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Algebraicity and transcendence of power series: combinatorial and computational aspects

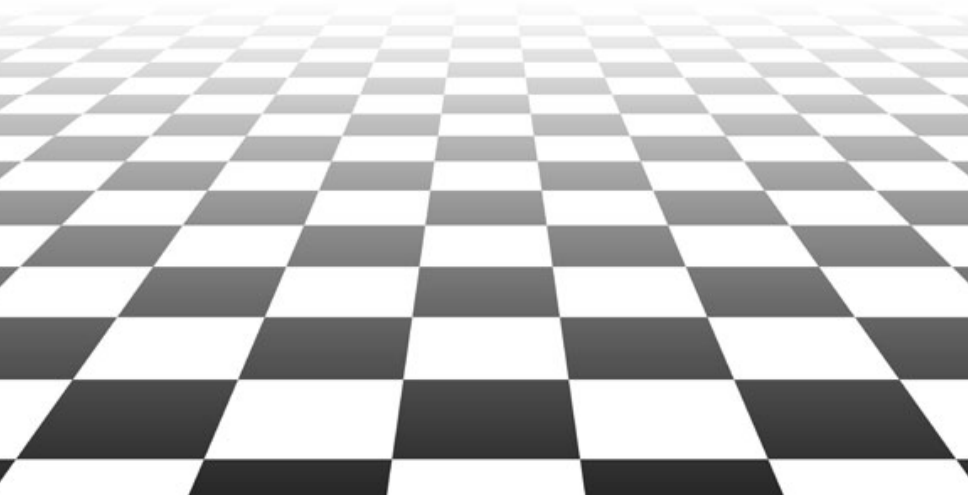
Alin Bostan



Algorithmic and Enumerative Combinatorics
RISC, Hagenberg, August 1–5, 2016

- ① Monday: Context and Examples
- ② Tuesday: Properties and Criteria (1)
- ③ Wednesday: Properties and Criteria (2)
- ④ Thursday: Algorithmic Proofs of Algebraicity
- ⑤ Friday: Transcendence in Lattice Path Combinatorics

Part V: Transcendence in Lattice Path Combinatorics

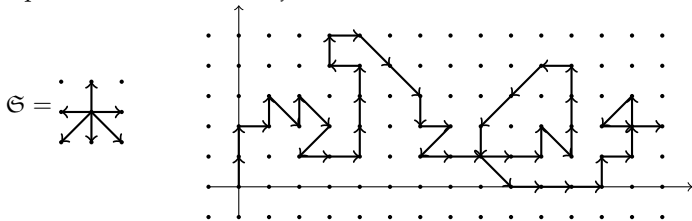


Lattice walks with small steps in the quarter plane

- ▷ We focus on **nearest-neighbor walks in the quarter plane**, i.e. walks in \mathbb{N}^2 starting at $(0,0)$ and using steps in a *fixed* subset \mathfrak{S} of

$$\{\swarrow, \leftarrow, \nearrow, \uparrow, \rightarrow, \searrow, \downarrow\}.$$

- ▷ Example with $n = 45$, $i = 14$, $j = 2$ for:

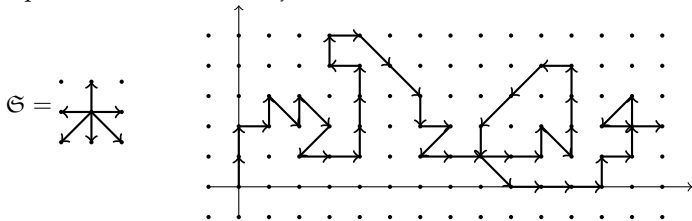


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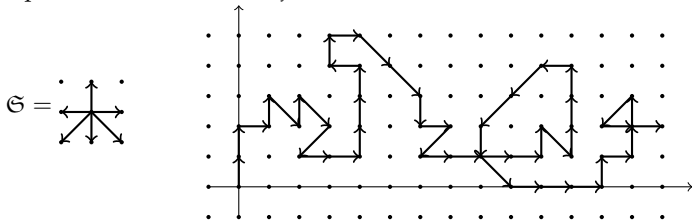
- ▷ Counting sequence: $f_{n;i,j}$ = number of **walks of length n ending at (i,j)** .

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- ▷ Example with $n = 45$, $i = 14$, $j = 2$ for:



- ▷ Counting sequence: $f_{n;i,j}$ = number of **walks of length n ending at (i,j)** .
- ▷ Specializations:
- $f_{n;0,0}$ = number of **walks of length n returning to origin** (“excursions”);
 - $f_n = \sum_{i,j \geq 0} f_{n;i,j}$ = number of **walks with prescribed length n** .

▷ Complete generating series:

$$F(t; x, y) = \sum_{n=0}^{\infty} \left(\sum_{i,j=0}^{\infty} f_{n,i,j} x^i y^j \right) t^n \in \mathbb{Q}[x, y][[t]].$$

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▷ Specializations:

- Walks returning to the origin (“excursions”):

$$F(t; 0, 0);$$

- Walks with prescribed length:

$$F(t; 1, 1) = \sum_{n \geq 0} f_n t^n;$$

- Walks ending on the horizontal axis:

$$F(t; 1, 0);$$

- Walks ending on the diagonal:

$$“F(t; 0, \infty)” := [x^0] F(t; x, 1/x).$$

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Combinatorial questions:

Given \mathfrak{S} , what can be said about $F(t; x, y)$, resp. $f_{n,i,j}$, and their variants?

- **Structure** of F : algebraic? transcendental?
- **Explicit form**: of F ? of f ?
- **Asymptotics** of f ?

Generating series and combinatorial problems

▷ Complete generating series:

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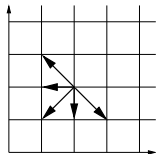
- **Structure** of F : algebraic? transcendental?
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Our goal: Use computer algebra to give computational answers.

From the 2^8 step sets $\mathfrak{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$, some are:

Small-step models of interest

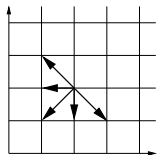
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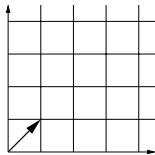
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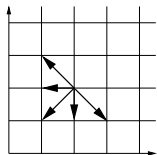
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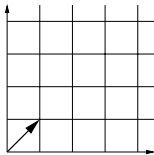
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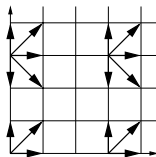
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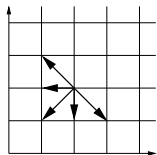
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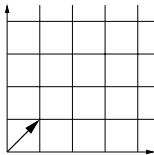
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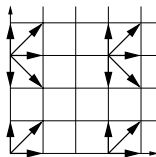
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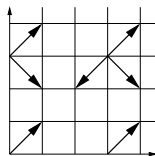
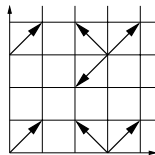
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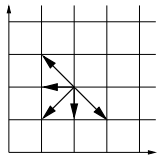
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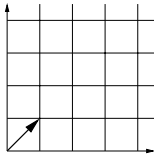
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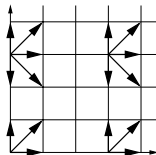
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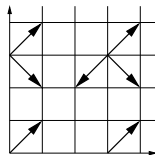
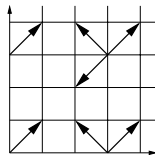
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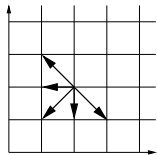


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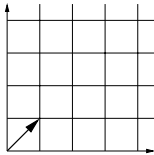
One is left with [79 interesting distinct models](#).

Small-step models of interest

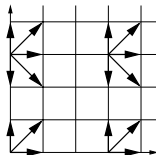
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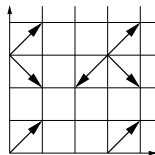
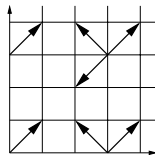
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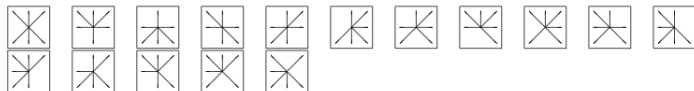
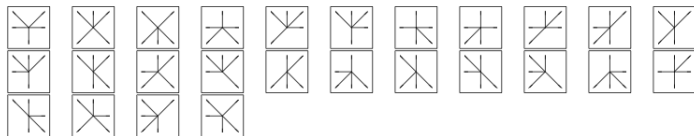
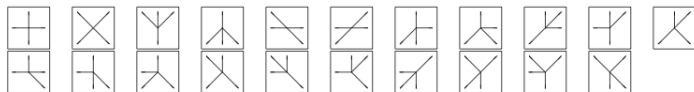
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Is any further classification possible?

The 79 models



Non-singular



Singular

The 79 models





Non-singular




Singular

“Special” models

Dyck: 

Motzkin: 

Pólya: 

Kreweras: 

Gessel: 

Gouyou-Beauchamps: 

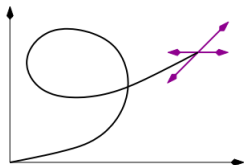
King: 

Algebraic reformulation: solving a functional equation

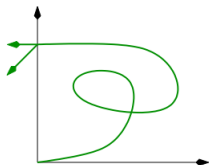
Generating function: $G(t; x, y) = \sum_{n=0}^{\infty} \sum_{i=0}^n \sum_{j=0}^n g(n; i, j) t^n x^i y^j \in \mathbb{Q}[x, y][[t]]$

“Kernel equation”:

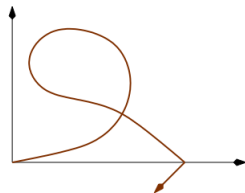
$$G(t; x, y) = 1 + t \left(xy + x + \frac{1}{xy} + \frac{1}{x} \right) G(t; x, y) \\ - t \left(\frac{1}{x} + \frac{1}{x} \frac{1}{y} \right) G(t; 0, y) - t \frac{1}{xy} (G(t; x, 0) - G(t; 0, 0))$$



⊖



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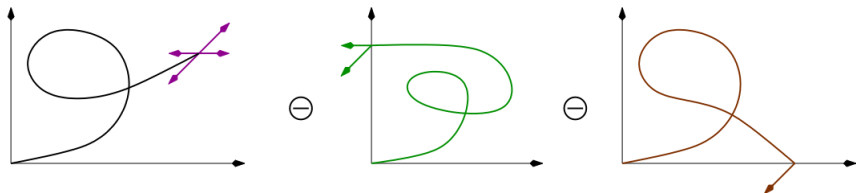


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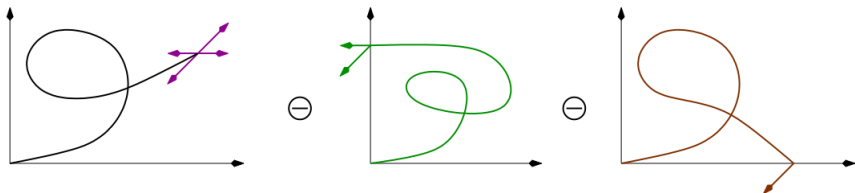
Task: Solve this functional equation!

Algebraic reformulation: solving a functional equation

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Task: For the other models: solve 78 similar equations!

Main results (I): algebraicity of Gessel walks

Theorem [Kreweras 1965; 100 pages long combinatorial proof!]

$$K(t;0,0) = {}_3F_2\left(\begin{matrix} 1/3 & 2/3 & 1 \\ 3/2 & 2 \end{matrix} \middle| 27t^3\right) = \sum_{n=0}^{\infty} \frac{4^n \binom{3n}{n}}{(n+1)(2n+1)} t^{3n}.$$

Theorem [Kauers, Koutschan & Zeilberger 2009: former Gessel's conj. 1]

$$G(t;0,0) = {}_3F_2\left(\begin{matrix} 5/6 & 1/2 & 1 \\ 5/3 & 2 \end{matrix} \middle| 16t^2\right) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (4t)^{2n}.$$

Question: What about the structure of $K(t; x, y)$ and $G(t; x, y)$?

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- ▷ Computer-driven discovery and proof.
- ▷ **Guess'n'Prove** method, using **Hermite-Padé approximants**[†] → Yesterday

[†] Minimal polynomial $P(x, y, t, G(t; x, y)) = 0$ has $> 10^{11}$ terms; ≈ 30 Gb (!)

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- ▷ **Guess'n'Prove** method, using **Hermite-Padé approximants**[†] → Yesterday
- ▷ New (human) proofs [B., Kurkova & Raschel 2013], [Bousquet-Mélou 2015]

[†] Minimal polynomial $P(x, y, t, G(t; x, y)) = 0$ has $> 10^{11}$ terms; ≈ 30 Gb (!)

Main results (II): Explicit form for $G(t; x, y)$

Theorem [B., Kauers & van Hoeij 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \dots$ be a root of

$$(V - 1)(1 + 3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \dots$ be a root of

$$\begin{aligned} & x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2 \\ & - xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0, \end{aligned}$$

let $W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \dots$ be a root of

$$y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.$$

Then $G(t; x, y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

▷ Computer-driven discovery and proof; no human proof yet.

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- ▷ Computer-driven discovery and proof; no human proof yet.
- ▷ Proof uses **guessed minimal polynomials** for $G(t; x, 0)$ and $G(t; 0, y)$.

Main results (II): Explicit form for $G(t; x, y)$

Theorem [B., Kauers & van Hoeij 2010]

Let $V = 1 + 4t^2 + 36t^4 + 396t^6 + \dots$ be a root of

$$(V - 1)(1 + 3/V)^3 = (16t)^2,$$

let $U = 1 + 2t^2 + 16t^4 + 2xt^5 + 2(x^2 + 83)t^6 + \dots$ be a root of

$$\begin{aligned} & x(V - 1)(V + 1)U^3 - 2V(3x + 5xV - 8Vt)U^2 \\ & - xV(V^2 - 24V - 9)U + 2V^2(xV - 9x - 8Vt) = 0, \end{aligned}$$

let $W = t^2 + (y + 8)t^4 + 2(y^2 + 8y + 41)t^6 + \dots$ be a root of

$$y(1 - V)W^3 + y(V + 3)W^2 - (V + 3)W + V - 1 = 0.$$

Then $G(t; x, y)$ is equal to

$$\frac{\frac{64(U(V+1)-2V)V^{3/2}}{x(U^2-V(U^2-8U+9-V))^2} - \frac{y(W-1)^4(1-Wy)V^{-3/2}}{t(y+1)(1-W)(W^2y+1)^2}}{(1+y+x^2y+x^2y^2)t - xy} - \frac{1}{tx(y+1)}.$$

- ▷ Computer-driven discovery and proof; ~~no human proof yet~~
- ▷ Recent (human) proofs [B., Kurkova, Raschel '13], [Bousquet-Mélou '15]

Main results (III): Conjectured D-Finite $F(t; 1, 1)$ [B. & Kauers 2009]

	OEIS	\mathfrak{G}	Pol size	ODE size		OEIS	\mathfrak{G}	Pol size	ODE size
1	A005566		—	3, 4	13	A151275		—	5, 24
2	A018224		—	3, 5	14	A151314		—	5, 24
3	A151312		—	3, 8	15	A151255		—	4, 16
4	A151331		—	3, 6	16	A151287		—	5, 19
5	A151266		—	5, 16	17	A001006		2, 2	2, 3
6	A151307		—	5, 20	18	A129400		2, 2	2, 3
7	A151291		—	5, 15	19	A005558		—	3, 5
8	A151326		—	5, 18					
9	A151302		—	5, 24	20	A151265		6, 8	4, 9
10	A151329		—	5, 24	21	A151278		6, 8	4, 12
11	A151261		—	4, 15	22	A151323		4, 4	2, 3
12	A151297		—	5, 18	23	A060900		8, 9	3, 5

Equation sizes = {order, degree}@{algeq, diffeq}

- ▷ Computerized discovery by enumeration + Hermite–Padé
- ▷ 1–22: Confirmed by human proofs in [Bousquet-Mélou & Mishna 2010]
- ▷ 23: Confirmed by a human proof in [B., Kurkova & Raschel 2015]

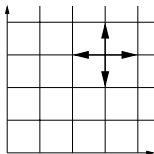
Main results (III): Conjectured D-Finite $F(t; 1, 1)$ [B. & Kauers 2009]

	OEIS	\mathfrak{G}	alg	asympt		OEIS	\mathfrak{G}	alg	asympt
1	A005566		N	$\frac{4}{\pi} \frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi} \frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi} \frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi} \frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi} \frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi} \frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}} \frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi} \frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}} \frac{6^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
9	A151302		N	$\frac{1}{3} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3} \sqrt{\frac{7}{3\pi}} \frac{7^n}{n^{1/2}}$	22	A151323		Y	$\frac{\sqrt{23}^{3/4}}{\Gamma(1/4)} \frac{6^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)} \frac{4^n}{n^{2/3}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi} \frac{(2B)^n}{n^2}$					

$$A = 1 + \sqrt{2}, \quad B = 1 + \sqrt{3}, \quad C = 1 + \sqrt{6}, \quad \lambda = 7 + 3\sqrt{6}, \quad \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$$

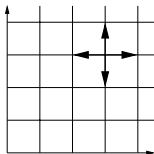
- ▷ Computerized discovery by enumeration + Hermite–Padé + LLL/PSLQ.
- ▷ Confirmed by human proofs in [Melzer & Wilson, 2015]

The group of a model: the simple walk case



The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$

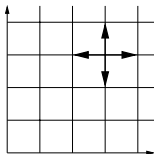
The group of a model: the simple walk case



The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \quad \phi(x, y) = \left(\frac{1}{x}, y\right),$$

The group of a model: the simple walk case



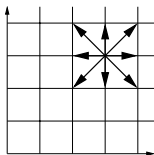
The characteristic polynomial $\chi_{\mathfrak{S}} := x + \frac{1}{x} + y + \frac{1}{y}$ is left invariant under

$$\psi(x, y) = \left(x, \frac{1}{y}\right), \quad \phi(x, y) = \left(\frac{1}{x}, y\right),$$

and thus under any element of the group

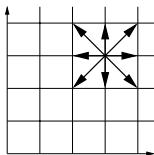
$$\langle \psi, \phi \rangle = \left\{ (x, y), \left(x, \frac{1}{y}\right), \left(\frac{1}{x}, \frac{1}{y}\right), \left(\frac{1}{x}, y\right) \right\}.$$

The group of a model: the general case



The polynomial $\chi_{\mathfrak{S}} := \sum_{(i,j) \in \mathfrak{S}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$

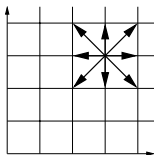
The group of a model: the general case



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$$\psi(x, y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right),$$

The group of a model: the general case



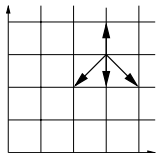
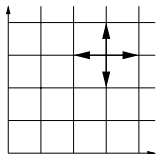
The polynomial $\chi_{\mathfrak{G}} := \sum_{(i,j) \in \mathfrak{G}} x^i y^j = \sum_{i=-1}^1 B_i(y) x^i = \sum_{j=-1}^1 A_j(x) y^j$ is left invariant under

$$\psi(x, y) = \left(x, \frac{A_{-1}(x)}{A_{+1}(x)} \frac{1}{y} \right), \quad \phi(x, y) = \left(\frac{B_{-1}(y)}{B_{+1}(y)} \frac{1}{x}, y \right),$$

and thus under any element of the group

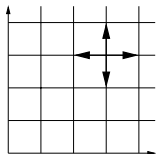
$$\mathcal{G}_{\mathfrak{G}} := \langle \psi, \phi \rangle.$$

Examples of groups

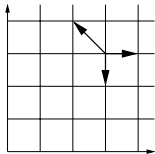


Order 4,

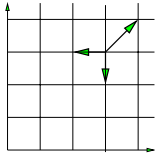
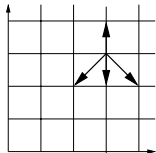
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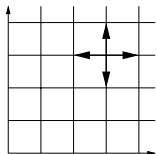
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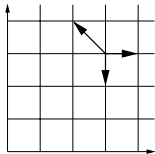
order 6,



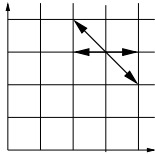
Examples of groups



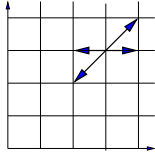
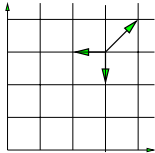
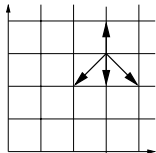
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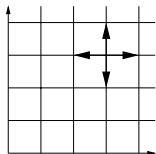
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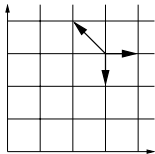
order 8,



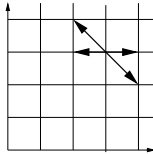
Examples of groups



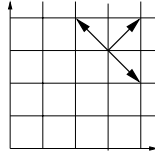
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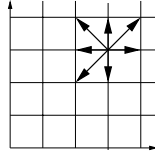
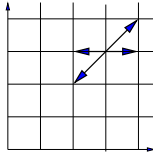
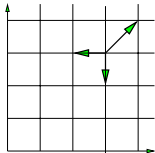
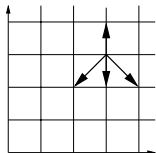
order 6,



order 8,



order ∞ .



An important concept: the orbit sum (OS)

The **orbit sum of a model** \mathfrak{G} is the following polynomial in $\mathbb{Q}[x, x^{-1}, y, y^{-1}]$:

$$\text{OrbitSum}(\mathfrak{G}) := \sum_{\theta \in \mathcal{G}_{\mathfrak{G}}} (-1)^{\theta} \theta(xy)$$

▷ E.g., for the simple walk:

$$\text{OS} \begin{array}{c} \nearrow \\ \leftarrow \\ \downarrow \\ \rightarrow \\ \nwarrow \end{array} = x \cdot y - \frac{1}{x} \cdot y + \frac{1}{x} \cdot \frac{1}{y} - x \cdot \frac{1}{y}$$

▷ For 4 models, the orbit sum is zero:

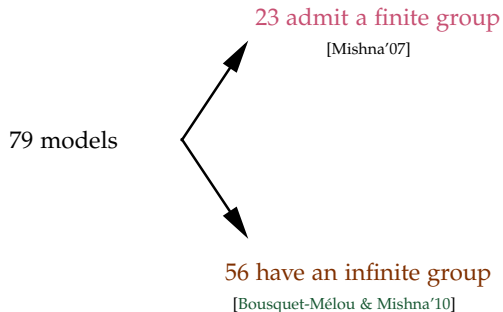


E.g. for the **Kreweras** model:

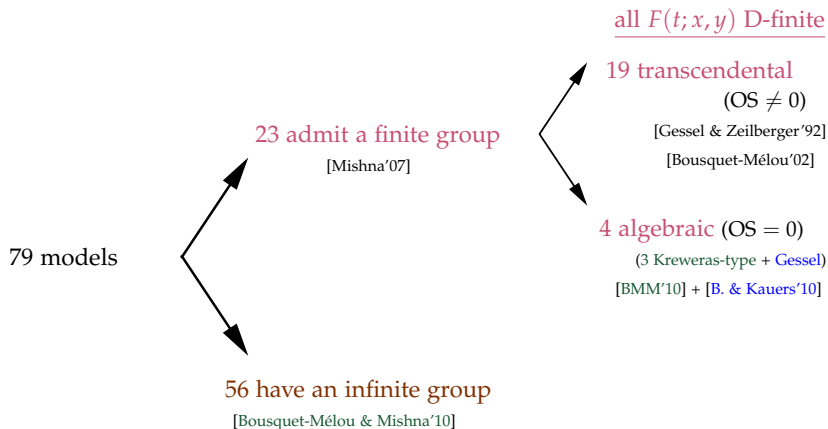
$$\text{OS} \begin{array}{c} \nearrow \\ \leftarrow \\ \downarrow \\ \rightarrow \\ \nwarrow \end{array} = x \cdot y - \frac{1}{xy} \cdot y + \frac{1}{xy} \cdot x - y \cdot x + y \cdot \frac{1}{xy} - x \cdot \frac{1}{xy} = 0$$

79 models

The 79 models: finite and infinite groups

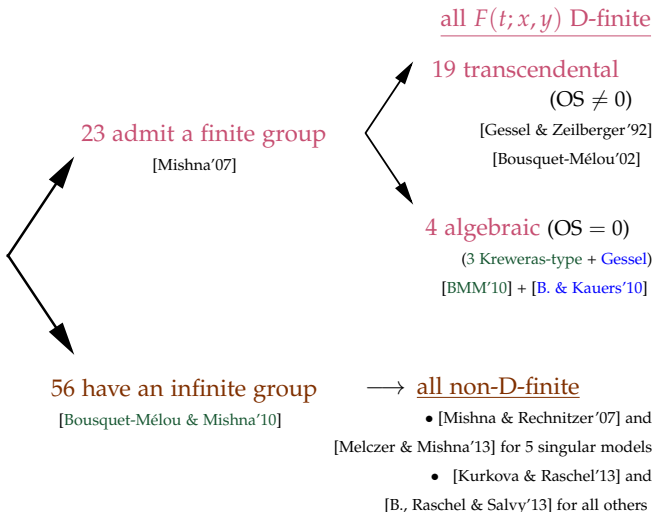


The 79 models: finite and infinite groups



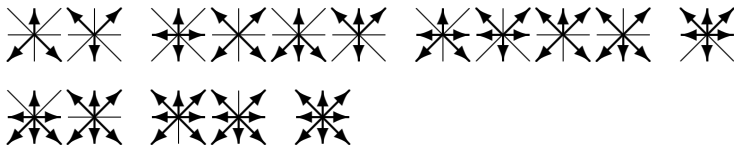
The 79 models: finite and infinite groups

79 models



The 23 models with a finite group

(i) 16 with a **vertical symmetry**, and group isomorphic to D_2



(ii) 5 with a **diagonal** or **anti-diagonal symmetry**, and group isomorphic to D_3



(iii) 2 with group isomorphic to D_4



(i): vertical symmetry; (ii)+(iii): zero drift $\sum_{s \in \mathfrak{G}} s$

In **red**, models with $OS = 0$ and **algebraic GF**

Main results (IV): explicit expressions for the 19 D-finite transcendental models

Theorem [B., Chyzak, van Hoeij, Kauers & Pech, 2016]

Let \mathfrak{S} be one of the 19 models with finite group $\mathcal{G}_{\mathfrak{S}}$, and non-zero orbit sum. Then

- $F_{\mathfrak{S}}$ is expressible using iterated integrals of ${}_2F_1$ expressions.
- Among the 19×4 specializations of $F_{\mathfrak{S}}(t; x, y)$ at $(x, y) \in \{0, 1\}^2$, only 4 are algebraic: for $\mathfrak{S} = \begin{matrix} \cdot & \cdot \\ \nearrow & \searrow \\ \cdot & \cdot \end{matrix}$ at $(1, 1)$, and $\mathfrak{S} = \begin{matrix} \cdot & \cdot \\ \nwarrow & \nearrow \\ \cdot & \cdot \end{matrix}$ at $(1, 0), (0, 1), (1, 1)$

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Example (King walks in the quarter plane, A025595)

$$F_{\begin{matrix} \nwarrow \\ \nearrow \\ \swarrow \\ \searrow \end{matrix}}(t; 1, 1) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\frac{3}{2} \quad \frac{3}{2} \mid \frac{16x(1+x)}{(1+4x)^2}\right) dx$$
$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \dots$$

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- ▷ Computer-driven discovery and proof; no human proof yet.
- ▷ Proof uses **creative telescoping**, **ODE factorization**, **ODE solving**.

Hypergeometric Series Occurring in Explicit Expressions for $F(t; 1, 1)$

	hyp ₁	hyp ₂	w		hyp ₁	hyp ₂	w
1	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{3}{2} \\ 2 \end{matrix} \middle w\right)$	$16t^2$	10	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{9}{4} & \frac{11}{4} \\ 3 \end{matrix} \middle w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
2	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle w\right)$		$16t^2$	11	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{3}{2} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{5}{2} \\ 3 \end{matrix} \middle w\right)$	$\frac{16t^2}{4t^2+1}$
3	${}_2F_1\left(\begin{matrix} \frac{3}{2} & \frac{3}{2} \\ 2 \end{matrix} \middle w\right)$		$\frac{16t}{(2t+1)(6t+1)}$	12	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{7}{4} \\ 1 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{7}{4} \\ 2 \end{matrix} \middle w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$
4	${}_2F_1\left(\begin{matrix} \frac{3}{2} & \frac{3}{2} \\ 2 \end{matrix} \middle w\right)$		$\frac{16t(1+t)}{(1+4t)^2}$	13	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 3 \end{matrix} \middle w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$
5	${}_2F_1\left(\begin{matrix} \frac{3}{4} & \frac{5}{4} \\ 1 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{7}{4} \\ 2 \end{matrix} \middle w\right)$	$64t^4$	14	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{9}{4} & \frac{11}{4} \\ 3 \end{matrix} \middle w\right)$	$\frac{64(t^2+t+1)t^2}{(12t^2+1)^2}$
6	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 3 \end{matrix} \middle w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$	15	${}_2F_1\left(\begin{matrix} \frac{1}{4} & \frac{3}{4} \\ 1 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{3}{4} & \frac{5}{4} \\ 2 \end{matrix} \middle w\right)$	$64t^4$
7	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 1 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{1}{2} & \frac{3}{2} \\ 1 \end{matrix} \middle w\right)$	$\frac{16t^2}{4t^2+1}$	16	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{9}{4} & \frac{11}{4} \\ 3 \end{matrix} \middle w\right)$	$\frac{64t^3(1+t)}{(1-4t^2)^2}$
8	${}_2F_1\left(\begin{matrix} \frac{5}{4} & \frac{7}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	$\frac{64t^3(2t+1)}{(8t^2-1)^2}$				
9	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 2 \end{matrix} \middle w\right)$	${}_2F_1\left(\begin{matrix} \frac{7}{4} & \frac{9}{4} \\ 3 \end{matrix} \middle w\right)$	$\frac{64t^2(t^2+1)}{(16t^2+1)^2}$	19	${}_2F_1\left(-\frac{1}{2} \middle \frac{1}{2} \middle w\right)$	${}_2F_1\left(\frac{1}{2} \middle \frac{1}{2} \middle w\right)$	$16t^2$

▷ All related to complete elliptic integrals!

Theorem [B., Rachel & Salvy 2013]

Let \mathfrak{G} be one of the 51 non-singular models with infinite group $\mathcal{G}_{\mathfrak{G}}$.
Then $F_{\mathfrak{G}}(t; 0, 0)$, and in particular $F_{\mathfrak{G}}(t; x, y)$, are non-D-finite.

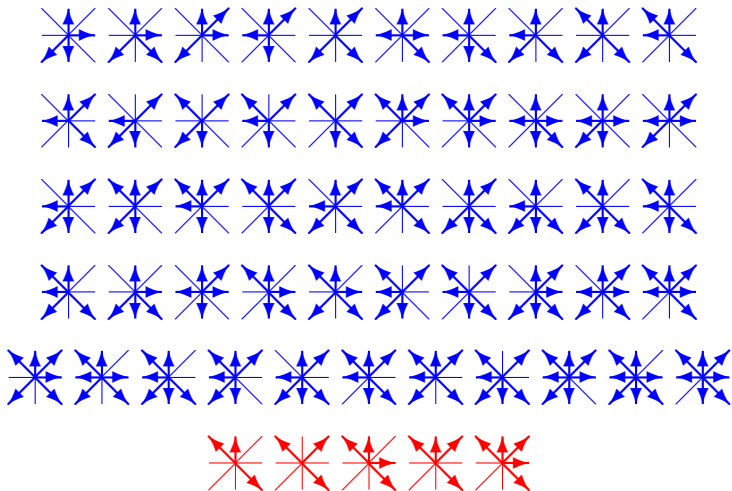
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- ▷ **Algorithmic proof.** Uses **Gröbner basis computations, polynomial factorization, cyclotomy testing.**
- ▷ Based on **two ingredients: asymptotics + irrationality.**

- ▷ [Kurkova & Raschel 2013] Human proof that $F_{\mathfrak{G}}(t; x, y)$ is non-D-finite.
- ▷ No human proof yet for $F_{\mathfrak{G}}(t; 0, 0)$ **non-D-finite.**

The 56 models with infinite group



In **blue**, non-singular models, solved by [B., Raschel & Salvy 2013]
In **red**, singular models, solved by [Melczer & Mishna 2013]

Example: the scarecrows

[B., Raschel & Salvy 2013]: $F_{\mathfrak{G}}(t;0,0)$ is not D-finite for the models



For the 1st and the 3rd, the excursions sequence $[t^n] F_{\mathfrak{G}}(t;0,0)$

$$1, 0, 0, 2, 4, 8, 28, 108, 372, \dots$$

is $\sim K \cdot 5^n \cdot n^{-\alpha}$, with $\alpha = 1 + \pi / \arccos(1/4) = 3.383396\dots$

The **irrationality** of α prevents $F_{\mathfrak{G}}(t;0,0)$ from being D-finite.

The Main Theorem Let \mathfrak{S} be one of the 74 non-singular models. The following assertions are equivalent:

- (1) The full generating series $F_{\mathfrak{S}}(t; x, y)$ is D-finite
- (2) the excursions generating series $F_{\mathfrak{S}}(t; 0, 0)$ is D-finite
- (3) the excursions sequence $[t^n] F_{\mathfrak{S}}(t; 0, 0)$ is $\sim K \cdot \rho^n \cdot n^\alpha$, with $\alpha \in \mathbb{Q}$
- (4) the group $\mathcal{G}_{\mathfrak{S}}$ is finite (and $|\mathcal{G}_{\mathfrak{S}}| = 2 \cdot \min\{\ell \in \mathbb{N}^* \mid \frac{\ell}{\alpha+1} \in \mathbb{Z}\}$)
- (5) the step set \mathfrak{S} has either an axial symmetry, or zero drift and cardinal different from 5.

Summary: Classification of 2D non-singular walks

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Moreover, under (1)–(5), $F_{\mathfrak{S}}(t; x, y)$ is **algebraic** if and only if the model \mathfrak{S} has **positive covariance** $\sum_{(i,j) \in \mathfrak{S}} ij - \sum_{(i,j) \in \mathfrak{S}} i \cdot \sum_{(i,j) \in \mathfrak{S}} j > 0$, and iff it has **OS = 0**.

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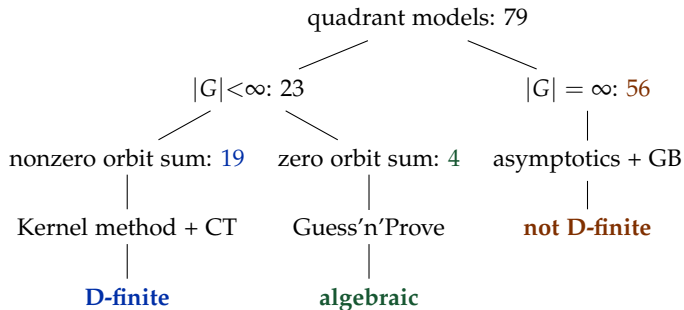
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In this case, $F_{\mathfrak{S}}(t; x, y)$ is expressible using **nested radicals**.

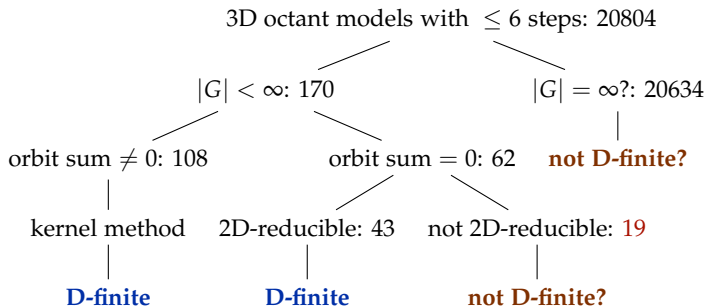
If not, $F_{\mathfrak{S}}(t; x, y)$ is expressible using **iterated integrals of ${}_2F_1$ expressions**.

Summary: Walks with unit steps in \mathbb{N}^2



Extensions: Walks with unit steps in \mathbb{N}^3

$2^{3^3-1} \approx 67$ millions models, of which ≈ 11 million inherently 3D

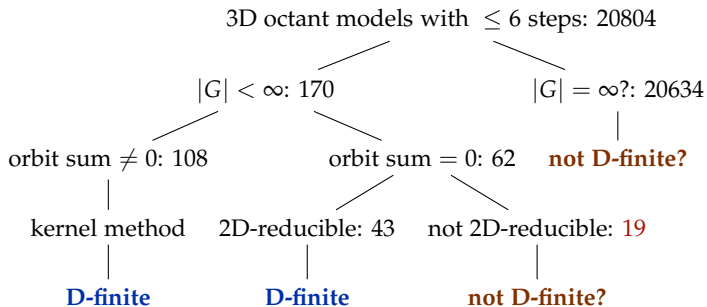


[B., Bousquet-Mélou, Kauers, Melczer 2015]

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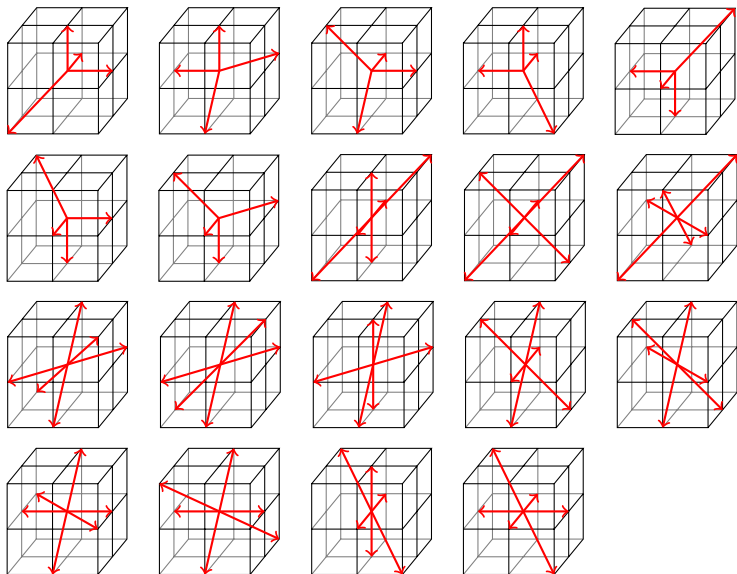
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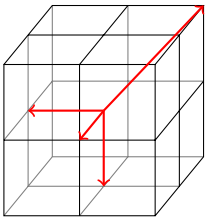


[B., Bousquet-Mélou, Kauers, Melczer 2015]

- ▷ Open question: **are there non-D-finite models with a finite group?**
- ▷ [Du, Hu, Wang, 2015]: proofs that groups are infinite in the 20634 cases
- ▷ [Bacher, Kauers, Yatchak, 2016]: extension to all 3D models; **170** models found with $|G| < \infty$ and orbit sum 0 (instead of **19**)

The 19 mysterious 3D-models





Two different computations suggest:

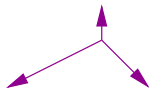
$$k_n \approx C \cdot 256^n / n^{3.3257570041744\dots},$$

so excursions are very probably transcendental
(and even non-D-finite)

Extensions: Walks in \mathbb{N}^2 with longer steps

- Define (and use) a group \mathcal{G} for models with larger steps?
- **Example:** When $\mathfrak{S} = \{(0,1), (1,-1), (-2,-1)\}$, there is an underlying group that is finite and

$$xyF(t; x, y) = [x^{>0}y^{>0}] \frac{(x - 2x^{-2})(y - (x - x^{-2})y^{-1})}{1 - t(xy^{-1} + y + x^{-2}y^{-1})}$$



[B., Bousquet-Mélou & Melczer, in preparation]

- ▷ Current status:
- 680 models with one large step, 643 **proved non D-finite**, 32 of 37 have differential equations **guessed**.
 - 5910 models with two large steps, 5754 **proved non D-finite**, 69 of 156 have differential equations **guessed**.

Conclusion



Computer algebra may solve difficult combinatorial problems



Classification of $F(t; x, y)$ **fully completed** for 2D small step walks



Robust algorithmic methods, based on efficient algorithms:

- **Guess'n'Prove**
- **Creative Telescoping**



Brute-force and/or use of naive algorithms = **hopeless**.

E.g. size of algebraic equations for $G(t; x, y) \approx 30\text{Gb}$.

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Lack of “purely human” proofs for some results.



Still missing a unified proof of: **finite group** \leftrightarrow **D-finite**.



Open: is $F(t; 1, 1)$ **non-D-finite** for all 56 models with infinite group?



Many open questions in dimension > 2 .

- Automatic classification of restricted lattice walks, with M. Kauers. Proc. FPSAC, 2009.
- The complete generating function for Gessel walks is algebraic, with M. Kauers. Proc. Amer. Math. Soc., 2010.
- Explicit formula for the generating series of diagonal 3D Rook paths, with F. Chyzak, M. van Hoeij and L. Pech. Séminaire Lotharingien de Combinatoire, 2011.
- Non-D-finite excursions in the quarter plane, with K. Raschel and B. Salvy. Journal of Combinatorial Theory A, 2013.
- On 3-dimensional lattice walks confined to the positive octant, with M. Bousquet-Mélou, M. Kauers and S. Melczer. Annals of Comb., 2015.
- A human proof of Gessel's lattice path conjecture, with I. Kurkova, K. Raschel, Transactions of the AMS, 2015.
- Explicit Differentiably Finite Generating Functions of Walks with Small Steps in the Quarter Plane, with F. Chyzak, M. van Hoeij, M. Kauers and L. Pech, 2016.

Thanks for your attention!