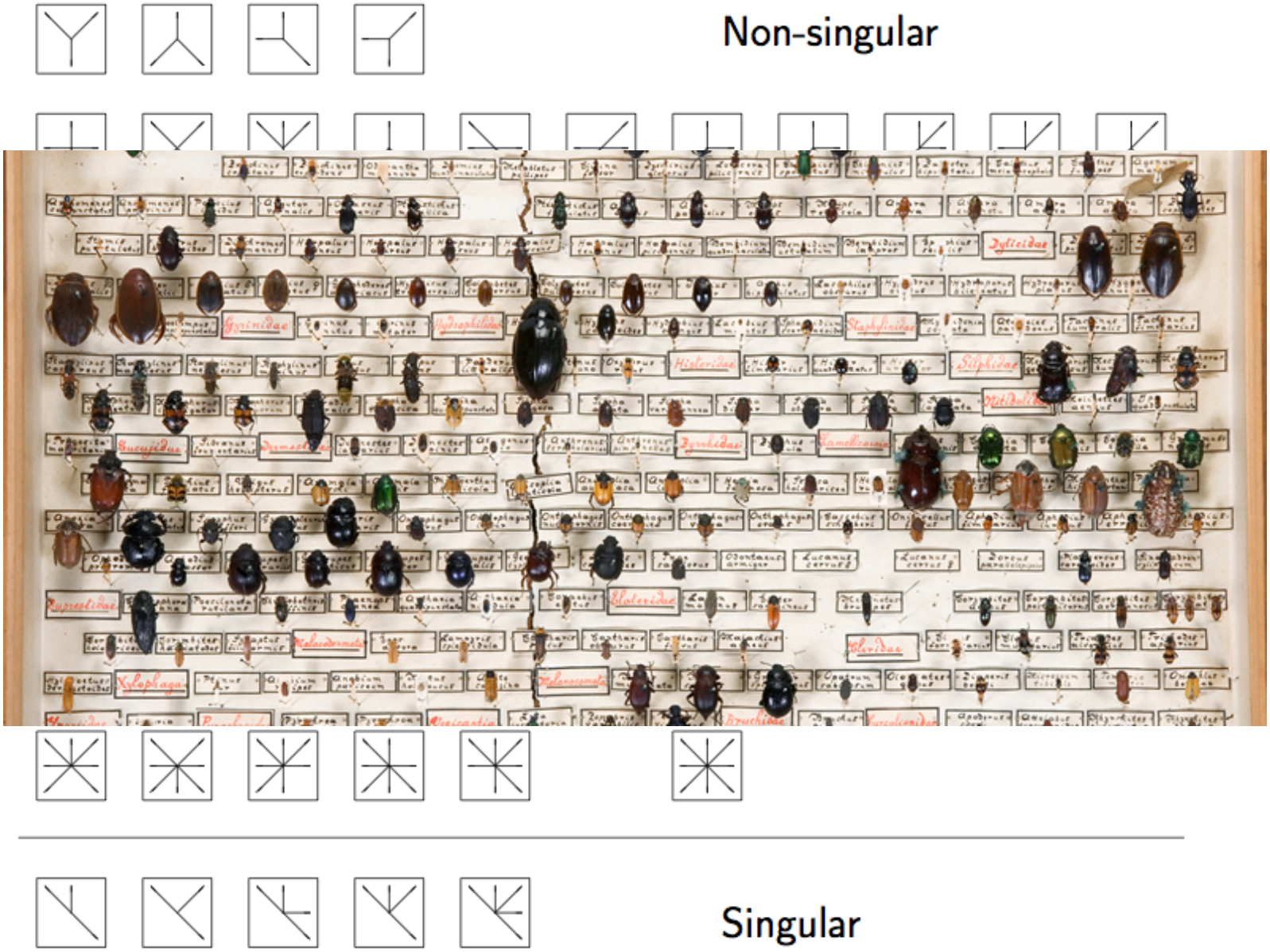
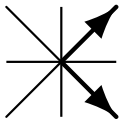
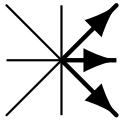


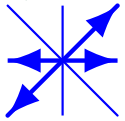
“Special” models

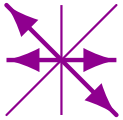
Dyck: 

Motzkin: 

Pólya:

Kreweras:

Gessel: 

Gouyou-Beauchamps: 

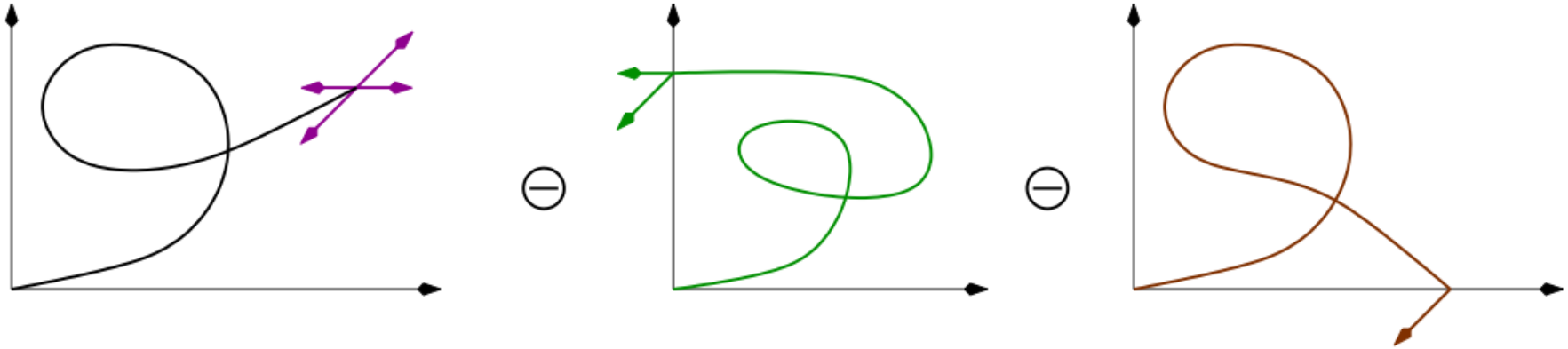
King:

Algebraic reformulation: solving a functional equation

Generating function: $G(t;x,y) = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{n} g(n;i,j)t^n x^i y^j \in \mathbb{Q}[x,y][[t]]$

"Kernel equation":

$$\begin{aligned} G(t;x,y) =& 1 + t\left(xy + x + \frac{1}{xy} + \frac{1}{x}\right)G(t;x,y) \\ & - t\left(\frac{1}{x} + \frac{1}{x}\frac{1}{y}\right)G(t;0,y) - t\frac{1}{xy}\left(G(t;x,0) - G(t;0,0)\right) \end{aligned}$$



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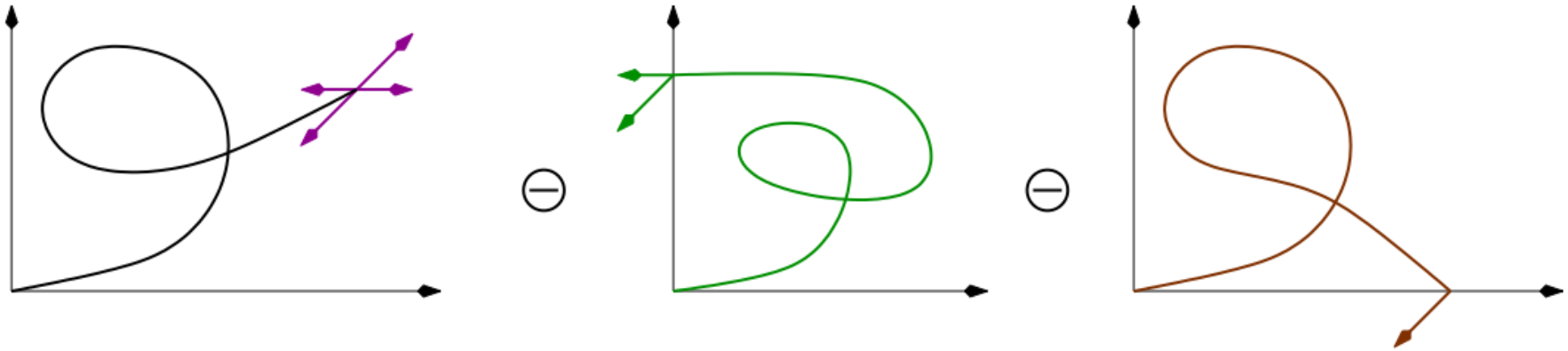

Task: Solve this functional equation!

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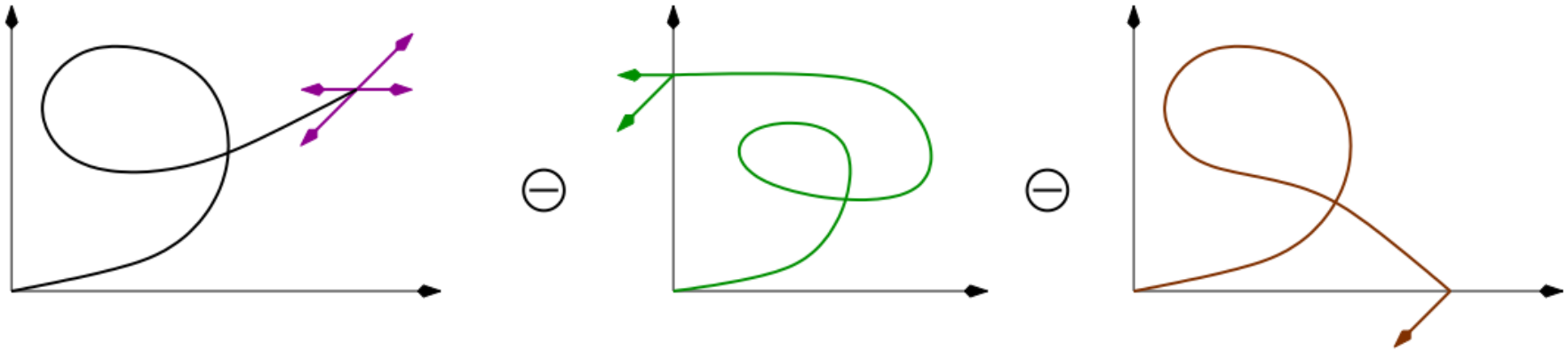

Task: For the other models: solve 78 similar equations!

Main results (I): algebraicity of Gessel walks

Theorem [Kreweras 1965; 100 pages long combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix}1/3 & 2/3 & 1\\ & 3/2 & 2\end{matrix}\middle|\, 27\,t^3\right) = \sum_{n=0}^{\infty} \frac{4^n\binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

Theorem [Kauers, Koutschan & Zeilberger 2009: former Gessel's conj. 1]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix}5/6 & 1/2 & 1\\ & 5/3 & 2\end{matrix}\middle|\, 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n(1/2)_n}{(5/3)_n(2)_n}(4t)^{2n}.$$

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Main results (II): Explicit form for $G(t;x,y)$

Theorem [B., Kauers & van Hoeij 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \cdots$ be a root of

$$(V-1)(1+3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2+83)t^6 + \cdots$ be a root of

$$\begin{aligned} x(V-1)(V+1)U^3 - 2V(3x+5xV-8Vt)U^2 \\ -xV(V^2-24V-9)U + 2V^2(xV-9x-8Vt) = 0, \end{aligned}$$

let $W = t^2 + (y+8)t^4 + 2(y^2+8y+41)t^6 + \cdots$ be a root of

$$y(1-V)W^3 + y(V+3)W^2 - (V+3)W + V - 1 = 0.$$

Then $G(t;x,y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

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- ▷ Proof uses guessed minimal polynomials for $G(t;x,0)$ and $G(t;0,y)$.

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- ▷ Recent (human) proofs [B., Kurkova, Raschel '13], [Bousquet-Mélou '15]

Main results (III): Conjectured D-Finite $F(t;1,1)$ [B. & Kauers 2009]

	OEIS	$\mathfrak{S}$	Pol size	ODE size		OEIS	$\mathfrak{S}$	Pol size	ODE size
1	A005566		—	3, 4	13	A151275		—	5, 24
2	A018224		—	3, 5	14	A151314		—	5, 24
3	A151312		—	3, 8	15	A151255		—	4, 16
4	A151331		—	3, 6	16	A151287		—	5, 19
5	A151266		—	5, 16	17	A001006		2, 2	2, 3
6	A151307		—	5, 20	18	A129400		2, 2	2, 3
7	A151291		—	5, 15	19	A005558		—	3, 5
8	A151326		—	5, 18					
9	A151302		—	5, 24	20	A151265		6, 8	4, 9
10	A151329		—	5, 24	21	A151278		6, 8	4, 12
11	A151261		—	4, 15	22	A151323		4, 4	2, 3
12	A151297		—	5, 18	23	A060900		8, 9	3, 5

Equation sizes = {order, degree}@(algeq, diffeq)

- Computerized discovery by enumeration + Hermite–Padé
- 1–22: Confirmed by human proofs in [Bousquet-Mélou & Mishna 2010]
- 23: Confirmed by a human proof in [B., Kurkova & Raschel 2015]

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	OEIS	$\mathfrak{S}$	alg	asympt		OEIS	$\mathfrak{S}$	alg	asympt
1	A005566		N	$\frac{4}{\pi}\frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi}\frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi}\frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi}\frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi}\frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi}\frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi}\frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi}\frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2}\sqrt{\frac{5}{2\pi}}\frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2}\sqrt{\frac{3}{\pi}}\frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}}\frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi}\frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}}\frac{6^n}{n^{1/2}}$					
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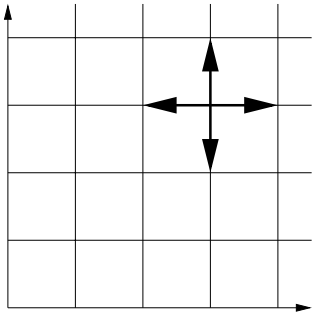
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi}\frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)}\frac{4^n}{n^{2/3}}$

$A = 1+\sqrt{2},\ B = 1+\sqrt{3},\ C = 1+\sqrt{6},\ \lambda = 7+3\sqrt{6},\ \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$

- ▷ Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.
- ▷ Confirmed by human proofs in [Melczer & Wilson, 2015]

The group of a model: the simple walk case



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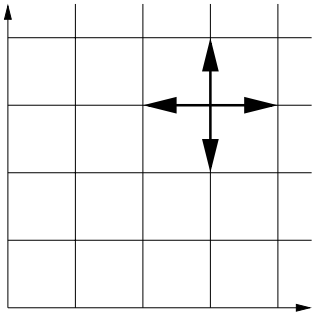

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$$\psi(x,y) = \left(x, \frac{1}{y}\right), \qquad \phi(x,y) = \left(\frac{1}{x}, y\right),$$

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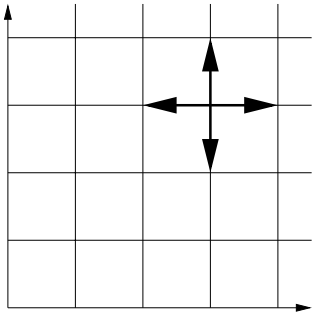

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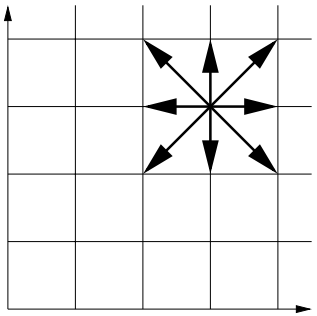
and thus under any element of the group

$$\langle \psi, \phi \rangle = \left\{ (x,y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

The group of a model: the general case



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j)\in\mathfrak{S}} x^i y^j = \sum_{i=-1}^{1} B_i(y) x^i = \sum_{j=-1}^{1} A_j(x) y^j$

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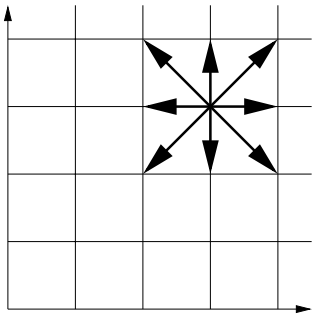

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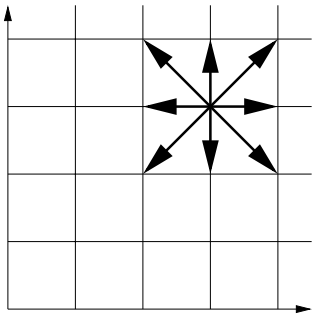

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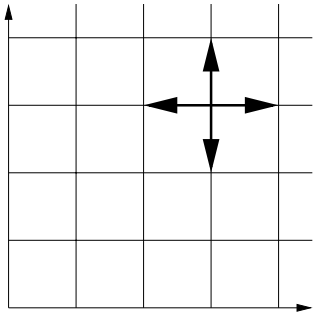
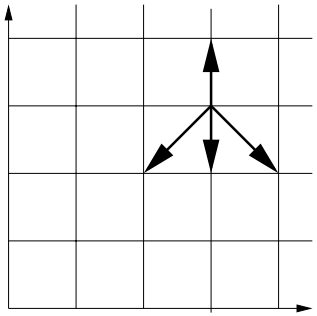
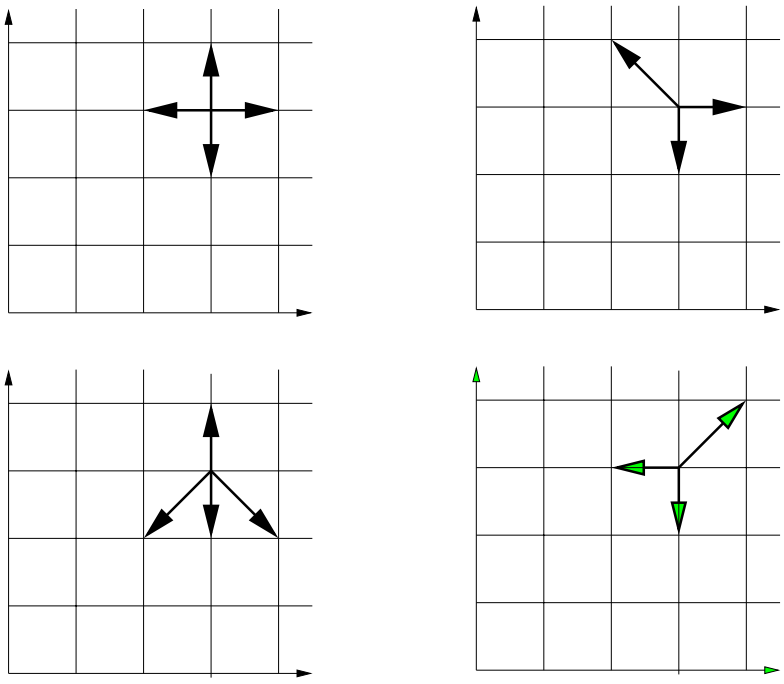
Examples of groups





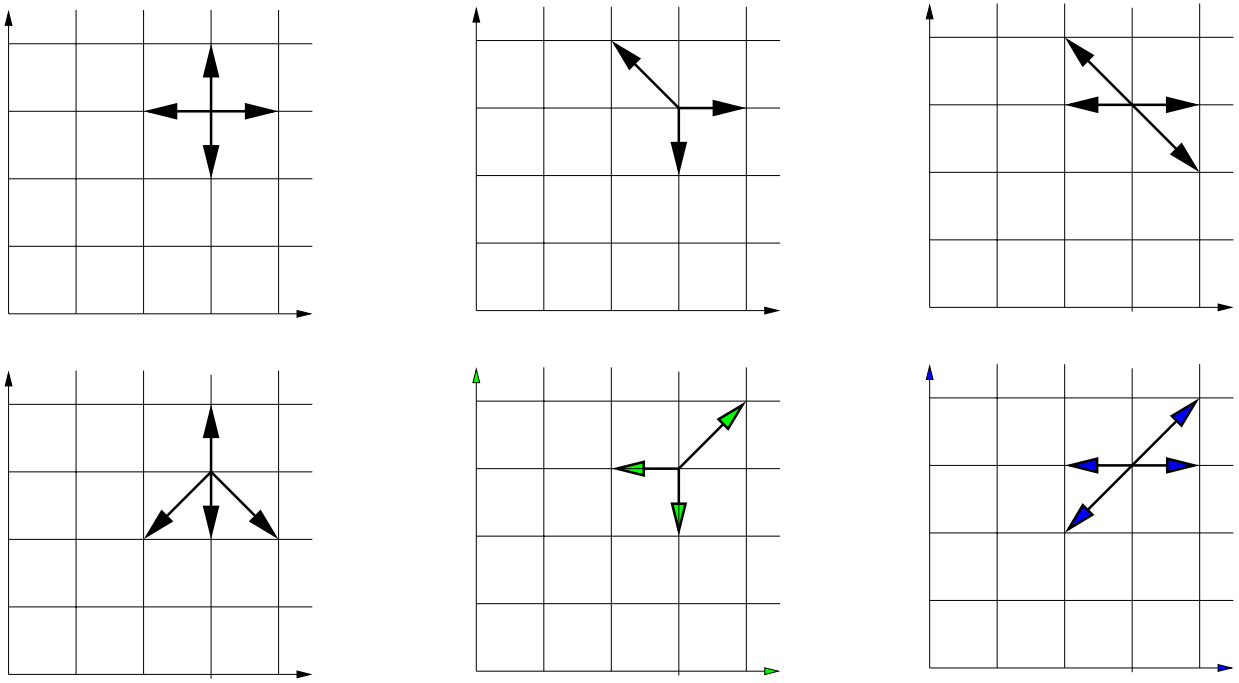
Order 4,

Examples of groups



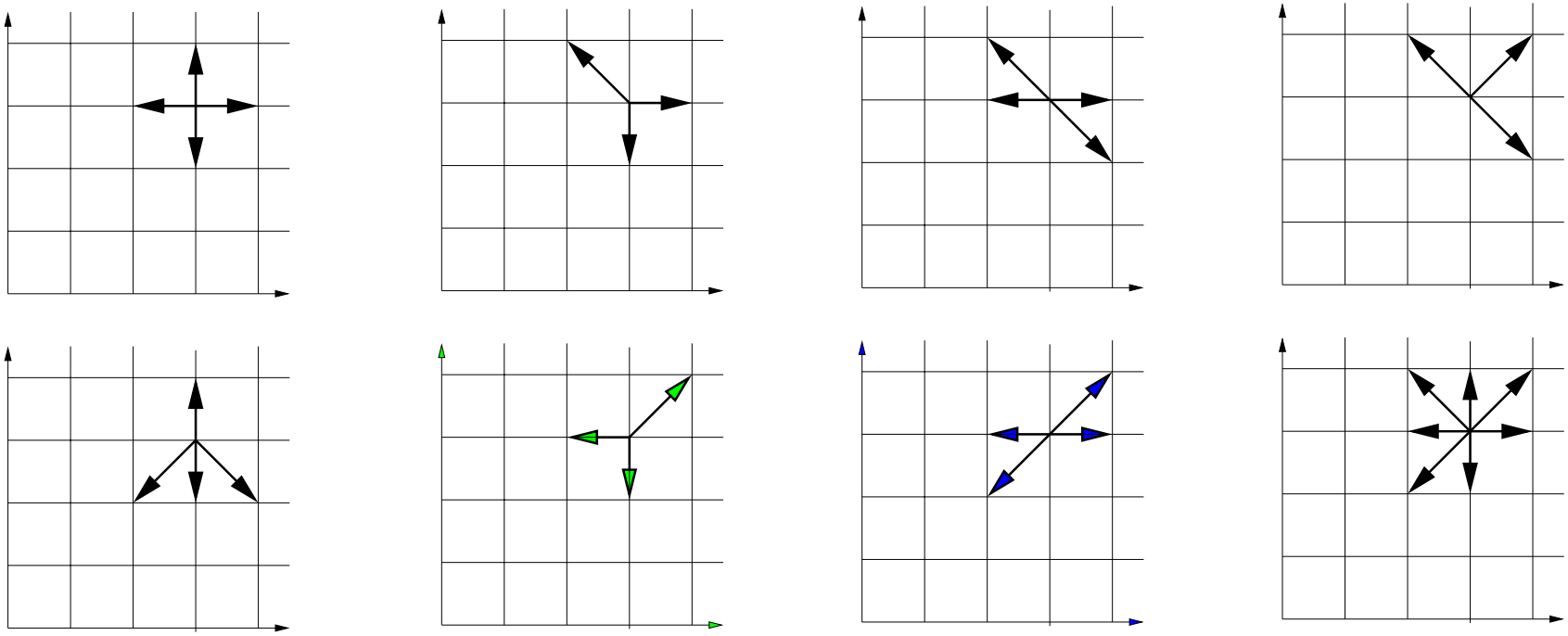
Order 4, order 6,

Examples of groups



Order 4, order 6, order 8,

Examples of groups



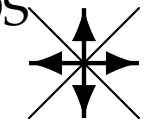
Order 4, order 6, order 8, order ∞.

An important concept: the orbit sum (OS)

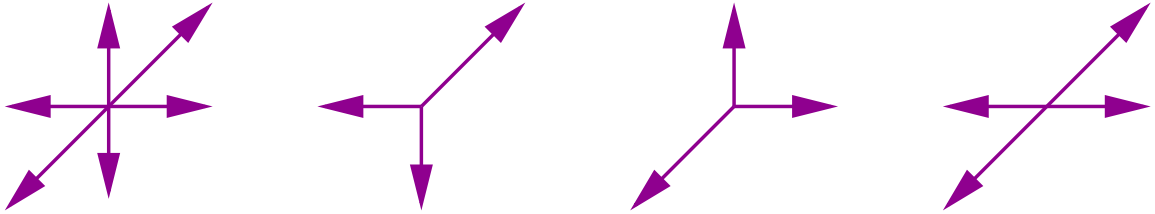
The orbit sum of a model $\mathfrak{S}$ is the following polynomial in $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$:

$$\mathrm{OrbitSum}(\mathfrak{S}) := \sum_{\theta \in \mathcal{G}_{\mathfrak{S}}} (-1)^{\theta} \theta(xy)$$

▷ E.g., for the simple walk:

OS  $= x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$

▷ For 4 models, the orbit sum is zero:



E.g. for the Kreweras model:

OS $= x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} \quad = 0$

The 79 models: finite and infinite groups

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23 admit a finite group

[Mishna'07]

56 have an infinite group

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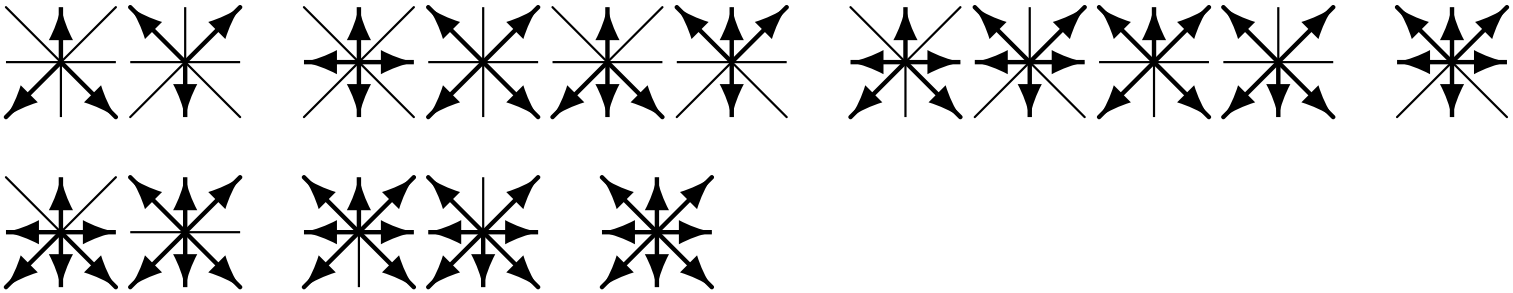
$\longrightarrow$ all non-D-finite

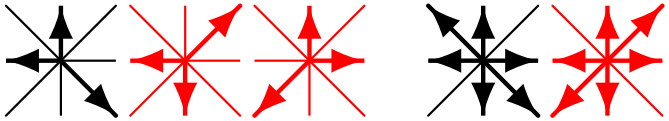
- [Mishna & Rechnitzer'07] and [Melczer & Mishna'13] for 5 singular models
- [Kurkova & Raschel'13] and [B., Raschel & Salvy'13] for all others

The 23 models with a finite group

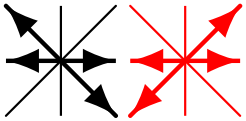
(i) 16 with a vertical symmetry, and group isomorphic to D_2



(ii) 5 with a diagonal or anti-diagonal symmetry, and group isomorphic to D_3



(iii) 2 with group isomorphic to D_4



(i): vertical symmetry; (ii)+(iii): zero drift $\sum_{s \in \mathfrak{S}} s$

In red, models with OS $= 0$ and algebraic GF

Main results (IV): explicit expressions for the 19 D-finite transcendental models

Theorem [B., Chyzak, van Hoeij, Kauers & Pech, 2016]

Let $\mathfrak{S}$ be one of the 19 models with finite group $\mathcal{G}_{\mathfrak{S}}$, and non-zero orbit sum. Then

- $F_{\mathfrak{S}}$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- Among the 19×4 specializations of $F_{\mathfrak{S}}(t; x, y)$ at $(x, y) \in \{0, 1\}^2$, only 4 are algebraic: for $\mathfrak{S} =$ at $(1, 1)$, and $\mathfrak{S} =$ at $(1, 0), (0, 1), (1, 1)$

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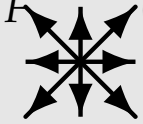
Example (King walks in the quarter plane, A025595)

F  $(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix} \frac{3}{2} & \frac{3}{2} \\ & 2 \end{matrix} \middle| \frac{16x(1+x)}{(1+4x)^2} \right) dx$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \cdots$$

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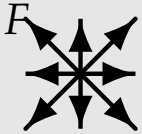
$F_{}$  $(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\begin{matrix} \frac{3}{2} & \frac{3}{2} \\ & 2 \end{matrix} \middle| \frac{16x(1+x)}{(1+4x)^2} \right) dx$

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▷ Computer-driven discovery and proof; no human proof yet.
▷ Proof uses creative telescoping, ODE factorization, ODE solving.

Hypergeometric Series Occurring in Explicit Expressions for $F(t;1,1)$

	hyp$_1$	hyp$_2$	w
1	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ & 2\end{matrix}\middle\vert w\right)$	$16t^2$
2	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\middle\vert w\right)$		$16t^2$
3	${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\middle\vert w\right)$		$\frac{16t}{(2t+1)(6t+1)}$
4	${}_2F_1\left(\begin{matrix}\frac{3}{2} & \frac{3}{2}\\ & 2\end{matrix}\middle\vert w\right)$		$\frac{16t(1+t)}{(1+4t)^2}$
5	${}_2F_1\left(\begin{matrix}\frac{3}{4} & \frac{5}{4}\\ & 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ & 2\end{matrix}\middle\vert w\right)$	$64t^4$
6	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ & 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ & 3\end{matrix}\middle\vert w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$
7	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ & 1\end{matrix}\middle\vert w\right)$	$\frac{16t^2}{4t^2+1}$
8	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ & 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ & 2\end{matrix}\middle\vert w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
9	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ & 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ & 3\end{matrix}\middle\vert w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
10	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ & 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ & 3\end{matrix}\middle\vert w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
11	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{3}{2}\\ & 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{5}{2}\\ & 3\end{matrix}\middle\vert w\right)$	$\frac{16t^2}{4t^2+1}$
12	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ & 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{5}{4} & \frac{7}{4}\\ & 2\end{matrix}\middle\vert w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
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16	${}_2F_1\left(\begin{matrix}\frac{7}{4} & \frac{9}{4}\\ & 2\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{9}{4} & \frac{11}{4}\\ & 3\end{matrix}\middle\vert w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$
19	${}_2F_1\left(\begin{matrix}-\frac{1}{2} & \frac{1}{2}\\ & 1\end{matrix}\middle\vert w\right)$	${}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{2}\\ & 2\end{matrix}\middle\vert w\right)$	$16t^2$

- All related to complete elliptic integrals!

Main results (V): non-D-finiteness in models with an infinite group

Theorem [B., Rachel & Salvy 2013]

Let $\mathfrak{S}$ be one of the 51 non-singular models with infinite group $\mathcal{G}_{\mathfrak{S}}$. Then $F_{\mathfrak{S}}(t;0,0)$, and in particular $F_{\mathfrak{S}}(t;x,y)$, are non-D-finite.

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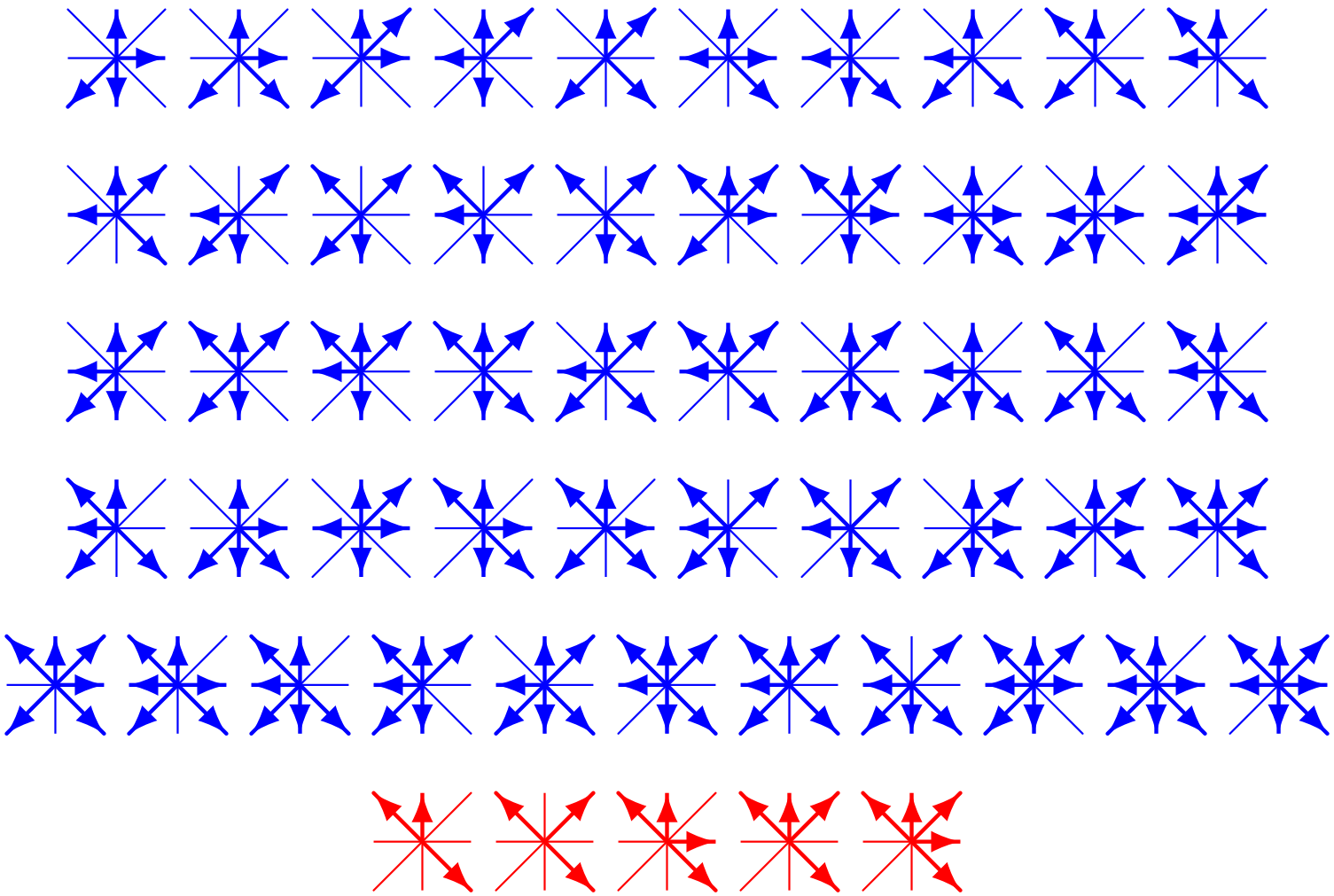
▷ Algorithmic proof. Uses Gröbner basis computations, polynomial factorization, cyclotomy testing.
▷ Based on two ingredients: asymptotics + irrationality.

▷ [Kurkova & Raschel 2013] Human proof that $F_{\mathfrak{S}}(t;x,y)$ is non-D-finite.
▷ No human proof yet for $F_{\mathfrak{S}}(t;0,0)$ non-D-finite.

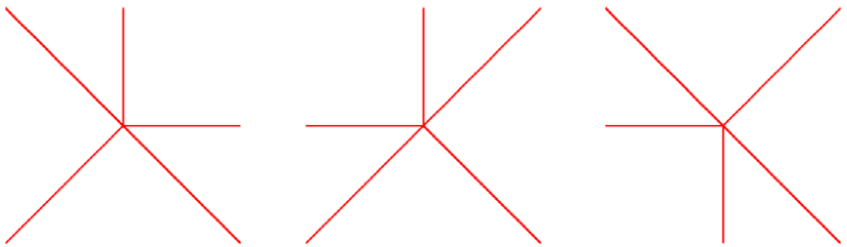
The 56 models with infinite group



In blue, non-singular models, solved by [B., Raschel & Salvy 2013]
In red, singular models, solved by [Melczer & Mishna 2013]

Example: the scarecrows

[B., Raschel & Salvy 2013]: $F_{\mathfrak{S}}(t;0,0)$ is not D-finite for the models



For the 1st and the 3rd, the excursions sequence $[t^n]\, F_{\mathfrak{S}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \ldots$$

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\ldots$

The irrationality of α prevents $F_{\mathfrak{S}}(t;0,0)$ from being D-finite.

Summary: Classification of 2D non-singular walks

The Main Theorem Let $\mathfrak{S}$ be one of the 74 non-singular models. The following assertions are equivalent:

(1) The full generating series $F_{\mathfrak{S}}(t;x,y)$ is D-finite

(2) the excursions generating series $F_{\mathfrak{S}}(t;0,0)$ is D-finite

(3) the excursions sequence $[t^n]\,F_{\mathfrak{S}}(t;0,0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$

(4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^\star \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)

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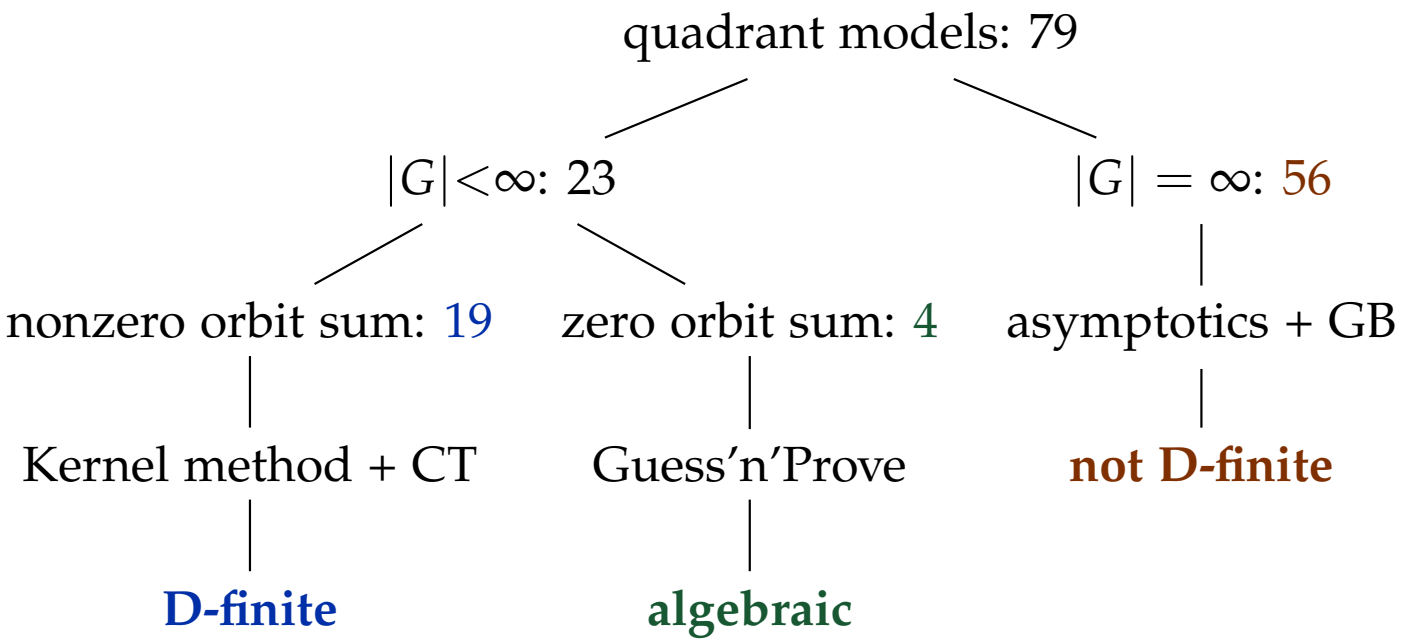
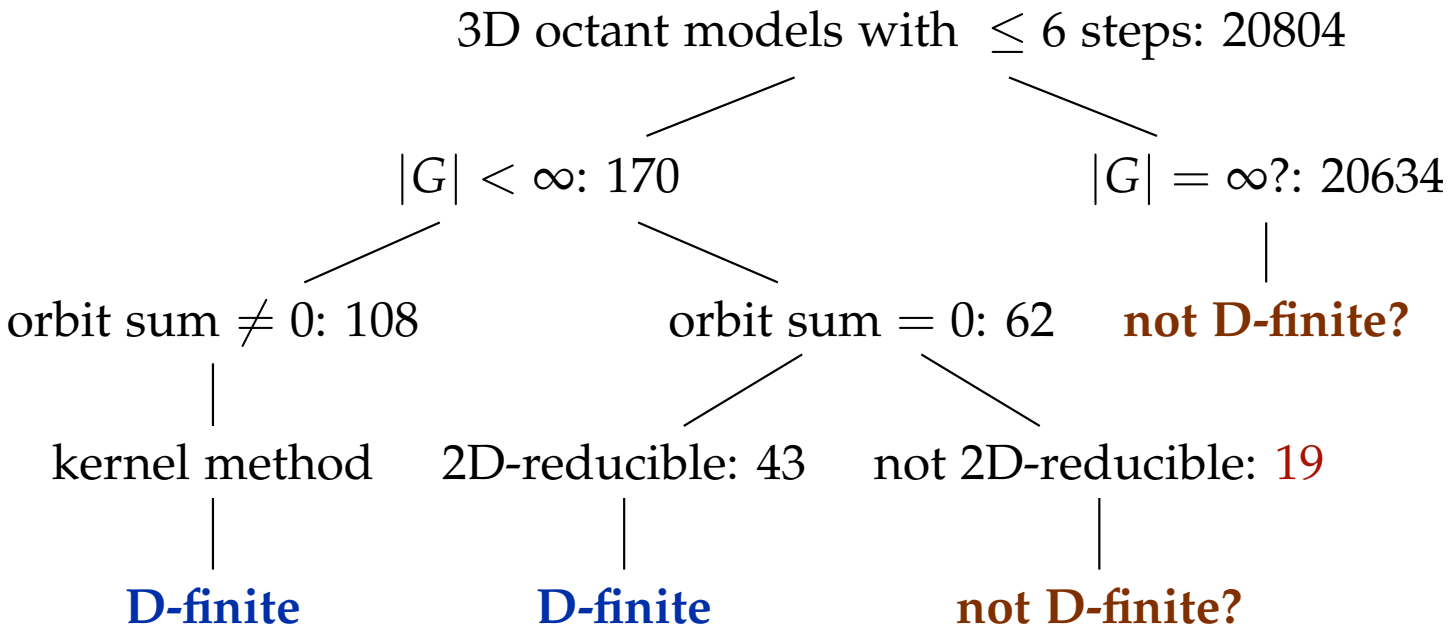
In this case, $F_{\mathfrak{S}}(t;x,y)$ is expressible using nested radicals.
If not, $F_{\mathfrak{S}}(t;x,y)$ is expressible using iterated integrals of ${}_2F_1$ expressions.

Summary: Walks with unit steps in $\mathbb{N}^2$



Extensions: Walks with unit steps in $\mathbb{N}^3$

$2^{3^3-1} \approx 67$ millions models, of which ≈ 11 million inherently 3D



[B., Bousquet-Mélou, Kauers, Melczer 2015]

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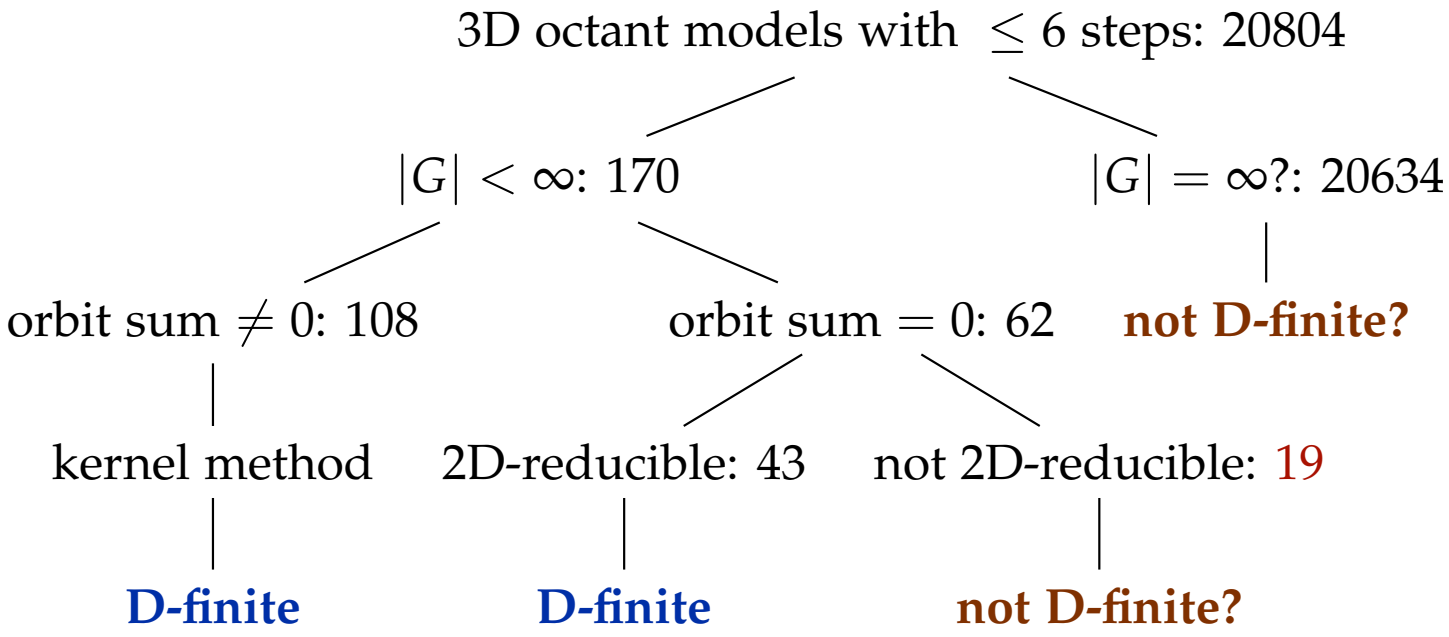

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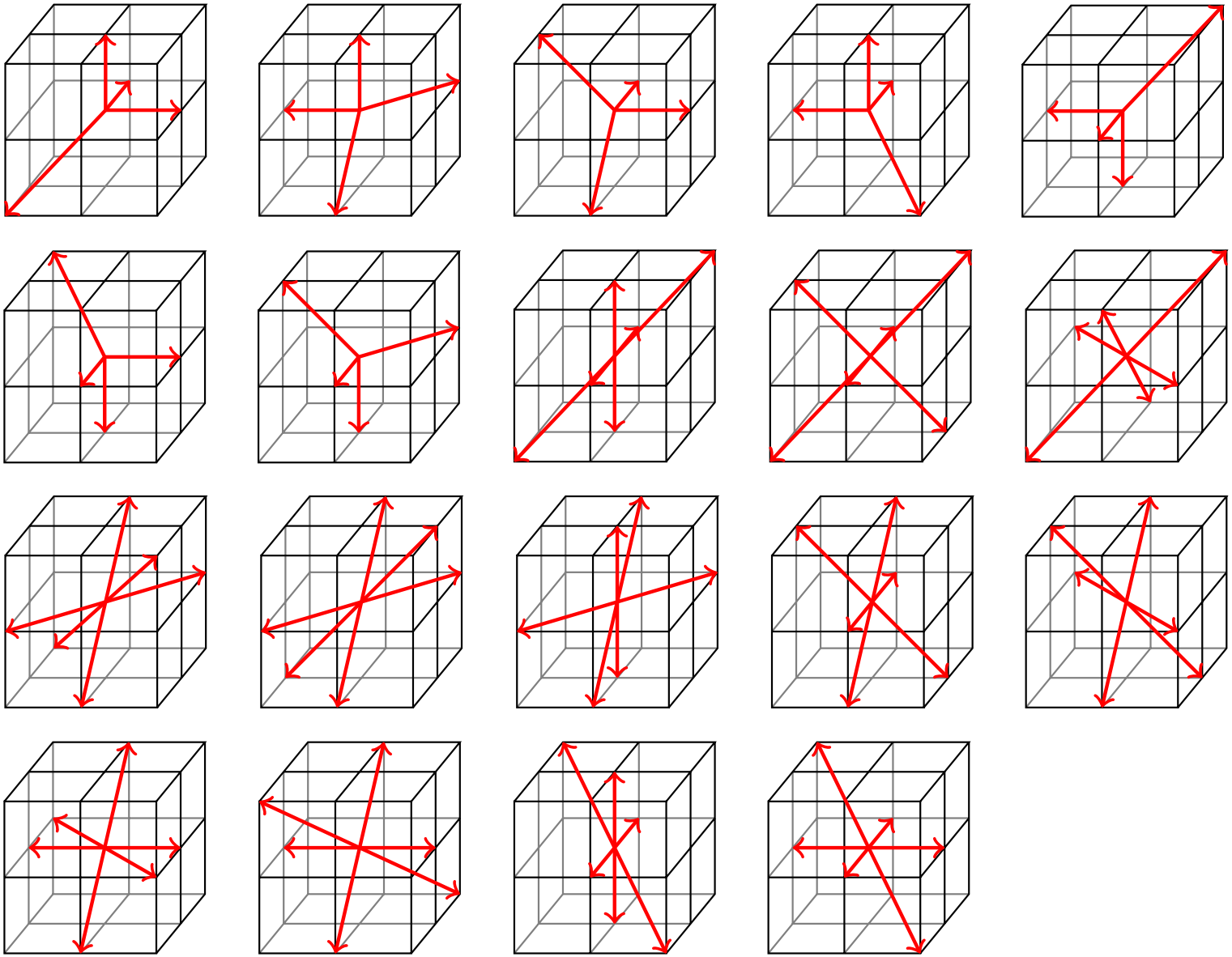
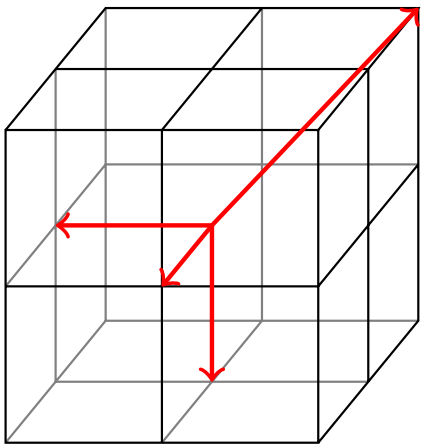
- Open question: are there non-D-finite models with a finite group?

- [Du, Hu, Wang, 2015]: proofs that groups are infinite in the 20634 cases
- [Bacher, Kauers, Yatchak, 2016]: extension to all 3D models; 170 models found with $|G| < \infty$ and orbit sum 0 (instead of 19)

The 19 mysterious 3D-models



Open question: 3D Kreweras



Two different computations suggest:

$$k_n \approx C \cdot 256^n / n^{3.3257570041744\ldots},$$

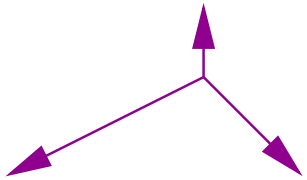
so excursions are very probably transcendental (and even non-D-finite)

Extensions: Walks in $\mathbb{N}^2$ with longer steps

- Define (and use) a group $\mathcal{G}$ for models with larger steps?

- Example: When $\mathfrak{S} = \{(0,1),(1,-1),(-2,-1)\}$, there is an underlying group that is finite and

$$xyF(t;x,y) = [x^{>0}y^{>0}]\frac{(x-2x^{-2})(y-(x-x^{-2})y^{-1})}{1-t(xy^{-1}+y+x^{-2}y^{-1})}$$



[B., Bousquet-Mélou & Melczer, in preparation]

▷ Current status:

- 680 models with one large step, 643 proved non D-finite, 32 of 37 have differential equations guessed.
- 5910 models with two large steps, 5754 proved non D-finite, 69 of 156 have differential equations guessed.

Conclusion

- Computer algebra may solve difficult combinatorial problems
- Classification of $F(t;x,y)$ fully completed for 2D small step walks
- Robust algorithmic methods, based on efficient algorithms:
  - Guess'n'Prove
  - Creative Telescoping
- Brute-force and/or use of naive algorithms = hopeless.
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Lack of "purely human" proofs for some results.

Still missing a unified proof of: finite group $\leftrightarrow$ D-finite.

Open: is $F(t;1,1)$ non-D-finite for all 56 models with infinite group?

Many open questions in dimension > 2.

Bibliography

- Automatic classification of restricted lattice walks, with M. Kauers. Proc. FPSAC, 2009.
- The complete generating function for Gessel walks is algebraic, with M. Kauers. Proc. Amer. Math. Soc., 2010.
- Explicit formula for the generating series of diagonal 3D Rook paths, with F. Chyzak, M. van Hoeij and L. Pech. Séminaire Lotharingien de Combinatoire, 2011.
- Non-D-finite excursions in the quarter plane, with K. Raschel and B. Salvy. Journal of Combinatorial Theory A, 2013.
- On 3-dimensional lattice walks confined to the positive octant, with M. Bousquet-Mélou, M. Kauers and S. Melczer. Annals of Comb., 2015.
- A human proof of Gessel's lattice path conjecture, with I. Kurkova, K. Raschel, Transactions of the AMS, 2015.
- Explicit Differentiably Finite Generating Functions of Walks with Small Steps in the Quarter Plane, with F. Chyzak, M. van Hoeij, M. Kauers and L. Pech, 2016.

End of Part V

Thanks for your attention!